Completely Positive Matrices Over Semirings and Their CP-rank

by

Preeti Mohindru

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ABSTRACT

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Preeti Mohindru
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Advisor:
Dr. Rajesh Pereira

An $n \times n$ real matrix A is called completely positive if it can be written as $A = BB^T$, where B is an $n \times m$ real nonnegative matrix for some positive integer m. The smallest such m is called the CP-rank of A. In 1994, Drew, Johnson and Loewy conjectured that the CP-rank of $n \times n$ real completely positive matrices of order $n \geq 4$ is bounded above by $[n^2/4]$. There was some evidence in support of this conjecture. However, Bomze and his co-workers (2014) disproved this conjecture for real completely positive matrices of order seven through eleven.

In this thesis, we initiate the study of completely positive matrices over special types of algebraic structures called semirings. Semirings satisfy all properties of unital rings except the existence of additive inverses. We formulate a notion of complete

positivity for matrices over semirings and show that this notion of complete positivity over special types of semirings has some important similarities with the standard notion of complete positivity of real matrices. We find some necessary and sufficient conditions for matrices over certain semirings to be completely positive. We prove the famous Drew-Johnson-Loewy conjecture for completely positive matrices over certain semirings, which include special types of inclines and Boolean algebras. Moreover, we show that in many cases the matrices of interest in graph theory are completely positive matrices over special types of semirings.

In addition, we define a new family of ranks of matrices over certain semirings and show that these ranks generalize some known rank functions over semirings such as the determinantal rank. We classify all bijective linear maps which preserve these ranks.

To My Husband, Vijay Kumar Kukreja

With Love

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Chapter 1

Introduction

A matrix is a rectangular array of numbers or symbols which are arranged in rows and columns. Matrices have applications in physics, economics, computer science, probability theory and statistics. A fundamental problem in all mathematical sciences is to analyze and solve m algebraic equations in n unknowns. A given system of simultaneous linear equations is sometimes difficult to solve in its original form. To find a solution of a given linear system matrix factorization technique is used. A matrix factorization or matrix decomposition is a factorization of a matrix into a product of two or more matrices. A matrix factorization converts the original problem into a series of easier subproblems. In the other words, a matrix factorization makes the original problem computationally convenient to obtain a solution. There are several different matrix factorizations, for example: LU factorization, UL factorization, Cholesky factorization UU^T, nonnegative factorization, etc.

If a real matrix A can be factored as

$$A = BB^T,$$

where B is any real matrix, then the matrix A is called a *positive semidefinite* matrix. If all the entries of B are nonnegative then the positive semidefinite matrix A is called a real *completely positive* matrix. Thus real completely positive matrices are symmetric matrices all of whose entries are nonnegative numbers.

Completely positive matrices have applications in the theory of inequalities, in the theory of block designs in combinatorics, in probability and statistics, in optimization theory and in economic modelling. Gray and Wilson [**41**] studied a mathematical model of energy demand for certain sectors of the U.S. economy, wherein $A = BB^T$, and elements of B are the parameters and satisfy nonnegativity constraints because of their physical interpretation.

Completely positive matrices are easy to construct. One can multiply an entrywise nonnegative matrix with its transpose to formulate a real completely positive matrix. On the other hand, it is very hard to determine whether a given real square matrix is completely positive or not. Moreover, if we have a real completely positive matrix A, it is generally hard to find a smallest possible nonnegative matrix B such that $A = BB^T$. The number of columns in that smallest possible choice of nonnegative matrix B is called the CP-rank of the given completely positive matrix.

There are two main problems in the theory of completely positive matrices:

1. Deciding if a given matrix is completely positive, and

2. Computing the CP-rank of a given completely positive matrix.

In the last 50 years, there has been a lot of research on real completely positive matrices. It has been shown [**2**] that the CP-rank of an $n \times n$ real completely positive matrix is less than or equal to n, for $n \leq 4$. However, determining least upper bound of the CP-rank of $n \times n$ completely positive matrices is still an open problem for $n \geq 6$.

The famous Drew-Johnson-Loewy conjecture (1994) [**30**] is by now twenty years old. It states that the CP-rank of any $n \times n$ real completely positive matrix is bounded above by $\left[\frac{n^2}{4}\right]$, where $n \geq 5$.

The Drew-Johnson-Loewy conjecture has been listed as a problem by Xingzhi Zhan in "Open Problems in Matrix Theory" [**88**]. This conjecture has been the subject of at least three dozen research papers and it has been proven for certain special classes of matrices. In 2013, N. Shaked-Monderer et al. [**82**] proved the Drew, Johnson and Loewy conjecture for completely positive matrices of order $n = 5$. For $n = 6$, N. Shaked-Monderer et al. [**81**] showed that the upper bound of the CP-rank of real completely positive matrices of order n is less than or equal to 15, but it is still unknown whether it is equal to 9 or not. However, Bomze et al. [**22**] gave counterexamples to the Drew-Johnson-Loewy conjecture for real completely positive matrices of order seven through eleven.

We consider the Drew-Johnson-Loewy conjecture for completely positive matrices over semirings. Semirings satisfy all properties of unital rings except the existence of additive inverses. A semiring is called an antinegative semiring if the only element with an additive inverse is the additive identity $\mathbf{0}$. In a semiring if the multiplication is commutative then the semiring is called a commutative semiring. The first natural example of an antinegative commutative semiring is the set of all nonnegative real integers. The set of all natural numbers, the nonnegative rational numbers and the two-element Boolean semiring are also antinegative commutative semirings.

There is a relation between matrices over the Boolean semiring and graphs. The Drew-Johnson-Loewy conjecture for completely positive matrices over the Boolean semiring is equivalent to a well known result in graph theory by Erdős, Goodman and Pósa; this is proved in chapter 3.

The Drew-Johnson-Loewy conjecture involves matrices over nonnegative real numbers. Since the set of all nonnegative numbers forms an antinegative commutative semiring, we formulate a question that generalizes the Drew-Johnson-Loewy conjecture.

Question 1.0.1. *For what antinegative commutative semirings S does every $n \times n$ completely positive matrix over S have its CP-rank less than or equal to the max of* $\{n, [n^2/4]\}$?

The aim of this thesis is to answer this question for completely positive matrices over various antinegative commutative semirings. To do so we determine:

1. Necessary and sufficient conditions for matrices over certain special types of antinegative commutative semirings to be completely positive, and

2. The upper bound on the CP-rank of completely positive matrices over certain types of antinegative commutative semirings.

We prove the Drew-Johnson-Loewy conjecture for completely positive matrices over various semirings. We also investigate those properties of real completely positive matrices which can be generalized to completely positive matrices over special semirings. In addition, we define a family of rank functions of matrices over special semirings and classify all bijective linear preservers of these ranks. We also discuss some applications of completely positive matrices over semirings to path finding problems in graph theory.

1.1 Organization of Thesis

In this chapter, we discuss some preliminary definitions, notations and results needed for the remainder of the thesis.

In chapter 2, we discuss the theory of real completely positive matrices and the theory of semirings. We define real completely positive matrices and their completely positive (CP) rank. We review some main properties and results on real completely positive matrices and their CP-rank. Then we discuss the Drew-Johnson-Loewy conjecture for real completely positive matrices in detail. Moreover, we introduce semirings and the theory of matrices over semirings. We also study orderings for semirings.

In chapter 3, we discuss completely positive matrices over semirings and their

CP-rank. We formulate a notion of complete positivity for matrices over semirings and show that this notion of complete positivity over special types of semirings has some important similarities with the standard notion of complete positivity of real matrices. In particular, we find necessary and sufficient conditions for matrices over special semirings to be completely positive. We also prove a semiring version of Markham's theorems which give sufficient conditions for a completely positive matrix over special types of semirings to have a triangular factorization. In addition, we formulate various CP-rank inequalities of completely positive matrices over special semirings.

In chapter 4, we discuss the Drew-Johnson-Loewy conjecture for completely positive matrices over Boolean algebras. We first examine the Drew-Johnson-Loewy conjecture for completely positive matrices over a two element Boolean algebra which is also called the Boolean semiring. Further, we use an isomorphism defined by Kirkland and Pullman in [**58**], to extend our results about the CP-rank of completely positive matrices over the Boolean semiring to completely positive matrices over Boolean algebras.

In chapter 5, we discuss completely positive matrices over special inclines, the max-plus semiring and the sign pattern semiring. We examine the upper bound on the CP-rank of completely positive matrices over special inclines. We use a characterization of completely positive matrices over the max-plus semiring given by Cartwright and Chen in [**26**] and explore that a result of Cartwright and Chen essentially answers the

Drew-Johnson-Loewy conjecture for completely positive matrices over the max-plus semiring. Moreover, we derive a characterization of completely positive matrices over the sign pattern semiring. We gave counterexamples to the Drew-Johnson-Loewy conjecture for completely positive matrices over the sign pattern semiring and the nonnegative interval subsemiring $\{-\infty\} \cup [0, \infty)$ of the max-plus semiring.

In chapter 6, we discuss some applications of completely positive matrices over semirings. We show that many path-finding problems can be formulated using certain completely positive matrices over special semirings. We also study the sequence of powers of completely positive matrices over special semirings.

In chapter 7, we discuss various rank functions of matrices over semirings. We define a new family of rank functions for matrices over semirings. We examine the properties of these rank functions as well as their relationship to some of the other rank functions found in the literature. We also compare the CP-rank of completely positive matrices over semirings with some other rank function of matrices over semirings.

In chapter 8, we discuss the novel results of the thesis.

1.2 Background

1.2.1 Matrix Theory

We begin with some matrix theoretical notations and terminology. For details see references [**17, 34, 44, 69**]. We denote the vector space of real column vectors of length n by $\mathbb{R}^n$. The entries of $x \in \mathbb{R}^n$ are denoted by $x_1, x_2, ..., x_n$. The *nonnegative orthant* of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, i.e.,

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \geq 0 \text{ for every } i = 1, ..., n\}.$$

If $x \in \mathbb{R}^n_+$, we say that x is a *nonnegative* vector and write $x \geq 0$. If all the entries of x are *positive*, we say that the vector x is positive and write $x > 0$.

A *Euclidean vector space* is a vector space V over the field of real numbers $\mathbb{R}$, endowed with an inner product $\langle \cdot, \cdot \rangle$. For $V = \mathbb{R}^n$ we denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}^n$, and by $\|.\|$ the norm it induces, the *2-norm*. That is, for $x, y \in \mathbb{R}^n$,

$$\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i,$$

$$\|x\| = \sqrt{x^T x} = \big(\sum_{i=1}^{n} x_i^2\big)^{1/2}.$$

We denote the set of all $m \times n$ matrices over $\mathbb{R}$ by $M_{m,n}(\mathbb{R})$. If $A \in M_{m,n}(\mathbb{R})$, then the $(i,j)^{th}$ element of A is denoted as a_{ij} or as $(A)_{ij}$. Sometimes we also write an $m \times n$ matrix A as $A = [a_{ij}]_{m \times n}$. A *line* of a matrix A is a row or a column of the matrix A. If we write a matrix A as:

$$A = \begin{array}{cc} & \begin{array}{cccccccc} \phantom{\mathbf{0}} & & k & & & l & & \phantom{\mathbf{0}} \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} \mathbf{0} & . & \mathbf{0} & . & . & \mathbf{0} & . & \mathbf{0} \\ \mathbf{0} & . & a_{ik} & . & . & a_{il} & . & \mathbf{0} \\ . & . & . & . & . & . & . & \\ \mathbf{0} & . & a_{jk} & . & . & a_{jl} & . & \mathbf{0} \\ \mathbf{0} & . & \mathbf{0} & . & . & \mathbf{0} & . & \mathbf{0} \end{pmatrix} \end{array}$$

then i and j represent the i^{th} and j^{th} row of the matrix A and k and l represent the k^{th} and l^{th} column of the matrix A.

A matrix $A \in M_{m,n}(\mathbb{R})$ is said to be a *nonnegative matrix* if $a_{ij} \geq 0$, for all i and j, and this is denoted by $A \geq 0$. In general, $A \geq B$ means that each $a_{ij} \geq b_{ij}$. Similarly, a matrix A is said to be a *positive matrix* if $a_{ij} > 0$, for all i and j, and this is denoted by $A > 0$. More generally, $A > B$ means that each $a_{ij} > b_{ij}$.

The *transpose* of $A \in M_{m,n}(\mathbb{R})$ is denoted by A^T, where $A^T \in M_{n,m}(\mathbb{R})$ and the $(j,i)^{th}$ entry of A^T is a_{ij}. If $A = A^T$ then A is called a *symmetric matrix*. The set of all $n \times n$ symmetric nonnegative matrices is denoted by $\mathbf{S}^n_+$.

Some special matrices: $0_{m\times n}$ is the $m \times n$ zero matrix, $J_{m\times n}$ is the $m \times n$ matrix of all ones; If $m = n$ we also write 0_n and J_n. I_n is the $n \times n$ identity matrix. A real square matrix all of whose entries are zero except for 1 in the ij^{th} position will be denoted by E_{ij}.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ real matrices and $C = [c_{ij}]$ be an $n \times m$ real matrix. Then the sum and the product of two matrices is defined as $A+B = [a_{ij}+b_{ij}]$

and $AC = [\sum_{k=1}^{n} a_{ik} \times c_{kj}]$. For any real number k, the matrix $kA = [k \times a_{ij}]$.

A *permutation* matrix P is a square matrix which has exactly one non-zero entry, equal to 1, in each row and each column. Multiplying a matrix A from the left by P results in a permutation of the rows of A, i.e., if $p_{1\sigma(1)}, p_{2\sigma(2)}, ..., p_{n\sigma(n)}$ are the nonzero elements in P, where σ is a permutation of order n, then PA is the matrix whose j-th row is the $\sigma(j)$-th row of A. Multiplying A from the right by P^T has the same effect on the columns of A. Hence PAP^T is the matrix obtained from A by permuting both the rows and columns of A the same way.

Given a vector $d \in \mathbb{R}^n$, an $n \times n$ diagonal matrix D with $d_1, ..., d_n$ as diagonal elements is denoted as $\text{diag}(d)$ or $\text{diag}(d_1, ..., d_n)$. A diagonal matrix D is called a *positive diagonal* matrix if all the diagonal elements of D are positive. If D is a positive diagonal matrix, we say that DAD is obtained from A by a *positive diagonal congruence.* The *trace* of an $n \times n$ matrix A is the sum of all the main diagonal elements of A.

Definition 1.2.1. (*Standard Determinant Expression*) *Let $A = [a_{ij}]$ be an $n \times n$ real matrix, then the standard determinant expression of A is:*

$$det(A) = \sum_{\sigma \in S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} ...... a_{n\sigma(n)}$$

where S_n is the symmetric group of order n and $sign(\sigma) = +$ if σ is even permutation and $sign(\sigma) = -$ if σ is odd permutation. Here $sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} ...... a_{n\sigma(n)}$ is called a term of the determinant.

A *minor* of a real matrix A is the determinant of a square submatrix of A. We remind the reader of the standard notation for submatrices. Let $\alpha = \{\alpha_1 < \alpha_2 < ... < \alpha_k\}$ and $\beta = \{\beta_1 < \beta_2 < ... < \beta_k\}$ be two subsets of $\{1, 2, ..., n\}$ of cardinality k. Then $A[\alpha|\beta]$ is the k by k submatrix of A whose (i, j)th entry is $a_{\alpha_i \beta_j}$ and $det(A[\alpha|\beta])$ is a minor of A. The set of all minors of A is $\{det(A[\alpha|\beta]) : \alpha, \beta \subseteq \{1, 2, ..., n\}, |\alpha| = |\beta|\}$. A minor $det(A[\alpha|\beta])$ is called a *principal minor* if $\alpha = \beta$.

Definition 1.2.2. (*Eigenvalues and Eigenvectors*) *Let A be an $n \times n$ matrix. A nonzero vector x of order $n \times 1$ is called an eigenvector of A if there exists a scalar λ such that $Ax = \lambda x$. The scalar λ is called the eigenvalue of A corresponding to the eigenvector x.*

Definition 1.2.3. (*Spectral Radius*) *Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of an $n \times n$ matrix A. Then its spectral radius $\rho(A)$ is defined as $\rho(A) = \max_i(|\lambda_i|)$.*

1.2.1.i Special Types of Matrices

In this subsection, we define some special types of real matrices. Some of these definitions will be generalized for matrices over semirings later on in this thesis.

Definition 1.2.4. (*Real Positive Semidefinite Matrices*) *A real matrix $A \in M_n(\mathbb{R})$ is positive semidefinite if it is a symmetric matrix and $x^T Ax \geq 0$, for all $x \in \mathbb{R}^n$.*

It is known that if A is an $n \times n$ real symmetric matrix then the following conditions are equivalent to A being positive semidefinite:

1. There exists an $n \times k$ real matrix B such that $A = BB^T$, where k is some positive integer.

2. There exists an $n \times n$ real lower triangular matrix L such that $A = LL^T$.

3. There exist k vectors $b_1, b_2, ..., b_k \in \mathbb{R}^n$ such that $A = \sum_{i=1}^{k} b_i b_i^T$.

4. There exists a k-dimensional Euclidean vector space V and vectors $v_1, v_2, ..., v_n \in V$ such that $A = Gram(v_1, v_2, ..., v_n)$. Here $Gram(v_1, v_2, ..., v_n)$ is a matrix whose $(i,j)^{th}$ entry is denoted as $\langle v_i, v_j \rangle$.

5. All eigenvalues of A are nonnegative.

6. All principal minors of A are nonnegative.

Definition 1.2.5. (*Copositive Matrices*) *A real matrix $A \in M_n(\mathbb{R})$ is copositive if it is a symmetric matrix and $x^T Ax \geq 0$, for all $x \in \mathbb{R}^n_+$.*

We note that every real positive semidefinite matrix is copositive, but the converse is not true.

If A is any real matrix, we denote by $|A|$ the matrix whose elements are the absolute values of the corresponding elements in A: $|A|_{ij} = |a_{ij}|$.

Definition 1.2.6. (*Comparison Matrices*) *Let A be a real square matrix. The comparison matrix of A, denoted by $M(A)$, is defined as:*

$$M(A) = \begin{cases} |a_{ij}|, & \text{if } i = j \\ -|a_{ij}|, & \text{if } i \neq j \end{cases}$$

where $|a_{ij}|$ is the absolute value of a_{ij}.

Definition 1.2.7. (*Real Diagonally Dominant Matrices*) *A real matrix A is diagonally dominant if* $|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ *for all* $i = 1, ..., n$.

It is strictly diagonally dominant if $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ for all $i = 1, ..., n$.

Totally nonnegative matrices are an important class of nonnegative matrices.

Definition 1.2.8. (*Real Totally Nonnegative* (*Positive*) *Matrices*) *A square real matrix is totally nonnegative if the determinant of every square submatrix is a nonnegative number, and totally positive if the determinant of every square submatrix is a positive number.*

Every zero matrix, identity matrix and J_n a matrix that has all entries equal to one, are totally nonnegative real matrices.

Definition 1.2.9. (*M-Matrices*) *A real square matrix* $A = sI - B, B \geq 0$, *is an M-matrix if* $s \geq \rho(B)$, *where* $\rho(B)$ *is the spectral radius of* B.

An M-matrix $A = sI - B, B \geq 0$, is nonsingular if and only if $s > \rho(B)$. In the following theorem we give some characterizations of nonsingular M-matrices.

Theorem 1.2.10. [**17**] *Let* $A = sI - B$ *be an* n *by* n *matrix,* $B \geq 0$. *Then the following statements are equivalent:*

1. *A is a nonsingular M-matrix.*

2. *All principal minors of A are positive.*

3. *There exist a lower triangular matrix L with positive diagonal entries and an upper triangular matrix U with positive diagonal entries such that $A = LU$.*

4. *There exists a positive diagonal matrix D such that AD is strictly diagonally dominant.*

1.2.1.ii Matrix Polynomials

Let $\mathbb{R}$ be the set of all real numbers and $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where $n \geq 0$ and $a_i \in \mathbb{R}$ for all $i = 0, 1, ..., n$ be a real polynomial. Two polynomials $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + ... + b_nx^n$ are said to be equal, i.e., $p(x) = q(x)$, if and only if $a_i = b_i$ for all $i \geq 0$. Addition and multiplication of two polynomials $p(x)$ and $q(x)$ is defined as:

$$p(x) + q(x) = c_0 + c_1x + ... + c_nx^n,$$

where $c_i = a_i + b_i$, for all $i \geq 0$.

$$p(x)q(x) = c_0 + c_1x + ... + c_nx^n,$$

where $c_i = a_0b_i + a_1b_{i-1} + ... + a_ib_0$, for all $i \geq 0$.

The set of all real polynomials in x, denoted as $\mathbb{R}[x]$, forms a commutative ring [**43**] and x is an indeterminate over $\mathbb{R}$. The ring $\mathbb{R}[x]$ is also called a polynomial ring. A real polynomial $p(x)$ is said to be a nonnegative polynomial if $p(x) \geq 0$ for all $x \in \mathbb{R}$ and we denote it as $p \geq 0$. A real polynomial $p(x)$ is said to be a positive polynomial if $p(x) > 0$ for all $x \in \mathbb{R}$ and we denote it as $p > 0$. For two real polynomials $p(x)$ and $q(x)$, $p \geq q$ means that $p(x) \geq q(x)$ for all $x \in \mathbb{R}$.

The concept of *sum of squares* plays an important role in this thesis. It is denoted as *SOS* in the literature. In [**62, 73**], the theory of sum of squares is discussed for polynomials. A real polynomial $p(x)$ is said to be a sum of squares of polynomials (i.e., $p(x)$ is *SOS*) if $p(x)$ can be

$$p(x) = \sum_{i=1}^{k} (q_i(x))^2,$$

where $q_i(x) \in \mathbb{R}[x]$ for all $i = 1, 2, ..., k$. It is evident that if $p(x)$ is *SOS* then $p(x) \geq 0$ for all x. The converse statement for real polynomials of a single variable is true as well. Every polynomial which is nonnegative on the real line can be written as a sum of squares of real polynomials. This result can be found in the collection of problems by Pólya and Szegö.

Theorem 1.2.11. [**73**] *If $p(x)$ is a nonnegative polynomial, then there exists polynomials $f(x)$ and $g(x)$ such that*

$$p(x) = f(x)^2 + g(x)^2, \qquad \text{for all } x.$$

Definition 1.2.12. (*Real Matrix Polynomial*) *A matrix all of whose entries are real polynomials in x, denoted as $A(x)$, is called a real matrix polynomial. We denote the $(i, j)^{th}$ entry of $A(x)$ by $p_{ij}(x)$.*

Definition 1.2.13. (*Pointwise Nonnegative Matrix Polynomial*) *A real matrix polynomial $A(x)$ is called pointwise nonnegative, if for all i, j, we have $p_{ij}(x) \geq 0$ for all*

$x \in \mathbb{R}$.

Definition 1.2.14. (*Pointwise Diagonally Dominant Matrix Polynomial*) *A real matrix polynomial $A(x)$ of order n is called pointwise diagonally dominant, if for all i, j, we have $|p_{ii}(x)| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |p_{ij}(x)|$, for all $x \in \mathbb{R}$.*

1.2.2 Abstract Algebra

In this section we review some of the fundamental structures of abstract algebra. For more information see [**51**].

Definition 1.2.15. (*Ring*) *A non empty set S together with two binary operations $(+)$ and $(\cdot)$ is called a ring if for all $a, b, c \in S$, the following laws hold:*

1. ***Laws of Addition**:*
   (a) *Closure Law:* $a + b \in S$.
   (b) *Associative Law:* $a + (b + c) = (a + b) + c$.
   (c) *Commutative Law:* $a + b = b + a$.
   (d) *Identity Law: There exists an element $\boldsymbol{0} \in S$ such that $a + \boldsymbol{0} = a = \boldsymbol{0} + a$, for all $a \in S$. Here $\boldsymbol{0}$ is called the additive identity.*
   (e) *Inverse Law: For each $a \in S$ there exists an element $-a \in S$ such that $a + (-a) = \boldsymbol{0} = (-a) + a$. Here $-a$ is called the additive inverse.*
2. ***Laws of multiplication**:*

(a) Closure Law: $a \cdot b \in S$.

(b) Associative Law: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

3. ***Distributive Law****:*

(a) $a \cdot (b + c) = ab + ac$

(b) $(a + b) \cdot c = ac + bc$

A ring $(S, +, \cdot)$ is called a *commutative ring* if $a \cdot b = b \cdot a$, for all $a, b \in S$

Definition 1.2.16. (*Unitary Ring*) *A ring $(S, +, \cdot)$ is called a unitary ring or a ring with unity if there exists an element $\boldsymbol{1} \in S$ such that for each $a \in S$*

$$a \cdot \boldsymbol{1} = a = \boldsymbol{1} \cdot a$$

Definition 1.2.17. (*Monoid*) *A non empty set S together with the given operation $(\cdot)$ and an element e is called a monoid if it satisfies the closure law, associative law and the identity law.*

Let R be a unital ring. An element a of the ring is said to have a multiplicative inverse if there exists an element $b \in R$ such that $a \cdot b = \mathbf{1}$. The element b is called the inverse of a and denoted as a^{-1}.

Definition 1.2.18. (*Field*) *A field is a nonzero commutative ring with unity that contains a multiplicative inverse for every nonzero element.*

We now define ideals for rings and fields.

Definition 1.2.19. (*Ideals*) *Let R be a commutative ring. A non empty subset I of R is called an ideal if the following two properties hold:*

1. $a + b \in I$, *for all* $a, b \in I$

2. $s \cdot a \in I$, *for all* $a \in I$ *and* $s \in R$.

Evidently, R itself and the zero ideal (0) are ideals in every commutative ring R. They are called the *trivial ideals* of R. An ideal I is called a *proper ideal* of R if it is a proper subset of R, that is, I is not equal R.

Note that if R is a field then the only ideals of R are R itself and the zero ideal (0). A proper ideal I of a commutative ring R is called a *prime ideal* if a and b are elements of R such that their product $a \cdot b \in I$, then either $a \in I$ or $b \in I$.

1.2.3 Convex Cones

In this section, we review some basic definitions and results on convex cones in a finite dimensional Euclidean space V, with an inner product $\langle , \rangle$. For more details we refer the reader to [**20**].

Definition 1.2.20. (*Convex Set*) *A set K in a Euclidean space V is called a convex set if for every $x, y \in K$ and $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda) y \in K$.*

Every linear subspace of V is clearly is a convex set. The closed unit ball in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n \quad | \quad ||x|| \leq 1\}$, is also an example of a convex set.

Definition 1.2.21. (*Cones*) *A set K in a vector space V is called a cone if for every $x \in K$ and $a \geq 0$, we have $ax \in K$.*

By combining above two definitions we get:

Definition 1.2.22. (*Convex Cones*) *A set K in a Euclidean space V is called a convex cone if for every $x, y \in K$ and $a, b \geq 0$, we have $ax + by \in K$.*

Examples of convex cones: Let $K_1 = \{(x, y) \in \mathbb{R}^2_+\}$ and $K_2 = \{(x, y) \in \mathbb{R}^2_+ | 0 \leq y \leq x\}$. Both K_1 and K_2 are convex cones in $\mathbb{R}^2$. Given a unit vector $u \in \mathbb{R}^n$, and an angle $0 \leq \theta < \pi/2$. The set of all $x \in \mathbb{R}^n$ such that the angle between x and u is at most θ is defined as:

$$K_{u,\theta} = \{x \in \mathbb{R}^n | \langle x, u \rangle \geq ||x|| cos(\theta)\}.$$

One can easily check that $K_{u,\theta}$ is a convex cone.

If $x \in V$, then the cone

$$\{x\} = \{ax | a \geq 0 \text{ and } a \in \mathbb{R}\}$$

is called *the ray generated* by x. If $x \in K$, where K is a convex cone, then the ray generated by x is contained in K.

Definition 1.2.23. (*Extreme Vectors and Extreme Rays*) *Let K be a convex cone and $x \in K$. If $x = y + z$ where $y, z \in K$, implies that y and z are both nonnegative scalar multiples of x, then x is called an extreme vector of K and the ray generated by x is called an extreme ray of K.*

The extreme rays of the cone K_1 defined above are the nonnegative x-axis, generated by $(1,0)$, and the nonnegative y-axis, generated by $(0,1)$. The extreme rays of K_2 are nonnegative x-axis, generated by $(1,0)$, and the ray generated by $(1,1)$.

A convex cone is said to be *closed* if it contains all its limit points. It has been shown that every closed convex cone is a convex hull of its extreme rays.

Theorem 1.2.24. (*Caratheodory's Theorem*) *If K is a closed convex cone in a Euclidean space of dimension n, then every $x \in K$ can be represented as a nonnegative combination of at most n elements of extreme rays of K.*

The proof of this theorem follows from the fact that if x is a positive combination of m linearly dependent vectors $x_1, x_2, ..., x_m$, then there exists a linearly independent subset of $\{x_1, x_2, ..., x_m\}$ such that x is a positive combination of its elements.

We now define dual cones.

Definition 1.2.25. (*Dual Cones*) *Let S be a cone in a Euclidean space V. The set*

$$S^* = \{y \in V | \langle x, y\rangle \geq 0, \textit{ for every } x \in S\},$$

is called the dual cone of S.

In other words, the dual cone of a cone S in V contains all $y \in V$ such that the angle between y and every element of S is at most $\pi/2$. It is easy to see that the cone K_1 is self dual, i.e., $K_1 = K_1^*$, and $K_2^* = \{(x, y)|0 \leq x, -x \leq y\}$.

We are interested in cones of matrices. Here we will discuss several sets of symmetric matrices S_n, which form convex cones. The space of all $n \times n$ symmetric matrices S_n is a subspace of $M_n(\mathbb{R})$ and the inner product of matrices in S_n is defined as: $\langle A, B\rangle = \text{trace}(AB)$. Let SNN_n denote the set of $n \times n$ symmetric nonnegative matrices, PSD_n denote the set of $n \times n$ positive semidefinite matrices.

Proposition 1.2.26. *The set SNN_n forms a closed convex cone in S_n. Its interior consists of all $n \times n$ symmetric positive matrices. Its extreme rays are all the rays generated by E_{ii}, $i = 1, 2, ..., n$, and generated by the matrices $E_{ij} + E_{ji}$, where $1 \leq i < j \leq n$. As a cone in S_n, SNN_n is self dual.*

SNN_n is a subset of NN_n - the set of al $n \times n$ nonnegative matrices, which forms a convex cone in $M_n(\mathbb{R})$.

Proposition 1.2.27. *The set PSD_n forms a closed convex cone in S_n. Its interior consists of all $n \times n$ positive definite matrices. Its extreme vectors are all rank 1 symmetric $n \times n$ matrices. As a cone in S_n, PSD_n is self dual.*

1.2.4 Binary Relations and Fuzzy Relations

In this section, we review some basic concepts of set theory and fuzzy set theory. We begin with the introduction of binary relations and the structure of their associated matrices. We then introduce fuzzy relations and some of their properties. For reference see [**75**].

A binary relation R between two sets S and T is a subset of the Cartesian product $S \times T$. If $(a, b) \in R$, we say a is in relation R with b. We denote this by aRb. The set S is called the *domain* of the relation and the set T the *codomain*. If $S = T$ we say R is a relation on S.

Some relations have special properties:

Definition 1.2.28. *Let R be a relation on a set S. Then R is called*

1. *Reflexive: if for all $x \in S$ we have $(x, x) \in R$;*

2. *Irreflexive: if for all $x \in S$ we have $(x, x) \notin R$;*

3. *Symmetric: if for all $x, y \in S$ we have xRy implies yRx;*

4. *Antisymmetric: if for all $x, y \in S$ we have that xRy and yRx implies $x = y$;*

5. *Transitive: if for all $x, y, z \in S$ we have that xRy and yRz implies xRz.*

A relation R on a set S is called an *equivalence relation* on S if and only if it is reflexive, symmetric and transitive.

Definition 1.2.29. (*The Adjacency Matrix of a Binary Relation*) *If $S = \{1, 2, ..., n\}$ and $T = \{1, 2, ..., m\}$ are finite sets and $R \subseteq S \times T$ is a binary relation, then the adjacency matrix A of the relation R is an $n \times m$ matrix whose rows are indexed by S and columns by T defined as:*

$$a_{ij} = \begin{cases} \mathbf{1} & \text{if } (i, j) \in R; \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

If the sets S and T are equal, then the adjacency matrix is a square matrix whose $(i, j)^{th}$ entry is $\mathbf{1}$ if iRj and $\mathbf{0}$ otherwise.

We now define the *composition* or *product* of two relations. Suppose that R_1 is a relation from S to T with the adjacency matrix A_1 and R_2 is a relation from T to U with the adjacency matrix A_2. Then the *composition* or *product* of R_1 and R_2 is denoted as $R = R_1; R_2$. We note that the composition of two relations $R = R_1; R_2$ is the relation between S and U defined by sRu if and only if there is a $t \in T$ with sR_1t and tR_2u. The adjacency matrix A of $R_1; R_2$ can be obtained by changing all nonzero entries by $\mathbf{1}$ in the matrix A_1A_2. Here A_1A_2 is the product of two matrices A_1 and A_2.

We now study the concept of fuzzy subsets and fuzzy relations. A *fuzzy subset* of S is a mapping

$$\mu : S \to [0, 1],$$

where $[0, 1]$ denotes the set $\{t \in \mathbb{R} | 0 \leq t \leq 1\}$. We think of μ as assigning to each element $x \in S$ a degree of membership, $0 \leq \mu(x) \leq 1$.

Let S and T be two sets and let μ and ν be fuzzy subsets of S and T, respectively. Then a *fuzzy relation* ρ from the fuzzy subset μ into the fuzzy subset ν is a fuzzy subset of $S \times T$ such that $\rho(x, y) \leq \mu(x) \wedge \nu(y)$, for all $x \in S$ and $y \in T$. Here $\mu(x) \wedge \nu(y)$ means the minimum of $\mu(x)$ and $\nu(y)$. That is, for ρ to be a fuzzy relation, we require that the degree of membership of a pair of elements never exceed the degree of membership of either of the elements themselves.

If $S = T$ and $\mu = \nu$ then ρ is said to be a fuzzy relation on μ. Note that ρ is a fuzzy subset of $S \times S$ such that $\rho(x, y) \leq \mu(x) \wedge \mu(y)$, for all $x, y \in S$. Let ρ and ω

are two fuzzy relations on a fuzzy subset μ of S, we write $\rho \subseteq \omega$ if $\rho(x,y) \leq \omega(x,y)$ for all $x, y \in S$.

It is quite natural to express a fuzzy relation as the fuzzy matrix.

Definition 1.2.30. (*The Adjacency Matrix of a Fuzzy Relation*) *Let* $S = \{1, 2, ..., n\}$ *and* $T = \{1, 2, ..., m\}$ *are finite sets and* $\rho : S \times T \longrightarrow [0, 1]$ *be a fuzzy relation on a fuzzy subset* μ *of* S *into a fuzzy subset* ν *of* T*. Then the adjacency matrix* A *of the fuzzy relation* ρ *is an* $n \times m$ *matrix whose rows are indexed by* S *and columns by* T *defined as:*

$$a_{ij} = \rho(i, j), \qquad \textit{for all } i \in S \textit{ and } j \in T.$$

If the sets S and T are equal, then the adjacency matrix is a square matrix.

We now introduce the *composition* or *product* of two fuzzy relations. Let

$$\rho : S \times T \longrightarrow [0, 1]$$

be a fuzzy relation from a fuzzy subset μ of S into a fuzzy subset ν of T. This implies that $\rho(x, y) \leq \mu(x) \wedge \nu(y)$, for all $x \in S$ and $y \in T$. Also let

$$\omega : T \times U \longrightarrow [0, 1]$$

be a fuzzy relation from a fuzzy subset ν of T to a fuzzy subset ξ of U. This implies that $\omega(y, z) \leq \nu(y) \wedge \xi(z)$, for all $y \in T$ and $z \in U$. Then the composition of ρ and ω is denoted as $\rho \circ \omega$, where

$$\rho \circ \omega : S \times U \longrightarrow [0, 1]$$

such that

$$\rho \circ \omega(x, z) = \bigvee_{y \in T} \{\rho(x, y) \wedge \omega(y, z)\},$$

for all $x \in S$ and $z \in U$. Here $\bigvee$ denotes the maximum. Let $A = [a_{ij}]$ be the adjacency matrix of the fuzzy relation ρ and $B = [b_{ij}]$ be the adjacency matrix of the fuzzy relation ω. Then the adjacency matrix of $\rho \circ \omega$, denoted as $C = [c_{ij}]$, is defined as:

$$c_{ij} = \max_k \{min(a_{ik}, b_{kj})\}, \text{ for all } i, j.$$

We use the notation ρ^2 to denote the composition $\rho \circ \rho$.

Note that the composition $(\rho \circ \omega)$ of two fuzzy relations ρ and ω is a fuzzy relation from a fuzzy subset μ of S into a fuzzy subset ξ of U, since for any $x \in S$ and $z \in U$,

$$\begin{aligned}
\rho \circ \omega(x, z) &= \bigvee_{y \in T} \{\rho(x, y) \wedge \omega(y, z)\} \\
&\leq \bigvee_{y \in T} \{(\mu(x) \wedge \nu(y)) \wedge (\nu(y) \wedge \xi(z))\} \\
&= \bigvee_{y \in T} \{\mu(x) \wedge \nu(y) \wedge \xi(z))\} \\
&\leq \bigvee_{y \in T} \{\mu(x) \wedge \xi(z)\} \\
&= \mu(x) \wedge \xi(z).
\end{aligned}$$

In the other words, for any $x \in S$ and $z \in U$, we have $\rho \circ \omega(x, z) \leq \mu(x) \wedge \xi(z)$. This implies that the composition of any two fuzzy relations is a fuzzy relation.

We now explain some properties that fuzzy relations may possess.

1. Reflexive: ρ is called reflexive (on μ) if $\rho(x,x) = \mu(x)$, for all $x \in S$. This implies that $\rho(x,y) \leq \mu(x) \wedge \mu(y) \leq \mu(x) = \rho(x,x)$. Hence if ρ is called reflexive then every diagonal entry of the adjacency matrix of ρ is greater than or equal to any other entry in the corresponding row and column. In other words, we can say ρ is reflexive if $max\{\rho(x,y), \rho(y,x)\} \leq \rho(x,x)$.

   If ρ and ω are two fuzzy relations on a fuzzy subset μ of S then the following properties hold:

   (a) If ρ is reflexive then $\omega \subseteq \omega \circ \rho$ and $\omega \subseteq \rho \circ \omega$.

   (b) If ρ is reflexive then $\rho \subseteq \rho^2$.

2. Symmetric: ρ is called symmetric if $\rho(x,y) = \rho(y,x)$, for all $x, y \in S$. In the other words, ρ is symmetric if the matrix representation of ρ is symmetric with respect to the diagonal.

3. Transitive: ρ is called transitive if $\rho^2 \subseteq \rho$, i.e., if

$$\max_k(min\{\rho(x,k), \rho(k,y)\}) \leq \rho(x,y).$$

A relation ρ on a fuzzy subset μ of S is called an *fuzzy equivalence relation* if and only if it is reflexive, symmetric and transitive.

1.2.5 Graphs and Fuzzy Graphs

In this section, we review some basic definitions and results of graph theory. We also study fuzzy graphs. For more details see [**7, 21**]. The main application of the graph theory in this thesis is to symmetric matrices. Therefore, the graphs we are using in this thesis are all undirected graphs.

Definition 1.2.31. *A (simple) graph G is a pair $G = (V(G), E(G))$ consisting of a finite set $V(G)$, whose elements are called vertices and a finite edge set $E(G)$, where each edge is an ordered pair of distinct vertices.*

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The graph G_1 is called a *subgraph* of the graph G_2 if $V(G_1)$ is a subset of $V(G_2)$ and $E(G_1)$ is a subset of $E(G_2)$. Let G be a graph and $v \in V(G)$ be a vertex of G. A subgraph of G obtained by deleting the vertex v along with all edges incident with v is denoted by $G - \{v\}$. The vertex set of the subgraph $G - \{v\}$ contains all vertices of G except v and it is denoted by $V(G)/v$. Let e be an edge of G. A subgraph of G obtained by deleting the edge e from G is denoted by $G - \{e\}$. The edge set of the subgraph $G - \{e\}$ contains all edges of G except e and it is denoted by $E(G)/e$.

Let G be a graph and G_1 and G_2 are two subgraphs of G. The *union* of two subgraphs $G_1 \cup G_2$ is a subgraph of G with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Two subgraphs G_1 and G_2 of G are called *disjoint* if $V(G_1) \cap V(G_2) = \phi$. If G_1 and G_2 have at least one vertex in common, then their *intersection* $G_1 \cap G_2$ is a subgraph of G with the vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$.

Definition 1.2.32. (*Weighted Graphs*) *A graph is said to be a weighted graph if a non-zero number* (*weight*) *is assigned to each edge.*

In a weighted graph weights represent, for example, costs, lengths or capacities depending on the problem. The weight of an edge, joining vertex i and vertex j in a weighted graph, is denoted as $w(i, j)$.

A sequence of edges $\{v_0, v_1\}, \{v_1, v_2\}, ..., \{v_{m-1}, v_m\}$ is called a *walk* of length m from vertex v_0 to vertex v_m. The walk is *closed* if $v_0 = v_m$ and *open* if $v_0 \neq v_m$. Such a walk from vertex v_0 to vertex v_m is called a *path* if vertices $v_0, ..., v_m$ are distinct. A *cycle* is a closed path. The weight of a path P from vertex v_0 to vertex v_m, denoted by $w(P)$, is defined as:

$$w(P) = \prod_{i=0}^{m-1} w(v_i, v_{i+1}).$$

Definition 1.2.33. (*Connected Graphs*) *A graph is said to be connected if for any two vertices v and w there is a path from v to w.*

A *tree* is a connected graph that contains no cycle.

Definition 1.2.34. (*Complete Graphs*) *A graph is said to be a complete graph if there is an edge between every two vertices.*

A graph $G = (V, E)$ can be covered by k complete subgraphs if there exist k not necessarily disjoint subsets S_i, $i = 1, 2, ..., k$, of V such that $\bigcup_{i=1}^{k} S_i = V$ and any two distinct vertices lying in the same subset S_i have an edge between them. Covering a graph by complete subgraphs is called a *clique covering* and each complete subgraph in a clique covering is called a *clique*.

Definition 1.2.35. (*Bipartite Graphs*) *A graph G is bipartite if the vertex set of G can be partitioned into two subsets X and Y, such that each edge has one vertex in X and one vertex in Y.*

From the definition it is clear that a bipartite graph is a graph that contains no cycle of odd length. In particular, every tree and every even cycle are bipartite, and every odd cycle is not bipartite.

Definition 1.2.36. (*Complete Bipartite Graphs*) *A complete bipartite graph is a bipartite graph in which every vertex in X is adjacent to each vertex in Y.*

While trees are bipartite graphs, bipartite graphs are part of a still large family - the triangle free graphs.

Definition 1.2.37. (*Triangle Free Graphs*) *A graph G is triangle free if G contains no triangle* (*cycle of length three*).

An odd cycle on 5 vertices or more is an example of a triangle free graph, which is not bipartite.

There is a relation between graphs and matrices. We define the correspondence between simple graphs and matrices.

Definition 1.2.38. (*The Graph of a Matrix*) *Let A be an $n \times n$ symmetric matrix. The graph of A, denoted by $G(A)$, is defined as follows:*

1. *The set of vertices of $G(A)$ is $\{1, 2, ..., n\}$, the set of indices of the rows or columns of A.*

2. *$\{i, j\}$ an edge of G if and only if $i \neq j$ and $a_{ij} \neq 0$.*

In the reverse direction, we associate a matrix with each graph.

Definition 1.2.39. (*The Adjacency Matrix*) *Let G be a simple undirected graph with n vertices. The adjacency matrix of G is an $n \times n$ symmetric matrix A, where $a_{ii} = 0$ for all $i \in V(G)$, and $a_{ij} = 1$ if there exists an edge from vertex i to vertex j and $a_{ij} = 0$ otherwise.*

We now define the correspondence between weighted undirected graphs and matrices.

Definition 1.2.40. (*The Weighted Graph of a Matrix*) *Let A be an $n \times n$ symmetric matrix. We define a weighed undirected graph $G(A)$ associated with A as follows:*

1. *The set of vertices of $G(A)$ is $\{1, 2, ..., n\}$, the set of indices of the rows or columns of A.*

2. *The set of edges of $G(A)$ is the set of ordered pair (i, j) corresponding to the terms $a_{ij} \neq 0$ of A and each edge (i, j) in the graph $G(A)$ has a value a_{ij}, $(a_{ij} \neq 0)$. If A contains a diagonal term $a_{ii} \neq 0$ then $G(A)$ contains an edge (i, i) also referred as a loop.*

Conversely, we associate a matrix with each weighted graph.

Definition 1.2.41. (*The Adjacency Matrix of a Weighted Graph*) *Let G be a weighted undirected graph with n vertices. The adjacency matrix of G is an $n \times n$ symmetric matrix A, where a_{ij} is equal to the weight of the edge joining vertex i and j for all i, j.*

We now review some basics of fuzzy graphs.

Definition 1.2.42. (*Fuzzy Graphs*) *Let S be a finite set which is equal to $\{1, 2, ..., n\}$. The triplet $G(S, \sigma, \mu)$ is called fuzzy graph on S where:*

$\sigma : S \to [0, 1]$, *stands for the weight (the degree of membership) of each vertex and,*

$\mu : S \times S \to [0, 1]$, *stands for the weight (the degree of membership) of each edge*

such that $\mu(i, j) \leq \sigma(i) \wedge \sigma(j)$, for all $i, j \in S$, where $\sigma(i) \wedge \sigma(j)$ denotes the minimum of $\sigma(i)$ and $\sigma(j)$.

That is $G(S, \sigma, \mu)$ is called fuzzy graph on S if the degree of membership of each edge never exceeds the degree of membership of either of the vertices joining the edge.

Definition 1.2.43. (*Complete Fuzzy Graphs*) *A fuzzy graph $G(S, \sigma, \mu)$ is said to be complete fuzzy graph if it satisfies $\mu(i, j) = \sigma(i) \wedge \sigma(j)$, for all $i, j \in S$.*

A complete fuzzy graph on a set S of two points is a line segment with edge weight equal to the minimum of weights on each vertex. If the set S contains three points then the complete fuzzy graph on S is just a triangle with weights on each edge equal to the minimum of weights on the corresponding vertices.

1.2.6 Lattice Theory

In this section, we introduce some basic definitions from lattice theory. For more details on lattice theory we refer the reader to [**9**]. We begin with some of the most fundamental notions of order theory.

Definition 1.2.44. (*Preordered Set*) *A preordered set is a set in which a binary relation $x \leq y$ is defined, which satisfies the following properties:*

- *$x \leq x$, for all x,* (*Reflexive*)
- *If $x \leq y$ and $y \leq z$, then $x \leq z$.* (*Transitivity*)

Definition 1.2.45. (*Partially Ordered Set*) *A preordered set with a binary relation $\leq$ is called a partially ordered set (or poset) if the binary relation $\leq$ satisfies the antisymmetric property, i.e.,*

$$\text{If } x \leq y \text{ and } y \leq x, \text{ then } x = y. \qquad \text{(Antisymmetry)}$$

In other words, a partially ordered set is an antisymmetric preordered set. If $x \leq y$ and $x \neq y$, we write $x < y$, and say that x is less than or properly contained in y. Examples of partially ordered sets are: the set of all real numbers where $x \leq y$ means that x is less than or equal to y, the set of positive integers where $x \leq y$ means that x divides y, the set of all subsets of a set I including I itself and an empty set ϕ where $x \leq y$ means that x is a subset of y.

Let P be a partially ordered set. The *least element* of P is an element of P that is smaller than every other element of P. We denote the least element of a partially ordered set by **0**. The *greatest element* of P is an element of P that is greater than every other element of P. We denote the greatest element of a partially ordered set by **1**. The elements **0** and **1** of a partially ordered set P may or may not exist and when they exist they are called *universal bounds* of P.

Definition 1.2.46. (*Totally Ordered Set*) *A partially ordered set P with a binary relation $\leq$ is said to be a totally ordered set if the binary relation $\leq$ satisfies the following property:*

$$\textit{For all } x, y \in P, \textit{ either } x \leq y \textit{ or } y \leq x.$$

A totally ordered set is also called a *chain*. It is clear from the definition that any two distinct elements in a chain are comparable, i.e., one element is less and the other is greater. Every totally ordered set is a partially ordered set but a partially ordered set may not be totally ordered, since partially ordered sets may contain pairs of elements x, y which are incomparable, i.e., neither $x \leq y$ nor $y \leq x$.

Let P be a partially ordered set and X be a subset of P. An upper bound of X is an element of $a \in P$ such that $x \leq a$ for all $x \in X$. The least upper bound is an upper bound which is contained in every other upper bound. The least upper bound of X is denoted as *l.u.b.* of X or $\sup X$. Note that the *l.u.b.* of X may not exist and if it exists then it is unique by antisymmetry. The concept of the lower bound and the greatest lower bound is defined dually. The greatest lower bound of X is denoted as *g.l.b.* of X or $\inf X$ and it is unique if exists.

Definition 1.2.47. (*Lattice*) *A lattice is a partially ordered set P any two of whose elements have a greatest lower bound or "meet" denoted by $x \wedge y$ and a least upper bound or "join" denoted by $x \vee y$.*

Any chain is a lattice, where $x \wedge y$ is the smaller of x and y and $x \vee y$ is the larger of x and y. The set of all subsets of a set I including I itself and an empty set ϕ forms a lattice where $\wedge$ denotes the intersection of sets and $\vee$ denotes the union of sets.

In lattice theory the binary operations $\wedge$ and $\vee$ have many important algebraic properties which are similar to those of usual multiplication and addition.

Theorem 1.2.48. [**9**, Theorem 8] *Any system L with two binary operations $\vee$ and $\wedge$ form a lattice if and only if it satisfies the following properties:*

- $x \wedge x = x, \quad x \vee x = x,$ *(Idempotence)*
- $x \wedge y = y \wedge x, \quad x \vee y = y \vee x,$ *(Commutativity)*
- $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee (y \vee z) = (x \vee y) \vee z,$ *(Associativity)*
- $x \wedge (x \vee y) = x \vee (x \wedge y) = x.$ *(Absorption)*

Definition 1.2.49. (*Semilattice*) *A system with a single binary operation which is idempotent, commutative and associative operation is called a semilattice.*

In other words, any partially ordered set P in which any two elements have a meet (or a join) is called a meet-semilattice (or join-semilattice respectively). In a lattice the following identities are equivalent:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad \text{for all } x, y, z. \tag{1.1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \qquad \text{for all } x, y, z. \tag{1.2}$$

Definition 1.2.50. (*Distributive Latttice*) *A lattice is called a distributive lattice if and only if the identity* (1.1) *holds in it.*

Every totally ordered set (chain) is a distributive lattice. Examples of partially ordered sets that we have given earlier are also examples of distributive lattices.

We now define complemented lattices. An element x of a lattice L with universal bounds $\mathbf{0}$ and $\mathbf{1}$ is said to have a complement if there exists an element $y \in L$ such that $x \wedge y = \mathbf{0}$ and $x \vee y = \mathbf{1}$. A lattice L is called a *complemented lattice* if all its elements have complements.

Definition 1.2.51. (*A Boolean Algebra*) *A complemented distributive lattice is called a Boolean algebra or a Boolean lattice.*

The concept of lattice ideals plays an important role in lattice theory. It was developed by Hashimoto [**42**].

Definition 1.2.52. (*Lattice Ideal*) *A nonempty subset J of a lattice L is called a lattice ideal if it satisfies the following properties:*

1. *$a \in J$ implies that $x \in J$ for all $x \leq a$, where $x \in L$.*

2. *$a \in J$ and $b \in J$ implies that $a \vee b \in J$.*

1.2.7 Ordered Fields and Ordered Rings

In this section, we study some of the fundamental properties of ordered fields and ordered rings. There are two equivalent definitions of an ordered field [**72**]. The first definition of an ordered field involves the existence of total order which behaves well with respect to the field operations.

Definition 1.2.53. (*Ordered Fields*) *Let* $(F, +, \times)$ *be a field together with a total order* $\leq$ *on* F. *The field* F *is called an ordered field if the order* $\leq$ *satisfies the following two properties, for* a, b *and* $c \in F$,

- *if* $a \leq b$ *then* $a + c \leq b + c$,
- *if* $0 \leq a$ *and* $0 \leq b$ *then* $0 \leq ab$.

In 1927, an ordered field was defined in terms of positive cone by Artin and Schreier [**6**]. A subset P of F is called a *positive cone* if the following conditions hold:

- $P + P \subseteq P$,
- $P \cdot P \subseteq P$,
- $P \cap -P = \{0\}$,
- $P \cup -P = F$.

The set $P = \{a \in F | 0 \leq a\}$ clearly satisfies all above properties and it is called a positive cone F.

Definition 1.2.54. (*Ordered Fields*) *A field* $(F, +, \times)$ *together with a positive cone* P *is called an ordered field, where an order is defined as:*

$$a \leq b \Leftrightarrow b - a \in P, \qquad \textit{for all } a, b \in F.$$

The set of all real numbers $\mathbb{R}$ together with $\mathbb{R}_+$ forms an ordered field, where $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. The set of all rational numbers $\mathbb{Q}$ together with $\mathbb{Q}_+$ forms an ordered field, where $\mathbb{Q}_+$ denotes the set of all nonnegative rational numbers.

To get an algebraic characterization of ordered fields, the notion of positive cones of fields was generalized to prepositive cones of fields. Prepositive cones are also called preorderings. A prepositive cone of a field is defined as follows.

Let $(F, +, \times)$ be a field and a subset P of F is called a *prepositive cone* or *preordering* of F if P has the following properties:

- $P + P \subseteq P$,
- $P \cdot P \subseteq P$,
- $\sum F^2 \subseteq P$, where $\sum F^2$ is a subset of F which consists of all the finite sums of perfect squares in F.
- $-1 \notin P$.

A field together with a prepositive cone P is called a *preordered field*, where an order is defined as:

$$a \leq b \Leftrightarrow b - a \in P, \qquad \text{for all } a, b \in F.$$

We note here that preorderings induce a partial order relation on the field F. Further, for any positive cone P of a field F, $a^2 \in P$ for all $a \in F$, since for any $a \in F$ if $a \in P$ then clearly $a^2 \in P$ or if $-a \in P$ then $a^2 = (-a)(-a) \in P$. In particular $1^2 = 1 \in P$ and this implies that $-1 \notin P$. Thus we get that every positive cone P of a field F is also a prepositive cone. The following two facts are easy to prove [**72**, Chapter 1].

1. A preordering P is an ordering if and only if P is maximal as a preordering;

2. Any preordering P can be enlarged into an ordering.

An important result in the theory of ordered fields is a theorem due to Artin and Schreier [**6**], that gives an algebraic characterization of fields which admit some ordering. These fields are called *formally real fields* or *orderable fields.*

Theorem 1.2.55. [**6**] *For a field F, the following are equivalent:*

1. *F is formally real,*

2. *$-1 \notin \sum F^2$, where $\sum F^2$ is a subset of F which consists of all the finite sums of perfect squares in F,*

3. *$\sum_i a_i^2 = 0$ implies that $a_i = 0$ for all i, where $a_i \in F$.*

Note that a field is an ordered field if and only if it is formally real. Finite field or fields of finite characteristic are not formally real, since in $\mathbf{Z}_p$ we have $-1 = \underbrace{1^2 + 1^2 + ... + 1^2}_{(p-1) times}$. In other words, any ordered field has characteristic zero.

Lam [**61**] extended the concept of ordering to rings. He defined preorderings for commutative rings in the same way as for fields and generalized the Artin-Schreier theorem to rings.

Definition 1.2.56. (*Preordering*) *A subset P of a commutative ring R is called a prepositive cone or preordering of R if $P + P \subseteq P$, $P \cdot P \subseteq P$, $\sum R^2 \subseteq P$ and $-1 \notin P$. A ring R together with a prepositive cone P is called a preordered ring.*

We note that if R is a field then $P \cap -P = \{0\}$. Otherwise there exist nonzero elements $a, b \in P$ such that $a = -b$. This implies that $-1 = a/b = ab(b^{-1})^2 \in P$, a contradiction. If R is a ring (not a field) then $P \cap -P$ need not be zero. It is easy to check that $I = P \cap -P$ is an additive subgroup of R. The group I is clearly a largest additive subgroup of R contained in P. Moreover, for any preordering P of R if $P \cup -P = R$, then $I = P \cap -P$ forms an ideal of R.

Lam [**61**] defined the notion of an ordering for an arbitrary commutative ring as follows:

Definition 1.2.57. (*Ordering*) *A preordering P of a ring R is called an ordering if it satisfies further two conditions:*

1. *$P \cup -P = R$, and*

2. *$I = P \cap -P$ is a prime ideal of R.*

Definition 1.2.58. (*Ordered Rings*) *A ring $(R, +, \times)$ together with an ordering P is called an ordered ring, where an order is defined as:*

$$a \leq b \Leftrightarrow b - a \in P, \qquad \textit{for all } a, b \in R.$$

Many results for ordered rings has been proved which are analogous to those of ordered fields. For example, If P is a maximal preordering of a ring R then P is an ordering.

We are now ready to state the generalization of the Artin-Schreier theorem for commutative rings.

Theorem 1.2.59. [**61**] *For a commutative ring R, the following are equivalent:*

1. *R is a formally real ring or an orderable ring,*
2. *$-1 \notin \sum R^2$, where $\sum R^2$ is a subset of R which consists of all the finite sums of perfect squares in R,*
3. *$\sum_i a_i^2 = 0$ implies that $a_i = 0$, for all i, where $a_i \in R$.*

Formally real rings have interesting properties in its own right; the results in [**83**] are a good example of this. In chapter 2, we review the theory of ordered semirings and in chapter 3, we examine completely positive matrices over formally real semirings.

Chapter 2

Completely Positive Matrices and Semirings

The aim of this chapter is to give some background information about real completely positive matrices and the theory of matrices over semirings. This chapter begins with a basic introduction to real completely positive matrices. We then give an overview of some central results about these matrices. We also discuss the CP-rank of real completely positive matrices and the Drew-Johnson-Loewy conjecture in detail. In addition we study the theory of semirings and matrices over semirings.

2.1 Completely Positive Matrices

Complete positivity is found in fields of mathematics that require some kind of nonnegativity, such as combinatorial analysis, optimization, probability and statistics.

Most of the definitions and results in this section can be found in [**20**].

Definition 2.1.1. (*Completely Positive Matrices*) *An* $n \times n$ *real matrix* A *is called completely positive (CP) if, for some positive integer* m*, there exists an* $n \times m$ *nonnegative matrix* B *such that*

$$A = BB^T. \tag{2.1}$$

Such a decomposition is called a *completely positive decomposition.* The set of all real completely positive $n \times n$ matrices is denoted by CP_n.

An example of a real completely positive matrix along with its factorization is

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Note that the completely positive decomposition of a completely positive matrix is not unique. We have more than one completely positive decompositions of a CP matrix. For example:

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 2\sqrt{3} & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix}.$$

It is clear that an $n \times n$ real matrix A is completely positive if and only if A can be written as

$$A = \begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} \begin{bmatrix} b_1^T \\ b_2^T \\ . \\ . \\ . \\ b_m^T \end{bmatrix},$$

i.e.,

$$A = b_1 b_1^T + b_2 b_2^T + ... + b_m b_m^T \tag{2.2}$$

where $b_i \in R_+^n$, for all $i = 1, ..., m$. In the equation (2.2), b_i corresponds to the i^{th} column of B (in the definition 2.1.1), for all i and all $B_i^{\prime}s = b_i b_i^T, (i = 1, 2...m)$ are called the rank one completely positive matrices. We refer to the equation (2.2) as a *rank 1 CP-representation of A*.

For $b_j \in R_+^n$, the support of b_j is defined as:

$$supp(b_j) = \{i : b_{ij} \neq 0\}. \tag{2.3}$$

The cardinality of supp(b_j) is denoted by $s(b_j)$. A representation

$$A = \sum_{j=1}^{k} b_j b_j^T, \qquad b_j \geq 0, \qquad s(b_j) \leq t \tag{2.4}$$

is called a *support t rank 1 CP-representation of A*.

We could also partition B (in the definition 2.1.1) into rows, obtaining yet an equivalent definition. Assume that an $n \times n$ real completely positive matrix A has a completely positive decomposition $A = BB^T$. Let v_i^T be the i^{th} row of B, for all $i = 1, 2, ...n$, then

$$A = \begin{bmatrix} v_1^T \\ v_2^T \\ . \\ . \\ v_n^T \end{bmatrix} \begin{bmatrix} v_1 & v_2 & . & . & . & v_n \end{bmatrix}$$

$$= \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & . & . & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & . & . & \langle v_2, v_n \rangle \\ . & . & . & . & . \\ . & . & . & . & . \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & . & . & \langle v_n, v_n \rangle \end{bmatrix} \tag{2.5}$$

This matrix is called the $Gram(v_1, v_2, ..., v_n)$. Thus we see that an equivalent condition for complete positivity of a real matrix A is that A is the Gram matrix of nonnegative vectors.

These equivalent definitions of real completely positive matrices are summarized in the following proposition:

Proposition 2.1.2. [**20**] *Let A be an $n \times n$ real matrix then the following three conditions are equivalent:*

1. *$A = BB^T$, for some nonnegative $n \times m$ matrix B,*

2. $A = \sum_{j=1}^{m} b_j b_j^T,$ *where* $b_j \geq 0$ *for all* $j = 1, 2, ..., m,$

3. $A = Gram(v_1, v_2, ..., v_n),$ *where* $v_j \in R_+^m$ *for all* $j = 1, 2, ..., n,$

The concept of complete positivity was first introduced in 1963, by Hall and Newman [**47**]. They defined complete positivity for quadratic forms used in combinatorial problems.

Definition 2.1.3. (*Copositive Quadratic Forms*) *A real quadratic form in* n *variables*

$$Q = Q(x_1, ..., x_n) = \sum a_{ij} x_i x_j$$

is called copositive if $Q(x_1, ..., x_n) \geq 0$ *whenever* $x_i \geq 0$ *for all* i*.*

Definition 2.1.4. (*Completely Positive Quadratic Forms*) *A real quadratic form* Q *in* n *variables is called completely positive if there exist nonnegative linear forms*

$$L_k = c_{1k} x_1 + ... + c_{nk} x_n \qquad (k = 1, 2, ..., t, \quad c_{ik} \geq 0)$$

such that

$$Q = L_1^2 + ... + L_t^2.$$

Hall and Newman showed that the set of completely positive matrices (i.e., the space of completely positive quadratic form) is a convex cone and the dual of this cone is the cone of copositive matrices (i.e., the cone of copositive quadratic forms). Surveys of both of these cones and their applications are provided in [**20**].

The cone of copositive matrices is defined as:

$$C^n = \{A \in S^n : x^T A x \geq 0 \text{ for all } x \in \mathbb{R}_+^n\},$$

and its dual cone, the cone of completely positive matrices is defined as:

$$C^{n*} = conv\{xx^T : x \in \mathbb{R}_+^n\}.$$

Note that if A is a real positive semidefinite matrices then we can write

$$A = \begin{bmatrix} v_1^T \\ v_2^T \\ . \\ . \\ v_n^T \end{bmatrix} \begin{bmatrix} v_1 & v_2 & . & . & . & v_n \end{bmatrix} = [\langle v_i, v_j \rangle],$$

where $v_1, v_2, ..., v_n$ are vectors in an m-dimensional Euclidean space V and these vectors may or may not be nonnegative. Moreover, if the real positive semidefinite matrix A is nonnegative then we get that $\langle v_i, v_j \rangle \geq 0$, for all i, j. In other words, $A \geq 0$ means that the distance between v_i and v_j is less than or equal to $\pi/2$. The real nonnegative positive semidefinite matrix A is called a completely positive matrix if there exists an isometry $T : V \longrightarrow \mathbb{R}^k$, for some natural number k such that $Tv_i \in \mathbb{R}_+^n$, for all $i = 1, 2, ..., n$. Therefore, every real completely positive matrix is doubly nonnegative, that is, it is both a nonnegative matrix and a positive semidefinite matrix. In some cases, the converse is also true, but not always. Maxfield and Minc [**74**] have shown by using matrix theory that if A is a real matrix of order $n \leq 4$, then the equation $A = XX^T$ has a solution X with nonnegative entries if and only if A has no negative entries and A is positive semidefinite (psd). In other words, for $n \leq 4$, every $n \times n$ real matrix is completely positive if and only if it is both a nonnegative and a positive semidefinite matrix. Further, for $n \geq 5$, there exist doubly nonnegative $n \times n$ real matrices which are not completely positive. Here is an example of a 5×5 doubly nonnegative matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 8 \\ 1 & 1 & 1 & 11 & 0 \\ 1 & 1 & 8 & 0 & 74 \end{bmatrix}$$

Gray and Wilson [**41**] have shown that A does not have a nonnegative factorization. Hence A is not completely positive.

There are several results on complete positivity of real matrices in the literature. Kaykobad [**54**] showed that every diagonally dominant symmetric nonnegative real matrix is a completely positive matrix. Ando [**2**] showed that every totally positive symmetric matrix is a completely positive matrix. Drew, Johnson and Loewy [**30**] showed that a symmetric nonnegative matrix A is completely positive if its comparison matrix is an M-matrix. Moreover, if A is a real completely positive matrix and $G(A)$ is a triangle free graph then the comparison matrix $M(A)$ is an M-matrix. However, it is still an open problem to decide whether a given real matrix is completely positive or not, in general.

2.1.1 Basic Results

In this subsection, we review some main results on completely positive matrices. We cover only those results which we use later, and mostly without proof. For reference see book [**20**].

Proposition 2.1.5. *Completely positive matrices are closed under addition and multiplication by nonnegative scalars.*

The proof of this proposition directly follows from the rank 1 CP-representation of completely positive matrices.

Proposition 2.1.6. *If A is an $n \times n$ real completely positive matrix, and C is an $m \times n$ real nonnegative matrix, then CAC^T is a real completely positive matrix.*

The following corollary contains two simple results of proposition 2.1.6, which are very useful in the study of completely positive matrices.

Corollary 2.1.7. *Let A be a symmetric $n \times n$ matrix. Then*

1. *If P be an $n \times n$ permutation matrix, then A is completely positive if and only if PAP^T is completely positive.*

2. *If D be a real positive $n \times n$ diagonal matrix, then A is completely positive if and only if DAD is completely positive.*

Theorem 2.1.8. *The m^{th} power of any $n \times n$ completely positive matrix A is completely positive for any positive integer m.*

Proof. If m is even, $m = 2s$ for some s, then $A^m = (A^s)^2 = (A^s)(A^s)^T$, which is a completely positive decomposition of A^m. If $m = 2s+1$, then $A^m = (A^s)A(A^s)^T$, Since A^s is an $n \times n$ nonnegative matrix, hence A^m is completely positive. ■

Corollary 2.1.9. *If $f(x) = \sum_{i=0}^{m} a_i x^i$ is a real polynomial with nonnegative coefficients and A is a real completely positive matrix, then $f(A) = \sum_{i=0}^{m} a_i A^i$ is also completely positive.*

The proof of the next proposition is similar to the corresponding proof in case of positive semidefinite matrices.

Proposition 2.1.10. *Principal submatrices of completely positive matrices are completely positive.*

There is another type of product of real matrices:

Definition 2.1.11. (*Hadamard Product*) *Let A and B are two $m \times n$ real matrices. The Hadamard product of A and B is the matrix $C = A \circ B$, defined as the entry by entry product. That is, $c_{ij} = a_{ij}b_{ij}$.*

Proposition 2.1.12. *The Hadamard product of completely positive matrices is completely positive.*

Proof. Let $A = \sum_{i=1}^{k} b_i b_i^T$, where $b_i \in \mathbb{R}^n_+$ for all i, and $B = \sum_{j=1}^{l} c_j c_j^T$, where $c_j \in \mathbb{R}^n_+$ for all j, be rank 1 CP-representations of A and B. Then

$$A \circ B = \sum_{i=1}^{k} \sum_{j=1}^{l} (b_i \circ c_j)(b_i \circ c_j)^T$$

■

Corollary 2.1.13. *The Hadamard powers* $A^{(k)} = A \circ A \circ ... \circ A$ *of completely positive matrix* A *are also completely positive.*

2.1.2 The Completely Positive Rank (The CP-rank)

The completely positive rank has received a lot of attention due to its application in optimization, probability and statistics, and combinatorial analysis.

Definition 2.1.14. (*The CP-rank*) *Let* A *be an* $n \times n$ *real completely positive matrix. The minimal integer* m *such that* $A = BB^T$ *for some nonnegative* $n \times m$ *real matrix* B, *is called the CP-rank of* A. *The CP-rank of* A *is denoted by CP-rank*(A) *and the decomposition* $A = BB^T$ *is called a minimal completely positive decomposition of* A.

In general, the minimal completely positive decomposition of a completely positive matrix is not unique. Let $A = BB^T$ be a minimal completely positive decomposition of A and the CP-rank$(A) = m$. Then for any $m \times m$ nonnegative orthogonal matrix P,

$$A = BPP^TB^T = (BP)(BP)^T = CC^T,$$

where $C = BP$, is also a minimal completely positive decomposition of A. Further, if the CP-rank$(A) = m$, then there exists a rank 1 CP-representation of A consisting of m rank one completely positive matrices, i.e.,

$$A = b_1b_1^T + b_2b_2^T + ... + b_mb_m^T.$$

Using this rank 1 CP-representation, it is easy to see that

Proposition 2.1.15. *If A and B are $n \times n$ completely positive matrices, then*

$$\text{CP-rank}(A+B) \leq \text{CP-rank}(A) + \text{CP-rank}(B).$$

If A is an $n \times n$ real completely positive matrix then $A \in conv\{bb^T : b \in \mathbb{R}^n_+\}$, which is a convex cone contained in $\mathbf{S}^n_+$, and by Caratheodory's theorem the CP-rank of A

$$\text{CP-rank}(A) \leq \frac{n(n+1)}{2}. \tag{2.6}$$

The rank of an $n \times n$ real matrix is always less than or equal to n. However, the CP-rank of an $n \times n$ real completely positive matrix can be bigger than n. It is evident from the definition of completely positive matrices that for any real $n \times n$ completely positive matrix A

$$\text{CP-rank}(A) \geq \ \text{rank}(A) \tag{2.7}$$

In the literature, the CP-rank of an $n \times n$ real completely positive matrix has been compared with its usual rank function. Barioli and Berman [**12**] showed that if A is a real completely positive matrix of rank r, then

$$\text{CP-rank}(A) \leq \frac{r(r+1)}{2} - 1, \qquad \text{if } r > 2 \tag{2.8}$$

and

$$\text{CP-rank}(A) = r, \qquad \text{if } r \leq 2. \tag{2.9}$$

It has been shown [**20**] that, for $n \leq 4$, if A is an $n \times n$ real completely positive matrix then

$$\text{CP-rank}(A) = \text{ rank}(A) \text{ if } n \leq 3. \tag{2.10}$$

and

$$\text{CP-rank}(A) \leq n \text{ if } n \leq 4. \tag{2.11}$$

There exists 4×4 real completely positive matrices whose CP-rank is strictly greater than actual rank. For example [**20**] the matrix

$$A = \begin{bmatrix} 6 & 3 & 3 & 0 \\ 3 & 5 & 1 & 3 \\ 3 & 1 & 5 & 3 \\ 0 & 3 & 3 & 6 \end{bmatrix}$$

satisfies that the rank$(A) = 3$ and the CP-rank$(A) = 4$

However, for $n \geq 5$, the CP-rank of an $n \times n$ completely positive matrix can be larger than n.

It would be of great interest to have an efficient algorithm to decide if a given matrix is completely positive or an efficient algorithm for computing the CP-rank of a given completely positive matrix. While there is no efficient way of solving either problem for real matrices, we will see later that for certain semiring cases there is an easy test for complete positivity.

2.1.3 Completely Positive Graphs

Many of the known results on completely positive matrices and their CP-rank are graph based. Here a graph means a simple undirected graph. If G is a simple undirected graph and A is a real matrix such that $G(A) = G$ then we say that A is a *matrix realization* of G. A real doubly nonnegative (DNN) matrix A such that $G(A) = G$ is called a *DNN matrix realization* of G.

Definition 2.1.16. (*Completely Positive Graphs*) *A graph G is completely positive if every DNN matrix realization of G is completely positive.*

We have seen in previous sections that every real doubly nonnegative (DNN) matrix of order less than or equal to four is completely positive. Therefore, graphs up to four vertices are completely positive. Completely positive graphs are characterized by Kogan and Berman [**56**]. They showed that a graph G is completely positive if and only if it does not contain an odd cycle of length greater than four. A real completely positive matrix A such that $G(A) = G$ is called a *CP matrix realization* of G.

Definition 2.1.17. (*The CP-rank of a Graph*) *Let G be a graph on n vertices. The CP-rank of G, denoted by CP-rank(G), is the maximal CP-rank of a completely positive (CP) matrix realization of G, that is,*

$$\textit{CP-rank}(G) = \max\{\textit{CP-rank}(A) | A \text{ is CP and } G(A) = G\}.$$

In 1987, Berman and Hershkowitz [**13**] showed that a completely positive matrix A satisfies that the CP-rank$(A) = \text{rank}(A)$ if $G(A)$ is a tree. In 2001, Shaked-Monderer

[**80**] gave a necessary and sufficient conditions for real completely positive matrices whose CP-rank is exactly equal to the rank. She proved that every completely positive matrix A satisfies cp-rank$(A) = $ rank(A) if and only if $G(A)$ contains no even cycle, and no triangle-free graph with more edges than vertices. Drew, Johnson and Loewy [**30**] proved that if A is a nonnegative symmetric matrix, $G(A)$ is a connected graph and $M(A)$ is a positive semidefinite matrix then the CP-rank of A is less than or equal to the maximum of number of vertices in $G(A)$ and number of edges on $G(A)$.

The most famous open problem in the theory of completely positive matrices is the following conjecture stated by Drew, Johnson and Loewy.

2.1.4 The Drew-Johnson-Loewy Conjecture

In [**30**], Drew, Johnson and Loewy stated the following conjecture.

Conjecture 2.1.18. *If A is a real completely positive matrix of order $n \geq 4$ then*

$$\textit{CP-rank(A)} \leq [n^2/4].$$

Here $[x]$ is the greatest integer function.

This conjecture has been proven only for matrices with special graphs, or for certain special classes of matrices.

The Drew-Johnson-Loewy conjecture can be rephrased [**80**] as: For every graph G on $n \geq 4$ vertices, CP-rank$(G) \leq [n^2/4]$. It has been proven for: Triangle free graphs in [**30**], Graphs which contain no odd cycle on 5 or more vertices in [**29**], All graphs on 5 vertices which are not the complete graph in [**66**], Nonnegative matrices with a positive semidefinite comparison matrix in [**19**], All 5×5 completely positive matrices in [**82**].

Recently Bomze et al. [**22**] have shown that the CP-rank of real completely positive matrices of order seven through eleven, does not satisfy the upper bound proposed by Drew, Johnson and Loewy. Such real completely positive matrices lie on the boundary of completely positive matrices cone and are orthogonal to copositive matrices. Bomze et al. studied copositive cyclically symmetric matrices (S) which generate a quadratic form with finitely many zeros (q_i) (i.e., $q_i S q_i^T = 0$) such that $e^T q_i = 1$. These copositive cyclically symmetric matrices are orthogonal to real completely positive matrices of type $\sum\limits_i y_i q_i q_i^T$, where $y_i \in \mathbb{R}_+$. They constructed a 7×7 copositive cyclically symmetric matrix S with 14 zeros (q_i) (i.e., $q_i S q_i^T = 0$) such that $e^T q_i = 1$ for all $i = 1, 2, ...14$. Then a real completely positive matrix A orthogonal to S is of type

$$A = \sum_{i=1}^{14} y_i q_i q_i^T. \tag{2.12}$$

where $y_i \in \mathbb{R}_+$. They proved that (2.12) is a unique minimal rank 1 CP-representation of A and hence the CP-rank$(A) = 14$, which is greater than $[7^2/4] = 12$. Similarly, they constructed counterexamples to the Drew, Johnson and Loewy conjecture for real completely positive matrices of order $n = 8, 9, 10, 11$.

In this thesis, we prove that the Drew-Johnson-Loewy conjecture is true for completely positive matrices over various semirings, specially over the Boolean semiring, the max-plus semiring, certain special types of inclines and Boolean algebras.

2.1.5 Almost Principal Minors

In this section, we look at some of the theory relating the nonnegativity of almost principal minors and triangular decomposition of real completely positive matrices.

A minor of a real matrix A is the determinant of a square submatrix of A. There are two other classes of minors which play a key role in the theory of completely positive matrices.

Definition 2.1.19. *Let $A \in M_n(\mathbb{R})$. Let $\alpha = \{\alpha_1 < \alpha_2 < ... < \alpha_k\}$ and $\beta = \{\beta_1 < \beta_2 < ... < \beta_k\}$ be two subsets of $\{1, 2, ..., n\}$ of the same cardinality. Then the $k \times k$ submatrix, $A[\alpha|\beta]$ is called a left almost principal submatrix of A if $\alpha_j = \beta_j$ for all $j : 2 \leq j \leq k$ but $\alpha_1 \neq \beta_1$. The determinant of a left almost principal submatrix is called a left almost principal minor.*

Definition 2.1.20. *Let $A \in M_n(\mathbb{R})$. Let $\alpha = \{\alpha_1 < \alpha_2 < ... < \alpha_k\}$ and $\beta = \{\beta_1 < \beta_2 < ... < \beta_k\}$ be two subsets of $\{1, 2, ..., n\}$ of the same cardinality. Then the $k \times k$ submatrix, $A[\alpha|\beta]$ is called a right almost principal submatrix of A if $\alpha_j = \beta_j$ for all $j : 1 \leq j \leq k-1$ but $\alpha_k \neq \beta_k$. The determinant of a right almost principal submatrix is called a right almost principal minor.*

The almost principal minors play a role in the theory of nonnegative LU and UL decompositions.

Definition 2.1.21. *We say that an n by n real matrix A is UL-completely positive if there exists an n by n upper triangular nonnegative matrix B such that $A = BB^T$. We say that A is LU-completely positive if there exists an n by n lower triangular nonnegative matrix C such that $A = CC^T$.*

The following results are due to Markham [**67**] and can also be found in the reference [**20**]

Theorem 2.1.22. *Let A be a doubly nonnegative matrix. If all of the left almost principal minors of A are nonnegative then A is UL-completely positive.*

Theorem 2.1.23. *Let A be a doubly nonnegative matrix. If all of the right almost principal minors of A are nonnegative then A is LU-completely positive.*

It has been shown in [**20**], that for $n \leq 3$, every $n \times n$ real completely positive matrix is either LU-completely positive or UL-completely positive or both LU-completely positive and UL-completely positive. However for $n \geq 4$, an $n \times n$ completely positive real matrix may be neither UL-completely positive nor LU-completely positive. An example is the matrix below given in [**20**, Example 2.17]. For example, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

is a real completely positive matrix, but it is neither UL-completely positive nor LU-completely positive.

Note that if a real matrix A is LU-completely positive, then it does not necessarily follow that A is UL-completely positive. For example, consider the matrix from [**20**, Example 2.19],

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

A is LU-completely positive and the LU-completely positive factorization of A is

$$A = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{5}}{\sqrt{2}} & 0 \\ \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{5}}{\sqrt{2}} & 0 \\ \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{bmatrix}^T$$

but it can be verified that A is not UL-completely positive.

The following example shows that a UL-completely positive real matrix may not be LU-completely positive. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 5 \end{bmatrix}$$

It is UL-completely positive and its UL-completely positive factorization of A is

$$A = \begin{bmatrix} \sqrt{\frac{5}{35}} & \sqrt{\frac{2}{35}} & \frac{3}{\sqrt{5}} \\ 0 & \sqrt{\frac{14}{5}} & \frac{1}{\sqrt{5}} \\ 0 & 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{5}{35}} & \sqrt{\frac{2}{35}} & \frac{3}{\sqrt{5}} \\ 0 & \sqrt{\frac{14}{5}} & \frac{1}{\sqrt{5}} \\ 0 & 0 & \sqrt{5} \end{bmatrix}^T$$

but it can be verified that A is not LU-completely positive.

A real matrix is called totally nonnegative if all of its minors are nonnegative. The class of nonnegative matrices is an area of great interest. We note that the following is an immediate consequence of Markham's theorems.

Corollary 2.1.24. *Any square symmetric totally nonnegative matrix is both LU and UL-completely positive.*

2.2 Semirings

Semirings naturally appear in different problems of optimization, communication complexity, Graph theory, mathematical modeling, etc. Semiring arithmetics allow us to describe and solve certain types of combinatorial problems by applying linear-algebraic approach, for instance see [**10**]. Matrix theory over semirings is an object of intensive study during the recent years. Different rank functions for matrices over various classes of semirings are intensively investigated during the last decades, for example: factor rank, tropical rank, nonnegative rank, determinantal rank and Gondran-Minoux rank. For details see [**3, 11**]. In this thesis, we will study completely positive matrices over various classes of semirings and their CP-rank. We also define a family of ranks of matrices over certain semirings and classify all bijective linear maps which preserve these ranks.

Semirings are a fairly natural generalization of rings. Semirings satisfy all properties of unital rings except the existence of additive inverses. H. S. Vandiver introduced the concept of semiring in [**86**], in connection with the axiomatization of the arithmetic of the natural numbers. Recent developments in both theory and applications of semirings have been studied by Gondran and Minoux, in [**39**] and by Golan in [**35, 36**].

Definition 2.2.1. (*Semirings*) *A semiring is a set S together with two operations $\oplus$ and $\otimes$ and two distinguished elements $\mathbf{0}, \mathbf{1}$ in S with $\mathbf{0} \neq \mathbf{1}$, such that*

1. *$(S, \oplus, \mathbf{0})$ is a commutative monoid,*
2. *$(S, \otimes, \mathbf{1})$ is a monoid,*
3. *$\otimes$ is both left and right distributive over $\oplus$,*
4. *The additive identity $\mathbf{0}$ satisfies the property $r \otimes \mathbf{0} = \mathbf{0} \otimes r = \mathbf{0}, \quad$ for all $r \in S$.*

In other words, semirings are unital rings without the requirement that each element has additive inverse. If (S, $\otimes$, **1**) is a commutative monoid then S is called a *commutative semiring.*

Note 1. *We use $\oplus$ for semiring addition and $\otimes$ for semiring multiplication.*

Definition 2.2.2. (*Antinegative Semiring*) *A semiring is said to be antinegative or zerosumfree if the only element with an additive inverse is the additive identity $\mathbf{0}$.*

Definition 2.2.3. (*Semiring with no zero-divisors*) *A semiring S is said to have no zero-divisors, if $ab = \mathbf{0}$ implies that either $a = \mathbf{0}$ or $b = \mathbf{0}$, for all $a, b \in S$.*

An antinegative semiring with no zero-divisors is called a *positive semiring* [**32**, p. 125]. An element of a semiring S is called a *unit* if it has a multiplicative inverse in S. If every element (except the additive identity) of a semiring S is a unit then S is called a *semifield*.

Definition 2.2.4. (*The Unique Square Root Property*) *A semiring S is said to have the unique square root property if for any $x \in S$ there exists a unique $c \in S$ such that $x = c^{\otimes 2}$, where $c^{\otimes 2} = c \otimes c$. We also write this as $\sqrt{x} = c$.*

Definition 2.2.5. (*Subsemiring*) *A subset T of a semiring $(S, \oplus, \otimes)$ is called a subsemiring of S if $\mathbf{0}, \mathbf{1} \in T$ and $(T, \oplus, \otimes)$ forms a semiring.*

An element $a \in S$ is said to be additively (resp. multiplicatively) *idempotent* if $a \oplus a = a$ (resp. $a \otimes a = a$). A semiring S is said to be additively (resp. multiplicatively) *idempotent* if every element of S is additively (resp. multiplicatively) idempotent.

2.2.1 Examples of Semirings

The set of all nonnegative real numbers $\mathbb{R}_+$ under the usual addition and multiplication forms a semiring, where 0 is the additive identity and 1 is the multiplicative identity. The set of all natural numbers including zero forms a semiring under the usual addition and multiplication. In this semiring 0 is the additive identity and 1 is the multiplicative identity.

A much studied example of a semiring is *the max-plus semiring*, where $\mathbb{R}_{max} = \mathbb{R} \bigcup \{-\infty\}$ with

$$a \oplus b = \max\{a, b\}$$

$$a \otimes b = a + b.$$

Note that [**71**] in this case $\mathbf{0}$ = -∞ and $\mathbf{1} = 0$. The set of elements less than or equal to the multiplicative identity $\mathbf{1} = 0$ is a subsemiring of the max-plus semiring. We call this subsemiring *the negative interval semiring* and denote it as $[-\infty, 0]$. The set of elements greater than or equal to the multiplicative identity $\mathbf{1} = 0$ together with $\{-\infty\}$ also forms a subsemiring of max-plus semiring. We call this subsemiring *the nonnegative interval semiring* and denote it as $\{-\infty\} \cup [0, \infty)$. The max-plus semiring provides a way of formulating many hard combinatorial optimization problems (like scheduling problems) in terms of a very elegant linear algebraic problems. The max-plus semiring has also applications in information technology and discrete-event dynamic systems [**25, 10**].

The *min-plus semiring* is isomorphic to the max-plus semiring and is defined as follows: $\mathbb{R}_{min} = \mathbb{R} \bigcup \{\infty\}$ with

$$a \oplus b = \min\{a, b\}$$

$$a \otimes b = a + b.$$

The name tropical semiring is sometimes used for the max-plus or min-plus semiring.

A totally ordered set S with greatest element $\mathbf{1}$ and least element $\mathbf{0}$ forms a semiring with

$$a \oplus b = max\{a, b\}$$

$$a \otimes b = min\{a, b\}.$$

This is called a *max-min semiring.* Max-min semirings are sometimes called *chain semirings* or fuzzy algebras. Clearly the additive identity of S is the least element of S and the multiplicative identity of S is the greatest element of S. The max-min semiring with two elements $\{\mathbf{0}, \mathbf{1}\}$ is called the *Boolean semiring.*

The set $\{+1, -1, 0, \#\}$, where # symbol denotes an unknown sign, forms a semiring, with the operations of addition and multiplication are defined as follows:

$\oplus$	0	+1	-1	#
0	0	+1	-1	#
+1	+1	+1	#	#
-1	-1	#	-1	#
#	#	#	#	#

$\otimes$	0	+1	-1	#
0	0	0	0	0
+1	0	+1	-1	#
-1	0	-1	+1	#
#	0	#	#	#

This is called the *sign pattern semiring.* Here 0 is the additive identity and +1 is the multiplicative identity. For more details see [**4**]. A subset $\{0, +1, \#\}$ of the sign pattern semiring S forms a subsemiring of S. We call this subsemiring *the nonnegative sign pattern semiring.*

If $(S, \oplus, \otimes)$ is a semiring and x is an indeterminate over S then the set of all polynomials

$$S[x] = \left\{ \bigoplus_{i=0}^{n} a_i x^i \mid n \geq 0, a_i \in S \right\}.$$

forms a semiring, called polynomial semiring [**50**] and it is denoted as $(S[x], \oplus, \otimes)$. Addition and multiplication in $S[x]$ is defined in the same way as defined for $\mathbb{R}[x]$ in

1.2.1.ii. The additive identity of $S[x]$ is the zero polynomial $O(x) = \mathbf{0}$ and the multiplicative identity of $S[x]$ is the identity polynomial, which is a constant polynomial equal to $\mathbf{1}$. If S is a commutative and antinegative semiring then so is $S[x]$.

A subset S of the polynomial ring $\mathbb{R}[x]$, where

$$S = \{p(x) \in \mathbb{R}[x] : \text{ such that } p(x) \geq 0, \text{ for all } x \in \mathbb{R}\}$$

forms a semirings under addition and multiplication of polynomials in $\mathbb{R}[x]$. The additive identity of S is the zero polynomial, $O(x) = 0$ and the multiplicative identity of S is a constant polynomial equal to 1.

Any distributive lattice with a unique minimal element $\mathbf{0}$ and a unique maximal element $\mathbf{1}$ forms a semiring under addition and multiplication defined as:

$$a \oplus b = a \vee b = l.u.b\{a, b\} \text{ and}$$
$$a \otimes b = a \wedge b = g.l.b\{a, b\}.$$

A Boolean algebra B with a unique minimal element $\mathbf{0}$ and a unique maximal element $\mathbf{1}$, forms a semiring where addition and multiplication is defined as follows:

$$a \oplus b = a \cup b$$
$$a \otimes b = a \cap b.$$

Here $\cap$ denotes the *intersection* operation and $\cup$ denotes the *union* operation. It is obvious that, for all $x, y \in B$

$$x \leq y \qquad \Leftrightarrow \qquad x = y \cap x \text{ and } y = x \cup y.$$

We denote a general Boolean algebra with a unique minimal element $\mathbf{0}$ and a unique maximal element $\mathbf{1}$ by $(B, \cup, \cap, *, \mathbf{0}, \mathbf{1})$. Here $*$ denotes the complement operation. A Boolean algebra with only two elements $\{\mathbf{0}, \mathbf{1}\}$ is called the *Boolean semiring* or the *binary Boolean algebra* and it is denoted by β. We denote a Boolean algebra of subsets of a k-element set S_k by β_k. In β_k, a unique minimal element $\mathbf{0}$ denotes the null set and a unique maximal element $\mathbf{1}$ denotes the set S_k.

Definition 2.2.6. (*Boolean Subalgebra*) [**37**] *A* (*Boolean*) *subalgebra of a Boolean algebra B is a subset C of B such that C, together with the distinguished elements $\boldsymbol{0}, \boldsymbol{1}$ and operations $\cup, \cap, *$, of B* (*restricted to the set C*), *is a Boolean algebra. The Boolean algebra B is called a Boolean extension of C.*

Note that the minimal element $\mathbf{0}$ and the maximal element $\mathbf{1}$ are essential parts of the structure of a Boolean algebra. If Y is a non-empty subset of a set X then both the power set of X, 2^X and the power set of Y, 2^Y are Boolean algebras. Evidently every element of 2^Y is an element of 2^X. Since, the maximal element $\mathbf{1}_{2^X}$ of 2^X is X, whereas the maximal element $\mathbf{1}_{2^Y}$ of 2^Y is Y, it is not true that 2^Y is a Boolean subalgebra of 2^X.

All these examples of semirings are both commutative and antinegative.

In section 1.2.7, we have seen that squares and their sums play an important role in ordering algebraic structures. We now construct subsemirings using squares and their sums. This construction will be very helpful in later chapters.

Let S be a commutative semiring. Let $P(S) = \sum S^{\otimes 2}$ be a subset of S which consists of all the finite sums of perfect squares in S, i.e., if $a \in P(S)$ then

$$a = \bigoplus_{k=1}^{m} b_k^{\otimes 2}, \tag{2.13}$$

where $b_k \in S$, for all $k = 1, 2, ..., m$ and $m \in \mathbb{N}$. It is easy to check that $P(S)$ forms a subsemiring of S and we call it *the positive subsemiring* of S. As an example, if S is a semiring of all real numbers then $P(S) = \mathbb{R}_+$ and if $S = \mathbb{R}[x]$ then $P(S)$ is the set of all nonnegative real polynomials, by theorem 1.2.11. Similarly, we define $P(P(S))$, the set of all the finite sums of perfect squares in $P(S)$. Thus we get a sequence of semirings:

$$S \supseteq P(S) \supseteq P(P(S)) \supseteq ..... \tag{2.14}$$

If in a semiring S every element has a square root, i.e., for all $a \in S$ there exists $c \in S$ such that $a = c^{\otimes 2}$, then clearly $S = P(S) = P(P(S)) = ...$. Examples of such semirings are the Boolean semiring, the max-plus semiring, max-min semirings, Boolean algebras, distributive lattices, etc. Sometimes we have a strict inequality at the first place of (2.14) and equality everywhere else. A semiring of all real numbers is an example of such kind of semirings, i.e.,

$$(S = \mathbb{R}) \supseteq (P(S) = \mathbb{R}_+) = P(P(S)) = ...$$

There are some semirings, for example the semiring of all real polynomials $\mathbb{R}[x]$, in which we have strict inclusion everywhere in (2.14).

We now study how new semirings can be built from old semirings. Let $(S, \oplus, \otimes)$

be a commutative semiring then $S^2 = \{(a,b)|a,b \in S\}$, where addition and multiplication is defined as follows, [**71**]: For all a, b, c and $d \in S$,

$$(a,b) \oplus (c,d) = (a \oplus c, b \oplus d)$$

$$(a,b) \otimes (c,d) = \big((a \otimes c) \oplus (b \otimes d), (b \otimes c) \oplus (a \otimes d)\big).$$

If S is a semiring then S^2 is called a *symmetrized semiring*.

2.2.2 Matrix Theory Over Semirings

The concepts of matrix theory are defined over a semiring as over a field. Let $M_{m,n}(S)$ denote the set of all m by n matrices over a semiring S and $M_n(S)$ denote the set of all n by n matrices over a semiring S. In a max-min semiring S, when $S = \{\mathbf{0}, \mathbf{1}\}$ we have (binary) Boolean matrices and when $S = [\mathbf{0}, \mathbf{1}]$ we have the fuzzy matrices. Addition and multiplication of matrices over semirings can be defined in the usual way. Let $A \in M_{m,n}(S)$, the element a_{ij} is called the (i,j)-entry of A. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be m by n matrices over a semiring $(S, \oplus, \otimes)$ and let $C = [c_{ij}]$ be an n by p matrix over the same semiring. Then $A + B = [a_{ij} \oplus b_{ij}]$ and $AC = [\bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj}]$.

If A $= (a_{ij})$ is an n by n matrix over a commutative ring, then the standard determinant expression of A is [**77**]:

$$det(A) = \bigoplus_{\sigma \in S_n} sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} ......a_{n\sigma(n)}$$

where S_n is the symmetric group of order n and $sgn(\sigma) = +1$ if σ is even permutation and $sgn(\sigma) = -1$ if σ is odd permutation. Here $sgn(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}......a_{n\sigma(n)}$ is called a term of the determinant.

Since we do not have subtraction in a semiring, we can not write the determinant of a matrix over a semiring in this form. We split the determinant into two parts, the positive determinant and the negative determinant.

Definition 2.2.7. (*The Positive and The Negative Determinant*) *Let A be an n by n matrix over a commutative semiring S, then we define the positive and the negative determinant as:*

$$det^{+}(A) = \bigoplus_{\sigma \in A_n} \bigotimes_{i=1}^{n} a_{i\sigma(i)}$$

$$det^{-}(A) = \bigoplus_{\sigma \in S_n \setminus A_n} \bigotimes_{i=1}^{n} a_{i\sigma(i)}$$

Where A_n is the alternating group of order n i.e, the set of all even permutations of order n and $S_n \setminus A_n$ is the set of all odd permutations of order n.

As such we note that the determinant of a matrix A over a ring takes the form:

$$det(A) = det^{+}(A) - det^{-}(A)$$

In the semiring case, one cannot subtract the negative determinant from the positive determinant and so the positive determinant and the negative determinant are listed as a pair $\left(det^{+}(A), det^{-}(A)\right)$. This pair is called the bideterminant of A. The properties of positive and negative determinants over semirings have been extensively studied; their properties can be found in [**77**] and the reference therein.

2.2.3 Orderings for Semirings

Craven [**24**] introduced a notion of orderings for commutative semirings, which is analogous to the notion of ordered fields and ordered rings. He defined preorderings and orderings for semirings as follows.

Definition 2.2.8. (*Preorderings or Prepositive Cones for Semirings*) *Let S be a commutative semiring and P be a subsemiring with $\sum S^{\otimes 2} \subseteq P \subseteq S$. A subsemiring P of S is called a preordering or a prepositive cone of S if $\boldsymbol{1} \oplus a \neq \boldsymbol{0}$ for any $a \in P$.*

Definition 2.2.9. (*Orderings or Positive Cones for Semirings*) *Let S be a commutative semiring and P be a subsemiring with $\sum S^{\otimes 2} \subseteq P \subseteq S$. A preordering P of S is called an ordering or a positive cone of S if the following conditions hold.*

1. *$s \in S$ and $s \notin P$ implies that there exists $t \in P$ such that $s \oplus t = \boldsymbol{0}$; and*

2. *$ab \oplus p = \boldsymbol{0}$ for some $ab, p \in P$ implies that either $a \in P$ or $b \in P$.*

Definition 2.2.10. (*Preordered Semirings*) *A semiring $(S, \oplus, \otimes)$ together with a preordering P is called a preordered semiring, where an order relation is defined as: for $a, b \in S$,*

$$a \leq b \Longleftrightarrow a \oplus c = b, \text{ where } c \in P.$$

Note that the order relation $\leq$ is reflexive and transitive but not antisymmetric in general. Therefore, the order relation $\leq$ is a preorder relation. Thus the preorderings for semirings have a weaker connection with partial order relations than in the case of fields and rings. Here we have an example of a semiring S in which a preordering induces a preorder order relation on S.

Example 2.2.11. *Let* $S = (\mathbb{N} \cup \{a, b, c\})$ *be a set of all natural numbers including* 0 *and three additional elements* a, b, c*. Where*

$\oplus$	a	b	c
a	c	c	a
b	c	c	a
c	a	a	c

$\otimes$	a	b	c
a	0	0	0
b	0	0	0
c	0	0	0

and for all $n \in \mathbb{N}$ *we have,*

1. $a \oplus n = n, \qquad b \oplus n = n \quad$ *and* $\quad c \oplus n = n.$

2. $a \otimes n = \underbrace{a \oplus a \oplus ... \oplus a}_{n \text{ times}} = \begin{cases} a & \text{if } n \text{ is odd,} \\ c & \text{if } n \text{ is even.} \end{cases}$

3. $b \otimes n = \underbrace{b \oplus b \oplus ... \oplus b}_{n \text{ times}} = \begin{cases} b & \text{if } n = 1, \\ a & \text{if } n > 1 \text{ and } n \text{ is odd,} \\ c & \text{if } n \text{ is even.} \end{cases}$

4. $c \otimes n = \underbrace{c \oplus c \oplus ... \oplus c}_{n \text{ times}} = c.$

5. *The addition* $(\oplus)$ *and the multiplication* $(\otimes)$ *of elements of* $\mathbb{N}$ *is defined as the usual addition and multiplication.*

Thus $S = ((\mathbb{N} \cup \{a, b, c\}), \oplus, \otimes)$ *forms an antinegative semiring with the additive identity* 0 *and the multiplicative identity* 1*. Let us assume that* $P = S$*. It is evident that* $\boldsymbol{1} \oplus a \neq \boldsymbol{0}$ *for all* $a \in P = S$*, since* S *is an antinegative semiring. Thus we get that* $P = S$ *is a preordering of* S *and it fails to determine a partial order relation, since we have*

$$b \oplus c = a \Longrightarrow c \leq a$$

$$a \oplus b = c \Longrightarrow a \leq c$$

but

$$c \neq a.$$

Hence the order relation $\leq$ *is a preorder relation on* S*.*

We can see that the antisymmetry in the binary relation $\leq$ depends only on the property of addition $(\oplus)$ in a semiring S. The most interesting cases when preorderings for semirings S induce a partial order relation on S are discussed in following propositions.

Proposition 2.2.12. [**60**] *Let* $(S, \oplus, \otimes)$ *be a commutative semiring. If the addition is idempotent* (*i.e., for all* $a \in S, a \oplus a = a$) *in* S *then the preordering induces a partial order relation on* S*.*

Proof. Let $a \leq b$ and $b \leq a$, this implies that $a = b \oplus c$ and $b = a \oplus d$, where $c, d \in P(S)$. Thus we get $b = a \oplus d = b \oplus c \oplus d$. By substituting this value of b in

$a = b \oplus c$ we get, $a = b \oplus c \oplus d \oplus c = b \oplus c \oplus d = b$, using the additive idempotent property ($c \oplus c = c$). ■

Proposition 2.2.13. [**40**] *Let* $(S, \oplus, \otimes)$ *be a commutative semiring. If the addition is selective* (*i.e., for all* $a, b \in S, a \oplus b = a$ *or* b) *in* S *then the preordering induces a partial order relation on* S.

This follows because selectivity implies idempotency. Moreover, if the addition is selective then for any pair $a, b \in S$, either $a \leq b$ or $b \leq a$ and thus the preorder relation $\leq$ is a *total order* relation.

Note that in the Boolean semiring, the max-plus semiring and max-min semirings the addition is selective, so the preorderings in these cases induce a total order relation.

Definition 2.2.14. (*Ordered Semirings*) *A semiring* $(S, \oplus, \otimes)$ *together with an ordering* P *is called a ordered semiring, where an order relation is defined as: for* $a, b \in S$,

$$a \leq b \Longleftrightarrow a \oplus c = b, \text{ where } c \in P.$$

We know that the orderings for ordered fields or ordered rings always determine a total order by definition. However, Craven [**24**] notes that an ordering for a semiring may not determine a total order relation. For example, let $S = \mathbb{R}_+[x]$ and $P = S$. It is clear that $a \oplus b \neq \mathbf{0}$ for all $a, b \in P = S$. Thus we get that $P = S$ is an ordering for S, but it does not induce a total order relation. Another example of a semiring in which an ordering does not induce a total order is $S = ((\mathbb{N} \cup \{a, b, c\}), \oplus, \otimes)$ in the example 2.2.11. Clearly $P = S$ forms an ordering for S and it does not induce a total order relation.

Craven proved a semiring version of Artin-Schreier theorem, which gives an algebraic characterization of semirings which admit some ordering. We call these semirings *formally real semirings* or *orderable semirings.*

Theorem 2.2.15. *For a commutative semiring S, the following are equivalent:*

1. *S is a formally real semiring or an orderable semiring,*

2. $\bigoplus_j a_j^{\otimes 2} = \mathbf{0}$ *implies that* $a_j = 0$ *for all* j.

From the above theorem we note that if S is a formally real semiring then $\bigoplus_{j=1}^{k} a_j^{\otimes 2} \neq \mathbf{0}$, for all nonzero $a_j \in S$. This implies that $\mathbf{1} \oplus a \neq \mathbf{0}$, for all $a \in P = \sum S^{\otimes 2}$. Thus $P = \sum S^{\otimes 2}$ always forms a preordering for a formally real semiring S and it is called the minimal preordering for S. The minimal preordering for S is denoted as $P(S)$. It is clear that the minimal preordering $P(S) = \sum S^{\otimes 2}$ forms an antinegative commutative subsemiring of a formally real semiring S. Moreover, if S is a commutative semiring and P is a preordering for S then for $a, b \in S$,

$$a \leq b \Longleftrightarrow ac \leq bc, \text{ for any } c \in P. \tag{2.15}$$

It will be useful to define the concept of diagonal dominance for matrices over formally real semirings.

Definition 2.2.16. (*Diagonally Dominant Matrices*) *Let S be a formally real semiring and P be a preordering for S. A matrix $A \in M_n(S)$ is called diagonally dominant if there exists $c_i \in P$ such that $a_{ii} = c_i \oplus \bigoplus_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}$, for all i,*

i.e., if

$$a_{ii} \geq \bigoplus_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}, \text{ for all } i.$$

If S is an antinegative commutative semiring then clearly S is a formally real semiring and $P = S$ forms a preordering for S. The preordering P such that $P = S$ is called the maximum preordering. Therefore, a matrix A over an antinegative commutative semiring is called diagonally dominant if there exists $c_i \in S$ such that $a_{ii} = c_i \oplus \bigoplus_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}$, for all i.

2.2.4 Semiring Homomorphism

Much of the standard terminology about ring homomorphisms can also be applied to semirings. For more information we refer the reader to [**1, 50**].

Definition 2.2.17. (*Semiring Homomorphism*) *Let $(R, \oplus_R, \otimes_R)$ and $(S, \oplus_S, \otimes_S)$ be two semirings. A semiring homomorphism is a mapping $\phi : (R, \oplus_R, \otimes_R) \to (S, \oplus_S, \otimes_S)$, such that*

$$\phi(\mathbf{0}_R) = \mathbf{0}_S \text{ and } \phi(\mathbf{1}_R) = \mathbf{1}_S$$

$$\phi(a \oplus_R b) = \phi(a) \oplus_S \phi(b) \text{ for all } a, b \in R.$$

$$\phi(a \otimes_R b) = \phi(a) \otimes_S \phi(b) \text{ for all } a, b \in R.$$

Consequently, a mapping $\phi : (R, \oplus_R, \otimes_R) \to (S, \oplus_S, \otimes_S)$ is a semiring homomorphism if and only if both, $\phi : (R, \oplus_R) \to (S, \oplus_S)$ and $\phi : (R, \otimes_R) \to (S, \otimes_S)$, are semigroup homomorphisms. This implies that $\phi(R) = \{\phi(a) | a \in R\} \subseteq S$ and it determines a subsemigroup $(\phi(R), \oplus_S)$ of $(S, \oplus_S)$ and $(\phi(R), \otimes_S)$ of $(S, \otimes_S)$. Moreover, the distributive law transfer from $(R, \oplus_R, \otimes_R)$ to $(\phi(R), \oplus_S, \otimes_S)$. For instance,

$$\begin{aligned}
\phi(a) \otimes_S (\phi(b) \oplus_S \phi(c)) &= \phi(a) \otimes_S (\phi(b \oplus_R c)) \\
&= \phi\left(a \otimes_R (b \oplus_R c)\right) \\
&= \phi\left((a \otimes_R b) \oplus_R (a \otimes_R c)\right) \\
&= \phi(a \otimes_R b) \oplus_S \phi(a \otimes_R c) \\
&= \left(\phi(a) \otimes_S \phi(b)\right) \oplus_S \left(\phi(a) \otimes_S \phi(c)\right).
\end{aligned}$$

Theorem 2.2.18. [**50**, Theorem 3.2] *Let $(R, \oplus_R, \otimes_R)$ and $(S, \oplus_S, \otimes_S)$ be semirings and $\phi : (R, \oplus_R, \otimes_R) \to (S, \oplus_S, \otimes_S)$ be a semiring homomorphism. Then the following statements hold:*

1. *The homomorphic image $(\phi(R), \oplus_S, \otimes_S)$ of $(R, \oplus_R, \otimes_R)$ is a subsemiring of $(S, \oplus_S, \otimes_S)$. If R is a commutative semiring then the same holds for $\phi(R)$.*

2. *If $(R, \oplus_R, \otimes_R)$ is a ring then the same holds for $(\phi(R), \oplus_S, \otimes_S)$. In other words, if an element $x \in R$ has an additive inverse $-x \in R$, then $\phi(-x) = -\phi(x)$ is an additive inverse of $\phi(x) \in \phi(R)$.*

3. *If $(R, \oplus_R, \otimes_R)$ is a semifield then the same holds for $(\phi(R), \oplus_S, \otimes_S)$. In other*

words, if an element $x \in R$ has a multiplicative inverse $x^{-1} \in R$, then $\phi(x^{-1}) = \phi(x)^{-1}$ is a multiplicative inverse of $\phi(x) \in \phi(R)$.

Definition 2.2.19. (*Semiring Isomorphism*) *Let $(R, \oplus_R, \otimes_R)$ and $(S, \oplus_S, \otimes_S)$ be two semirings. A semiring homomorphism $\phi : R \to S$, is called a semiring isomorphism if the homomorphism ϕ is bijective.*

If there is an isomorphism between semirings R and S, we write $R \cong S$. If $(R, \oplus_S, \otimes_S)$ be a semiring then the identity map $I : R \to R$ is a semiring isomorphism. If $\phi : (R, \oplus_R, \otimes_R) \to (S, \oplus_S, \otimes_S)$ is an isomorphism then the same holds for the inverse mapping $\phi^{-1} : (S, \oplus_S, \otimes_S) \to (R, \oplus_R, \otimes_R)$.

Definition 2.2.20. (*Semiring Monomorphism*) *Let R and S are be two semirings. A semiring homomorphism $\phi : R \to S$ is called a semiring monomorphism if the homomorphism ϕ is injective.*

We will use semiring homomorphisms and isomorphisms in chapter 4 to extend our results for matrices over the Boolean semiring to matrices over finite Boolean algebras.

2.2.5 Inclines

An incline is a commutative semiring which satisfies two additional properties: the addition is idempotent and the product of two elements is always less than or equal to either factor.

The concept of slope was introduced by Cao [**23**] and later Cao, Kim and Roush [**27**] assigned it a new name called incline. The theory of inclines and their applications are outlined completely in [**27**]. Kim and Roush [**59**] have surveyed and described algebraic properties of inclines and matrices over inclines.

Definition 2.2.21. [**27**] (*Inclines*) *A nonempty set L with two binary operations $\oplus$ and $\otimes$ is called an incline if it satisfies the following conditions;*

1. $(L, \oplus = l.u.b)$ *is a semilattice.*

2. $(L, \otimes)$ *is a semigroup.*

3. $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z), \qquad$ *for all* $x, y, z \in L$.

4. $x \oplus (x \otimes y) = x, \qquad$ *for all* $x, y \in L$.

An incline L is called a *commutative incline* if $(L, \otimes)$ is a commutative semigroup. We assume that very incline L has the additive identity $\mathbf{0}$ and the multiplicative identity $\mathbf{1}$. It follows that $\mathbf{0}$ is the least element of L and $\mathbf{1}$ is the greatest element of L, satisfying $\mathbf{0} \otimes x = \mathbf{0}$ and $\mathbf{1} \oplus x = \mathbf{1}$ for all $x \in L$.

Note that for any incline L, if $x \oplus y = \mathbf{0}$ for any $x, y \in L$ then we get $x = \mathbf{0} = y$, since $\mathbf{0}$ is the least element of L. This implies that every incline is an antinegative semiring. Therefore, $P = L$ is a preordering for L and we define a relation $\leq$ in L by

$$x \leq y \qquad \Leftrightarrow \qquad x \oplus y = y.$$

Since the addition is idempotent in L, the order relation $\leq$ is a partial order relation by proposition 2.2.12. For all $x, y \in L$, we have $xy \leq x$, since $x \oplus xy = x$. The relation $\leq$ is a monotonically increasing relation, i.e., if $x \leq y$ then $x^{\otimes 2} \leq y^{\otimes 2}$.

Definition 2.2.22. [**28**] (*Totally Ordered Inclines*) *An incline is said to be linearly ordered or totally ordered if the partial order relation $\leq$ is a total order relation.*

Examples of totally ordered inclines include the two element Boolean semiring $(\{\mathbf{0}, \mathbf{1}\})$, the max-min fuzzy algebra $([\mathbf{0}, \mathbf{1}], max(x, y), min(x, y))$, the negative interval subsemiring of the max-plus semiring $([-\infty, 0], max(x, y), x + y)$, Max-min semirings, the max-times semiring $([\mathbf{0}, \mathbf{1}], max(x, y), xy)$ where xy is the ordinary real multiplication.

Note that distributive lattices and Boolean algebras are also inclines but they are not totally ordered inclines.

There are two notions of ideal in incline theory [**5, 59**]: ideals in the semiring sense and ideals in a lattice sense.

Definition 2.2.23. (*r-Ideal or Ideal in Semiring Sense*) *An r-ideal J of a commutative incline L is a nonempty subset of L satisfying the following conditions:*

1. *$a \in J$ and $x \in L$ implies that $x \otimes a \in J$ and $a \otimes x \in J$.*

2. *$a \in J$ and $b \in J$ implies that $a \oplus b \in J$.*

Definition 2.2.24. (*Lattice Ideal or Ideal in Lattice Sense*) *A lattice ideal J of a commutative incline L is a nonempty subset of L satisfying the following conditions:*

1. $a \in J$ implies that $x \in J$ for all $x \leq a$, where $x \in L$.

2. $a \in J$ and $b \in J$ implies that $a \oplus b \in J$.

It is easy to check that every lattice ideal of a commutative incline is an r-ideal. However, an r-ideal of a commutative incline may not be a lattice ideal. For example, suppose that $L = \{\mathbb{N} \cup \{\infty\}, min, \times\}$, where $\mathbb{N}$ is the set of all natural numbers not including zero. Evidently, L forms a commutative incline where ∞ is the additive identity and 1 is the multiplicative identity. Note that in this incline L, the order relation is reversed. The set of all even natural numbers not including zero forms an r-ideal but it is not a lattice ideal.

In this thesis, we are particularly interested in those special inclines which have a property that r-ideals are same as lattice ideals.

Definition 2.2.25. (*The LI-Property*) *A commutative incline L is said to have the LI-property if all r-ideals of L are lattice ideals of L.*

We know that every lattice ideal of a commutative incline L is an r-ideal of L. If in a commutative incline L, all r-ideals are lattice ideals then we get that all r-ideals of L are same as lattice ideals of L.

In the following proposition we characterize those commutative inclines which have the LI-property.

Proposition 2.2.26. *A commutative incline L has the LI-property, i.e., r-ideals of L are same as lattice ideals of L if and only if for $x, y \in L$, $x \leq y$ implies that there* *exists $z \in L$ (not necessarily unique) such that $x = yz$.*

Proof. The forward direction of the proposition is obvious. Now suppose that $x \leq y$ implies that there exists $z \in L$ such that $x = yz$, for $x, y \in L$. We will show that L has the LI-property. From definition it is clear that every lattice ideal of L is an r-ideal of L. Let J be an r-ideal of L and $a \in J$. Suppose $y \leq a$, then there exists $z \in L$ such that $y = az$. Therefore, $y = az \in J$. This is true for all $y \leq a$ and for $a \in J$ and hence J is a lattice ideal of L. Thus every r-ideal is a lattice ideal. ■

Note that if L is a commutative incline with the unique square root property then the square root function in L is multiplicative, i.e.,

$$\sqrt{x}\sqrt{y} = \sqrt{xy}.$$

Let $x, y, c, d \in L$, such that $x = c^{\otimes 2}$ and $y = d^{\otimes 2}$. This implies that

$$xy = c^{\otimes 2} d^{\otimes 2} = (cd)^{\otimes 2}, \text{ i.e. }, \sqrt{xy} = cd. \tag{2.16}$$

We also have $\sqrt{x} = c$ and $\sqrt{y} = d$. This implies that

$$\sqrt{x}\sqrt{y} = cd. \tag{2.17}$$

From (2.16) and (2.17) we get that

$$\sqrt{x}\sqrt{y} = \sqrt{xy}. \tag{2.18}$$

Further, we note that if a commutative incline L with the unique square root property and the LI-property then the square root function is an increasing function, i.e.,

$$\text{if } x \leq y \quad \Longrightarrow \quad \sqrt{x} \leq \sqrt{y}.$$

Let $x, y \in L$, such that $x \leq y$. This implies that

$$x = yz, \qquad \text{where } z \in L, \qquad \text{(using the LI-property).} \tag{2.19}$$

Taking square root on both sides of (2.19), we get

$$\sqrt{x} = \sqrt{yz}$$

$$\Rightarrow \sqrt{x} = \sqrt{y}\sqrt{z}, \qquad \text{(using (2.18))}$$

$$\Rightarrow \sqrt{x} \leq \sqrt{y} \qquad \text{(using the LI-property).}$$

Definition 2.2.27. (*Arithmetic Geometric Property*) *A commutative incline L is said to have the arithmetic geometric property (AG-property) if $x \otimes y \leq x^{\otimes 2} \oplus y^{\otimes 2}$.*

Note that every totally ordered commutative incline has the arithmetic geometric property. Since in every totally ordered commutative incline L either $x \leq y$ or $y \leq x$, for all $x, y \in L$. This implies that either $x \otimes y \leq y^{\otimes 2}$ or $x \otimes y \leq x^{\otimes 2}$. Therefore, $x \otimes y \leq l.u.b\{x^{\otimes 2}, y^{\otimes 2}\} = x^{\otimes 2} \oplus y^{\otimes 2}$. Moreover, commutative inclines in which the multiplication is idempotent also have the arithmetic geometric property, since $x \otimes y \leq x = x^{\otimes 2}$ and $x \otimes y \leq y = y^{\otimes 2}$. Therefore, $x \otimes y \leq l.u.b\{x^{\otimes 2}, y^{\otimes 2}\} = x^{\otimes 2} \oplus y^{\otimes 2}$.

Proposition 2.2.28. *Let L be a commutative incline with the arithmetic geometric property. Then $\bigoplus_{i=1}^{k} x_i^{\otimes 2} = \left(\bigoplus_{i=1}^{k} x_i\right)^{\otimes 2}$, where $x_i \in L$ for all i.*

Proof. It is evident that the result is true for $k = 1$. For $k = 2$, we have to prove that $x_1^{\otimes 2} \oplus x_2^{\otimes 2} = (x_1 \oplus x_2)^{\otimes 2}$. Since L has the arithmetic geometric property, the $l.u.b\{x_1^{\otimes 2}, x_2^{\otimes 2}, (x_1 \otimes x_2)\} = l.u.b\{x_1^{\otimes 2}, x_2^{\otimes 2}\}$. Thus we have $(x_1 \oplus x_2)^{\otimes 2} = x_1^{\otimes 2} \oplus x_2^{\otimes 2} \oplus (x_1 \otimes x_2) \oplus (x_1 \otimes x_2) = l.u.b\{x_1^{\otimes 2}, x_2^{\otimes 2}, (x_1 \otimes x_2)\} = l.u.b\{x_1^{\otimes 2}, x_2^{\otimes 2}\} = x_1^{\otimes 2} \oplus x_2^{\otimes 2}$. Hence the result is true for $k = 2$. A simple induction argument shows that the result is true for all k. ■

Definition 2.2.29. (*Normal Incline*) *A commutative incline L is called a normal incline if it has the LI-property, the unique square root property and the AG-property.*

Inclines which are normal and totally ordered are called *totally ordered normal* inclines. All above examples of totally ordered inclines are totally ordered normal inclines.

We use these properties of inclines in section 3.5 to study completely positive matrices over normal inclines.

Chapter 3

CP Matrices Over Semirings and Their CP-rank

In this chapter, we define a notion of complete positivity for matrices over semirings. We characterize completely positive matrices over special semirings and derive basic results such as every symmetric diagonally dominant matrix over special semirings is completely positive. In particular, we show that the notion of complete positivity over special types of semirings has some important similarities with the standard notion of complete positivity of real matrices.

We use the characterizations of completely positive matrices over special semirings to find the upper bound of their CP-rank. Another purpose of characterizing completely positive matrices over semirings is that the class of completely positive matrices over certain semirings has applications in graph theory.

The later part of this chapter concentrates on the CP-rank of completely positive matrices over special semirings. We examine the upper bound of the CP-rank of completely positive matrices over certain special class of semirings. We also prove a semiring version of Markham's theorems which give sufficient conditions for a completely positive matrix over special types of semirings to have a triangular factorization. In addition, we use semiring homomorphisms to formulate various CP-rank inequalities of completely positive matrices over different semirings.

3.1 Positive Semidefinite Matrices and CP matrices over semirings

We begin by defining the concept of positive semidefinite matrices over semirings and completely positive matrices over semirings.

Definition 3.1.1. (*Positive Semidefinite Matrices*) *Let S be a commutative semiring and A be an $n \times n$ matrix over S. The matrix A is called positive semidefinite if it can be written as $A = BB^T$, where $b_{ij} \in S$ for all i, j.*

We use the concept of squares and their sums in the theory of completely positive matrices over semirings. Let S be a commutative semiring and a subsemiring of S which consists of all the finite sums of perfect squares in S, the positive subsemiring of S, is denoted by $P(S)$. If S is a formally real semiring then by theorem 2.2.15, the positive subsemiring $P(S)$ is a preordering for S and it gives us a preorder on $P(S)$.

For $a, b \in P(S)$, we say that $a \geq b$ if there exists $c \in P(S)$ such that $a = b \oplus c$.

Now we are ready to define completely positive matrices over semirings. Our definition is a natural generalization of real completely positive matrices.

Definition 3.1.2. (*Completely Positive Matrices*) *Let S be a commutative semiring and $P(S)$ be the positive subsemiring of S. An $n \times n$ matrix A over S is called completely positive if A can be written as $A = BB^T$, where $b_{ij} \in P(S)$ for all i, j.*

Definition 3.1.3. (*The CP-rank*) *Let S be a commutative semiring and $P(S)$ be the positive subsemiring of S. The CP-rank of an $n \times n$ completely positive matrix A over S is the smallest number k such that $A = BB^T$, where B is an $n \times k$ matrix over $P(S)$.*

Lemma 3.1.4. *Let S be a formally real semiring and $A \in M_n(S)$ be a completely positive matrix. If the matrix A has a zero on the main diagonal then the corresponding row and column have all entries equal to zero.*

Proof. Let A be an $n \times n$ completely positive matrix over a formally real semiring S. This implies that A can be written as $A = BB^T$, where B is an $n \times m$ matrix over $P(S)$ for some positive integer m. Suppose that the i^{th} diagonal entry of A is zero. This implies that

$$\mathbf{0} = \bigoplus_{j=1}^{m} (b_{ij})^{\otimes 2}, \text{ where } b_{ij} \in P(S).$$

Since S is a formally real semiring, $\bigoplus_{j=1}^{m} (b_{ij})^{\otimes 2} = \mathbf{0}$ implies that $b_{ij} = \mathbf{0}$ for all $j = 1, 2, ..., m$. Thus we get that $a_{ij} = \bigoplus_{k=1}^{m} b_{ik} b_{jk} = \mathbf{0}$, for all j.

■

Throughout in this work we assume that S is a formally real semiring and thus the positive subsemiring $P(S)$ is always a preordering for S.

It is evident from the definition of completely positive matrices that every completely positive matrix over a commutative semiring S is a matrix all of whose entries are the elements of $P(S)$ and it is a positive semidefinite matrix over S. However, the converse may not true, i.e., every positive semidefinite matrix over S all of whose entries are the elements of $P(S)$ may not be completely positive. Here we have an example.

Example 3.1.5. *Let S be a semiring of real polynomials and $P(S)$ be the positive subsemiring of S which consists of all the finite sums of perfect squares in S, i.e., $P(S)$ contains all nonnegative real polynomials, by theorem 1.2.11. Let*

$$A(x) = \begin{bmatrix} (x-1)^2 & (x-1)^2 \\ (x-1)^2 & (x-1)^2 \end{bmatrix}$$

be a 2×2 matrix polynomial over S all of whose entries are squares of polynomials , i.e., $a_{ij} \in P(S)$ for $i, j = 1, 2$. The matrix polynomials $A(x)$ can be written as:

$$A(x) = \begin{bmatrix} (x-1)^2 & (x-1)^2 \\ (x-1)^2 & (x-1)^2 \end{bmatrix} = \begin{bmatrix} (x-1) \\ (x-1) \end{bmatrix} \begin{bmatrix} (x-1) & (x-1) \end{bmatrix}.$$

This implies that A is positive semidefinite matrix, since $(x-1) \in S$. However, $(x-1) \notin P(S)$ and we do not have any other factorization BB^T of $A(x)$ such that $b_{ij} \in P(S)$. Thus we get that $A(x)$ is not a completely positive matrix over S.

3.2 Characterization of CP Matrices over Semirings

Hannah and Laffey [**45**] remarked that no general necessary and sufficient conditions for a real matrix A to be completely positive are known. Some special results in this respect were obtained by Markham [**67**] and Lau and Markham [**65**]. In particular, M. Kaykobad [**54**] has shown that diagonal dominance is a sufficient condition for real nonnegative symmetric matrices to be completely positive. In our work, we generalize Kaykobad's result for completely positive matrices over certain special types of semirings. We also examine those properties of such semirings for which the diagonally dominance condition is sufficient as well as necessary for symmetric matrices to be completely positive.

The following theorem gives us a sufficient condition for symmetric matrices over special semirings to be completely positive.

Theorem 3.2.1. *Let S be a commutative semiring and $P(S)$ be the positive subsemiring of S. Then every symmetric diagonally dominant matrix A over $P(S)$ is positive semidefinite. Furthermore, if $P(S) = P(P(S))$, then every symmetric diagonally dominant matrix A over $P(S)$ is completely positive.*

Proof. Let $A \in M_n(P(S))$ be a symmetric diagonally dominant matrix. This implies that

$$a_{ii} \geq \bigoplus_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}, \qquad \text{for all } i.$$

Therefore there exists $c_i \in P(S)$ such that

$$a_{ii} = c_i \bigoplus \left(\bigoplus_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} \right), \qquad \text{for all } i.$$

Thus we get that if A is a symmetric diagonally dominant matrix over a commutative semiring S such that $a_{ij} \in P(S)$ for all i, j, then A can be written as

$$A = C \oplus D.$$

Here the matrix

$$C = \begin{bmatrix} c_1 & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} \\ \mathbf{0} & c_2 & \mathbf{0} & . & . & . & \mathbf{0} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & . & . & . & c_n \end{bmatrix},$$

where $c_i \in P(S)$ for all $i = 1, 2, ..., n$ and the matrix

$$D = \begin{bmatrix} \bigoplus_{\substack{j=1 \\ j \neq 1}}^{n} a_{1j} & a_{12} & a_{13} & . & . & . & a_{1n} \\ a_{12} & \bigoplus_{\substack{j=1 \\ j \neq 2}}^{n} a_{2j} & a_{23} & . & . & . & a_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{1n} & a_{2n} & a_{3n} & . & . & . & \bigoplus_{\substack{j=1 \\ j \neq n}}^{n} a_{nj} \end{bmatrix},$$

To prove that the matrix A is positive semidefinite, we show that the matrices C and D are positive semidefinite over the semiring S.

Since $C \in M_n(P(S))$, we assume that for $i = 1, 2, ..., n$

$$c_i = \bigoplus_{k=1}^{m_i} (b_{i,k})^{\otimes 2},$$

where $b_{i,k} \in S$ for all $k = 1, 2, ..., m_i$. Thus the matrix

$$C = \begin{bmatrix} \bigoplus\limits_{k=1}^{m_1} (b_{1,k})^{\otimes 2} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \bigoplus\limits_{k=1}^{m_2} (b_{2,k})^{\otimes 2} & \mathbf{0} & \cdots & \mathbf{0} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \bigoplus\limits_{k=1}^{m_n} (b_{n,k})^{\otimes 2} \end{bmatrix}$$

can be written as:

$$C = \bigoplus_{i=1}^{n} C_i,$$

where

$$C_i = \bigoplus_{k=1}^{m_i} \begin{array}{cc} & i \\ i & \begin{pmatrix} \mathbf{0} & . & \mathbf{0} & . & \mathbf{0} \\ . & . & . & . & . \\ \mathbf{0} & . & (b_{i,k})^{\otimes 2} & . & \mathbf{0} \\ . & . & . & . & . \\ \mathbf{0} & . & \mathbf{0} & . & \mathbf{0} \end{pmatrix} \end{array}$$

is a bordered matrix whose $(i,i)^{th}$ entry is equal to $\bigoplus\limits_{k=1}^{m_i}(b_{i,k})^{\otimes 2}$ and all other entries are equal to $\mathbf{0}$. Here i on the left border denotes the i^{th} row and i on the top denotes the i^{th} column of C_i. Thus we get that

$$C_i = \bigoplus_{k=1}^{m_i} \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} \begin{pmatrix} \mathbf{0} \\ . \\ \mathbf{0} \\ b_{i,k} \\ \mathbf{0} \\ . \\ \mathbf{0} \end{pmatrix} \begin{array}{c} \begin{array}{ccccccc} \phantom{\mathbf{0}} & & \phantom{\mathbf{0}} & i & \phantom{\mathbf{0}} & & \phantom{\mathbf{0}} \end{array} \\ \begin{pmatrix} \mathbf{0} & . & \mathbf{0} & b_{i,k} & \mathbf{0} & . & \mathbf{0} \end{pmatrix} \end{array}$$

This implies that C_i is a positive semidefinite matrix over the semiring S, for all $i = 1, 2, ..., n$ and hence C is a positive semidefinite matrix over the semiring S, since it is a sum of positive semidefinite matrices over S.

Further, the matrix D can be written as:

$$D = \bigoplus_{\substack{i,j=1 \\ i<j}}^{n} D_{ij},$$

where

$$D_{ij} = \begin{array}{cc} & \begin{array}{cccccccc} \phantom{\mathbf{0}} & & i & & & j & & \phantom{\mathbf{0}} \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} \mathbf{0} & . & \mathbf{0} & . & . & \mathbf{0} & . & \mathbf{0} \\ . & . & . & . & . & . & . & . \\ \mathbf{0} & . & a_{ij} & . & . & a_{ij} & . & \mathbf{0} \\ . & . & . & . & . & . & . & \\ \mathbf{0} & . & a_{ij} & . & . & a_{ij} & . & \mathbf{0} \\ . & . & . & . & . & . & . & . \\ \mathbf{0} & . & \mathbf{0} & . & . & \mathbf{0} & . & \mathbf{0} \end{pmatrix} \end{array}$$

Since $a_{ij} \in P(S)$, we suppose that

$$a_{ij} = \bigoplus_{k=1}^{m} (d_k^{(ij)})^{\otimes 2},$$

where $d_k^{(ij)} \in S$ for all $k = 1, 2, ..., m$. Then D_{ij} can be written as:

$$D_{ij} = \bigoplus_{k=1}^{m} \begin{array}{cc} \begin{array}{c} \\ \\ i \\ \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} \mathbf{0} \\ . \\ d_k^{(ij)} \\ \mathbf{0} \\ . \\ d_k^{(ij)} \\ . \\ \mathbf{0} \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{cccccccc} \phantom{\mathbf{0}} & & i & \phantom{\mathbf{0}} & & j & & \phantom{\mathbf{0}} \end{array} \\ \begin{pmatrix} \mathbf{0} & . & d_k^{(ij)} & \mathbf{0} & . & d_k^{(ij)} & . & \mathbf{0} \end{pmatrix} \end{array}$$

This implies that D_{ij} is a positive semidefinite matrix over the semiring S, for all i, j and hence D is a positive semidefinite matrix over the semiring S, since it is a sum of positive semidefinite matrices over S. Thus $A = C \oplus D$ is a positive semidefinite matrix over the semiring S.

Moreover, if $P(S) = P(P(S))$ then we can write every $c_i \in P(S)$ for all $i = 1, 2, ..., n$, as:

$$c_i = \bigoplus_{k=1}^{m_i} (b_{i,k})^{\otimes 2},$$

where $b_{i,k} \in P(S)$ for all $k = 1, 2, ..., m_i$. Therefore, we can write C_i as:

$$C_i = \bigoplus_{k=1}^{m_i} \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} \begin{pmatrix} \mathbf{0} \\ . \\ \mathbf{0} \\ b_{i,k} \\ \mathbf{0} \\ . \\ \mathbf{0} \end{pmatrix} \begin{array}{c} \begin{array}{ccccccc} & & & i & & & \end{array} \\ \begin{pmatrix} \mathbf{0} & . & \mathbf{0} & b_{i,k} & \mathbf{0} & . & \mathbf{0} \end{pmatrix} \end{array},$$

where $b_{i,k} \in P(S)$ for all $k = 1, 2, ..., m_i$. This implies that C_i is a completely positive matrix over the semiring S, for all $i = 1, 2, ..., n$ and hence C is a completely positive matrix over the semiring S, since it is a sum of completely positive matrices over S.

Similarly, all a_{ij} can be written as:

$$a_{ij} = \bigoplus_{k=1}^{m} (d_k^{(ij)})^{\otimes 2},$$

where $b_k^{(ij)} \in P(S)$ for all $k = 1, 2, ..., m$. Thus D_{ij} can be written as:

$$D_{ij} = \bigoplus_{k=1}^{m} \begin{array}{c} \\ \\ i \\ \\ \\ j \\ \\ \\ \end{array} \begin{pmatrix} \mathbf{0} \\ . \\ d_k^{(ij)} \\ \mathbf{0} \\ . \\ d_k^{(ij)} \\ . \\ \mathbf{0} \end{pmatrix} \begin{array}{c} \begin{array}{ccccccc} \phantom{\mathbf{0}} & & i & \phantom{\mathbf{0}} & & j & \phantom{.\ \mathbf{0}} \end{array} \\ \begin{pmatrix} \mathbf{0} & . & d_k^{(ij)} & \mathbf{0} & . & d_k^{(ij)} & . & \mathbf{0} \end{pmatrix} \end{array},$$

where $b_k^{(ij)} \in P(S)$ for all $k = 1, 2, ..., m$. This implies that D_{ij} is a completely positive matrix over the semiring S, for all i, j and hence D is a completely positive matrix over the semiring S, since it is a sum of completely positive matrices over S. Thus $A = C \oplus D$ is a completely positive matrix over the semiring S.

■

We note that the Kaykobad's result is a special case of our result. If $S = \mathbb{R}$, then we know that $P(S) = \mathbb{R}_+$ and $P(S) = P(P(S))$. Therefore, every symmetric nonnegative diagonally dominant real matrix is completely positive.

Corollary 3.2.2. *Let $A(x)$ be a real matrix polynomial. If $A(x)$ is a symmetric, pointwise nonnegative and pointwise diagonally dominant matrix then $A(x)$ is a positive semidefinite matrix polynomial, i.e., $A(x)$ can be written as $A(x) = B(x)B(x)^T$, where $B(x)$ is a rectangular matrix polynomial.*

Proof. Let $A(x)$ be a symmetric, pointwise nonnegative and pointwise diagonally dominant real matrix polynomial. This implies that every entry of $A(x)$ is a nonnegative real polynomial and hence can be written as a finite sum of perfect squares of real polynomials, i.e., for all i, j, $p_{ij}(x) = f_1^2(x) \oplus f_2^2(x) \oplus ... \oplus f_k^2(x)$, where $f_i(x)$ are real polynomials, using theorem 1.2.11. In other words, every entry of $A(x)$ is an element of $P(S)$, where $S = \mathbb{R}[x]$. We are also given that the matrix polynomial $A(x)$ is symmetric and pointwise diagonally dominant. Therefore, $A(x)$ is a symmetric diagonally dominant matrix polynomial over $P(S)$ and hence by theorem 3.2.1, we get that $A(x)$ is a positive semidefinite matrix, i.e., $A(x) = B(x)B(x)^T$, where $B(x)$ is a rectangular matrix polynomial.

■

Note that in corollary 3.2.2, the real matrix polynomial $A(x)$ is a symmetric, pointwise nonnegative and pointwise diagonally dominant matrix. Therefore, it is pointwise completely positive. However, the real matrix polynomial $A(x)$ is not completely positive in general as a real matrix polynomial. We know that $p_{ij}(x) = f_1^2(x) \oplus f_2^2(x) \oplus ... \oplus f_k^2(x)$, where $f_i(x)$ are real polynomials which may not be nonnegative for some i. In other words, nonnegative real polynomials may not be generated by the perfect squares of nonnegative real polynomials. Thus the matrix polynomial $B(x)$ may not be pointwise nonnegative. Hence we get that the pointwise complete positivity of a real matrix polynomial does not imply complete positivity in general as a real matrix polynomial. For instance, see example 3.1.5.

3.3 Characterization of CP matrices over Inclines

In this section, we study completely positive matrices over special types of semirings called inclines.

The set of all $m \times n$ matrices over an incline L is denoted by $M_{m,n}(L)$. Note that if an incline L has the unique square root property then the positive subsemiring $P(L)$ of L, which consists of all the finite sums of perfect squares in L, is in fact equal to L, i.e., $P(L) = L$. For such inclines (semirings) every symmetric diagonally dominant matrix is completely positive matrix, by theorem 3.2.1. The following theorem characterizes completely positive matrices over normal inclines.

Theorem 3.3.1. *Let L be a normal incline and $A \in M_n(L)$ be a symmetric matrix. Then the following are equivalent.*

1. *A is completely positive.*

2. *Every 2 by 2 principal submatrix has the positive determinant greater than or equal to the negative determinant.*

*3. There exists a diagonal matrix $D \in M_n(L)$ and a symmetric matrix $M \in M_n(L)$ all of whose diagonal entries equal to **1**, such that $A = DMD$.*

Proof. (1) $\implies$ (2) Let A be a completely positive matrix over a given incline L. This implies that there exists a matrix B over the incline L such that $A = BB^T$. Now

$$a_{ii} = \bigoplus_k (b_{ik} \otimes b_{ik}) = l.\underset{k}{u}.b \left\{b_{ik}^{\otimes 2}\right\}$$

$$a_{jj} = \bigoplus_k (b_{jk} \otimes b_{jk}) = l.\underset{k}{u}.b \left\{b_{jk}^{\otimes 2}\right\}$$

$$a_{ij} = \bigoplus_k (b_{ik} \otimes b_{jk}) = l.\underset{k}{u}.b \left\{b_{ik} \otimes b_{jk}\right\}$$

Clearly

$$\begin{aligned} a_{ii} \otimes a_{jj} &= l.\underset{k}{u}.b \left\{b_{ik}^{\otimes 2}\right\} l.u.b_k \left\{b_{jk}^{\otimes 2}\right\} \\ &\geq l.\underset{k}{u}.b \left\{b_{ik}^{\otimes 2} b_{jk}^{\otimes 2}\right\} \\ &= l.\underset{k}{u}.b \left\{(b_{ik} \otimes b_{jk})^{\otimes 2}\right\} \\ &= (l.\underset{k}{u}.b \left\{b_{ik} \otimes b_{jk}\right\})^{\otimes 2}, \text{(by proposition 2.2.28)} \\ &= a_{ij}^{\otimes 2}, \end{aligned}$$

and this is true for all i, j. Hence every 2×2 principal submatrix of A has $det^+ \geq det^-$.

(2) $\implies$ (3) Let $D = diag(\sqrt{a_{11}}, \sqrt{a_{22}}, \sqrt{a_{33}}, ..., \sqrt{a_{nn}})$. Since $a_{ii} \otimes a_{jj} \geq a_{ij}^{\otimes 2}$, we have $\sqrt{a_{ii}} \otimes \sqrt{a_{jj}} \geq a_{ij}$. Hence by the LI-property, for any $i \neq j$, there exists $m_{ij} \in L$ such that $\sqrt{a_{ii}} \otimes m_{ij} \otimes \sqrt{a_{jj}} = a_{ij}$. Let M be the matrix whose main diagonal

entries are $\mathbf{1}$ and whose off-diagonal entries are m_{ij}. A simple calculation shows us that $A = DMD$.

(3) $\implies$ (1) Let us suppose that $M \in M_n(L)$ be a symmetric matrix with all diagonal entries equal to $\mathbf{1}$. This implies that M is a symmetric diagonally dominant matrix over L and thus by theorem 3.2.1, the matrix M is a completely positive matrix, since $L = P(L)$. Hence $A = DMD$ is a completely positive matrix.

■

It is clear from theorem 3.3.1 that the diagonal dominance is a sufficient condition for symmetric matrices over normal inclines to be completely positive. It is not a necessary condition for symmetric matrices over normal inclines to be completely positive, since we have matrices over normal inclines which are completely positive but not diagonally dominant. One can easily check that the semiring $\big([-\infty, 0], max, +\big)$ forms a normal incline. Here is an example of such a matrix over the semiring $[-\infty, 0]$:

$$A = \begin{bmatrix} -4 & -5 \\ -5 & -6 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \begin{bmatrix} -2 & -3 \end{bmatrix}$$

Clearly A is not a diagonally dominant matrix $[-\infty, 0]$, but it is completely positive with the CP-rank equal to one.

Now we will examine a special class of normal inclines in which the diagonal dominance is necessary and sufficient condition for symmetric matrices to be completely positive.

Definition 3.3.2. (*Regular Incline*) [**48**] *An incline L is said to be regular if every element of L is multiplicatively idempotent, i.e., for every $a \in L$, $a \otimes a = a$.*

Examples of regular inclines include the Boolean semiring, max-min semirings and distributive lattices. It has been shown [**48**, Corollary 3.3] that every regular incline is commutative. Furthermore, every element in a regular incline is a unique square root of itself. Thus regular inclines also satisfy the unique square root property.

We also note that every regular incline has the AG-property, since $x \otimes y \leq x = x^{\otimes 2}$ and $x \otimes y \leq y = y^{\otimes 2}$. Therefore, $x \otimes y \leq l.u.b\{x^{\otimes 2}, y^{\otimes 2}\} = x^{\otimes 2} \oplus y^{\otimes 2}$. This implies that the theorem 3.3.1 holds for all symmetric matrices over regular inclines having the LI-property.

In the next theorem we will prove that the diagonally dominant condition is necessary and sufficient for symmetric matrices over regular inclines to be completely positive.

Our result is formulated as follows:

Theorem 3.3.3. *Let L be a regular incline and A be an $n \times n$ symmetric matrix over L. Then the matrix A is completely positive if and only if A is diagonally dominant.*

Proof. Let us suppose that A is a symmetric diagonally dominant matrix over L. Since L is a regular incline, we have $L = P(L)$. Therefore, A is a completely positive matrix over L, by theorem 3.2.1.

For the other direction, we consider that A is a completely positive matrix over a regular incline L. This implies that there exists a matrix B over the incline L such that $A = BB^T$. Thus we get,

$$\begin{aligned} a_{ii} &= \bigoplus_{k=1}^{n} (b_{ik} \otimes b_{ik}) \\ &= \bigoplus_{k=1}^{n} b_{ik} \\ &\geq \bigoplus_{k=1}^{n} (b_{ik} \otimes b_{jk}) \\ &= a_{ij} \end{aligned}$$

and this is true for all i, j. This implies that $a_{ii} \geq \underset{j \neq i}{l.u.b}\{a_{ij}\} = \underset{j \neq i}{\oplus} a_{ij}$. Hence A is diagonally dominant.

■

3.4 LU & UL Factorization of CP Matrices Over Inclines

In this section, we examine an important family of completely positive matrices over normal inclines. We generalize various results of completely positive matrices over reals to completely positive matrices over normal inclines. In particular, we discuss the conditions which guarantee that a completely positive matrix over a normal incline has a square factorization BB^T especially one where B is a triangular matrix.

We begin by defining UL and LU completely positive matrices over commutative semirings. Our definition is a natural generalization of real UL and LU completely positive matrices.

Definition 3.4.1. (*UL and LU-completely positive matrix*) *Let S be a commutative semiring and $P(S)$ be the positive subsemiring of S. A matrix A over S is called a UL-completely positive matrix if there exists an upper triangular matrix U over $P(S)$ such that $A = UU^T$. A matrix A over S is called a LU-completely positive matrix if there exists a lower triangular matrix L over $P(S)$ such that $A = LL^T$.*

Example 3.4.2. *We note that every 2×2 completely positive matrix over a normal incline L is LU-completely positive and UL-completely positive. This can be proved directly. Let*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

be a completely positive matrix over a normal incline L. Then by theorem 3.3.1, the positive determinant of A is greater than or equal to the negative determinant of A, i.e., $a_{11}a_{22} \geq a_{12}^{\otimes 2}$. Thus we get $\sqrt{a_{11}}\sqrt{a_{22}} \geq a_{12}$, since in a normal incline the square root function is increasing and multiplicative. Using the LI-property, we get that there exists $c \in L$ such that $c\sqrt{a_{11}}\sqrt{a_{22}} = a_{12}$. Now the matrix A can be written as:

$$A = \begin{bmatrix} \sqrt{a_{11}} & c\sqrt{a_{11}} \\ \boldsymbol{0} & \sqrt{a_{22}} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & c\sqrt{a_{11}} \\ \boldsymbol{0} & \sqrt{a_{22}} \end{bmatrix}^T$$

if $a_{11} > \boldsymbol{0}$, and

$$A = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \sqrt{a_{22}} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \sqrt{a_{22}} \end{bmatrix}^T$$

if $a_{11} = \mathbf{0}$

and similarly,

$$A = \begin{bmatrix} \sqrt{a_{11}} & \mathbf{0} \\ c\sqrt{a_{22}} & \sqrt{a_{22}} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & \mathbf{0} \\ c\sqrt{a_{22}} & \sqrt{a_{22}} \end{bmatrix}^T$$

Further, we will see that 3×3 completely positive matrices over normal inclines are either UL-completely positive or LU-completely positive, but not necessarily both. However, for $n \geq 4$, $n \times n$ completely positive matrices over normal inclines may be neither UL-completely positive nor LU-completely positive.

We begin by proving incline analogs of Markham theorems (theorem 2.1.22 and theorem 2.1.23) relating almost principal minors with the triangular factorizations.

Theorem 3.4.3. *If A is an $n \times n$ completely positive matrix over a normal incline L, $n \geq 3$, and all its left almost principal 2×2 submatrices have $det^+ \geq det^-$, then A is UL-completely positive.*

Proof. Let us suppose that A is an $n \times n$ completely positive matrix over a normal incline L. Because of theorem 3.3.1, we assume that A is a symmetric matrix over L whose diagonal entries are equal to $\mathbf{1}$. We will prove this theorem by constructing an upper triangular matrix U such that $A = UU^T$. Let

$$u_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ \mathbf{1} & \text{if } i = j \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then we have

$$(UU^T)_{ii} = \bigoplus_{k=1}^{n} (u_{ik} \otimes u_{ik}).$$

It is clear from the construction of U that u_{ik} is nonzero ($\neq \mathbf{0}$) only if $k \geq i$. We split the above sum into two parts. In the first part the sum is over all $k > i$ and the second part consists of the single term where $k = i$. Clearly the second part contains a single entry of the sum. Thus we have

$$\begin{aligned}(UU^T)_{ii} &= \bigoplus_{k>i} (u_{ik} \otimes u_{ik}) \quad \oplus \quad (u_{ii} \otimes u_{ii}) \\ &= \underset{k>i}{l.u.b} \quad (a_{ik} \otimes a_{ik}) \quad \oplus \quad (\mathbf{1} \otimes \mathbf{1}) \\ &= \mathbf{1}, \text{(since } \mathbf{1} \text{ is the greatest element of the incline).}\end{aligned}$$

Now for $i \neq j$,

$$(UU^T)_{ij} = \bigoplus_{k=1}^{n} (u_{ik} \otimes u_{jk}).$$

Without loss of generality, we assume that $i < j$. It is clear from the construction of U that u_{ik} is nonzero only if $k \geq i$ and u_{jk} is nonzero only if $k \geq j$. This implies that $(u_{ik} \otimes u_{jk})$ is nonzero only if $k \geq j$. Now we split the above sum into two parts. In the first part the sum is over all $k > j$ and the second part consists of the single term where $k = j$. Clearly the second part contains a single entry of the sum. Thus we have,

$$(UU^T)_{ij} = \bigoplus_{k>j}(u_{ik} \otimes u_{jk}) \quad \oplus \quad (u_{ij} \otimes u_{jj})$$

$$= \underset{k>j}{l.u.b}(a_{ik} \otimes a_{jk}) \quad \oplus \quad (a_{ij} \otimes \mathbf{1})$$

$$= \underset{k>j}{l.u.b}(a_{ik} \otimes a_{jk}) \quad \oplus \quad a_{ij}.$$

Since all the left almost principal 2×2 submatrices of A have $det^+ \geq det^-$, we have for all k, where $k > j > i$, $det^+(A[i,k|j,k]) \geq det^-(A[i,k|j,k])$, i.e., $a_{ij} \otimes a_{kk} \geq a_{ik} \otimes a_{jk}$. This implies that $a_{ij} \otimes \mathbf{1} = a_{ij} \geq a_{ik} \otimes a_{jk}$ and this is true for all $k > j$. Thus the

$$\underset{k>j}{l.u.b}(a_{ik} \otimes a_{jk}) \oplus a_{ij} = a_{ij}.$$

Therefor $(UU^T)_{ij} = a_{ij}$. This proves that $A = UU^T$. Hence A is an UL-completely positive matrix.

■

Analogous results hold for LU-completely positive matrices. By a similar argument to the previous theorem, we have:

Theorem 3.4.4. *If A is an $n \times n$ completely positive matrix over a normal incline L and all its right almost principal 2×2 submatrices have $det^+ \geq det^-$, then A is LU-completely positive.*

The converses of the theorem 3.4.3 and theorem 3.4.4 are not true. We give an example of a UL-completely positive matrix over a normal incline L which has a left almost principal 2×2 submatrix that does not satisfy the inequality $det^+ \geq det^-$.

Since every max-min semiring is a normal incline, we use matrices over a max-min semiring in the following example.

Example 3.4.5. *Let*

$$A = \begin{bmatrix} 6 & 2 & 4 & \boldsymbol{0} \\ 2 & 5 & 4 & \boldsymbol{0} \\ 4 & 4 & 5 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

be a 4×4 *matrix over a max-min semiring. The matrix* A *is UL-completely positive because there exists an upper triangular matrix* U *such that* $A = UU^T$*, where*

$$U = \begin{bmatrix} \boldsymbol{0} & 6 & \boldsymbol{0} & 4 \\ \boldsymbol{0} & 2 & 5 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & 4 & 5 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

However, the left almost principal 2×2 *submatrix* $A[2,3|1,3]$ *of* A *has* $det^+ = 2 \leq 4 = det^-$*.*

Theorems 3.4.3 and 3.4.4 are incline generalizations of theorems 2.1.22 and 2.1.23. We note that in an incline case, we need only 2×2 left or right almost principal submatrices, i.e., if all 2×2 left or right almost principal submatrices of an $n \times n$ completely positive matrix have $det^+ \geq det^-$ then the matrix is UL or LU completely positive respectively.

Remark 3.4.6. *The CP-rank of an $n \times n$ completely positive matrix over a normal incline L is less than or equal to n if either all its left almost principal 2×2 submatrices or all of its right almost principal 2×2 submatrices have $det^+ \geq det^-$.*

Now we will prove that all 3×3 completely positive matrices over normal inclines are either LU-completely positive matrices or UL-completely positive matrices (or both). We need the following lemma:

Lemma 3.4.7. *Let L be a normal incline. If A is a 3×3 completely positive matrix over L, then at least two of the following inequalities holds:*

$$a_{11} \otimes a_{23} \geq a_{12} \otimes a_{13}$$

$$a_{22} \otimes a_{13} \geq a_{12} \otimes a_{23}$$

$$a_{33} \otimes a_{12} \geq a_{13} \otimes a_{23}$$

Proof. Suppose that two of the inequalities do not hold, say,

$$a_{11} \otimes a_{23} \ngeq a_{12} \otimes a_{13}$$

$$a_{22} \otimes a_{13} \ngeq a_{12} \otimes a_{23}.$$

Then

$$a_{11} \otimes a_{23} \otimes a_{22} \otimes a_{13} \ngeq a_{12} \otimes a_{13} \otimes a_{12} \otimes a_{23}. \tag{3.1}$$

Given that A is a 3×3 completely positive matrix over a normal incline L. Therefore, by theorem 3.3.1, every 2×2 principal submatrix of A has $det^+ \geq det^-$, i.e., $a_{11} \otimes a_{22} \geq a_{12}^{\otimes 2}$. Therefore, $a_{11} \otimes a_{23} \otimes a_{22} \otimes a_{13} \geq a_{12} \otimes a_{13} \otimes a_{12} \otimes a_{23}$, where $a_{23}, a_{13} \in L = P(L)$. Thus we get a contradiction to equation (3.1).

■

Now we will relate inequalities of the above lemma with the positive and the negative determinant of 2×2 submatrices of the given matrix A. In the above lemma, the first inequality $a_{11} \otimes a_{23} \geq a_{12} \otimes a_{13}$ implies that $det^{+}(A[1,2|1,3]) \geq det^{-}(A[1,2|1,3])$. In other words, this inequality implies that the right almost principal 2×2 submatrix $A[1,2|1,3]$ of A has the positive determinant greater than or equal to the negative determinant.

The third inequality $a_{33} \otimes a_{12} \geq a_{13} \otimes a_{23}$ implies that $det^{+}(A[1,3|2,3]) \geq det^{-}(A[1,3|2,3])$, or stated in words that the left almost principal 2×2 submatrix $A[1,3|2,3]$ of A has the positive determinant greater than or equal to the negative determinant. However the second inequality $a_{22} \otimes a_{13} \geq a_{12} \otimes a_{23}$ has no relation with any left or right almost principal 2×2 submatrix of A.

Theorem 3.4.8. *Let L be a normal incline. If A is a 3×3 completely positive matrix over L, then A is either LU-completely positive or UL-completely positive or both.*

Proof. If any diagonal entry of A is $\mathbf{0}$, then all the entries in the corresponding row and column will be $\mathbf{0}$. In this case the result follows from example 3.4.2. Now suppose that $a_{ii} > \mathbf{0}$ for all $i = 1, 2, 3$. The only left almost principal 2×2 submatrices of A are

$$A_1 = A[1,3|2,3] \qquad\qquad A_2 = A[2,3|1,3]$$

and

$$det^+(A_1) = det^+(A_2) = a_{33} \otimes a_{12}$$

$$det^-(A_1) = det^-(A_2) = a_{13} \otimes a_{23}$$

and the only right almost principal 2×2 submatrices of A are

$$A_3 = A[1,3|1,2] \qquad\qquad A_4 = A[1,2|1,3]$$

and

$$det^+(A_3) = det^+(A_4) = a_{11} \otimes a_{23}$$

$$det^-(A_3) = det^-(A_4) = a_{13} \otimes a_{12}$$

By lemma 3.4.7, either the left almost principal 2×2 submatrices have $det^+ \geq det^-$ or the right almost principal 2×2 submatrices have $det^+ \geq det^-$ or both. Thus by theorem 3.4.3 and 3.4.4, A is either UL-completely positive or LU-completely positive or both. ■

Example 3.4.9. *Let L be a normal incline. A 3×3 matrix A over L need not be both LU-completely positive and UL-completely positive. For example, consider a matrix A over a max-min semiring, where*

$$A = \begin{bmatrix} 5 & \boldsymbol{0} & 3 \\ \boldsymbol{0} & 4 & 4 \\ 3 & 4 & 8 \end{bmatrix}$$

All right almost principal 2×2 submatrices of A has the positive determinant greater than or equal to the negative determinant. Thus by throrem 3.4.4, A is LU-completely *positive. However, we can check that there does not exists any upper triangular matrix U over the max-min semiring such that $A = UU^T$.*

Definition 3.4.10. (TN_2 *Matrix*) *A matrix over any commutative semiring is called a TN_2 matrix if $det^+ \geq det^-$ for all its two by two submatrices, i.e., for all i, j, k, l with $i < k$ and $j < l$, we have*

$$a_{ij}a_{kl} \geq a_{il}a_{kj}.$$

The class of TN_2 real matrices has interesting properties in its own right; the results in [**53**] are a good example of this. For TN_2 matrices over normal inclines, we have an analog of corollary 2.1.24.

Corollary 3.4.11. *Every square symmetric TN_2 matrix over a normal incline is both LU- and UL- completely positive.*

3.5 The CP-rank of CP Matrices Over Semirings

In this section, we provide an algorithm of factorizing symmetric diagonally dominant matrices over special semirings. We use the same techniques as used by Kaykobad [**54**] to factor real symmetric diagonally dominant matrices. We also give an upper bound of the CP-rank of diagonally dominant completely positive matrices over a special class of semirings.

Let $\langle n \rangle = \{1, 2, ..., n\}$ and A be an $n \times n$ symmetric matrix. The set of all nonzero non-diagonal positions in the i^{th} row of A is denoted by $N(i) = \{j \in \langle n \rangle \,|j \neq i, a_{ij} \neq$

$\mathbf{0}\}$. The cardinality of $N(i)$ is denoted by $d(i)$, i,e, the total number of nonzero non-diagonal entries in the i^{th} row is denoted by $d(i)$. $A(l|m)$ is a matrix obtained by deleting the l^{th} row and m^{th} column of the matrix A. The matrix obtained by deleting the l^{th} row and the l^{th} column of A is denoted by $A(l)$.

Theorem 3.5.1. *Let L be a normal incline and $A \in M_n(L)$ be a symmetric diagonally dominant matrix. Then the CP-rank of A is less than or equal to the CP-rank$(A(l))+d(l)+1$, where $l \in \langle n \rangle$.*

Proof. Since L is a normal incline, we get $L = P(L)$ and hence by theorem 3.2.1, every symmetric diagonally dominant matrix $A \in M_n(L)$ is completely positive. Without loss of generality we assume that $l = n$ and $A(n)$ is an $(n-1) \times (n-1)$ upper left principal submatrix of A. Since $A \in M_n(L)$ is completely positive, every principal submatrix of A is also completely positive. This implies that $A(n)$ is completely positive and let us suppose that the CP-rank$(A(n)) = k$, for some positive integer k. Therefore, there exists an $(n-1) \times k$ matrix D over $P(L) = L$ such that $A(n) = DD^T$. Let $N(n) = \{k_1, k_2, ..., k_{d(n)}\}$ and $m = k + d(n) + 1$. We now construct an $n \times m$ matrix $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ where

$$B_1 = \begin{bmatrix} D \\ O \end{bmatrix} \text{ is an } n \times k \text{ matrix over } P(L) = L,$$

here O denotes a $1 \times k$ zero matrix over L, and $B_2 = [(b_2)_{ij}]$ is an n by $(d(n)+1)$ matrix over L, such that

$$(b_2)_{k_i,i} = c_{k_in}, \qquad i \in \langle d(n) \rangle,$$

$$(b_2)_{n,i} = c_{k_in}, \qquad i \in \langle d(n) \rangle,$$

$$(b_2)_{n,(d(n)+1)} = c_{nn},$$

and all other entries of B are equal to $\mathbf{0}$. Here $c_{k_in}^{\otimes 2} = a_{k_in}$ and $c_{nn}^{\otimes 2} = a_{nn}$, since every element of L has a unique square root in L. Thus we get that $A = BB^T$, where B is an $n \times m$ matrix over L and $m = k + d(l) + 1$. This implies that the CP-rank of A is less than or equal to the CP-rank$(A(n)) + d(n) + 1$.

■

Kaykobad presented an algorithm in [**54**] which finds a completely positive factorization of an $n \times n$ real symmetric diagonally dominant matrix and showed that the CP-rank of such a matrix is always less than or equal to $\frac{1}{2}n(n+1) - N$, where $2N$ is the number of off-diagonal entries which are equal to zero. We use the same notations and technique to prove that the exact same upper bound also holds for the CP-rank of completely positive matrices over a special class of semirings.

Definition 3.5.2. (*Vertex-Edge Incidence Matrix*) *Let G be a graph with n vertices. An $n \times n$ matrix B is said to be a vertex-edge incidence matrix of the graph G if*

$$b_{k(i,j)} = \begin{cases} 1 & \textit{if the vertex } k \textit{ is incident at edge (i,j) in } G, (k = i \textit{ or } j) \\ 0 & \textit{otherwise} \end{cases}$$

BB^T is a symmetric $(0,1)$ matrix. If the entries of B are any numbers, then B is said to be a weighted vertex-edge incidence matrix.

Theorem 3.5.3. *Let S be a commutative semiring and $P(S)$ be the positive subsemiring of S. If every element in $P(S)$ has a unique square root in $P(S)$ then every symmetric diagonally dominant matrix $A \in M_n(P(S))$ has the CP-rank less than or equal to $\frac{1}{2}n(n+1) - N$, where $2N$ is the number of off-diagonal entries which are equal to zero.*

Proof. Suppose that every element of $P(S)$ has a unique square root in $P(S)$. This implies that $P(P(S)) = P(S)$, and hence by theorem 3.2.1, every symmetric diagonally dominant matrix $A \in M_n(P(S))$ is completely positive. We also know that

$$a_{ii} \geq \bigoplus_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}, \qquad \text{for all } i.$$

This implies that there exists $d_i \in P(S)$ such that

$$a_{ii} = d_i \bigoplus \left(\bigoplus_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} \right), \qquad \text{for all } i.$$

We now construct a factorization of A. We relate a multigraph $G = (V, E)$ to the matrix in the following way: Row i and column i of the matrix A correspond to vertex $i \in V$. Vertex i is connected to vertex j if $a_{ij} \neq \mathbf{0}$, $i \neq j$. Vertex i is connected to itself by a loop if $a_{ii} \neq \mathbf{0}$. For each nonzero a_{ij} there is an edge $(i, j) \in E$. The set E also includes loops. Now construct the weighted vertex-edge incidence matrix B (vertex corresponds to row, and edge corresponds to column) of the graph G as follows:

$$b_{k(i,j)} = \begin{cases} c_{ij} & \text{if } k = i \text{ or } j, \text{ and } i \neq j \\ l_i & \text{if } k = i = j, \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Here $c_{ij}, l_i \in P(S)$ such that $c_{ij}^{\otimes 2} = a_{ij}$ for all i, j, and $l_i^{\otimes 2} = d_i$, since every element of $P(S)$ has a unique square root in $P(S)$. Clearly, we have exactly two entries in each column of B corresponding to edges which are not loops. For loops we have a single entry, since a loop is incident at a single vertex. Thus B is a matrix the order $n \times m$ over $P(S)$, where $m = \frac{1}{2}n(n+1) - N$, $2N$ is the number of off-diagonal entries which equal zero. Now we only have to show that $A = BB^T$. We know that

$$(BB^T)_{ij} = \bigoplus_{k(i,k)\in E} b_{i(i,k)} \otimes b_{j(i,k)}$$

Now $b_{j(i,k)} \neq \mathbf{0}$ only if edge (i, k) is incident on vertex j, that is, either $i = j$ or $k = j$. Thus for $i \neq j$,

$$\begin{aligned} (BB^T)_{ij} &= b_{i(i,j)} \otimes b_{j(i,j)} \\ &= c_{ij} \otimes c_{ij} \\ &= c_{ij}^{\otimes 2} = a_{ij}, \end{aligned}$$

and for $i = j$

$$
\begin{aligned}
(BB^T)_{ii} &= \Bigg(\bigoplus_{\substack{k(i,k)\in E \\ k\neq i}} (b_{i(i,k)} \otimes b_{i(i,k)}) \Bigg) \bigoplus (b_{i(i,i)} \otimes b_{i(i,i)}), \\
&= \Bigg(\bigoplus_{\substack{k=1 \\ k\neq i}}^{n} (c_{ik} \otimes c_{ik}) \Bigg) \bigoplus (l_i \otimes l_i) \\
&= \Bigg(\bigoplus_{\substack{k=1 \\ k\neq i}}^{n} a_{ik} \Bigg) \bigoplus d_i \\
&= a_{ii}.
\end{aligned}
$$

This proves that $A = BB^T$.

■

Note that in the above theorem if a diagonal entry of A is equal to the sum of all off-diagonal entries in the corresponding row and column (i.e., $a_{ii} = \bigoplus_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}$) then we can delete a column of B having a single nonzero entry and which fixes that (the i^{th}) diagonal entry of A. Thus the dimension of the matrix B which we have constructed is $n \times m$, where $m = \frac{1}{2}n(n+1) - N - I$, $2N$ is the number of off-diagonal entries which are equal to zero, and I is the number of diagonal entries which are equal to the sum of all off-diagonal entries in the corresponding row and column.

Further, if we remove the condition that every element of $P(S)$ has a unique square root in $P(S)$ from theorem 3.5.3, then the CP-rank of a symmetric diagonally dominant matrix $A \in M_n(P(S))$, where $P(S) = P(P(S))$, can be greater than $\frac{1}{2}n(n+1) - N$. For example, let $S = \mathbb{N}$, the set of all natural numbers including zero. It is clear that $\mathbb{N} = P(\mathbb{N}) = P(P(\mathbb{N})) = ...$, and not every element of $\mathbb{N} = P(\mathbb{N})$ has a unique square root in $\mathbb{N} = P(\mathbb{N})$. Consider a symmetric diagonally dominant matrix A over $\mathbb{N}$, where

$$A = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}.$$

By theorem 3.2.1, the matrix A is a completely positive matrix over $\mathbb{N}$. A rank 1 CP-representation of A over $\mathbb{N}$ is:

$$A = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix}$$

This implies that the CP-rank of A is less than or equal to 4. However, one can easily check that there does not exist any rank 1 CP-representation of A over $\mathbb{N}$ consisting of three or less rank one completely positive matrices over $\mathbb{N}$. Hence, the CP-rank of A is equal to $4 > \frac{2(2+1)}{2} - 1 = 2$.

3.6 The CP-rank Inequalities

We now establish some general results about the CP-rank of completely positive matrices over semirings. We use the same notations used in [**14**] to compare the column rank and the factor rank of matrices over semirings.

Suppose that $\mathbf{K}$ and $\mathbf{L}$ are two commutative semirings and that $\xi : \mathbf{K} \to \mathbf{L}$ is a semiring homomorphism. Let $P(\mathbf{K})$ and $P(\mathbf{L})$ be the positive subsemirings of $\mathbf{K}$ and $\mathbf{L}$, which consists of all the finite sums of perfect squares in $\mathbf{K}$ and $\mathbf{L}$, respectively.

For all $a \in P(\mathbf{K})$, we have $a = \bigoplus_{i=1}^{m} k_i^{\otimes 2}$, where $k_i \in \mathbf{K}$ and m is any positive integer. Thus we have

$$\begin{aligned}\xi(a) &= \xi\left(\bigoplus_{i=1}^{m} k_i^{\otimes 2}\right)\\ &= \bigoplus_{i=1}^{m} (\xi(k_i))^{\otimes 2}.\end{aligned}$$

This implies that $\xi(P(\mathbf{K})) \subseteq P(\mathbf{L})$. Let

$$\Xi : M_n(\mathbf{K}) \to M_n(\mathbf{L}),$$

where $(\Xi(A))_{ij} = \xi(a_{ij})$ for all $A = [a_{ij}] \in M_n(\mathbf{K})$. It can be easily seen that $\Xi(M_n(P(\mathbf{K}))) \subseteq M_n(P(\mathbf{L}))$. Since we are using two semirings, we denote the CP-rank of a matrix A over a semiring $\mathbf{K}$ and $\mathbf{L}$ by CP-rank$_{\mathbf{K}}(A)$ and CP-rank$_{\mathbf{L}}(A)$ respectively.

We state our result as follows:

Theorem 3.6.1. *Let $\boldsymbol{K}$ and $\boldsymbol{L}$ be two semirings and $\xi : \boldsymbol{K} \to \boldsymbol{L}$ be a semiring homomorphism. Then $A \in M_n(\boldsymbol{K})$ is completely positive implies that $\Xi(A) \in M_n(\boldsymbol{L})$ is completely positive and CP-rank$_{\boldsymbol{K}}(A) \geq$ CP-rank$_{\boldsymbol{L}}(\Xi(A))$.*

Proof. Let $A \in M_n(\mathbf{K})$ be a completely positive with CP-rank$_{\mathbf{K}}(A) = k$. Then there exists a matrix $B \in M_{n,k}(P(\mathbf{K}))$ such that $A = BB^T$. Since ξ is a semiring homomorphism,

$$\begin{aligned}\Xi(B)\Xi(B)^T &= \left[\bigoplus_{r=1}^{k} \xi(b_{ir}) \otimes \xi(b_{jr})\right]\\ &= \left[\xi\left(\bigoplus_{r=1}^{k} (b_{ir} \otimes b_{jr})\right)\right] = \Xi(BB^T) = \Xi(A)\end{aligned}$$

Since $\Xi(B)$ is an $n \times k$ matrix over the positive subsemiring $P(\mathbf{L})$, we have $\Xi(A) \in M_n(\mathbf{L})$ is a completely positive matrix and the CP-$\text{rank}_{\mathbf{L}}(\Xi(A)) \leq k = \text{CP-rank}_{\mathbf{K}}(A)$.

■

If $\mathbf{K}$ is a subsemiring of a commutative semiring $\mathbf{L}$, then the injection map from $\mathbf{K}$ to $\mathbf{L}$ is a semiring homomorphism, and hence by theorem 3.6.1 every completely positive matrix over the semiring $\mathbf{K}$ is completely positive over the semiring $\mathbf{L}$ and

$$\text{CP-rank}_{\mathbf{K}}(A) \geq \text{CP-rank}_{\mathbf{L}}(A),$$

for all completely positive matrices $A \in M_n(\mathbf{K})$.

In this case, we abbreviate the above to CP-$\text{rank}_{\mathbf{K}} \geq$ CP-$\text{rank}_{\mathbf{L}}$.

Corollary 3.6.2. *If $\boldsymbol{K}$ is a subsemiring of a commutative semiring $\boldsymbol{L}$, then every completely positive matrix over $\boldsymbol{K}$ is completely positive over $\boldsymbol{L}$ and CP-$\text{rank}_{\boldsymbol{K}} \geq$ CP-$\text{rank}_{\boldsymbol{L}}$. In particular,*

1. *If C is a Boolean subalgebra of a Boolean algebra B then CP-$\text{rank}_C \geq$ CP-rank_B.*

2. *If S_1 is a subsemiring of a max-min semiring S_2 then CP-$\text{rank}_{S_1} \geq$ CP-rank_{S_2}.*

3. *CP-$\text{rank}_{\mathbb{N}} \geq$ CP-$\text{rank}_{\mathbb{R}_+}$, where $\mathbb{N}$ is the set of all natural numbers including zero.*

Here we have an example of a completely positive matrix over $\mathbb{N}$ along with its factorization in $\mathbb{N}$ and in $\mathbb{R}_+$, which shows that there is a strict inequality between the CP-$\text{rank}_{\mathbb{N}}$ and the CP-$\text{rank}_{\mathbb{R}_+}$.

Example 3.6.3. *Let*

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

be a completely positive matrix over $\mathbb{N}$*. We know that* $\mathbb{N}$ *is a subsemiring of* $\mathbb{R}_+$*. This implies that* A *is also completely positive over* $\mathbb{R}_+$*. The CP-rank*$_{\mathbb{R}_+}(A) = 1$ *, since we have a following factorization of* A *over* $\mathbb{R}^+$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \end{bmatrix}$$

However, a rank 1 CP-representation of A *over* $\mathbb{N}$ *is*

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

One can easily check that there does not exist any rank 1 CP-representation of A *over* $\mathbb{N}$ *consisting of only one rank one completely positive matrix over* $\mathbb{N}$*. Hence the CP-rank*$_{\mathbb{N}}(A) = 2$*.*

Equality may also hold in certain cases. We now discuss the cases when the CP-rank of completely positive matrices over different semirings is equal.

Lemma 3.6.4. *Let* S *be a max-min semiring and* $A \in M_n(S)$*. Construct a max-min semiring* $S(A)$ *consisting of* $\mathbf{0}, \mathbf{1}$*, and the entries of the matrix* A*. Then* A *is a completely positive matrix over the max-min semiring* S *if and only if* A *is a completely positive matrix over the max-min semiring* $S(A)$ *and the CP-rank*$_{S(A)}(A) =$ *CP-rank*$_S(A)$*.*

Proof. It is clear that $S(A)$ is a subsemiring of S and identity map from $S(A)$ to S is a semiring homomorphism. Therefore by corollary 3.6.2, if A is a completely positive matrix over the max-min semiring $S(A)$ then A is also completely positive over the max-min semiring S and CP-rank$_{S(A)}(A) \geq$ CP-rank$_S(A)$. For other direction, let us consider a map $\xi : S \to S(A)$ such that

$$\xi(x) = \bigoplus_{y \in C(x)} y, \quad \text{where,}$$

$$C(x) = \{y \in S(A) : y \leq x\}$$

If $x_1, x_2 \in S$ with $x_1 \leq x_2$, we find that $C(x_1) \subseteq C(x_2)$ and hence $\xi(x_1) \leq \xi(x_2)$. It now follows that ξ is a homomorphism from S to $S(A)$ and $\Xi(A) = A$. Hence by corollary 3.6.2, if A is a completely positive matrix over the max-min semiring S then $\Xi(A) = A$ is completely positive over the max-min semiring $S(A)$ and the CP-rank$_S(A) \geq$ CP-rank$_{S(A)}(A)$. ■

Theorem 3.6.5. *Suppose that S_1 and S_2 are max-min semirings and that S_1 is a subsemiring of S_2. If $A \in M_n(S_1)$ is completely positive then A is completely positive over S_2 and the CP-rank$_{S_1}(A) =$ CP-rank$_{S_2}(A)$.*

Proof. We are given that $S_1 \subseteq S_2$ and A is a completely positive matrix over S_1. Thus by corollary 3.6.2, A is a completely positive matrix over S_2. Now construct a subsemiring $S_1(A)$ of S_1 consisting of $\mathbf{0}, \mathbf{1}$, and the entries of the matrix A. By lemma 3.6.4, we get that

$$\text{CP-rank}_{S_1(A)}(A) = \text{CP-rank}_{S_1}(A). \tag{3.2}$$

Now construct a subsemiring $S_2(A)$ of S_2 consisting of $\mathbf{0}, \mathbf{1}$, and the entries of the matrix A. By lemma 3.6.4, we get that

$$\text{CP-rank}_{S_2(A)}(A) = \text{CP-rank}_{S_2}(A). \tag{3.3}$$

Note that the subsemiring $S_1(A) = S_2(A)$. Thus from equations (3.2) and (3.3) we get that $\text{CP-rank}_{S_1}(A) = \text{CP-rank}_{S_2}(A)$.

■

Let $A = [a_{ij}]$ be an $n \times n$ matrix whose entries belong to a commutative semiring S. We define the *pattern* matrix of A to be the $n \times n$ matrix $\bar{A} = [\bar{a_{ij}}]$ where

$$\bar{a_{ij}} = \mathbf{0}, \qquad \text{if } a_{ij} = \mathbf{0} \qquad \text{and}$$

$$\bar{a_{ij}} = \mathbf{1}, \qquad \text{if } a_{ij} \neq \mathbf{0}.$$

From this construction we get that the pattern matrix of any matrix over a commutative semiring is a matrix over the Boolean semiring.

Corollary 3.6.6. *Let S be an antinegative semiring having no zero divisors with the additive identity $\mathbf{0}$ and the multiplicative identity $\mathbf{1}$. Then the pattern of every completely positive matrix $A \in M_n(S)$ is completely positive over the Boolean semiring β and the CP-rank$_\beta(\bar{A}) \leq$ CP-rank$_S(A)$.*

Proof. Let $\xi : S \to \beta$ be a mapping defined by

$$\xi(a) = \mathbf{0}, \qquad \text{if } a = \mathbf{0}, \qquad \text{and}$$

$$\xi(a) = \mathbf{1}, \qquad \text{otherwise.}$$

Clearly it is a semiring homomorphism, since S is an antinegative semiring having no zero divisors. Thus the result follows from theorem 3.6.1.

■

We note that the equality holds in the corollary 3.6.6, for all rank one completely positive matrices A over an antinegative commutative semiring S with no zero divisors. However, for $n \geq 2$ there exist completely positive matrices over an antinegative commutative semiring S with no zero divisors such that the CP-rank$_\beta(\bar{A}) <$ CP-rank$_S(A)$. Here we have an example that shows the strict inequality between the CP-rank of a completely positive matrix A over the semiring of nonnegative numbers and the CP-rank of $\bar{A}$ over the Boolean semiring.

Example 3.6.7. *Let*

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 5 \end{bmatrix}$$

be a 2×2 matrix over $\mathbb{R}_+$. The rank of A is 2, this implies that the CP-rank of A is also 2, (using (2.9)). However,

$$\bar{A} = \begin{bmatrix} \mathit{1} & \mathit{1} \\ \mathit{1} & \mathit{1} \end{bmatrix} = \begin{bmatrix} \mathit{1} \\ \mathit{1} \end{bmatrix} \begin{bmatrix} \mathit{1} & \mathit{1} \end{bmatrix}$$

and hence the CP-rank$_\beta(\bar{A}) = 1$

If S is a max-min semiring in corollary 3.6.6 and A is any $(\mathbf{0},\mathbf{1})$ completely positive matrix over S then the CP-rank$_\beta(A) =$ CP-rank$_S(A)$, using theorem 3.6.5.

Now we will give a graph theoretic interpretation to the homomorphism defined in corollary 3.6.6.

Definition 3.6.8. (*The CP-adjacency Matrix*) *Let $G = (V, E)$ be a simple undirected graph with n vertices. The CP-adjacency matrix of this graph is an $n \times n$ Boolean matrix B where*

$$b_{ii} = \boldsymbol{1} \qquad \textit{for all } i \in V, \qquad \textit{and}$$

$$b_{ij} = \begin{cases} \boldsymbol{1}, & \textit{if there is an edge between vertex } i \textit{ and vertex } j, \\ \boldsymbol{0}, & \textit{if there is no edge between vertex } i \textit{ and vertex } j. \end{cases}$$

Let A be an $n \times n$ symmetric matrix. The graph of A, denoted by $G(A)$, is a graph on vertices $1, 2, ..., n$ with $\{i, j\}$ an edge if and only if $i \neq j$ and $a_{ij} \neq 0$. If A is a real completely positive matrix and $G(A) = G$, we say that A is a *CP matrix realization* of G.

In corollary 3.6.6, if $S = \mathbb{R}_+$ and A is any completely positive matrix over $\mathbb{R}_+$ such that $G(A) = G$, then ξ maps A to the CP-adjacency matrix of G. In general, ξ maps every CP matrix realization of G to the CP-adjacency matrix of G. As a result, the CP-rank of an CP-adjacency matrix of a graph G is always less than or equal to the CP-rank of all CP matrix realizations of G. We use this result in chapter four to analyze the lower bound of the CP-rank of a graph.

Chapter 4

The DJL Conjecture for CP Matrices Over Boolean Algebras

In this chapter, we consider the Drew-Johnson-Loewy conjecture for completely positive matrices over Boolean algebras. We first examine completely positive matrices over the two element Boolean algebra called the Boolean semiring. In particular, we formulate a correspondence between completely positive matrices over the Boolean semiring and simple undirected graphs. Then we show that the Drew-Johnson-Loewy conjecture for completely positive matrices over the Boolean semiring is equivalent to a well-known result by Erdős, Goodman and Pósa in graph theory. We also use a semiring isomorphism defined by Kirkland and Pullman in [**58**], to prove the truth of the Drew-Johnson-Loewy conjecture for completely positive matrices over Boolean algebras.

4.1 CP Boolean Matrices and Graphs

A graph $G(V, E)$ can be covered by k complete graphs if there exist k not necessarily disjoint subsets S_i, $i = 1, 2, ..., k$, of V such that $\bigcup_{i=1}^{k} S_i = V$ and any two distinct vertices lying in the same subset S_i have an edge between them.

In 1964, Erdős, Goodman and Pósa [**33**] discovered an upper bound on the covering of an undirected graph with n vertices by complete graphs. Their result can be stated as follows.

Theorem 4.1.1. [**33**] *Any undirected graph G of order $n \geq 2$ with no isolated points can be covered by at most $[n^2/4]$ complete graphs. Further, in the covering with complete graphs we need to use only single edges and triangles.*

We will show that this graph theoretical result is equivalent to the matrix result that the CP-rank of $n \times n$ completely positive matrices over the Boolean semiring is bounded above by $[n^2/4]$.

We first describe completely positive matrices over the Boolean semiring.

Proposition 4.1.2. *A square matrix over the Boolean semiring is completely positive if and only if it is symmetric and diagonally dominant.*

This proposition follows from Theorem 3.3.3, where this result is proved for the more general class of regular inclines.

There is a connection between completely positive matrices over the Boolean semiring and graphs. Recall that if $G = (V, E)$ is a simple undirected graph with n

vertices then the Boolean matrix corresponding to this graph is a matrix B where $b_{ii} = \mathbf{1}$, for all $i \in V$, and $b_{ij} = \mathbf{1}$, if there is an edge between vertex i and vertex j, and $b_{ij} = \mathbf{0}$, if there is no edge between vertex i and vertex j.

Note that the Boolean matrix B corresponding to the graph G is a symmetric and diagonally dominant matrix. Therefore, it is a completely positive Boolean matrix by proposition 4.1.2. Thus every matrix over the Boolean semiring which corresponds to a simple undirected graph is a completely positive Boolean matrix. Moreover, corresponding to every completely positive matrix B over the Boolean semiring there exists a simple undirected graph $G(B)$ with n vertices and there is an edge from vertex i to vertex j if $i \neq j$ and $b_{ij} = \mathbf{1}$.

We use this relation between completely positive matrices over the Boolean semiring and simple undirected graphs to prove the equivalence of the Drew-Johnson-Loewy conjecture for completely positive matrices over the Boolean semiring and the result of Erdős, Goodman and Pósa.

4.2 The DJL Conjecture and The Boolean Semiring

Let $G(V, E)$ be a simple undirected graph with n vertices and B be an $n \times n$ completely positive Boolean matrix corresponding to the graph G. A rank 1 CP-representation of B is

$$B = \bigoplus_i b_i b_i^T$$

where $b_i = \begin{pmatrix} b_{i1} \\ b_{i2} \\ . \\ . \\ b_{in} \end{pmatrix}$ and $b_{ij} = \mathbf{1}$ if vertex j is in the i^{th} clique covering and $\mathbf{0}$ otherwise.

In other words, each complete subgraph in an edge clique covering of G corresponds to a rank one completely positive matrix over the Boolean semiring in a rank 1 CP-representation of B. Therefore, any rank 1 CP-representation of a matrix over the Boolean semiring is equivalent to an edge clique covering of the corresponding graph.

Thus the result of Erdős, Goodman and Pósa implies the following bound on the CP-rank of completely positive matrices over the Boolean semiring.

Theorem 4.2.1. *Let $A = [a_{ij}]$ be an $n \times n$, for $n \geq 2$, completely positive matrix over the Boolean semiring. If every row/column of A has at least two non-zero entries then A can be written as the sum of at most $[n^2/4]$ rank one completely positive matrices $B_i = b_i b_i^T$ with support less than or equal to 3.*

Proof. Let A be an $n \times n$ matrix over the Boolean semiring and $G(A)$ be a graph of A with the vertex set $V = 1, 2, ..., n$ and there is an edge between vertex i and vertex j if and only if $i \neq j$ and $a_{ij} = \mathbf{1}$. Since A is a symmetric matrix and every row/column of A has at least two non-zero entries, the graph $G(A)$ is a simple undirected graph with no isolated points. Therefore, by theorem 4.1.1, $G(A)$ can be covered by at most

$[n^2/4]$ complete graphs and each complete graph is either an edge or a triangle. We also know that any rank 1 CP-representation of A is equivalent to an edge clique covering of the corresponding graph $G(A)$. Thus we get a rank 1 CP-representation of A with at most $[n^2/4]$ rank one completely positive matrices. Moreover, every rank one completely positive matrix $B_i = b_i b_i^T$ either corresponds to an edge or a triangle. Hence each rank one completely positive matrix $B_i = b_i b_i^T$ has support less than or equal to 3.

■

If we remove the condition that every row/column of A has at least two non-zero entries in the above theorem, then for $n \leq 3$, the CP-rank of A is less than or equal to n and for $n \geq 4$, the CP-rank of A is less than or equal to $[n^2/4]$. This has been proved in the following corollary.

Corollary 4.2.2. *Let $A = [a_{ij}]$ be an $n \times n$ completely positive matrix over the Boolean semiring. Then A can be written as the sum of at most $max\{n, [n^2/4]\}$ rank one completely positive matrices $B_i = b_i b_i^T$ with support less than or equal to* 3.

Proof. If A has a zero row and column then we can delete that row and the corresponding column of A and the resulting matrix is a completely positive matrix over the Boolean semiring and its CP-rank is equal to the CP-rank of the matrix A. Thus we can assume that A has no zero row and column. Suppose that for a nonnegative integer $m \leq n$, there are m rows and columns of A having only one non-zero entry.

Since A is a completely positive matrix over the Boolean semiring, if a row/column has only one non-zero entry then that will the diagonal entry. Without loss of generality, we can assume that

$$A = \begin{bmatrix} B_{n-m} & O_{n-m,m} \\ O_{m,n-m} & I_m \end{bmatrix},$$

where B_{n-m} is an $(n-m) \times (n-m)$ completely positive Boolean matrix having at least two non-zero entries in every row and column, I_m is an $m \times m$ identity Boolean matrix and $O_{n-m,m}$ is an $(n-m) \times m$ matrix of all zeros. Clearly the CP-rank of A is equal to the sum of the CP-rank of B_{n-m} and the CP-rank of I_m. Hence the CP-rank$(A) \leq [(n-m)^2/4] + m$, using the theorem 4.2.1 and the fact that the CP-rank if an $m \times m$ identity Boolean matrix is equal to m. A simple induction now shows that $[(n-m)^2/4] + m \leq max\{n, [n^2/4]\}$, for all n and $m \leq n$. Furthermore, we get a support 3 rank 1 CP-representation of A, since we have a support 3 rank 1 CP-representation of B_{n-m} (using the theorem 4.2.1) and support 1 rank 1 CP-representation of I_m.

■

The corollary 4.2.2 proves the truth of the Drew-Johnson-Loewy conjecture for completely positive matrices over the Boolean semiring.

The following lemma shows that in covering by complete graphs, $[n^2/4]$ can not be replaced by any smaller number, since there are some special types of graphs which need exactly $[n^2/4]$ complete graphs for covering.

Lemma 4.2.3. [**30**, Lemma 7] *The number of edges in a triangle free graph on n vertices is never more than $[n^2/4]$. Equality is attained if and only if the graph is complete bipartite with independent vertex sets are as balanced as possible, i.e., the graph is complete bipartite with the two vertex sets being of cardinality $(n/2)$ and $(n/2)$ (when n is even) or $((n-1)/2)$ and $((n+1)/2)$ (when n is odd).*

Remark 4.2.4. *From the above lemma we conclude that Boolean matrices of type*

$$\begin{pmatrix} I_m & J_m \\ J_m & I_m \end{pmatrix}, \text{ if } n = 2m, \text{ i.e., } n \text{ is even}$$

and

$$\begin{pmatrix} I_m & J_{m,m+1} \\ J_{m+1,m} & I_{m+1} \end{pmatrix}, \text{ if } n = 2m+1, \text{ i.e., } n \text{ is odd,}$$

where J_m is an $m \times m$ matrix of all ones and $J_{m+1,m}$ is an $(m+1) \times m$ matrix of all ones, achieve the upper bound of the CP-rank. Thus the upper bound for the CP-rank in the Drew-Johnson-Loewy conjecture for completely positive matrices over the Boolean semiring can not be replaced by any smaller number.

In [**58**], Kirkland and Pullman have shown that there is an isomorphism between the matrices over a Boolean algebra of subsets of a k-element set and the k-tuples of Boolean (binary) matrices. In section 4.4, we will study this isomorphism and use the results of this section to prove the Drew-Johnson-Loewy conjecture for completely positive matrices over Boolean algebras. In chapter 5, we will extend the results of this section to completely positive matrices over totally ordered normal inclines.

4.3 The CP-rank of Graphs and CP Boolean matrices

In this section, we study the lower bound of the CP-rank of a graph. We begin with the definition of the CP-rank of a graph.

Definition 4.3.1. (*The CP-rank of A Graph*) *Let G be a graph with n vertices and let $S(G) \subseteq \mathbb{N}$ be defined as follows:*

$$S(G) = \{\textit{CP-rank}(A) \quad | \quad A \in M_n(\mathbb{R}) \textit{ is a CP matrix and } G(A) = G\}.$$

The maximum value of $S(G)$ is called the CP-rank of the graph G.

We are most interested in the maximal value in $S(G)$. However, in the following theorem we show that the minimum value in $S(G)$ is equal to the CP-rank of the completely positive Boolean matrix associated with the graph G. In other words, the CP-rank of a graph G is bounded below by the CP-rank of its CP-adjacency matrix.

Theorem 4.3.2. *Let G be a graph with n vertices and $S(G) = \{\textit{CP-rank}(A) | A \in M_n(\mathbb{R})$ is a CP matrix and $G(A) = G\}$. Then $\min(S(G))$ is exactly the CP-rank of the Boolean matrix B such that $G(B) = G$.*

Proof. Let A be an $n \times n$ real completely positive matrix with $G(A) = G$ and B be a Boolean completely positive matrix with $G(B) = G$. Clearly the zero-nonzero patterns of A and B are the same. Let CP-rank$(A) = k$, this implies that $A = \sum_{i=1}^{k} b_i b_i^T$, where $b_i \geq 0$, for all i, are nonzero real column vectors of order n.

Now replace all nonzero entries of b_i with the Boolean element $\mathbf{1}$ and all zero entries of b_i with the Boolean element $\mathbf{0}$ and denote this Boolean vector by $\hat{b}_i$, for all b_i. Observe that $B = \sum_{i=1}^{k} \hat{b}_i \hat{b}_i^T$, since A and B have same zero nonzero pattern. This implies that CP-rank$(B) \leq k =$ CP-rank(A) and this is true for all $n \times n$ real completely positive matrices A such that G(A) = G, i.e., the CP-rank$(B) \leq$ $min(S(G))$.

Further, let B be a Boolean CP matrix with $G(B) = G$ and CP-rank$(B) = k$. This implies that there exists an $n \times k$ Boolean matrix C such that $B = CC^T$. Now let M be an $n \times k$ real matrix which such that $m_{ij} = 1$ (the real number) if $c_{ij} = \mathbf{1}$ (the Boolean) and $m_{ij} = 0$ (the real number) if $c_{ij} = \mathbf{0}$ (the Boolean). Then MM^T has graph G and CP-rank less than or equal to k. This implies that the CP-rank$(B) \geq min(S(G))$. Thus $min(S(G))$ is exactly the CP-rank of the Boolean matrix associated with G.

■

Corollary 4.3.3. *Let G be a complete bipartite graph with n vertices, whose sets of independent vertices are as balanced as possible. Then $min(S(G)) = max(S(G))$ and all real completely positive matrices associated with this graph have the same CP-rank.*

Proof. For a complete bipartite graph G with n vertices, whose sets of independent vertices are as balanced as possible we have $min(S(G)) = [n^2/4]$, by remark 4.2.4 and theorem 4.3.2. Also we know [**30**, Corollary 8] that the CP-rank of every real matrix A such that $G(A)$ is triangle free graph is less than or equal to $[n^2/4]$ and equality holds when we have a complete bipartite with independent vertex sets are as balanced as possible. Therefore, the $max(S(G)) = [n^2/4]$. Thus we get that $min(S(G)) = max(S(G))$ and all real completely positive matrices associated with this graph have the same CP-rank equal to $[n^2/4]$. ■

4.4 CP Matrices over Boolean Algebras

Although there is a great deal of literature on the study of matrices over finite Boolean algebras, many results in Boolean matrix theory are stated only for Boolean matrices. The fact is that most Boolean algebras contain zero divisors, but the two element Boolean algebra (the Boolean semiring) has no zero divisors. Kim pointed out in his extensive survey of the subject [**55**], that the isomorphism between matrices over β_k, a Boolean algebra of subsets of a k-element and k-tuples of Boolean matrices, allows many problems concerning matrices over an arbitrary finite Boolean algebra to follow from the results about matrices over Boolean semirings.

4.4.1 An Isomorphism Between Matrices Over Finite Boolean Algebras and The Boolean Semiring

We begin with introducing a method, as shown in [**57**], of representing a matrix over a Boolean algebra of subsets of a k-element set as a linear combination of (binary) Boolean matrices. It is found to be a computationally convenient way to use an isomorphism linking the nonbinary case to the binary case.

Let $\sigma_1, \sigma_2, ..., \sigma_k$ denote the singleton subsets of a k-element set S_k. For each $m \times n$ matrix A over a Boolean algebra β_k, the l^{th} constituent of A, denoted as $A^{(l)}$, is an $m \times n$ Boolean matrix whose ij^{th} entry is $\mathbf{1}$ if and only if $\sigma_l \subseteq a_{ij}$. Clearly $A = \bigcup_{l=1}^{k} \sigma_l A^{(l)}$.

Here we have an example of a 2×2 matrix over β_3.

Example 4.4.1. *Let*

$$A = \begin{bmatrix} \{1,3\} & \mathbf{0} \\ \{2,3\} & \{1\} \end{bmatrix}$$

be a 2×2 *matrix over* β_3*, where*

$$\sigma_1 = \{1\}, \qquad \sigma_2 = \{2\}, \qquad \text{and } \sigma_3 = \{3\}$$

and

$$A^{(1)} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \qquad A^{(2)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad A^{(3)} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}.$$

Clearly A *can be written as the linear combination of Boolean matrices* $A^{(i)}$*, for* $i = 1, 2, 3$ *as follows:*

$$A = \{1\} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \bigoplus \{2\} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \bigoplus \{3\} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}.$$

The next two propositions follow directly from the definitions and the fact that for any singletons σ and τ, $\sigma \otimes \tau = \sigma$ or $\mathbf{0}$ according as $\tau = \sigma$ or not.

Proposition 4.4.2. [**57**] *If* $A = \bigcup_{l=1}^{k} \sigma_l C^{(l)}$ *and the* $C^{(l)}$ *are all Boolean matrices, then* $C^{(l)} = A^{(l)}$ *for all* $1 \leq l \leq k$. *That is, the constituents of* A *are the unique Boolean matrices which satisfy* $A = \bigcup_{l=1}^{k} \sigma_l X_l$.

Proposition 4.4.3. [**57**] *For all* $p \times q$ *matrices* A*, all* $q \times r$ *matrices* B *and* C*, and all* α *over a Boolean algebra* β_k*, we have*

1. $(AB)^{(l)} = A^{(l)} B^{(l)}$,

2. $(B \cup C)^{(l)} = B^{(l)} \cup C^{(l)}$, *and*

3. $(\alpha A)^{(l)} = \alpha^{(l)} A^{(l)}$, *for all* $1 \leq l \leq k$.

If X and Y are row vectors of length k whose entries x_i and y_i are m by n Boolean matrices, let $XY = [x_1 y_1, x_2 y_2, ..., x_k y_k]$, the *Schur or Hadamard* product of the vectors. Now for each $m \times n$ matrix A over a Boolean algebra β_k define,

$$[A] = [A^{(1)}, A^{(2)}, ..., A^{(k)}],$$

where $A^{(l)}$ is the l^{th} constituent of A. The following proposition shows an isomorphism between matrices over β_k, a Boolean algebra of subsets of a k-element set and the k-tuples of Boolean matrices [**58**].

Proposition 4.4.4. *Let* β_k *be a Boolean algebra of subsets of a* k*-element set and* β *be the Boolean semiring. Let* $\sigma_1, \sigma_2, ..., \sigma_k$ *denote the singleton subsets of a* k*-element set. The mapping*

$$\phi : \beta_k \rightarrow [\beta]^k,$$

where $[\beta]^k$ *is the set of all* k*-tuples of Boolean elements, such that*

$$\phi(x) = [x_1, x_2, ..., x_k],$$

where $x_i = \mathbf{1}$ *if and only if* $\sigma_i \subseteq x$ *and* $x_i = \mathbf{0}$ *otherwise, is an isomorphism.*

We identify an $n \times m$ matrix $A = [a_{ij}]$ over a Boolean algebra β_k, with the $n \times m$ matrix $\Xi(A)$ whose $(i, j)^{th}$ entry equals $\phi(a_{ij})$. Thus

$$\Xi : M_{m,n}(\beta_k) \rightarrow [M_{m,n}(\beta)]^k,$$

where $[M_{m,n}(\beta)]^k$ is the set of all k-tuples of $m \times n$ Boolean matrices, such that

$$\Xi(A) = [A], \text{ for all } A \in M_{m,n}(\beta_k)$$

is an isomorphism.

Note that the isomorphism takes scalars (1×1 matrices) of β_k to row vectors of length k over the Boolean semiring, where a union of subsets of a k-set becomes the Boolean sum of such vectors and intersection becomes the Hadamard product of Boolean vectors.

We will use this isomorphism to prove the Drew-Johnson-Loewy conjecture for completely positive matrices over finite Boolean algebras.

4.4.2 CP Matrices Over Boolean Algebras and The DJL conjecture

It has been shown in section 4.1, that the Drew-Johnson-Loewy conjecture over the two element Boolean algebra (the Boolean semiring) is equivalent to a well-known result in graph theory. In this section, we show the truth of the Drew-Johnson-Loewy conjecture for completely positive matrices over Boolean algebras in general. To prove this we use an isomorphism defined by Kirkland and Pullman [**58**] between matrices over β_k, a Boolean algebra of subsets of a k-element set and k-tuples of Boolean matrices.

We note that in a Boolean algebra we have;

$$a \oplus a = a \cup a = a \text{ for all } a,$$

i.e., the addition is idempotent and

$$a \otimes b = a \cap b \leq a, b,$$

i.e., the product of two elements is always less than or equal to either factor. This implies that every every Boolean algebra is an incline. Moreover, any Boolean algebra is multiplicatively idempotent and if $a \leq b$ then $a = a \otimes b$. Thus every Boolean algebra is a regular incline with LI-property.

Since $\oplus$ is the union operation, an n by n matrix A over a Boolean algebra is diagonally dominant if $a_{ii} \geq a_{ij}$ for all $1 \leq i, j \leq n$. The following theorem characterizes completely positive matrices over Boolean algebras.

Theorem 4.4.5. *Let B be a Boolean algebra and A be an $n \times n$ symmetric matrix over B. Then the matrix A is completely positive if and only if A is diagonally dominant.*

The proof of this theorem follows directly from theorem 3.3.3.

We first prove the truth of the Drew-Johnson-Loewy conjecture for completely positive matrices over finite Boolean algebras.

Theorem 4.4.6. *Let A be an $n \times n$ matrix over β_k, a Boolean algebra of subsets of a k-element set. If A is completely positive then the CP-rank of A is less than or equal to $max\{n, [n^2/4]\}$.*

Proof. Let $\sigma_1, \sigma_2, ..., \sigma_k$ denote the singleton subsets of a k-element set. We know that there is an isomorphism

$$\phi : \beta_k \rightarrow [\beta]^k, \qquad \text{such that}$$

$$\phi(x) = [x_1, x_2, ..., x_k],$$

where $x_i = \mathbf{1}$ if and only if $\sigma_i \subseteq x$ and $x_i = \mathbf{0}$ otherwise.

We identify any matrix $A = [a_{ij}]$ over a semiring β_k, with the matrix $\Xi(A)$ whose $(i, j)^{th}$ entry equals $\phi(a_{ij})$. Thus the mapping Ξ that sends each matrix A over a finite Boolean algebra β_k to $[A]$, its vector of constituents, is an isomorphism.

Let $A \in M_n(\beta_k)$ be a completely positive matrix. Then by theorem 3.6.1, $[A] = [A^{(1)}, A^{(2)}, ..., A^{(k)}] \in [M_n(\beta)]^k$ is completely positive. This implies that $A^{(l)}$ is a completely positive Boolean matrix, for all $1 \leq l \leq k$. By corollary 4.2.2, the CP-rank$(A^{(l)}) \leq max\{n, [n^2/4]\}$ for all $1 \leq l \leq k$. This implies that, for all $1 \leq l \leq k$, there exist $n \times k_l$ Boolean matrices B_l, where $k_l \leq max\{n, [n^2/4]\}$, such that $A^{(l)} = B_l B_l^T$. Thus

$$[A] = [B_1 B_1^T, B_2 B_2^T, ..., B_k B_k^T],$$
$$= [B_1, B_2, ..., B_k][B_1, B_2, ..., B_k]^T.$$

Suppose that $max_{l=1}^{k}\{k_l\} = K$, clearly $K \leq max\{n, [n^2/4]\}$. Now make all the matrices B_l of order $n \times K$ by introducing zero columns and denote them as $\hat{B}_l$, for all $1 \leq l \leq k$. Thus we get

$$[A] = [\hat{B}_1, \hat{B}_2, ..., \hat{B}_k][\hat{B}_1, \hat{B}_2, ..., \hat{B}_k]^T.$$

We know that the mapping Ξ is an isomorphism from matrices over β_k to k-tuples of Boolean matrices. Thus for $[\hat{B}_1, \hat{B}_2, ..., \hat{B}_k] \in [M_{n,K}(\beta)]^k$ there exists an $n \times K$ matrix $B \in M_{n,K}(\beta_k)$ such that $\Xi(B) = [\hat{B}_1, \hat{B}_2, ..., \hat{B}_k]$, where

$$b_{ij} = \bigcup\{\sigma_l \quad | l \in (1, ..., k) \text{ and } (\hat{B}_l)_{ij} = \mathbf{1}\}.$$

Thus we get $\Xi(A) = [A] = [\hat{B}_1, \hat{B}_2, ..., \hat{B}_k][\hat{B}_1, \hat{B}_2, ..., \hat{B}_k]^T = \Xi(BB^T)$. This implies that $A = BB^T$, where $B \in M_{n,K}(\beta_k)$ and $K \leq max\{n, [n^2/4]\}$, since the mapping Ξ is an isomorphism. Hence the CP-rank$_{\beta_k}(A) \leq K \leq max\{n, [n^2/4]\}$.

■

A subalgebra of a Boolean algebra B is said to be *finitely generated* if it is generated by a finite subset of B. It has been shown in [**37**], the if the set E of generators has n elements, then there are 2^n possible intersections of elements of E and their complements, which are called atoms of the subalgebra C generated by E (i.e., if $E = \{p, q\}$ then the possible intersections of elements of E and their complements are $p \cap q$, $p^* \cap q$, $p \cap q^*$ and $p^* \cap q^*$). As a result, there can be at most 2^n distinct atoms. Suppose C has m atoms, where $m \leq 2^n$. The join of every set of atoms is an element of C, and every element of C can be written in exactly one way as the join of a set of atoms. There are 2^m subsets of the set of atoms, so there must be 2^m elements in C.

Corollary 4.4.7. [**37**, corollary 2, p. 82] *Every finitely generated Boolean algebra B is finite, and the number of its elements is 2^m, where m is the number of atoms in B. If a generating set of B has n elements, then B has at most 2^n atoms, and hence it has at most 2^{2^n} elements.*

Thus from theorem 4.4.6 and corollary 4.4.7, we conclude that the Drew-Johnson-Loewy conjecture is true for completely positive matrices over finite Boolean algebras. We now generalize this result for completely positive matrices over general Boolean algebras (finite or infinite).

We first need the following lemma.

Lemma 4.4.8. *Let B be a Boolean algebra (finite or infinite) and $A \in M_n(B)$. Construct a finite Boolean subalgebra $B(A)$ consisting of $\mathbf{0}$, $\mathbf{1}$, and generated by the entries of the matrix A. Then A is a completely positive matrix over the Boolean algebra B if and only if A is a completely positive matrix over the finite Boolean algebra $B(A)$.*

Proof. If A is a completely positive matrix over the finite Boolean algebra $B(A)$ then by corollary 3.6.2, we get that A is a completely positive matrix over the Boolean algebra B. For the other direction, let us suppose that $A \in M_n(B)$ is a completely positive matrix. By theorem 4.4.5, A is a symmetric diagonally dominant matrix over B. Therefore, one can easily see that A is also symmetric diagonally dominant matrix over the finite Boolean algebra $B(A)$ and hence it is completely positive over the Boolean algebra $B(A)$, by theorem 4.4.5.

■

Theorem 4.4.9. *Let A be an $n \times n$ matrix over a Boolean algebra B (finite or infinite). If A is completely positive then the CP-rank of A is less than or equal to $max\{n, [n^2/4]\}$.*

Proof. Let A be an $n \times n$ completely positive matrix over a Boolean algebra B. Now construct a Boolean subalgera $B(A)$ consisting of $\mathbf{0}, \mathbf{1}$, and generated by the entries of the matrix A. Then by lemma 4.4.8, A is a completely positive matrix over the finite Boolean algebra $B(A)$. By theorem 4.4.6 and corollary 3.6.2, we get that the CP-rank$_B(A) \leq$ CP-rank$_{B(A)}(A) \leq max\{n, [n^2/4]\}$. ■

Chapter 5

The CP-rank of Completely Positive Matrices Over Special Semirings

In this chapter, we study completely positive matrices over totally ordered normal inclines, over the max-plus semiring and over the sign pattern semiring. We give a combinatorial proof of the Drew-Johnson-Loewy conjecture for completely positive matrices over totally ordered normal inclines. We prove a generalization of the Erdős, Goodman and Pósa result for fuzzy graphs. We also use a characterization of completely positive matrices over the max-plus semiring by Cartwright and Chen [**26**] to show that a result of Cartwright and Chen is essentially equivalent to the truth of the Drew-Johnson-Loewy conjecture for completely positive matrices over the max-plus semiring. In addition, we prove analogs of the results relating the nonnegativity of almost principal minors and triangular decomposition for completely positive matrices over the max-plus semiring.

Moreover, we derive a characterization of completely positive matrices over the sign pattern semiring. We gave counterexamples to the Drew-Johnson-Loewy conjecture for completely positive matrices over the sign pattern semiring and the nonnegative interval subsemiring $\{-\infty\} \cup [0, \infty)$ of the max-plus semiring.

5.1 The DJL Conjecture for CP matrices over Special Inclines

An incline is a commutative semiring in which the addition is idempotent and the product of two elements is always less than or equal to either factor. We have the relation $\leq$ in an incline L, defined as $x \leq y \Leftrightarrow x \oplus y = y$. The order relation $\leq$ is a partial order relation. If the partial order relation $\leq$ in an incline is a total order relation then the incline is called totally ordered incline. Totally ordered inclines were studied in [**28**].

In this section, we prove the truth of the Drew-Johnson-Loewy conjecture for completely positive matrices over totally ordered normal inclines. The two element Boolean semiring, max-min semirings, the negative interval subsemiring of the max-plus semiring, max-$\times$ semirings, etc. are examples of totally ordered inclines.

Note that the theorem 3.3.1 holds for all totally ordered normal inclines. We start with the following lemma.

Lemma 5.1.1. *Let L be a totally ordered normal incline and A be an $n \times n$ completely positive matrix over L. Then the CP-rank$(A) \leq n$ for $n = 2, 3$.*

Proof. Let A be an $n \times n$ completely positive matrix over a totally ordered normal incline L. Because of theorem 3.3.1, we need to consider only symmetric matrices over L all of whose diagonal entries are $\mathbf{1}$. For $n = 2$, let us consider

$$A = \begin{bmatrix} \mathbf{1} & a_{12} \\ a_{12} & \mathbf{1} \end{bmatrix}$$

Where $a_{12} \in L$, then the rank 1 CP-representation of A is

$$A = \begin{bmatrix} \mathbf{1} \\ a_{12} \end{bmatrix} \begin{bmatrix} \mathbf{1} & a_{12} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & a_{12} \\ a_{12} & a_{12}^{\otimes 2} \end{bmatrix} \bigoplus \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Since $\mathbf{1}$ is the maximal element of the incline L, so $a_{12}^{\otimes 2} \oplus \mathbf{1} = \mathbf{1}$. Hence the CP-rank$(A) \leq 2$.

Now for $n = 3$, let us consider

$$A = \begin{bmatrix} \mathbf{1} & a_{12} & a_{13} \\ a_{12} & \mathbf{1} & a_{23} \\ a_{13} & a_{23} & \mathbf{1} \end{bmatrix}$$

then the rank 1 CP-representation of A is

$$A = \begin{bmatrix} \mathbf{1} \\ a_{12} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & a_{12} & \mathbf{0} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ a_{23} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{1} & a_{23} \end{bmatrix} \oplus \begin{bmatrix} a_{13} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} a_{13} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & a_{12} & \mathbf{0} \\ a_{12} & a_{12}^{\otimes 2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & a_{23} \\ \mathbf{0} & a_{23} & a_{23}^{\otimes 2} \end{bmatrix} \oplus \begin{bmatrix} a_{13}^{\otimes 2} & \mathbf{0} & a_{13} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ a_{13} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Since $\mathbf{1}$ is the maximal element of the incline L, the diagonal entries of A will not be affected when we will add all rank one completely positive matrices in rank 1 CP-representation of A. Hence the CP-rank$(A) \leq 3$, Furthermore, the equality holds for nonzero diagonal matrices of order 2×2 and 3×3. Thus the CP-rank$(A) \leq n$ for $n = 2, 3$. ■

Theorem 5.1.2. *Let L be a totally ordered normal incline and A be an $n \times n$ completely positive matrix over L. Then the CP-rank$(A) \leq max\{n, [n^2/4]\}$. Further, A has an $[n^2/4]$ - support 3 rank 1 CP-representation for $n \geq 4$.*

Proof. Let us suppose that A is a completely positive matrix over a totally ordered normal incline L. Because of theorem 3.3.1, we need to consider only symmetric matrices over L all of whose diagonal entries are $\mathbf{1}$. If $n \leq 3$ then $max\{n, [n^2/4]\} = n$ and by Lemma 5.1.1, the CP-rank$(A) \leq n$. Now we will prove the theorem for $n \geq 4$. First, we will show that the result is true for $n = 4$ and 5, then we will use induction going from index n to index $n + 2$.

For $n = 4$, let us consider

$$A = \begin{bmatrix} \mathbf{1} & a_{12} & a_{13} & a_{14} \\ a_{12} & \mathbf{1} & a_{23} & a_{24} \\ a_{13} & a_{23} & \mathbf{1} & a_{34} \\ a_{14} & a_{24} & a_{34} & \mathbf{1} \end{bmatrix}$$

Now choose the smallest non-diagonal entry in the matrix and then choose the largest non-diagonal entry in the same row. Without loss of generality, we assume that a_{13} be the smallest non-diagonal entry of the entire matrix and a_{12} be the largest non-diagonal entry in the first row. Then the rank 1 CP-representation of A is

$$A = \begin{bmatrix} \mathbf{1} \\ a_{12} \\ a_{13} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & a_{12} & a_{13} & \mathbf{0} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ a_{23} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{1} & a_{23} & \mathbf{0} \end{bmatrix}$$

$$\oplus \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ a_{34} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & a_{34} \end{bmatrix} \oplus \begin{bmatrix} a_{14} \\ a_{24} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} a_{14} & a_{24} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & a_{12} & a_{13} & \mathbf{0} \\ a_{12} & a_{12}^{\otimes 2} & a_{12}a_{13} & \mathbf{0} \\ a_{13} & a_{12}a_{13} & a_{13}^{\otimes 2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \bigoplus \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & a_{23} & \mathbf{0} \\ \mathbf{0} & a_{23} & a_{23}^{\otimes 2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\bigoplus \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & a_{34} \\ \mathbf{0} & \mathbf{0} & a_{34} & a_{34}^{\otimes 2} \end{bmatrix} \bigoplus \begin{bmatrix} a_{14} & a_{14}a_{24} & \mathbf{0} & a_{14} \\ a_{14}a_{24} & a_{24}^{\otimes 2} & \mathbf{0} & a_{24} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ a_{14} & a_{24} & \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

Since $a_{12}a_{13} \leq a_{13} \leq a_{23}$ and $a_{14}a_{24} \leq a_{14} \leq a_{12}$, so it will not effect the $(2,3)^{th}$ and $(1,2)^{th}$ entry of A respectively. Hence CP-rank$(A) \leq 4 = [4^2/4]$ with support 3.

Now for $n = 5$, let us consider

$$A = \begin{bmatrix} \mathbf{1} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & \mathbf{1} & a_{23} & a_{24} & a_{25} \\ a_{13} & a_{23} & \mathbf{1} & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & \mathbf{1} & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & \mathbf{1} \end{bmatrix}$$

Without loss of generality, let us suppose that a_{13} be the smallest non-diagonal entry of the entire matrix and a_{12} be the largest non-diagonal entry in the first row. We need the following rank one completely positive matrices for the rank 1 CP-representation of A.

- $B_1 = \begin{bmatrix} \mathbf{1} \\ a_{12} \\ a_{13} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & a_{12} & a_{13} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & a_{12} & a_{13} & \mathbf{0} & \mathbf{0} \\ a_{12} & a_{12}^{\otimes 2} & a_{12}a_{13} & \mathbf{0} & \mathbf{0} \\ a_{13} & a_{12}a_{13} & a_{13}^{\otimes 2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

- $B_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ a_{23} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{1} & a_{23} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & a_{23} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & a_{23} & a_{23}^{\otimes 2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

- $B_3 = \begin{bmatrix} a_{14} \\ a_{24} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} a_{14} & a_{24} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} a_{14}^{\otimes 2} & a_{14}a_{24} & \mathbf{0} & a_{14} & \mathbf{0} \\ a_{14}a_{24} & a_{24}^{\otimes 2} & \mathbf{0} & a_{24} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ a_{14} & a_{24} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

- $B_4 = \begin{bmatrix} a_{15} \\ a_{25} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} a_{15} & a_{25} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} a_{15}^{\otimes 2} & a_{15}a_{25} & \mathbf{0} & \mathbf{0} & a_{15} \\ a_{15}a_{25} & a_{25}^{\otimes 2} & \mathbf{0} & \mathbf{0} & a_{25} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ a_{15} & a_{25} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

Clearly $A = B_1 \bigoplus B_2 \bigoplus B_3 \bigoplus B_4 \bigoplus C$, where

$$C = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & a_{34} & a_{35} \\ \mathbf{0} & \mathbf{0} & a_{34} & * & a_{45} \\ \mathbf{0} & \mathbf{0} & a_{35} & a_{45} & ** \end{bmatrix}.$$

Here $*$ and $**$ can be any element of the incline L. Now choose the smallest of a_{34}, a_{35}, and a_{45}. We have the following cases:

Case 1: If a_{34} or a_{35} is the smallest of all the entries $\{a_{34}, a_{35}, a_{45}\}$ then C can be written as the sum of two rank one completely positive matrices C_1 and C_2, where

$$C_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ a_{34} \\ a_{35} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & a_{34} & a_{35} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & a_{34} & a_{35} \\ \mathbf{0} & \mathbf{0} & a_{34} & a_{34}^{\otimes 2} & a_{34}a_{35} \\ \mathbf{0} & \mathbf{0} & a_{35} & a_{34}a_{35} & a_{35}^{\otimes 2} \end{bmatrix}$$

and

$$C_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ a_{45} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{45} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{45}^{\otimes 2} & a_{45} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{45} & \mathbf{1} \end{bmatrix}$$

Note that either we have $a_{34}a_{35} \leq a_{34} \leq a_{45}$ or $a_{34}a_{35} \leq a_{35} \leq a_{45}$, so it will not affect the $(4,5)^{th}$ entry of A when we will add C_1 and C_2 to the rank 1 CP-representation of A.

Case 2: If a_{45} is the smallest of all the entries $\{a_{34}, a_{35}, a_{45}\}$ then C can be written as the sum of two rank one completely positive matrices C_1 and C_2, where

$$C_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ a_{34} \\ a_{45} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & a_{34} & a_{45} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & a_{34} & a_{45} \\ \mathbf{0} & \mathbf{0} & a_{34} & a_{34}^{\otimes 2} & a_{34}a_{45} \\ \mathbf{0} & \mathbf{0} & a_{45} & a_{34}a_{45} & a_{45}^{\otimes 2} \end{bmatrix}$$

and

$$C_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ a_{35} \\ a_{45} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & a_{35} & a_{45} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & a_{35}^{\otimes 2} & a_{35}a_{45} & a_{35} \\ \mathbf{0} & \mathbf{0} & a_{35}a_{45} & a_{45}^{\otimes 2} & a_{45} \\ \mathbf{0} & \mathbf{0} & a_{35} & a_{45} & \mathbf{1} \end{bmatrix}$$

Note that we have $a_{34}a_{45} \leq a_{45} \leq a_{34}$, so it will not effect the $(3,4)^{th}$ entry of A and $a_{45} \leq a_{35}$, so it will not affect the $(3,5)^{th}$ entry of A when we will add C_1 and C_2 to rank 1 CP-representation of A.

In both cases C can be written as the sum of two rank one completely positive matrices. Thus the rank 1 CP-representation of A is

$$A = B_1 \bigoplus B_2 \bigoplus B_3 \bigoplus B_4 \bigoplus C_1 \bigoplus C_2$$

Hence the CP-rank$(A) \leq 6 = [5^2/4]$ with support 3.

Thus the theorem is true for $n = 4, 5$. Further, we note that for any integer n,

$$[(n+2)^2/4] = [n^2/4] + n + 1.$$

Now we are ready to prove the induction step. Let A_{n+2} be a symmetric matrices over L all of whose diagonal entries are $\mathbf{1}$. Without loss of generality, let us suppose that a_{13} be the smallest non-diagonal entry of the entire matrix and a_{12} be the largest non-diagonal entry in the first row. Let A_n be a submatrix of A_{n+2} obtained by deleting 1^{st} and 2^{nd} row and 1^{st} and 2^{nd} column of A_{n+2}. Since A_n is a principal submatrix of A_{n+2}, A_n is also a symmetric matrix over L all of whose diagonal entries are $\mathbf{1}$. By the induction hypothesis, A_n is the sum of at most $[n^2/4]$ rank 1 CP-matrices with support 3.

Now for each i, where $i = 4, 5, \ldots.n+2$, we introduce a rank 1 CP-matrix

$$B_i = b_i b_i^T = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \mathbf{0} \\ . \\ . \\ \mathbf{0} \\ a_{ii} = \mathbf{1} \\ \mathbf{0} \\ . \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} a_{1i} & a_{2i} & \mathbf{0} & . & . & \mathbf{0} & a_{ii} = \mathbf{1} & \mathbf{0} & . & \mathbf{0} \end{bmatrix}$$

Note that here we get the $(1,2)^{th}$ entry of $(B_i B_i^T)$ is $a_{1i}a_{2i}$, which is less than or equal to $a_{1i} \leq a_{12}$. Thus it will not affect the $(1,2)^{th}$ entry of A_{n+2} when we will add the $b_i b_i^T$ term to the rank 1 CP representation of A_{n+2}. However this rank one completely positive matrix fixes a_{1i}, a_{2i} and $a_{ii} = \mathbf{1}$ in A_{n+2}. Hence we have at most $(n+2) - 3$ rank one completely positive matrices.

Finally, we need 2 rank 1 CP-matrices:

- $B_1 = b_1 b_1^T = \begin{bmatrix} \mathbf{1} \\ a_{12} \\ a_{13} \\ \mathbf{0} \\ . \\ . \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & a_{12} & a_{13} & \mathbf{0} & . & . & \mathbf{0} \end{bmatrix}.$

Since a_{13} is the smallest non-diagonal entry, it will not affect the $(2,3)^{th}$ entry of A_{n+2} when we will add the $b_1 b_1^T$ term to the rank 1 CP representation of A_{n+2}. However this rank one completely positive matrix fixes $a_{11} = \mathbf{1}$, a_{12} and a_{13} in A_{n+2}.

- $B_2 = b_2 b_2^T = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ a_{23} \\ \mathbf{0} \\ . \\ . \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{1} & a_{23} & \mathbf{0} & . & . & \mathbf{0} \end{bmatrix}.$

This rank one completely positive matrix will fix $a_{22} = \mathbf{1}$ and a_{23} in A_{n+2}.

Thus A_{n+2} is the sum of at most

$$[n^2/4] + (n+2) - 3 + 2 = [n^2/4] + n + 1$$

rank 1 CP-matrices with support 3.

Further we will show that $[n^2/4]$ can not be replaced by any smaller number. Since the Boolean semiring is a totally ordered normal incline with only two elements and it has been shown in Remark 4.2.4 that the upper bound of the CP-rank is achieved for special type of Boolean matrices. Hence $[n^2/4]$ can not be replaced by any smaller number. ■

Our proof generalizes the results proved in [**70**] for completely positive matrices over max-min semirings. A min-max semiring is a totally ordered set with greatest **1** and least element **0**, and where the sum of two elements is defined as their maximum and the product of two elements is defined as their minimum. Note that in a max-min semiring we have;

$$a \oplus a = max\{a, a\} = a, \text{ for all } a,$$

i.e., the addition is idempotent and

$$a \otimes b = min\{a, b\} \leq a, b,$$

i.e., the product of two elements is always less than or equal to either factor. This implies that every max-min semiring is a totally ordered commutative incline. Moreover, every max-min semiring is multiplicatively idempotent and if $a \leq b$ then $a = a \otimes b$. Thus every max-min semiring is a totally ordered regular incline with LI-property.

Since $\oplus$ is the maximum operation, an n by n matrix A over a max-min semirng S is diagonally dominant if $a_{ii} \geq a_{ij}$ for all $1 \leq i, j \leq n$. The following theorem characterizes completely positive matrices over max-min semirings and the proof of this theorem follows directly from theorem 3.3.3.

Theorem 5.1.3. *Let S be a max-min semiring and A be an $n \times n$ symmetric matrix over S. Then the matrix A is completely positive if and only if A is diagonally dominant.*

We also know that every totally ordered regular incline with LI-property is a totally ordered normal incline. Therefore, theorem 5.1.2 implies the following result proved in [**70**], which shows the truth of the Drew-Johnson-Loewy conjecture for completely positive matrices over max-min semirings.

Theorem 5.1.4. *Let S be a max-min semiring and A be an $n \times n$ completely positive matrix over S. Then the CP-rank$(A) \leq max\{n, [n^2/4]\}$. Further, A has an $[n^2/4]$ -support 3 rank 1 CP-representation for $n \geq 4$.*

5.2 Erdős, Goodman and Pósa Result and Fuzzy Graphs

In this section, we define a relationship between fuzzy graphs and completely positive matrices over the max-min semiring $([0, 1], Max, Min)$. We prove the Erdős, Goodman and Pósa [**33**] result for fuzzy graphs. In addition, we relate complete fuzzy graphs with rank one completely positive matrices over the max-min semiring $([0, 1], Max, Min)$.

Let $G(S, \sigma, \mu)$ be a fuzzy graph, where $S = \{1, 2, ..., n\}$, $\sigma : S \to [0, 1]$ and $\mu : S \times S \to [0, 1]$, and $\mu(i, j) \leq \sigma(i) \wedge \sigma(j)$, for all i, j in S, where $\sigma(i) \wedge \sigma(j)$ denotes the minimum of $\sigma(i)$ and $\sigma(j)$. For all $i, j \in S$, $\sigma(i)$ denotes the weight on vertex i and $\mu(i, j)$ denotes the weight on edge (i, j).

There is a connection between fuzzy graphs and matrices over the max-min semiring $([0, 1], Max, Min)$. We can define a matrix over the max-min semiring $([0, 1], Max, Min)$ corresponding to a fuzzy graph as follows.

Definition 5.2.1. (*The CP-adjacency Matrix of a Fuzzy Graph*) *Let $G(S, \sigma, \mu)$ be a fuzzy graph on $S = \{1, 2, ..., n\}$. The CP-adjacency matrix of this fuzzy graph is an $n \times n$ matrix A over the max-min semiring $([0, 1], Max, Min)$ where*

$$a_{ii} = \sigma(i) \text{ for all } i \in S, \text{ and}$$

$$a_{ij} = \begin{cases} \mu(i, j) & \text{if there is an edge between vertex } i \text{ and vertex } j, \\ \mathbf{0} & \text{if there is no edge between vertex } i \text{ and vertex } j \end{cases}$$

The CP-adjacency matrix A of a fuzzy graph G is also called the fuzzy matrix.

Proposition 5.2.2. *The CP-adjacency matrix of a fuzzy graph is a completely positive matrix over the max-min semiring $([0, 1], Max, Min)$. Furthermore, any completely positive matrix over the max-min semiring $([0, 1], Max, Min)$ corresponds to a fuzzy graph.*

Proof. Let $G(S, \sigma, \mu)$ be a fuzzy graph with $S = \{1, 2, ..., n\}$ and A be the CP-adjacency matrix of $G(S, \sigma, \mu)$. Clearly A is an $n \times n$ symmetric matrix over the max-min semiring $([0,1], Max, Min)$ with $a_{ii} = \sigma(i)$, for all $i \in S$ and $a_{ij} = \mu(i,j)$ if and only if there is an edge between vertex i and vertex j. We also know that $\mu(i,j) \leq min\{\sigma(i), \sigma(j)\}$, for all $i, j \in S$. This implies that $a_{ii} = \sigma(i) \geq \mu(i,j) = a_{ij}$, for all $i, j \in S$. Therefore, $a_{ii} \geq \bigoplus_{j=1}^{n} a_{ij}$, for all $i \in S$. Hence A is a diagonally dominant matrix over the max-min semiring $([0,1], Max, Min)$ and by theorem 5.1.3, we get that A is a completely positive matrix over the max-min semiring $([0,1], Max, Min)$.

Now let A be an $n \times n$ completely positive matrix over the max-min semiring $([0,1], Max, Min)$. We will show that there exists a fuzzy graph G such that A is the CP-adjacency matrix of G. By theorem 5.1.3, A is a diagonally dominant matrix over the max-min semiring $([0,1], Max, Min)$, i.e., $a_{ii} \geq a_{ij}$ for all i and j. Construct a graph $G(S, \sigma, \mu)$ with $S = \{1, 2, ..., n\}$, where $\sigma(i) = a_{ii}$ and $\mu(i,j) = a_{ij}$. Evidently $\mu(i,j) = a_{ij} \leq min\{a_{ii}, a_{jj}\} = min\{\sigma(i), \sigma(j)\}$. Thus we get that G is a fuzzy graph corresponds to the completely positive matrix A over the max-min semiring $([0,1], Max, Min)$.

$\blacksquare$

Corollary 5.2.3. *The CP-rank of the CP-adjacency matrix of a fuzzy graph with* n *vertices is less than or equal to* $max\{n, [n^2/4]\}$. *Moreover, the CP-adjacency matrix has an* $[n^2/4]$ *- support 3 rank 1 CP-representation for* $n \geq 4$.

The proof of corollary 5.2.3 follows directly from proposition 5.2.2 and the theorem 5.1.4.

We now examine the correspondence between complete fuzzy graphs and rank one completely positive matrices over a max-min semiring $([0,1], Max, Min)$.

Proposition 5.2.4. *Let G be a fuzzy graph and A be the CP-adjacency matrix of G. The fuzzy graph G is a complete fuzzy graph if and only if the CP-adjacency matrix A is a rank one completely positive matrix over the max-min semiring* $([0,1], Max, Min)$.

Proof. Let G be a complete fuzzy graph with n vertices and $A = [a_{ij}]$ be its CP-adjacency matrix. By proposition 5.2.2, A is a completely positive matrix over the max-min semiring $([0,1], Max, Min)$. We also note that $\mu(i,j) = min\{\sigma(i), \sigma(j)\}$, since G is a complete fuzzy graph, and this implies that $a_{ij} = min\{a_{ii}, a_{jj}\}$, for all i, j. Thus we get that $A = bb^T$, where $b = [a_{11}, a_{22}, ..., a_{nn}]^T$ and hence the CP-rank of A is equal to one.

Conversely, suppose that A is a rank one $n \times n$ completely positive matrix over the max-min semiring $([0,1], Max, Min)$. We can write $A = bb^T$, where b is a column vector over a max-min semiring $([0,1], Max, Min)$ of order n. Now construct a graph G on a set S, where $S = \{j : b_j \neq 0\}, \sigma(j) = b_j$, for all $j \in S$ and $\mu(j,k) = min\{b_j, b_k\}$, for all $j, k \in S$. Evidently $\mu(i,j) = min\{\sigma(i), \sigma(j)\}$. Thus we get that G is a complete fuzzy graph and A is the CP-adjacency matrix of G.

■

Erdős, Goodman and Pósa [**33**] showed that any undirected graph G of order $n \geq 2$ with no isolated points can be covered by at most $[n^2/4]$ complete graphs. Further, in the covering with complete graphs we need to use only single edges and triangles. We now prove this result for fuzzy graphs. Firstly we define a covering of a fuzzy graph. Let G_1 and G_2 be two fuzzy graphs. We assume that the weight on a vertex i of a fuzzy graph is equal to zero if the vertex i does not belong to the graph and the weight on an edge (i, j) of a fuzzy graph is equal to zero if the edge (i, j) does not belong to the graph. The sum (union) $G = G_1 \oplus G_2$ of two fuzzy graphs is defined as follows:

1. i is a vertex of G if it is a vertex of G_1 or G_2.

2. The weight on the vertex i of G is equal to the maximum of the weights on the vertex i in G_1 and in G_2

3. The vertex i and the vertex j are connected by an edge in G if they are connected by an edge in G_1 or in G_2.

4. The weight on the edge (i, j) of G is equal to the maximum of the weights on the edge (i, j) in G_1 and in G_2.

Evidently G is a fuzzy graph and G_1 and G_2 are fuzzy subgraph of G.

Definition 5.2.5. (*Cover of a Fuzzy Graph*) *Let G be a fuzzy graph and $G_1, G_2, ..., G_k$ be fuzzy subgraphs of G. We say that the fuzzy subgraphs G_i cover G or G is covered by fuzzy subgraphs G_i, for $i = 1, 2, ..., k$, if $G = G_1 \oplus G_2 \oplus ... \oplus G_k$.*

If $G_1, G_2, ..., G_k$ are complete fuzzy subgraphs of G, then we say that G is covered by complete fuzzy graphs G_i. Covering a fuzzy graph by complete fuzzy subgraphs is called a *clique covering* and each complete fuzzy subgraph in a clique covering is called a *clique*.

The following theorem gives us an upper bound on the number of cliques in an edge clique covering of fuzzy graphs which is an analog of the famous result by Erdős, Goodman and Pósa [**33**] for simple undirected graphs.

Theorem 5.2.6. *Let G be a fuzzy graph on a set S of $n \geq 2$ points. Then G can be covered by at most $max\{n, [n^2/4]\}$ complete fuzzy graphs. Further, in covering with complete fuzzy graphs we need only single edges and triangles.*

Proof. Let A be the CP-adjacency matrix of the fuzzy graph G on a set S of $n \geq 2$ points. By corollary 5.2.3 A can be written as a sum of at most $max\{n, [n^2/4]\}$ rank one completely positive matrices over a max-min semiring $([0, 1], Max, Min)$. Suppose that

$$A = \bigoplus_{i=1}^{k} b_i b_i^T,$$

where $k \leq max\{n, [n^2/4]\}$ and $b_i b_i^T$ is a rank one completely positive matrix over the max-min semiring $([0, 1], Max, Min)$, for all $i = 1, 2, ...k$. By proposition 5.2.4, each $b_i b_i^T$ corresponds to a complete fuzzy graph G_i, on a set S_i where $S_i = \{j : b_{ji} \neq 0\}, \sigma_i(j) = b_{ji}$, for all $j \in S_i$ and $\mu_i(j, k) = min(b_{ji}, b_{ki})$, for all $j, k \in S_i$. Moreover, $A = \bigoplus_{i=1}^{k} b_i b_i^T$ implies that $G = \bigoplus_{i=1}^{k} G_i$, where G_i is a complete fuzzy graph corresponds to a rank one completely positive matrix $b_i b_i^T$ over the max-min semiring $([0, 1], Max, Min)$ and $k \leq max\{n, [n^2/4]\}$. Thus we get that the fuzzy graph G can be covered by at most $max\{n, [n^2/4]\}$ complete fuzzy graphs. Further, in the rank 1 CP-representation of the fuzzy matrix A the support of all rank one completely positive matrices is less than or equal to three, by corollary 5.2.3. Thus the corresponding complete fuzzy graphs are either single edges or triangles.

■

It follows from the proof of the theorem 5.2.6 that an edge clique covering of a fuzzy graph is equivalent to a rank 1 CP-representation of the CP-adjacency matrix.

5.3 CP Matrices Over the Max-plus Semiring

In this section, we study completely positive matrices over the max-plus semiring. One can easily check that the max-plus semiring is not an incline. However, the negative interval subsemiring $[-\infty, 0]$ of the max-plus semiring forms a commutative incline. We show that completely positive matrices over the max-plus semiring have some important similarities with the standard notion of complete positivity of real matrices. We also show that the Drew-Johnson-Loewy conjecture is true for matrices over the max-plus semiring. We first review the characterization of completely positive matrices over the max-plus semiring given by D. Cartwright and M. Chan.

In [**26**], D. Cartwright and M. Chan have studied three different rank functions for symmetric matrices over the min-plus semiring. One of these which they called *the symmetric Barvinok rank* is identical to the CP-rank of a matrix over the min-plus semiring. They have investigated those symmetric matrices over the min-plus semiring for which the symmetric Barvinok rank (the CP-rank) exists. To remain consistent with the rest of our thesis we will state their result in max-plus semiring terms. We note that the characterization of completely positive matrices over the max-plus semiring given by D. Cartwright and M. Chan is similar to the characterization of completely positive matrices over special inclines.

Their result can be formulated as follows:

Theorem 5.3.1. [**26**] *Let A be an $n \times n$ symmetric matrix over the max-plus semiring. Then the following are equivalent.*

1. *A is completely positive.*
2. *There exists an invertible diagonal matrix D such that DAD is a diagonally dominant matrix whose main diagonal consists entirely of $-\infty$ and 0 entries.*
3. *Every 2 by 2 principal submatrix of A has its positive determinant greater than or equal to its negative determinant.*

Proof. $1 \Longrightarrow 2$ Suppose $A = BB^T$ for some n by m matrix B over the max-plus semiring. Let D be the n by n diagonal matrix whose $(i,i)^{th}$ entry is $-max_{1 \leq j \leq n} b_{ij}$ if the ith row of B contains at least one entry not equal to $-\infty$ and whose $(i,i)^{th}$ entry is equal to 0 otherwise. Then $C = DB$ is a matrix all of whose entries lie in the max-plus negative interval $[-\infty, 0]_{max}$. Furthermore C has at least one entry equal to $0 = \mathbf{1}$ in each row that does not consist entirely of $-\infty$ entries. Now $DAD = CC^T$

and hence if the ith row of B does not consist entirely of $-\infty$ entries then the $(i,i)^{th}$ entry of DAD is equal to $m_{ii} = \bigoplus_{k=1}^{m} c_{ik} \otimes c_{ik} = \max_{1\leq k\leq m} 2c_{ik} = 0$ and the $(i,j)^{th}$ entry of DAD is $m_{ij} = \bigoplus_{k=1}^{m} c_{ik} \otimes c_{jk} \leq 0$. If the ith row of B consists entirely of $\mathbf{0} = -\infty$ entries, then $m_{ii} = m_{ij} = -\infty$. In either case, $\bigoplus_{j\neq i} m_{ij} \leq 0 = m_{ii}$.

$2 \Longrightarrow 3$ Let $D = diag\{d_{11}, d_{22}, ..., d_{nn}\}$ be a diagonal matrix and DAD is a diagonally dominant matrix. Every two by two principal submatrix of the matrix DAD is equal to

$$\begin{pmatrix} a_{ii} + d_{ii} + d_{ii} & a_{ij} + d_{ii} + d_{jj} \\ a_{ij} + d_{jj} + d_{ii} & a_{jj} + d_{jj} + d_{jj} \end{pmatrix}$$

Since DAD is a diagonally dominant matrix over the max-plus semiring, every diagonal entry of the matrix DAD is greater than or equal to every off-diagonal entry of the matrix in the same row and in the same column. Hence every two by two principal submatrix of the matrix DAD has the positive determinant greater than or equal to the negative determinant, i.e.,

$$a_{ii} + a_{jj} + 2d_{ii} + 2d_{jj} \geq 2a_{ij} + 2d_{ii} + 2d_{jj}$$

This implies that

$$a_{ii} + a_{jj} \geq 2a_{ij} \qquad \text{for all } i, j.$$

Thus every two by two principal submatrix of A has the positive determinant greater than or equal to the negative determinant.

$3 \Longrightarrow 1$ Let A be an $n \times n$ matrix over the max-plus semiring and every 2×2 principal submatrix of A has $det^+ \geq det^-$. For $1 \leq k < l \leq n$, construct $n \times n$ matrices A_{kl} such that $(k,k), (k,l), (l,k)$ and (l,l) entry of A_{kl} is a_{kk}, a_{kl}, a_{lk} and a_{ll} respectively and all other entries are $-\infty$. Clearly $A = \bigoplus_{1 \leq k < l \leq n} A_{kl}$. Further given that every 2×2 principal submatrix of A has $det^+ \geq det^-$, so for all $1 \leq k < l \leq n$, we can write $A_{kl} = BB^T$, where B is an $n \times 2$ matrix over the max-plus semiring whose

$$(k,1) \text{ entry is } = \tfrac{a_{kk}}{2},$$

$$(l,1) \text{ entry is } = a_{kl} - \tfrac{a_{kk}}{2},$$

$$(l,2) \text{ entry is } = \tfrac{a_{ll}}{2},$$

and all other entries are $-\infty$. Thus all the matrices A_{kl}, $1 \leq k < l \leq n$, are completely positive. Hence A is completely positive.

■

While diagonal dominance is a sufficient condition for symmetric matrices over the max-plus semiring to be completely positive, it is not a necessary condition for symmetric matrices over the max-plus semiring to be completely positive, since we have matrices over the max-plus semiring which are completely positive but not diagonally dominant. Here is an example of such a matrix:

$$A = \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 5 \end{bmatrix}$$

Clearly A is not a diagonally dominant matrix but it is completely positive with the CP-rank equal to one. We note that if D is an invertible diagonal matrix and A is a completely positive matrix, then CP-rank$(DAD) =$ CP-rank(A).

Cartwright and Chan have also found an upper bound of the symmetric Barvinok rank (the CP-rank) for special type of symmetric matrices of order n over the min-plus semiring. To remain consistent we will state their result in max-plus semiring terms.

Theorem 5.3.2. [**26**, Theorem 4] *Suppose that M is a symmetric $n \times n$ matrix over the max-plus semiring with $M_{ii} \otimes M_{jj} \geq 2M_{ij}$ for all i and j. Then the symmetric Barvinok rank of M is at most max$\{n, [\frac{n^2}{4}]\}$, and this bound is tight.*

The Drew-Johnson-Loewy conjecture for completely positive matrices over the max plus semiring now follows immediately from theorem 5.3.1 and theorem 5.3.2. We formally state it as follows:

Corollary 5.3.3. *Let A be an $n \times n$ completely positive matrix over the max-plus semiring then the CP-rank of A is less than or equal to the max$\{n, [\frac{n^2}{4}]\}$.*

There exists some matrices over the max-plus semiring having the CP-rank equal to $[n^2/4]$. Thus the upper bound $[n^2/4]$ of the CP-rank of completely positive matrices over the max-plus semiring can not be replaced by any smaller number. We are using an example given in [**26**], which is also similar to one given in remark 4.2.4.

Example 5.3.4. *Consider the block matrices over the max-plus semiring*

$$A_n = \begin{bmatrix} I_m & J_m \\ \\ J_m & I_m \end{bmatrix}, \textit{ if } n = 2m, \textit{ i.e., } n \textit{ is even}$$

and

$$A_n = \begin{bmatrix} I_m & J_{m,m+1} \\ \\ J_{m+1,m} & I_{m+1} \end{bmatrix}, \textit{ if } n = 2m+1, \textit{ i.e., } n \textit{ is odd.}$$

Here J_m *is an* $m \times m$ *all of whose entries are* $0 = \mathbf{1}$ *and* $J_{m+1,m}$ *is an* $(m+1) \times m$ *matrix all of whose entries are also* 0. *The CP-rank of* A_n *is* $[n^2/4]$.

5.3.1 LU & UL Factorization of CP Matrices Over the Max-plus Semiring

In this section, we prove the max-plus version of Markham's theorems which give sufficient conditions for completely positive matrices over the max-plus semiring to have a triangular factorization.

In chapter 3, we have proved Markham's theorems for special inclines. We have examined sufficient conditions for a completely positive matrix over special inclines to have a triangular factorization.

The max-plus semiring does not form an incline, so the results proved for completely positive matrices over special inclines can not be applied to completely positive matrices over the max-plus semiring. However, the subsemiring of the max-plus semiring consisting of all the elements less than or equal to the multiplicative identity $\mathbf{1} = 0$, (i.e.,the negative interval semiring $[-\infty, 0]$), forms an incline. We show that the negative interval semiring $[-\infty, 0]$ satisfies all the properties of a totally ordered normal incline. In particular, we generalize the results about completely positive matrices over totally ordered normal inclines to completely positive matrices over the max-plus semiring using the characterization of completely positive matrices over the max-plus semiring given in theorem 5.3.1.

Proposition 5.3.5. *The negative interval semiring* $[-\infty, 0]$ *forms a totally ordered normal incline.*

Proof. We know that the negative interval semiring $[-\infty, 0]$ is a commutative semiring, where the operation of addition and multiplication is defined as follows:

$$a \oplus b = \max\{a, b\}$$

$$a \otimes b = a + b.$$

We also note that in the negative interval semiring $[-\infty, 0]$, we have, for all $a, b \in S$,

$$a \oplus a = max\{a, a\} = a, \text{ and}$$

$$a \otimes b = a + b \leq a, b,$$

since both a and b are negative. In other words, in the negative interval semiring $[-\infty, 0]$ the addition is idempotent and the product of two elements is always less than or equal to either factor. This implies that the negative interval semiring $[-\infty, 0]$ is a commutative incline. Moreover, for every $a \in [-\infty, 0]$, we have $a/2 \in [-\infty, 0]$ and we can write $a = a/2 \otimes a/2$. Therefore, every element in the negative interval semiring $[-\infty, 0]$ has a unique square root. Furthermore, if $a, b \in [-\infty, 0]$ and $a \leq b$ then there exists $c \in [-\infty, 0]$ such that $a = c + b = c \otimes b$. We also note that the negative interval semiring $[-\infty, 0]$ is a totally ordered semiring and hence it has the arithmetic geometric property. Thus the negative interval semiring $[-\infty, 0]$ forms a totally ordered normal incline. ■

Corollary 5.3.6. *Every symmetric diagonally dominant matrix over the negative interval semiring* $[-\infty, 0]$ *is a completely positive matrix.*

The proof of this corollary follows directly from the proposition 5.3.5 and theorem 3.3.1.

Remark 5.3.7. *Note that the theorem 5.3.2 by Cartwright and Chan is also a special case of theorem 5.1.2. If A is an $n \times n$ completely positive matrix over the max-plus semiring S then by theorem 5.3.1, there exists an invertible diagonal matrix D over the max-plus semiring S such that $DAD = M$, where M is an $n \times n$ diagonally dominant matrix whose main diagonal consists entirely of $-\infty$ and 0 entries. By corollary 5.3.6, M is a completely positive matrix over the negative interval semiring $\hat{S} = [-\infty, 0]$ and by theorem 5.1.2 the CP-rank of M is less than or equal to $max\{n, [n^2/4]\}$. Let us assume that $M = BB^T$, where B is an $n \times m$ matrix over $P(\hat{S}) = \hat{S}$ and $m \leq max\{n, [n^2/4]\}$. Thus we get*

$$A = D^{-1}BB^TD^{-1} = (D^{-1}B)(D^{-1}B)^T.$$

Evidently, $D^{-1}B$ is an $n \times m$ matrix over S and $m \leq max\{n, [n^2/4]\}$. Hence the

CP-rank of A is less than or equal to the maximum of n and $[n^2/4]$.

From proposition 5.3.5, it is clear that every completely positive matrix over the negative interval semiring $[-\infty, 0]$ satisfies all the results relating the nonnegativity of almost principal minors and triangular decomposition for completely positive matrices over special inclines proved in section 3.4.

We now prove the max-plus analogs of Markham theorems (theorem 2.1.22 and theorem 2.1.23) relating almost principal minors with the triangular factorizations. We note that in case of max-plus semiring every 2×2 completely positive matrix is LU-completely positive and UL-completely positive. This can be proved by the following examples.

Example 5.3.8. *Let* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ *be a completely positive matrix over the max-plus semiring, then*

$$A = \begin{bmatrix} \frac{a_{11}}{2} & a_{12} - \frac{a_{22}}{2} \\ -\infty & \frac{a_{22}}{2} \end{bmatrix} \begin{bmatrix} \frac{a_{11}}{2} & a_{12} - \frac{a_{22}}{2} \\ -\infty & \frac{a_{22}}{2} \end{bmatrix}^T$$

if $a_{11} > -\infty$*, and*

$$A = \begin{bmatrix} -\infty & -\infty \\ -\infty & \frac{a_{22}}{2} \end{bmatrix} \begin{bmatrix} -\infty & -\infty \\ -\infty & \frac{a_{22}}{2} \end{bmatrix}^T$$

if $a_{11} = -\infty$

and similarly,

$$A = \begin{bmatrix} \frac{a_{11}}{2} & -\infty \\ a_{12} - \frac{a_{11}}{2} & \frac{a_{22}}{2} \end{bmatrix} \begin{bmatrix} \frac{a_{11}}{2} & -\infty \\ a_{12} - \frac{a_{11}}{2} & \frac{a_{22}}{2} \end{bmatrix}^T$$

Theorem 5.3.9. *If A is an $n \times n$ completely positive matrix over the max-plus semiring, $n \geq 3$, and all its right almost principal 2×2 submatrices have $det^+ \geq det^-$, then A is LU-completely positive.*

Proof. Given that A is an $n \times n$, $(n \geq 3)$, completely positive matrix over the max-plus semiring. This implies that there exists an invertible diagonal matrix D over the max-plus semiring such that DAD is a completely positive matrix over the negative interval semiring $[-\infty, 0]$. We are also given that all right almost principal 2×2 submatrices of A have $det^+ \geq det^-$. Therefore, all right almost principal 2×2 submatrices of DAD have $det^+ \geq det^-$. Thus by proposition 5.3.5 and theorem 3.4.4, we get that DAD is LU-completely positive, i.e., $DAD = LL^T$ for some lower triangular matrix L over the negative interval semiring $[-\infty, 0]$. Since D is an invertible diagonal matrix, $A = D^{-1}LL^TD^{-1} = (D^{-1}L)(D^{-1}L)^T = \hat{L}\hat{L}^T$, where $\hat{L}$ is a lower triangular matrix over the max-plus semiring. ■

Analogous results hold for UL-completely positive matrices. By a similar argument to the previous theorem, we have:

Theorem 5.3.10. *If A is an $n \times n$ completely positive matrix over the max-plus semiring, $n \geq 3$, and all its left almost principal 2×2 submatrices have $det^+ \geq det^-$, then A is UL-completely positive.*

We have shown that in case of special inclines the converses of the above theorems are not true. The same also holds in case max-plus semiring, i.e., the converse of theorem 5.3.9 and theorem 5.3.10 is not true. We give an example of a UL-completely positive matrix which has a left almost principal 2×2 submatrix which does not satisfy the inequality $det^+ \geq det^-$.

Example 5.3.11. *Let*

$$A = \begin{bmatrix} 6 & 5 & 4 & -\infty \\ 5 & 12 & 4 & -\infty \\ 4 & 4 & 2 & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{bmatrix}$$

be a 4×4 matrix over the max-plus semiring. The matrix A is UL-completely positive because there exists an upper triangular matrix U such that $A = UU^T$, where

$$U = \begin{bmatrix} -\infty & 3 & -\infty & 3 \\ -\infty & 2 & 6 & -\infty \\ -\infty & -\infty & -2 & 1 \\ -\infty & -\infty & -\infty & -\infty \end{bmatrix}$$

However, the left almost principal 2×2 submatrix $A[2,3|1,3]$ of A has $det^+ = 7 \leq 8 = det^-$.

Theorems 5.3.9 and 5.3.10 are the max-plus generalizations of theorems 3.4.4 and 3.4.3. For $n = 3$, we have proved that every $n \times n$ completely positive matrix over special inclines is either UL-completely positive or LU-completely positive, but not necessarily both. A similar result also holds for completely positive matrices over the max-plus semiring. We first need the following lemma.

Lemma 5.3.12. *If A is a 3×3 completely positive matrix over the max-plus semiring, then at least two of the following inequalities hold:*

$$a_{11} + a_{23} \geq a_{12} + a_{13}$$

$$a_{22} + a_{13} \geq a_{12} + a_{23}$$

$$a_{33} + a_{12} \geq a_{13} + a_{23}$$

Proof. Suppose that two of the inequalities do not hold, say,

$$a_{11} + a_{23} \leq a_{12} + a_{13}$$

$$a_{22} + a_{13} \leq a_{12} + a_{23}.$$

Then

$$a_{11} + a_{23} + a_{22} + a_{13} \leq a_{12} + a_{13} + a_{12} + a_{23}.$$

This implies that $a_{11} + a_{22} \leq 2a_{12}$, which is not possible because A is completely positive so every 2×2 principal submatrix has $det^+ \geq det^-$. ∎

Let A be an $n\times n$ matrix over the max-plus semiring. The positive and the negative determinant of 2×2 submatrices of the matrix A are related with the inequalities of the above lemma. We note that $det^+(A[1,2|1,3]) \geq det^-(A[1,2|1,3])$ is equivalent to the first inequality $a_{11} + a_{23} \geq a_{12} + a_{13}$ of the above lemma and $det^+(A[1,3|2,3]) \geq det^-(A[1,3|2,3])$ is equivalent to the third inequality $a_{33} + a_{12} \geq a_{13} + a_{23}$ of the above lemma. However, the second inequality is not related to any left or right almost principal submatrix of A of order two.

Theorem 5.3.13. *If A is a 3×3 completely positive matrix over the max-plus semiring, then A is either LU-completely positive or UL-completely positive or both.*

Proof. If any diagonal entry of A is $-\infty$, then all the entries in the corresponding row and column will be $-\infty$. In this case the result follows from example 5.3.8. Now suppose that $a_{ii} > -\infty$ for all $i = 1, 2, 3$. The only left almost principal 2×2 submatrices of A are

$$A_1 = A[1,3|2,3] \qquad\qquad A_2 = A[2,3|1,3]$$

and

$$det^+(A_1) = det^+(A_2) = a_{33} + a_{12}$$

$$det^-(A_1) = det^-(A_2) = a_{13} + a_{23}$$

and the only right almost principal 2×2 submatrices of A are

$$A_3 = A[1,3|1,2] \qquad\qquad A_4 = A[1,2|1,3]$$

and

$$det^+(A_3) = det^+(A_4) = a_{11} + a_{23}$$

$$det^-(A_3) = det^-(A_4) = a_{13} + a_{12}$$

By lemma 5.3.12, either the left almost principal 2×2 submatrices have $det^+ \geq det^-$ or the right almost principal 2×2 submatrices have $det^+ \geq det^-$ or both. Thus by theorem 5.3.9 and 5.3.10, A is either LU-completely positive or UL-completely positive or both. ■

Example 5.3.14. *A three by three matrix A over the max-plus semiring need not be both LU-completely positive and UL-completely positive. For example, consider the matrix*

$$A = \begin{bmatrix} 4 & 3 & 3 \\ 3 & 4 & 3 \\ 3 & 3 & 2 \end{bmatrix}$$

All right almost principal 2×2 submatrices of A has the positive determinant greater than or equal to the negative determinant. Thus by theorem 5.3.9, A is LU-completely positive. However, we can check that there does not exist any upper triangular matrix U over the max-plus semiring such that $A = UU^T$.

Corollary 5.3.15. *Every square symmetric TN_2 matrix over the max-plus semiring is both LU- and UL- completely positive.*

Remark 5.3.16. *The CP-rank of an $n \times n$ completely positive matrix over the max-plus semiring is less than or equal to n if either all its left almost principal 2×2 submatrices or all of its right almost principal 2×2 submatrices have $det^+ \geq det^-$.*

5.4 CP Matrices Over the Sign Pattern Semiring

In this section, we study completely positive matrices over the sign pattern semiring. We find a necessary and sufficient condition for symmetric matrices over the sign pattern semiring to be completely positive. We also examine whether the Drew-Johnson-Loewy conjecture holds for completely positive matrices over the sign pattern semiring. In addition, we show that there exists a homomorphism from the nonnegative interval semiring $\{-\infty\} \cup [0, \infty)$ and the positive subsemiring $P(S)$ of the sign pattern semiring S. Further, we use this homomorphism to examine the Drew-Johnson-Loewy conjecture for completely positive matrices over the nonnegative interval semiring $\{-\infty\} \cup [0, \infty)$.

The set $S = \{0, +1, -1, \#\}$, where $\#$ symbol denotes and unknown sign, forms a semiring with the operations of addition and multiplication defined in chapter 2. It is called the sign pattern semiring. In the sign pattern semiring 0 is the additive identity and $+1$ is the multiplicative identity. The positive subsemiring of the sign pattern semiring S equal to $P(S) = \{0, +1, \#\}$ and in $P(S)$ we have $0 \leq +1 \leq \#$.

From the definition of completely positive matrices over semirings we get that every completely positive matrix over the sign pattern semiring S has all its entries belonging to $P(S)$. In other words, completely positive matrices over the sign pattern semiring has no entry equal to -1. Moreover, we have $P(S) = P(P(S)) = ...$, where S is the sign pattern semiring. Thus we have the following corollary, which gives us a sufficient condition for a symmetric matrix over the $P(S)$, where S is the sign pattern semiring, to be completely positive and the proof of this corollary follows from the proof of theorem 3.2.1.

Corollary 5.4.1. *Let S be the sign pattern semiring and $P(S)$ be the positive sub-semiring of S. Then every symmetric diagonally dominant matrix A over $P(S)$ is completely positive.*

Note that diagonal dominance is not a necessary condition for symmetric matrices over $P(S)$, where S is the sign pattern semiring, to be completely positive. There exist some completely positive matrices over the sign pattern semiring which are not diagonally dominant. For example, let

$$A = \begin{bmatrix} \# & \# \\ \# & +1 \end{bmatrix} = \begin{bmatrix} \# \\ +1 \end{bmatrix} \begin{bmatrix} \# & +1 \end{bmatrix}$$

Clearly A is a 2×2 completely positive matrix over the sign pattern semiring and it is not diagonally dominant.

The sign pattern semiring S is a formally real semiring, since it is an antinegative semiring. Thus by lemma 3.1.4, if a completely positive matrix over the sign pattern semiring has a zero on the main diagonal then the corresponding row and column have all entries equal to zero. We now give a necessary and sufficient condition for symmetric matrices over the sign pattern semiring to be completely positive.

Theorem 5.4.2. *Let S be the sign pattern semiring and $A \in M_n(S)$ be a symmetric matrix. The matrix A is a completely positive matrix over the sign pattern semiring*

S if and only if $A \in M_n(P(S))$ and every 2×2 principal submatrix of A has the positive determinant greater than or equal to the negative determinant.

Proof. Suppose that A is a completely positive matrix over the sign pattern semiring S. Then clearly $A \in M_n(P(S))$ and A can be written as $A = BB^T$, where B is an $n \times m$ matrix over $P(S)$ for some positive integer m. Firstly suppose that $a_{ij} = \#$ for $i \neq j$. This implies that

$$\# = a_{ij} = \bigoplus_{k=1}^{n} b_{ik}b_{jk}.$$

Thus we get that at least one of b_{ik} and b_{jk} is equal to $\#$ for some k. Without loss of generality we assume that b_{ik} is equal to $\#$ for some k. Therefore,

$$a_{ii} = \bigoplus_{k=1}^{n} (b_{ik})^{\otimes 2} = \#.$$

Similarly if b_{jk} is equal to $\#$ for some k then $a_{jj} = \#$. Hence $a_{ii}a_{jj} = \# = (a_{ij})^{\otimes 2}$, i.e., every 2×2 submatrix of A containing $a_{ij} = \#$ has $det^+ = \# = det^-$. Now if $a_{ij} \neq \#$ for some i and j, then a_{ij} is equal to $+1$ or $\mathbf{0}$. This implies that $a_{ii}a_{jj} \geq (a_{ij})^{\otimes 2}$ and equality holds if at least one of a_{ii} and a_{jj} are equal to zero or $a_{ii} = a_{jj} = a_{ij} = +1$. Thus every 2×2 principal submatrix of A has the positive determinant greater than or equal to the negative determinant.

Conversely, suppose that $A \in M_n(P(S))$ is a symmetric matrix, where S is the sign pattern semiring, and every 2×2 principal submatrix of A has the positive determinant greater than or equal to the negative determinant. If A is diagonally dominant then by corollary 5.4.1, A is completely positive. Now suppose that A is not a diagonally dominant matrix over the sign pattern semiring. To prove that A is a completely positive matrix over the sign pattern semiring S, we will factor A as a sum of rank one completely positive matrices over $P(S)$. We introduce n rank one completely positive matrices $b_i b_i^T$, for $i = 1, 2, ..., n$, where the i^{th} entry b_i is equal to a_{ii} and all other entries are equal to 0. Note that all $b_i b_i^T$, for $i = 1, 2, ..., n$, in a rank 1 CP-representation of A fix the diagonal entries of A. For off-diagonal entries we have the following cases:

1. If $a_{ij} = \#$, then at least one of a_{ii} and a_{jj} is equal to $\#$, since every 2×2 principal submatrix of A has the positive determinant greater than or equal to the negative determinant. In this case we have the following options:

   (a) If both a_{ii} and a_{jj} is equal to $\#$, then we introduce a rank one completely positive matrix $c_i c_i^T$ in the rank 1 CP-representation of A, where i^{th} and j^{th} entry of c_i is equal to $\#$ and all other entries are equal to 0.

   (b) If $a_{ii} = \#$ and $a_{jj} \neq \#$, then a_{jj} has to be equal to $+1$. In this case we introduce a rank one completely positive matrix $c_i c_i^T$ in the rank 1 CP-representation of A, where i^{th} entry of c_i is equal to $\#$, j^{th} entry of c_i is equal to $+1$ and all other entries are equal to 0.

   (c) If $a_{jj} = \#$ and $a_{ii} \neq \#$, then a_{ii} has to be equal to $+1$. In this case we introduce a rank one completely positive matrix $c_i c_i^T$ in the rank 1 CP-representation of A, where i^{th} entry of c_i is equal to $+1$, j^{th} entry of c_i is equal to # and all other entries are equal to 0.

Note that $c_ic_i^T$ fixes the $(i,j)^{th}$ entry of A in a rank 1 CP-representation of A, where $a_{ij} = \#$.

2. If $a_{ij} = +1$, then we introduce a rank one completely positive matrix $d_id_i^T$ in the rank 1 CP-representation of A, where i^{th} and j^{th} entry of d_i is equal to $+1$ and all other entries are equal to 0. Here $d_id_i^T$ fixes the $(i,j)^{th}$ entry of A in a rank 1 CP-representation of A, where $a_{ij} = +1$.

3. If $a_{ij} = 0$, then we do not add any rank one completely positive matrix to the rank 1 CP-representation of A.

Thus we get that $A = \left(\bigoplus_{i=1}^{n} b_ib_i^T\right) \oplus \left(\bigoplus c_ic_i^T\right) \oplus \left(\bigoplus d_id_i^T\right)$, the sum of all rank one completely positive matrices over $P(S)$. Hence A is a completely positive matrix.

■

We know that if S is the sign pattern semiring then $P(S) = \{+1, 0, \#\}$. One can easily see that every element of $P(S)$ has a unique square root in $P(S)$. Hence by theorem 3.5.3, we have the following corollary which gives us an upper bound on the CP-rank of all diagonally dominant completely positive matrices over the sign pattern semiring.

Corollary 5.4.3. *Let S be the sign pattern semiring and $P(S)$ be the positive subsemiring of S. Then every diagonally dominant completely positive matrix A over*

S has the CP-rank less than or equal to $\frac{1}{2}n(n+1)-N$, where $2N$ is the number of off-diagonal entries which are equal to zero.

We note that the upper bound on the CP-rank of diagonally dominant completely positive matrices over the sign pattern matrix can not be replaced by any smaller number. There exist some diagonally dominant completely positive matrices over the sign pattern matrix whose CP-rank is exactly equal to $\frac{1}{2}n(n+1)-N$, where $2N$ is the number of off-diagonal entries which are equal to zero.

Example 5.4.4. *Let*

$$A = \begin{bmatrix} \# & +1 \\ +1 & \# \end{bmatrix}$$

Clearly A is a 2×2 symmetric diagonally dominant matrix over $P(S)$, where S is the sign pattern semiring. This implies that A is a completely positive matrix over the sign pattern semiring. A rank 1 CP-representation of A is

$$A = \begin{bmatrix} \# \\ 0 \end{bmatrix} \begin{bmatrix} \# & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \# \end{bmatrix} \begin{bmatrix} 0 & \# \end{bmatrix} \oplus \begin{bmatrix} +1 \\ +1 \end{bmatrix} \begin{bmatrix} +1 & +1 \end{bmatrix}$$

Therefore, the CP-rank of A is less than or equal to three. However, one can easily check that there does not exist any rank 1 CP-representation of A having only two rank one completely positive matrices. Thus we get that the CP-rank of A is $3 = \frac{2(2+1)}{2}$.

Remark 5.4.5. *In general, matrices of type:*

$$\begin{bmatrix} \#(I_m) & J_m \\ J_m & \#(I_m) \end{bmatrix}, \text{ if } n = 2m, \text{ i.e., } n \text{ is even,}$$

and

$$\begin{bmatrix} \#(I_m) & J_{m,m+1} \\ J_{m+1,m} & \#(I_{m+1}) \end{bmatrix}, \text{ if } n = 2m+1, \text{ i.e., } n \text{ is odd,}$$

where J_m *is an* $m \times m$ *matrix all of whose entries are equal to* $+1$ *and* $\#(I_m)$ *is an* $m \times m$ *diagonal matrix all of whose diagonal entries are equal to* $\#$*, has the CP-rank exactly equal to* $\frac{1}{2}n(n+1) - N$*, where* $2N$ *is the number of off-diagonal entries which are equal to zero.*

Remark 5.4.6. *From corollary 5.4.3 and remark 5.4.5, it follows that if* S *is the sign pattern semiring and* A *be an* $n \times n$ *completely positive matrix over* S*, then the CP-rank of* A *may not be less than or equal to the max*$\{n, [n^2/4]\}$*. Therefore, the Drew-Johnson-Loewy conjecture is not true for completely positive matrices over the sign pattern semiring.*

We now show that there exists a homomorphism between the nonnegative interval subsemiring $\{-\infty\} \cup [0, \infty)$ of the max-plus semiring and the positive subsemiring $P(S)$ of the sign pattern semiring S. This homomorphism is then used to prove that the Drew-Johnson-Loewy conjecture is not true for completely positive matrices over the nonnegative interval semiring $\{-\infty\} \cup [0, \infty)$. Note that the nonnegative interval subsemiring $\{-\infty\} \cup [0, \infty)$ of the max-plus semiring is not an incline.

Lemma 5.4.7. *Let* $P(S)$ *be the positive subsemiring of the sign pattern semiring* S *and* $\big(\{-\infty\} \cup [0, \infty)\big)$ *be a subsemiring of the max-plus semiring. Then a mapping*

$$\phi : \{-\infty\} \cup [0, \infty) \longrightarrow P(S),$$

such that

$$\phi(-\infty) = 0$$

$$\phi(0) = +1$$

$$\phi(a) = \#, \qquad \textit{for all } a \in (0, \infty),$$

is a semiring homomorphism.

It is evident that the positive subsemiring $P(S)$ of the nonnegative interval semiring $S = \{-\infty\} \cup [0, \infty)$ is equal to S. Thus we have the following corollary which gives us a sufficient condition for symmetric matrices over $\{-\infty\} \cup [0, \infty)$ to be completely positive and the proof follows immediately from theorem 3.2.1.

Corollary 5.4.8. *Every symmetric diagonally dominant matrix over $\{-\infty\} \cup [0, \infty)$ is completely positive.*

Theorem 5.4.9. *Let $\{-\infty\} \cup [0, \infty)$ be the nonnegative interval subsemiring of the max-plus semiring and $A \in M_n(\{-\infty\} \cup [0, \infty))$ be a completely positive matrix. Then the CP-rank of A is not be bounded above by max$\{n, [n^2/4]\}$.*

Proof. Let S be a sign pattern semiring and $P(S)$ be the positive subsemiring of S. Now consider a mapping

$$\Xi : M_n(\{-\infty\} \cup [0, \infty)) \to M_n(P(S)),$$

where

$$(\Xi(A))_{ij} = \phi(a_{ij}), \text{ for all } A = [a_{ij}] \in M_n(\{-\infty\} \cup [0, \infty)).$$

Here ϕ is a semiring homomorphism $\phi : \{-\infty\} \cup [0, \infty) \longrightarrow P(S)$, as defined in lemma 5.4.7. By theorem 3.6.1, if $A \in M_n(\{-\infty\} \cup [0, \infty))$ is completely positive then $\Xi(A) \in M_n(P(S))$ is also completely positive. Further the CP-rank$_{\{-\infty\}\cup[0,\infty)}(A) \geq$ CP-rank$_{P(S)}(\Xi(A))$. Let us assume that

$$A = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix},$$

is an $n \times n$ symmetric matrix over $\{-\infty\} \cup [0, \infty)$, where $n = 2m$. Here A_1 is an $m \times m$ diagonal matrix over $\{-\infty\} \cup [0, \infty)$, all of whose diagonal entries belong to the interval $(0, \infty)$ and A_2 is an $m \times m$ matrix over $\{-\infty\} \cup [0, \infty)$ all of whose entries are equal to 0. Since the operation of addition is defined as maximum in the semiring $\{-\infty\} \cup [0, \infty)$, the matrix A is a symmetric diagonally dominant matrix over $\{-\infty\} \cup [0, \infty)$. Thus by corollary 5.4.8, A is a completely positive matrix over $\{-\infty\} \cup [0, \infty)$. Moreover,

$$\Xi(A) = \begin{bmatrix} \#(I_m) & J_m \\ J_m & \#(I_m) \end{bmatrix},$$

where J_m is an $m \times m$ matrix all of whose entries are equal to $+1$ and $\#(I_m)$ is an $m \times m$ diagonal matrix all of whose diagonal entries are equal to $\#$. By remark 5.4.5, the CP-rank$_{P(S)}(\Xi(A)) = \frac{1}{2}n(n+1) - N$, where $2N$ is the number of off-diagonal entries which equal zero. Thus we get that the CP-rank$_{\{-\infty\}\cup[0,\infty)}(A) \geq \frac{1}{2}n(n+1) - N$.

Here $n = 2m$ and $N = m(m-1)$. Therefore,

$$
\begin{aligned}
\frac{1}{2}n(n+1) - N &= \frac{1}{2}2m(2m+1) - m(m-1) \\
&= m(2m+1-m+1) \\
&= m(m+2) \\
&> 2m, \qquad \text{if } m \leq 4 \qquad [\because max\{n, [n^2/4]\} = n \text{ if } n \leq 4] \\
&> [(2m)^2/4] = m^2, \qquad \text{if } m > 5 \qquad [\because max\{n, [n^2/4]\} = [n^2/4] \text{ if } n > 4]
\end{aligned}
$$

$$
\text{since } m(m+2) = m^2 + 2m > \begin{cases} 2m & \text{always if } m \geq 1; \\ m^2 & \text{always if } m \geq 1. \end{cases}
$$

Hence the CP-rank$_{\{-\infty\}\cup[0,\infty)}(A) \geq \frac{1}{2}n(n+1) - N > max\{n, [n^2/4]\}$.

$\blacksquare$

Therefore, we find that the Drew-Johnson-Loewy conjecture is also not true for completely positive matrices over the nonnegative interval semiring $\{-\infty\} \cup [0, \infty)$.

Chapter 6

Applications of Completely Positive Matrices over Semirings

Semirings frequently arise in a wide variety of fields such as: path finding problems in graphs; fuzzy sets; operation research; and discrete event systems. In this chapter, we examine some applications of completely positive matrices over semirings. We show that completely positive matrices over different semirings can be used to formulate a wide variety of path-finding problems. We also relate special types of completely positive matrices over certain semirings to matrices corresponding to the binary relations, fuzzy relations and metric matrices. In addition, we explore some properties of sequences of powers of completely positive matrices over special inclines.

6.1 Path-Finding Problems in Graphs and CP Matrices Over Special Semirings

In most path-finding problems, we study various types of graphs where each edge of a graph has a weight assigned to it, which represents some quantity reflecting the relationship between the two incident vertices such as distance, transit cost, capacity or reliability. We have to find a path for which some function of the edge weights is either maximized or minimized. For example, in transportation problems weight of each edge represents the transit cost from one place to another place or the distance between two places. We find the least cost or the shortest path from one place to another place. In transmitting a message through a communication network, weight of each edge represents the reliability between two points. We find a maximum reliable path between two points. Here the reliability of a path is denoted as the minimum reliability of an edge in the path. Sometimes, we encounter path connectivity problems where the goal is to find whether there is any path from one point to another or not, such problems arise in testing computer logical circuits.

In this section, we study the relationship between completely positive matrices over special semirings and path-finding problems in graphs. We show that completely positive matrices over different semirings are used to formulate a wide variety of path-finding problems. We consider weighted graphs, where the weights are the elements of special semirings. We then show that many path-finding problems can be solved using powers of special completely positive matrices over certain semirings.

Let A be an $n \times n$ matrix over a semiring and $G(A)$ be the graph of the matrix A. For any $k \in \mathbb{N}$, let us denote:

- P_{ij}^k, the set of paths of $G(A)$ joining vertex i to vertex j and containing exactly k edges.
- $P_{ij}^{(k)}$, the set of paths of $G(A)$ joining vertex i to vertex j and containing at most k edges.

Any path $\mu \in P_{ij}^k$ is composed of the sequence of vertices $(i, i_1, ... i_{k-1}, j)$. Now we can state the following properties:

1. Each term (i, j) of the matrix A^k is equal to:

$$(A^k)_{ij} = \bigoplus_{\mu \in P_{ij}^k} (a_{i,i_1} \otimes a_{i_1,i_2} \otimes ... \otimes a_{i_{k-1},j}).$$

2. Each term (i, j) of the matrix $\hat{A}^k = A \oplus A^2 \oplus ... \oplus A^k$ is equal to:

$$(\hat{A}^k)_{ij} = \bigoplus_{l=1}^{k} \bigoplus_{\mu \in P_{ij}^l} (a_{i,i_1} \otimes a_{i_1,i_2} \otimes ... \otimes a_{i_{l-1},j}).$$

We first consider the Boolean semiring and completely positive matrices over the Boolean semiring. The Boolean semiring $(\{\mathbf{0}, \mathbf{1}\}, Max, Min)$ is the basis for modeling and solving *connectivity problems* in graph theory. We know that if $G(V, E)$ be a simple undirected graph with n vertices then the CP-adjacency matrix of G is a matrix B over the Boolean semiring, where $b_{ii} = \mathbf{1}$ for all $i \in V$ and $b_{ij} = \mathbf{1}$ if there is an edge between vertex i and vertex j and $b_{ij} = \mathbf{0}$ otherwise. Here $\mathbf{1}$ denotes that there is a connection from one vertex (point) to another and $\mathbf{0}$ denotes that there is no connection.

We have shown that the CP-adjacency matrix of a simple undirected graph is completely positive as a Boolean matrix and every completely positive matrix over the Boolean semiring is an CP-adjacency matrix of a simple undirected graph.

Here we obtain the following properties relating existence and connectivity of paths:

1. $(B^k)_{ij} = \mathbf{1}$, if and only if there exists a walk containing k edges between vertex i and j.

2. $(\hat{B^k})_{ij} = \mathbf{1}$, if and only if there exists a walk containing at most k edges between vertex i and j.

The min-plus semiring $(\mathbb{R} \cup \{+\infty\}, Min, +)$ and any max-min semiring (S, Max, Min), where S is a totally ordered set, provide natural models for the *shortest path problems* and for the *maximum capacity path problems* respectively.

Let us suppose that weights in an undirected weighted graph G represent the length of each edge. The adjacency matrix of a such weighted graph is a matrix A, where a_{ij} = length of the edge (i, j), if there is an edge between vertex i and vertex j and $a_{ij} = 0$ if there is no edge between vertex i and vertex j and $a_{ii} = 0$ for all $i \in V$, since the distance from a vertex to itself is zero. The adjacency matrix A

in this case is a matrix over the min-plus semiring, whose diagonal entries are equal to 0 and all off diagonal entries are nonnegative. We know that such matrices are completely positive over the min-plus semiring.

In this case, we obtain the following property relating the shortest path problems:

$$(A^k)_{ij} = \min_{\mu \in P_{ij}^k}(a_{i,i_1} + a_{i_1,i_2} + ... + a_{i_{k-1},j}),$$

represents the length of the shortest path between vertex i and vertex j containing k edges in the graph G. In other words, the length of the shortest path between two vertices i and j containing k edges in the graph G is the $(i,j)^{th}$ entry of A^k, where A is a completely positive matrix over the min-plus semiring corresponding to the graph G.

Now suppose that weights in an undirected weighted graph G represent the capacity of each edge. Here the capacity of an edge (i,j) means the maximum flow that can be sent through the edge (i,j). The capacity of a path from vertex i to vertex j or the maximum flow that can be sent through a path from vertex i to vertex j is equal to the minimum capacity of an edge in the path. Moreover, the best path of some length k from vertex i to vertex j is a path of length of k from vertex i to vertex j having maximum capacity. The adjacency matrix of a such weighted graph is the matrix A, where $a_{ij} =$ capacity of the edge (i,j) if there is an edge between vertex i and vertex j and $a_{ij} = \mathbf{0}$ if there is no edge between vertex i and vertex j and $a_{ii} =$ capacity of the vertex i, for all $i \in V$. The adjacency matrix in this case is a matrix over the max-min semiring. The capacity of each vertex is greater than or equal to the capacity of the corresponding edges, since the maximum flow that can be sent from one vertex to another can not be more than the capacity of either vertex. Thus the adjacency matrix of such a graph is a symmetric diagonally dominant matrix over the max-min semiring. We have shown that such matrices are completely positive over the max-min semiring.

In this case, we obtain the following property relating the minimum capacity of a path:

$$(A^k)_{ij} = \max_{\mu \in P_{ij}^k} \{min(a_{i,i_1}, a_{i_1,i_2}, ..., a_{i_{k-1},j})\}$$

represents the capacity of the best path containing k edges between vertex i and vertex j. In other words, the capacity of the best path between vertex i and vertex j containing k edges in the graph G is the $(i,j)^{th}$ entry of A^k, where A is a completely positive matrix over a max-min semiring corresponding to the graph G.

We now consider a Boolean algebra $(\beta_k, \cup, \cap)$ of subsets of a k-element set $\{1, 2, ..., k\} = [K]$ and completely positive matrices over β_k. Let G be an undirected weighted graph with n vertices, where weights are the elements of β_k and the graph G has k simple undirected subgraphs S_l for $1 \leq l \leq k$, where

i is a vertex of S_l if $l \subseteq \{$weight on the vertex i in $G\}$ and

(i,j) is an edge of S_l if $l \subseteq \{$weight on the edge (i,j) in $G\}$.

The matrix corresponding to this graph G is a matrix B where

$$b_{ii} = \bigcup \{l \in [K] \quad | \text{ such that vertex } i \in S_l\}, \quad \text{and}$$

$$b_{ij} = \bigcup\{l \in [K] \quad | \text{ such that edge } (i,j) \in S_l\}, \quad \text{and}$$

$$b_{ij} = \{\mathbf{0} \quad | \text{ if there is no edge } (i,j) \in S_l \text{ , for all } l\}.$$

Clearly the matrix B is a matrix over a Boolean algebra β_k. Note that the matrix B corresponding to the graph G is symmetric and diagonally dominant as a matrix over a Boolean algebra β_k. Hence it is a completely positive matrix over a Boolean algebra β_k. In this case, we obtain the following properties relating existence and connectivity of paths:

1. $(B^k)_{ij} \supseteq l$ if and only if the subgraph S_l has a path containing k edges between vertex i and j .

2. $(\hat{B^k})_{ij} \supseteq l$ if and only if the subgraph S_l a path from vertex i to j containing at most k edges.

6.2 Symmetric Idempotent Matrices over Special Semirings

In this section, we will discuss a very important class of matrices called idempotent matrices. A square matrix A is said to be *idempotent* if

$$A = A^2.$$

Identity matrices I_n and null matrices O_n where all entries are equal to $\mathbf{0}$ are simple examples of idempotent matrices. Idempotent matrices play a significant role in many fields of mathematics and statistics. Symmetric idempotent matrices over semirings are special types of completely positive matrices. If A is a symmetric idempotent matrix over a semiring, then A can be written as $A = AA^T$. In other words, symmetric idempotent matrices over semirings are completely positive matrices that have square factorizations. Therefore, the CP-rank of any $n \times n$ symmetric idempotent matrix over a given semiring is less than or equal to n.

In the following subsections, we will study the structure of symmetric idempotent matrices over different semirings specially over the Boolean semiring, the min-plus semiring and max-min semirings.

6.2.1 Symmetric Idempotent Matrices over The Boolean Semiring

In this section, we study the relationship between the adjacency matrix of an equivalence relation and a special type of idempotent Boolean matrices. We use the theory of binary relations and the structure of their associated matrices. We know that if $S = \{1, 2, ..., n\}$ and $T = \{1, 2, ..., m\}$ are finite sets and $R \subseteq S \times T$ is a binary relation, then the *adjacency matrix* A of the relation R is an $n \times m$ matrix where $a_{ij} = \mathbf{1}$, if $(i, j) \in R$ and zero otherwise. If $S = T$ then the binary relation R is said to be a relation on the set S. Moreover, the composition of two binary relations R_1 and R_2 is denoted as $R = R_1; R_2$ and the adjacency matrix A of $R_1; R_2$ can be obtained by multiplying the adjacency matrix A_1 of the binary relation R_1 with the adjacency matrix A_2 of the binary relation R_2 and then changing all nonzero entries by $\mathbf{1}$ in the matrix A_1A_2.

If we consider the adjacency matrix of a binary relation R as an $n \times m$ Boolean matrix, then the adjacency matrix A of $R = R_1; R_2$ can be obtained by multiplying A_1 and A_2 as Boolean matrices, i.e.,

$$A = A_1A_2,$$

where A_1 is the Boolean matrix corresponding to the binary relation R_1 and A_2 is the Boolean matrix corresponding to the binary relation R_2.

The following result uses the adjacency matrix (the Boolean matrix) of a binary relation to explain the properties of an equivalence relation.

Proposition 6.2.1. *Let B be a Boolean matrix corresponding to a binary relation R on a set $S = \{1, 2, ..., n\}$.*

1. *R is reflexive if and only if all the diagonal entries of the Boolean matrix B are equal to **1**.*
2. *R is symmetric if and only if the Boolean matrix B is symmetric.*
3. *R is transitive if and only if the Boolean matrix B satisfies $B \geq B^2$.*

Proof. The proof of 1 and 2 is obvious. To prove 3, let us suppose that $B \geq B^2$. We will show that R is a transitive relation on a set S. Let iRk and kRj. This implies that $(B)_{ik} = \mathbf{1}$ and $(B)_{kj} = \mathbf{1}$. Now consider the $(i,j)^{th}$ entry of B^2,

$$\begin{aligned}(B^2)_{ij} &= \bigoplus_{k=1}^{n}((B)_{ik} \otimes (B)_{kj}) \\ &= \bigoplus_{l \neq k}((B)_{il} \otimes (B)_{lj}) \oplus ((B)_{ik} \otimes (B)_{kj}) \\ &= \mathbf{1}.\end{aligned}$$

Since $B \geq B^2$, this implies that $(B)_{ij} = \mathbf{1}$. Thus we get that iRj, and this is true for all $i, j, k \in S$. Hence R is a transitive relation on S. Conversely, let B be the adjacency matrix of the transitive relation R on a set S. Therefore, $(B)_{ik} = (B)_{kj} = \mathbf{1}$ implies that $(B)_{ij} = \mathbf{1}$. Now if the $(i,j)^{th}$ entry of $B^2 = \mathbf{1}$ then there exists at least one k such that $(B)_{ik} \otimes (B)_{kj} = \mathbf{1}$ and thus by transitivity of R we get $(B)_{ij} = \mathbf{1}$. This implies that $B^2 \leq B$. ■

Remark 6.2.2. *Note that if a binary relation R is reflexive then the corresponding Boolean matrix B satisfies $B \geq I$, where I is the identity Boolean matrix. This implies that $B^2 \geq B$. Therefore, a binary relation R is reflexive and transitive if and only if the corresponding Boolean matrix is an idempotent Boolean matrix with all diagonal entries equal to* $\mathbf{1}$.

From proposition 6.2.1 and remark 6.2.2, it is clear that a binary relation is an equivalence relation if and only if the matrix corresponding to the binary relation is a symmetric idempotent Boolean matrix with all diagonal entries equal to $\mathbf{1}$.

We have shown in chapter 2 that completely positive matrices over the Boolean semiring are symmetric and diagonally dominant. If we have a diagonal entry zero in the completely positive Boolean matrix then the corresponding row and the corresponding column has all the entries equal to zero, so we can ignore that row and column. Here we will consider only those completely positive Boolean matrices which have non-zero diagonal entries. In the following corollaries, we establish a correlation between completely positive matrices over the Boolean semiring and binary relations with special properties.

Corollary 6.2.3. *Every completely positive matrix over the Boolean semiring is an adjacency matrix of a relation which is reflexive and symmetric.*

Corollary 6.2.4. *Every completely positive matrix over the Boolean semiring corresponds to an adjacency matrix of an equivalence relation if and only if it is an idempotent matrix.*

The proof of corollary 6.2.3 and 6.2.4 follows directly from proposition 6.2.1.

6.2.2 Symmetric Idempotent Matrices over The Min-Plus Semiring

There is a well known correspondence between the triangle inequality for a distance function on a finite set and the idempotency of an associated matrix over the min-plus semiring. Given a finite order set X and a function

$$d : X \times X \to \mathbb{R}.$$

We say d is a *pseudometric* on X (or equivalently, X is a pseudometric space with respect to d) if d satisfies the following conditions:

1. $d(x, y) \geq 0$ for all $x, y \in X$;

2. $d(x, x) = 0$ for all $x \in X$;

3. $d(x, y) = d(y, x)$ for all $x, y \in X$;

4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The function d is a *metric* on X if d is a pseudometric on X satisfying the following condition:

$$d(x, y) > 0 \text{ for all } x \neq y.$$

Let $X = \{1, 2, ..., n\}$ and $d : X \times X \to \mathbb{R}$. We may consider the $|X| \times |X|$ matrix D whose $(i, j)^{th}$ entry is $d(i, j)$, as a matrix over the min-plus semiring. It is obvious from the construction of D that:

1. If $d(x, y) \geq 0$, for all $x, y \in X$, then all the entries of the matrix D are nonnegative.

2. If $d(x, x) = 0$, for all $x \in X$, then all the diagonal entries of D are 0.

3. If $d(x, y) = d(y, x)$, for all $x, y \in X$, then D is a symmetric matrix.

Moreover, we have the following theorem by Johnson and Kambites.

Theorem 6.2.5. [**52**, lemma 4.1] *Let n be a positive integer. Given a function*

$$d : [n] \times [n] \longrightarrow \mathbb{R}.$$

Let D denote the $n \times n$ matrix given by $D = d(i, j)$. Then the following are equivalent:

1. *d satisfies $d(i, i) = 0$ and $d(i, j) \le d(i, k) + d(k, j)$ for all $i, j, k \in X$.*

2. *D is an idempotent matrix over the min-plus semiring with all diagonal entries equal to 0.*

It follows from the above theorem that if d is a pseudometric then the matrix D is a symmetric idempotent matrix over the min-plus semiring with nonnegative entries and all diagonal entries are equal to 0. We also know that completely positive matrices over the min-plus semiring are diagonally similar to a non-negative matrices with all diagonal entries equal to 0. Thus we have the following characterization for completely positive idempotent matrices over the min-plus semiring in terms of metric matrices.

Corollary 6.2.6. *Completely positive matrices over the min-plus semiring having all entries zero on the main diagonal are pseudometric matrices if and only if they are idempotent. Moreover, if only the main diagonal entries are zero for a completely positive matrix over the min-plus semiring then this matrix corresponds to a metric matrix if and only if it is idempotent.*

6.2.3 Symmetric Idempotent Matrices over Max-Min Semirings

In this section, we discuss a well known correspondence between the reflexive and transitive properties of a fuzzy relation and the idempotency of the corresponding matrix over a max-min semiring. We also characterize completely positive idempotent matrices over a max-min semiring with respect to fuzzy equivalence relations.

We know that if $S = \{1, 2, ..., n\}$ and $T = \{1, 2, ..., m\}$ are finite sets and $\rho : S \times T \longrightarrow [0, 1]$ is a fuzzy relation on a fuzzy subset μ of S into a fuzzy subset ν of T, then the matrix A corresponding to the fuzzy relation ρ is an $n \times m$ matrix where $a_{ij} = \rho(i, j)$, for all $i, j \in S$. The matrix A is also called the adjacency matrix of the fuzzy relation ρ. Furthermore, the composition of two fuzzy relations $\rho : S \times T \longrightarrow [0, 1]$, such that $\rho(x, y) \leq \mu(x) \wedge \nu(y)$, for all $x \in S$ and $y \in T$ and $\omega : T \times U \longrightarrow [0, 1]$, such that $\omega(y, z) \leq \nu(y) \wedge \xi(z)$, for all $y \in T$ and $z \in U$ is a fuzzy relation and it is denoted by $\rho \circ \omega$. The adjacency matrix of $\rho \circ \omega$, denoted as $C = [c_{ij}]$, is defined as:

$$c_{ij} = \underset{k}{max}\{min(a_{ik}, b_{kj})\}, \text{ for all } i, j,$$

where $A = [a_{ij}]$ is the adjacency matrix of the fuzzy relation ρ and $B = [b_{ij}]$ is the adjacency matrix of the fuzzy relation ω.

If we consider the adjacency matrix of a fuzzy relation $\rho \circ \omega$ as an $n \times m$ matrix over the max-min semiring $([0, 1], Max, Min)$, then the adjacency matrix C of $\rho \circ \omega$

can be obtained by multiplying A and B as matrices over the max-min semiring $([0,1], Max, Min)$, where A is the adjacency matrix of the fuzzy relation ρ and B is the adjacency matrix of the fuzzy relation ω.

Moreover, we know that if a fuzzy relation ρ is reflexive and transitive then $\rho^2 = \rho$. In other words, if any fuzzy relation ρ is reflexive and transitive then the matrix representation of ρ is idempotent as a matrix over the max-min semiring $([0,1], Max, Min)$. In the following corollary, we shed light on the structure of completely positive idempotent matrices over a max-min semiring in terms of fuzzy relations.

Corollary 6.2.7. *Every completely positive matrix over a max-min semiring* $([0,1], Max, Min)$ *is a matrix representation of a fuzzy equivalence relation if and only if it is a symmetric idempotent matrix over the max-min semiring* $([0,1], Max, Min)$.

6.3 Kleene Star of Matrices over Special Inclines

In this section, we study the sequence of powers of completely positive matrices over special inclines. In particular, we study the Kleene star of matrices over special inclines. We also give a graph theoretic interpretation of the Kleene star of a matrix over a special inclines. We start this section with some necessary definitions and results proved by Raymond Cuninghame-Green in [**25**]. We note that if A and B are two matrices of order m by n then $A \geq B$ means that each $a_{ij} \geq b_{ij}$.

Definition 6.3.1. (*Increasing Matrices*) *An* $n \times n$ *matrix* A *over a semiring* S *is* *called an increasing matrix if*

$$A \otimes B \geq B, \text{ for all } B \in M_{n,m}(S) \text{ for any integer } m \geq 1.$$

It is obvious that if A is an increasing matrix over a semiring then $A = A \otimes I \geq I$. This implies that the diagonal entries of A are greater than or equal to $\mathbf{1}$ and all the off diagonal entries of A are greater than or equal to $\mathbf{0}$. Moreover, it has been shown [**25**, lemma 27-1] that the converse also holds for matrices over the max-plus semiring, max-min semirings and the Boolean semiring. We generalize this result to matrices over commutative inclines.

Lemma 6.3.2. *Let $A = [a_{ij}]$ be a square matrix over a commutative incline. Then A is an increasing matrix if and only if $a_{ii} = \mathbf{1}$, for all i.*

The proof of this lemma is similar to the proof of lemma 27-1 of [**25**].

Corollary 6.3.3. *Let L be a commutative incline with a unique square root property. Then every symmetric increasing matrix over L is completely positive.*

Proof. Let A be a symmetric increasing matrix over L. This implies that all the diagonal entries of A are equal to $\mathbf{1}$, by lemma 6.3.2. Therefore, A is a symmetric diagonally dominant matrix over L. We also know that the incline L has a unique square root property. This implies that $L = P(L)$. Hence, A is a completely positive matrix over L, by theorem 3.2.1. ∎

By corollary 6.3.3, all symmetric increasing matrices over commutative inclines with a unique square root property are completely positive matrices. However, the converse is not true, i.e., a completely positive matrix over a commutative incline L with a unique square root property may not be an increasing matrix. We know that the Boolean semiring is a commutative incline with a unique square root property. Here we have an example of a completely positive matrix over the Boolean semiring which is not an increasing matrix.

Example 6.3.4. *Let* $A = \begin{bmatrix} \mathit{1} & \mathit{1} & \mathit{0} & \mathit{0} \\ \mathit{1} & \mathit{1} & \mathit{0} & \mathit{0} \\ \mathit{0} & \mathit{0} & \mathit{1} & \mathit{0} \\ \mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} \end{bmatrix}$ *be a* 4×4 *matrix over the Boolean semiring. It is clear that* A *is a symmetric diagonally dominant matrix. This implies that* A *is a completely positive matrix over the Boolean semiring. However,* A *is not an increasing matrix, i.e.,* $A \not\geq I$*, since* $a_{44} = \mathbf{0}$.

Let $A \in M_n(\mathbb{R}_+)$ then the series

$$A^* = I \oplus A \oplus A^2 \oplus A^3 \oplus .... \tag{6.1}$$

converges to a finite limit if and only if $\rho(A) < 1$, where $\rho(A)$ is the maximum eigenvalue of A. The matrix A^* is called the *Kleene star* of A. In [**25**], it has been shown that any matrix A over the max-plus semiring has a maximum eigenvalue $\rho(A)$, which can be calculated as the maximum mean weight of a path from one node to itself in a weighted graph corresponding to the matrix A. Johnson and Kambites [**52**] generalized the concept of Kleene star to matrices over the max-plus semiring. They noted that if the maximum eigenvalue of a matrix over the max-plus semiring is nonpositive (i.e., less than or equal to $\mathbf{1} = 0$) then equation (6.1) converges to a finite limit. Moreover, they studied the role of Kleene stars in tropical mathematics.

One of the most classical combinatorial optimization problems is: Given an $n \times n$ matrix A of direct distances between n places, find the matrix A^* of shortest distances (that is the matrix of the lengths of shortest paths between any pair of places). We may assume without loss of generality that A is a matrix over the min-plus semiring with all diagonal entries equal to 0. It is known that the shortest-distances matrix exists if and only if all off diagonal entries of A are nonnegative. Thus $A \leq I$, where I is an identity matrix of the min-plus semiring, i.e., all the diagonal entries of I are 0 and all off diagonal entries are equal to ∞. This implies that $I \geq A \geq A^2 \geq ... \geq A^{n-1} \geq A^n \geq ....$, also it has been shown by Raymond Cuninghame-Green in [**25**], that if $A \leq I$ as a min-plus semiring then $A^{n-1} = A^n = A^{n+1} = ...$ Thus $A^* = A^{n-1}$.

Note that if A is an $n \times n$ matrix over an incline then $(A^k)_{ij} = \underset{\mu \in P_{ij}^k}{l.u.b}\{a_{i,i_1}a_{i_1,i_2}...a_{i_{k-1},j}\}$, for any positive integer k. Since the product of elements in an incline is always less than or equal to all the factors, the matrix A^k always exists, where the $(i,j)^{th}$ entry denotes the best path from vertex i to vertex j of length k. We also know that the addition of elements in an incline is defined as the least upper bound of the elements. Therefore, the Kleene star of a matrix A, A^*, over an incline always exists and it is the matrix of the values of the optimal path from one place to another place. If G is a graph with n vertices then the maximum length of a path from vertex i to vertex j with all distinct edges is $n-1$. In other words, every path of length greater than or equal to n from vertex i to vertex j will repeat at least one edge. This implies that if $G = G(A)$ then for $m \geq n$, $(A^m)_{ij} = \underset{\mu \in P_{ij}^m}{l.u.b}\{a_{i,i_1}a_{i_1,i_2}...a_{i_{m-1},j}\}$ is less than or equal to the $(i,j)^{th}$ entry of A^{n-1}, since every μ repeats at least one edge and the multiplication is decreasing. Therefore, A^m is less than or equal to (A^{n-1}), and hence the Kleene stars of a matrix A over an incline is equal to

$$A^* = I \oplus A \oplus A^2 \oplus A^3 \oplus .... \oplus A^{n-1}.$$

We now examine Kleene stars of matrices over special inclines in terms of completely positive matrices.

Proposition 6.3.5. *Let L be a commutative incline with a unique square root property and A be an $n \times n$ symmetric matrix over L. Then the Kleene star of A is an idempotent completely positive matrix over L.*

Proof. Let A be a symmetric matrix over a commutative incline L with the unique square root property then the Kleene star of A is

$$A^* = I \oplus A \oplus A^2 \oplus A^3 \oplus .... \oplus A^{n-1}.$$

Evidently, the Kleene star of A, A^*, is a symmetric matrix over L. Moreover, A^* is a diagonally dominant matrix over L, since all the diagonal entries of A^* are **1** and **1** is the greatest element of L. Hence A^* is a completely positive matrix over L. It is also easy to check that $(A^*)^2 = A^*$. Thus we get that the Kleene star of A is an idempotent completely positive matrix over a commutative incline L with a unique square root property. ∎

Yoeli [**87**] has shown that if A is an $n \times n$ increasing matrix over a commutative incline then $I \leq A \leq A^2 \leq .... \leq A^{n-1} = A^n = A^{n+1} = ...$ The corresponding result on Boolean matrices is due to Lunts [**63**] and can be found in reference [**49**]. Thus we obviously have the following proposition:

Proposition 6.3.6. [**87**] *If A is an $n \times n$ increasing matrix over a commutative incline then $A^* = A^{n-1}$, i.e., If $A \geq I$ then $A^* = A^{n-1}$.*

We are interested in the Kleene star of completely positive matrices over special semirings. From example 6.3.4, it is also clear that every completely positive matrix over the Boolean semiring is not an increasing matrix. We now prove a result similar to the proposition 6.3.6 for completely positive matrices over the Boolean semiring.

Proposition 6.3.7. *If B is an $n \times n$ completely positive matrix over the Boolean semiring then $B^* = I_n \oplus B^{n-m-1}$, where m is the number of zeros on the main diagonal of B. Moreover if $m = 0$ then $B^* = B^{n-1}$*

Proof. Since B is an $n \times n$ completely positive Boolean matrix, so B is a diagonally dominant Boolean matrix. First consider the case when all the diagonal entries of B are nonzero, i.e., $B \geq I$. Thus the Kleene star of B, $B^* = B^{n-1}$ by proposition 6.3.6.

Now consider the case when some of the diagonal entries of B are equal to zero. Then the corresponding rows and columns of B are all zero. Without loss of generality, we can assume that last m rows and m columns of B are all zero, for some nonnegative integer $m < n$. Let $\hat{B}$ is a submatrix of B obtained by deleting last m zero rows and columns. Clearly $\hat{B}$ is an $(n-m)\times(n-m)$ completely positive Boolean matrix with all diagonal entries nonzero. Thus we get that the Kleene star of $\hat{B}$, $\hat{B}^* = \hat{B}^{n-m-1} = \hat{B}^{n-1}$. Further, the Kleene star of B,

$$
\begin{aligned}
B^* &= I \oplus B \oplus \ldots \oplus B^{n-m-1} \oplus \ldots. \\
&= \begin{bmatrix} I_{n-m} & \mathbf{0} \\ \mathbf{0} & I_m \end{bmatrix} \oplus \begin{bmatrix} \hat{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \oplus \ldots. \oplus \begin{bmatrix} \hat{B}^{n-m-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \oplus \ldots
\end{aligned}
$$

Since $I_{n-m} \leq \hat{B} \leq \hat{B}^2 \leq \ldots. \leq \hat{B}^{n-m-1} = \hat{B}^{n-m} = \ldots..$, we get

$$
\begin{bmatrix} \hat{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \leq \begin{bmatrix} \hat{B}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \leq \ldots \leq \begin{bmatrix} \hat{B}^{n-m-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \hat{B}^{n-m} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \ldots..
$$

Thus the Kleene star of B,

$$
\begin{aligned}
B^* &= \begin{bmatrix} I_{n-m} & \mathbf{0} \\ \mathbf{0} & I_m \end{bmatrix} \oplus \begin{bmatrix} \hat{B}^{n-m-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \ldots. \\
&= I_n \oplus B^{n-m-1}.
\end{aligned}
$$

■

Definition 6.3.8. [**55**] (*The Reachability Matrix*) *The reachability matrix of a graph $G = (V, E)$ is a Boolean matrix A such that $a_{ij} = \mathbf{1}$, if the vertex i is reachable from the vertex j by some path in the graph G and $a_{ij} = \mathbf{0}$ otherwise. Every point is considered reachable from itself.*

If a given graph G with n vertices is connected then clearly the reachability matrix of G is equal to J_n, where J_n is an $n \times n$ all $\mathbf{1}'^s$ matrix over the Boolean semiring.

Suppose A is a square matrix over a commutative incline L, whose entries represent the edge weights in a weighted graph $G(A)$. Then the Kleene star of A, A^*, represents weights of the paths in the graph $G(A)$ of arbitrary (finite) length from one vertex to another. In other words, the $(i,j)^{th}$ entry of A^* is defined as follows:

$$(A^*)_{ij} = \bigoplus_{\mu \in P_{ij}} \{w(A,\mu)\},$$

where P_{ij} is the set of all paths of finite length from vertex i to vertex j in $G(A)$, and $w(A,\mu)$ is the weight of the path μ.

If L is the Boolean semiring then the $(i,j)^{th}$ entry of A denotes the existence of an edge from the vertex i to the vertex j. The $(i,j)^{th}$ entry of A^* is $\mathbf{1}$ if there is a path of arbitrary (finite) length from vertex i to vertex j and $\mathbf{0}$ otherwise. Thus the Kleene star of A, A^* is the reachability matrix of the graph $G(A)$.

If L is the nonnegative subsemiring $[0,\infty]$ of the min-plus semiring then the $(i,j)^{th}$ entry of A denotes the distance or length or cost of the edge from vertex i to vertex j. The $(i,j)^{th}$ entry of A^* is the shortest distance from vertex i to vertex j or least cost of going from vertex i to vertex j.

If L is a max-min semiring then the $(i,j)^{th}$ entry of A denotes the capacity of an edge (the maximum flow that can be sent) from vertex i to vertex j. The $(i,j)^{th}$ entry of A^* is the minimum capacity of the best route from vertex i to vertex j.

A graph theoretic interpretation of proposition 6.3.7 is: If G is a simple undirected graph with n vertices and B be the CP-adjacency Boolean matrix of G. Then the reachability matrix of G is B^{n-1}. Since B is a completely positive matrix over the

Boolean semiring with all diagonal entries equal to **1**. Therefore, we get that $B^{n-1} = B^*$ and B^* is the reachability matrix of G.

We now define the convergent and oscillatory Boolean matrices, for reference see [**55**]. An $n \times n$ matrix A over the Boolean semiring is said to be *convergent* (in its powers) if and only if there exists $m \in \mathbb{N}$ such that $A^m = A^{m+1}$. It is said that the sequence of powers of A converges to the matrix A^m. An $n \times n$ matrix A over the Boolean semiring is said to be *oscillatory or periodic* (in its powers) if there exists $m, p \in \mathbb{N}$ with $p \geq 1$, such that $A^m = A^{m+p}$.

We know that if A is an $n \times n$ completely positive matrix over the Boolean semiring then A^k is also an $n \times n$ completely positive matrix over the Boolean semiring, for any positive integer k. This implies that the number of distinct matrices in the sequence of powers of completely positive Boolean matrices is at most 2^r , where $r = \frac{n(n-1)}{2}$. Thus a completely positive Boolean matrix must be either finitely convergent or finitely oscillatory. It is obvious that the sequence of powers of a convergent completely positive Boolean matrix converges to an idempotent matrix.

In the following corollary, we show that the sequence of powers of special completely positive matrices over the Boolean semiring always converges to their Kleene stars.

Corollary 6.3.9. *The sequence of powers of every completely positive matrix over the Boolean semiring with all diagonal entries equal to **1** converges to its Kleene star.*

Proof. Let B be an $n \times n$ completely positive Boolean matrix with all diagonal entries equal to **1**. This implies that $I \leq B$, also we get that $B \leq B^2 \leq \ldots \leq B^{n-1} = B^n \ldots$, using Lunts [**63**]. This implies that the sequence of powers of the Boolean matrix B converges to the Boolean matrix B^{n-1}, which is the Kleene star of B by proposition 6.3.7. ■

Combining the result of the proposition 6.3.5 and the corollary 6.3.9 we get:

Theorem 6.3.10. *The Kleene star of any symmetric Boolean matrix is a completely positive Boolean matrix with all diagonal entries equal to **1** and the sequence of powers of every completely positive Boolean matrix with all diagonal entries equal to **1** converges to its Kleene star.*

Corollary 6.3.11. *The sequence of powers of every symmetric increasing matrix over a commutative incline converges to its Kleene star.*

Proof. Let A be an $n \times n$ symmetric increasing matrix over a commutative incline. This implies that $I \leq A$ and we get $I \leq A \leq A^2 \leq \ldots \leq A^{n-1} = A^n \ldots$ Thus the symmetric increasing matrix A converges to a matrix A^{n-1}. Further, by proposition 6.3.6 we get $A^* = A^{n-1}$. Hence, the sequence of powers of A converges to A^*. ■

In the following theorem we characterize those completely positive matrices over special semirings which are equal to their Kleene stars.

Theorem 6.3.12. *Let B be a completely positive matrix over the Boolean semiring, with no row and column completely equal to zero. Then the following are equivalent:*

1. *B corresponds to the adjacency matrix of an equivalence relation.*

2. *B is an idempotent Boolean matrix.*

3. *$B = B^* = B^T$.*

Proof. Equivalence of (1) and (2) follows from corollary 6.2.4.

Now let us suppose that B is an idempotent Boolean matrix. this implies that $B = B^2 = ... = B^{n-1} = ...$ We are given that no row and column of B is completely equal to zero. Therefore, we get that $B \geq I$ and $B^* = I \oplus B \oplus B \oplus .... = B$. Further, $B = B^T$, since B is a completely positive matrix.

Conversely, let us suppose that $B = B^* = B^T$, i.e., $B = I \oplus B \oplus B^2.....$ Since no row and column of B is completely equal to zero, $B \geq I$, i.e., B is an increasing Boolean matrix. By proposition 6.3.6, we get $B \leq B^2 \leq .... \leq B^{n-1} = B^* = B$. Hence the equality holds everywhere and we get $B = B^2$.

■

Theorem 6.3.13. *Let A be a symmetric increasing matrix over a max-min semiring, then the following are equivalent:*

1. *A is an idempotent matrix.*

2. *A is a matrix representation of a fuzzy equivalence relation.*

3. *$A = A^* = A^T$.*

Proof. Equivalence of (1) and (2) follows directly from corollary 6.2.7 and the proof for (1) $\iff$ (3) is similar to the proof of (2) $\iff$ (3) in theorem 6.3.12. ■

Theorem 6.3.14. *Let n be a positive integer. Given a function*

$$d : [n] \times [n] \longrightarrow \mathbb{R}.$$

Let D denote the $n \times n$ matrix given by $D = d(i, j)$. Then the following are equivalent:

1. *D is a completely positive idempotent matrix over the min-plus semiring with zero diagonal.*
2. *D is a pseudometric matrix.*
3. *$D = D^* = D^T$.*

Proof. The equivalence of (1) and (2) is given by corollary 6.2.6 and the equivalence of (2) and (3) is shown by Johnson and Kambites in [**52**, lemma 4.1]. ■

Chapter 7

Ranks of Matrices Over Semirings

There are many equivalent ways of defining the rank of a matrix over the field. The rank of an m by n matrix A can be defined as the largest k for which there exists a k by k submatrix of A with nonzero determinant, or the dimension of the row space of A, or the dimension of the column space of A or the smallest k for which there exists an m by k matrix B and a k by n matrix C with $A = BC$. For matrices over semirings, all of these definitions are no longer equivalent and each of these generalizes to a distinct rank function for matrices over semirings. There are many different rank functions for matrices over semirings and their properties and the relationships between them have been much studied. (See [**3, 11, 16**] for examples).

In this chapter, we first review some background material on rank functions of matrices over semirings and ϵ-determinant. We then use the ϵ-determinant to introduce a new family of rank functions called the (ϵ, I)-rank functions and show that these rank functions satisify some of the usual inequalities such as the rank-sum inequality and the Sylvester inequality. We also classify all bijective linear preservers of ϵ-rank (which is a particular type of (ϵ, I)-rank function) for all matrices over antinegative semirings with no zero divisors. Moreover, we show that our rank function generalizes determinantal rank and in case of the sign pattern semiring it is equal to the dimension of the largest sign-nonsingular submatrix. In addition, we define the nonnegative rank function for matrices over semirings and compare it with the CP-rank of completely positive matrices over semirings.

We begin with some important definitions of rank functions for matrices over semirings. We also list the definition and some fundamental properties of ϵ-determinant.

Definition 7.0.15. (*The Factor Rank*) *Let $A \in M_{m,n}(S)$. Then the factor rank (or Schein rank) of A is the smallest integer r for which there exists an m by r matrix B over S and an r by n matrix C over S such that $A = BC$. The factor rank is denoted by $f(A)$.*

Definition 7.0.16. (*The Determinantal Rank*) *Let A be an m by n matrix over a semiring S. Then the determinantal rank of A is the largest k for which there exists B a k by k submatrix of A with $det_+(B) \neq det_-(B)$.*

The determinantal rank has been much studied. (See [**3**] for instance). Our results in next section generalize the known results on the determinantal rank of max-plus matrices. The following proposition compares the factor rank of a matrix over the max-plus semiring with its determinantal rank and the proof of this can be found in [**3**].

Proposition 7.0.17. [**3**] *Let A be a matrix over the max-plus semiring S. The determinantal rank of A is always less than or equal to the factor rank of A.*

Proposition 7.0.18. *Let A be a matrix over a commutative semiring S. The factor rank of A equal to one implies that the determinantal rank of A is also equal to one.*

Proof. Suppose that the factor rank of A is equal to one. This implies that $A = de^T$ and $a_{ij} = d_i e_j$. Now any two by two submatrix of A is equal to

$$\begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix} = \begin{bmatrix} d_i e_k & d_i e_l \\ d_j e_k & d_j e_l \end{bmatrix}, \text{ where } i \neq j \text{ and } k \neq l.$$

Evidently, the positive determinant of any two by two submatrix of A is equal to its negative determinant. Hence the determinantal rank of A is equal to one. ∎

A matrix whose entries are from the set $\{+1, -1, 0\}$ is called a *sign pattern matrix*. If $A = [a_{ij}]$ is a real matrix. Then the sign pattern of A is obtained from A, by replacing each entry by its signs [**46**]. The sign pattern of A is denoted by $Sg(A) = [sg(a_{ij})]$, where

$$sg(a_{ij}) = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ +1 & \text{if } a_{ij} > 0 \\ -1 & \text{if } a_{ij} < 0. \end{cases}$$

Thus in a sign pattern matrix all we know is the sign of each entry. We do not know the exact values of the entries. We denote the set of all n by n sign pattern matrices by Q_n. Sometimes we may not know the signs of certain entries, so a new symbol, #, has been introduced to denote such entries.

Definition 7.0.19. (*The Generalized Sign Pattern Matrices*) [**64**] *The generalized sign pattern matrices are the matrices over the sign pattern semiring* $\{+1, -1, 0, \#\}$, *where # corresponds to entries where the sign is unknown.*

Let A be a real matrix. The qualitative class of A is $Q(A)$, the set of all real matrices with the same sign pattern as A.

Definition 7.0.20. (*Sign-nonsingular Matrices (SNS)*) [**18**] *A sign pattern matrix A is called sign-nonsingular (SNS) if every matrix in its qualitative class is nonsingular.*

In [**84**] and [**85**], Tan introduced a new type of determinant for semirings. We begin with a concept formulated independently by Akian, Gaubert and Guterman in [**3**] and Tan in [**84**].

Definition 7.0.21. [**3, 84, 85**] *Let S be a semiring. A bijection $\tau : S \to S$ is called a symmetry if $\tau(\tau(a)) = a$ for all $a \in S$ and if $\tau(a \otimes b) = a \otimes \tau(b) = \tau(a) \otimes b$ for all $a, b \in S$.*

The term symmetry is from [**3**]; this same concept is called an ϵ-function in [**84, 85**]. We note that all of these references also required a symmetry to be additive (i.e. $\tau(a \oplus b) = \tau(a) \oplus \tau(b)$). In [**3**], a symmetry τ must further satisfy $\tau(\mathbf{0}) = \mathbf{0}$. We have removed these conditions from the definition as we will show they follow from other properties of a symmetry. One can easily characterize all symmetries in a semiring.

Proposition 7.0.22. *Let S be a commutative semiring. A function $\tau : S \to S$ is an symmetry if and only if there exists an $\epsilon \in S$ such that $\epsilon^2 = \mathbf{1}$ and $\tau(x) = \epsilon \otimes x$ for all $x \in S$.*

Proof. Suppose $\tau : S \to S$ is a symmetry. Let $\epsilon = \tau(\mathbf{1})$, then $\tau(x) = \tau(\mathbf{1} \otimes \mathbf{x}) = \epsilon \otimes \mathbf{x}$. Furthermore $\epsilon^2 = \epsilon \otimes \tau(\mathbf{1}) = \tau(\epsilon \otimes \mathbf{1}) = \mathbf{1}$. The other direction is a straightfoward verification. ∎

It follows easily from the previous proposition that if $\tau : S \to S$ is a symmetry and $x, y \in S$ then $\tau(x \oplus y) = \tau(x) \oplus \tau(y)$, $\tau(\mathbf{0}) = \mathbf{0}$ and $\tau(x) \otimes \tau(y) = x \otimes y$.

We use this observation to slightly restate the definition of an ϵ-determinant given in [**84, 85**]. We will use ϵ to denote the element whose square is the identity rather than a symmetry or ϵ-function as [**84, 85**]; this allows us to use the same terminology and notation as in [**84**] and [**85**] while taking advantage of the characterization of symmetries given in Proposition 7.0.22.

Definition 7.0.23. *Let S be a commutative semiring and let $\epsilon \in S$ satisfy $\epsilon^2 = \mathbf{1}$. Then $det_\epsilon : M_n(S) \to S$ is defined as the following function: $det_\epsilon(A) = det_+(A) \oplus (\epsilon \otimes det_-(A))$.*

We will get a distinct symmetry on S and hence a distinct det_ϵ from every choice of $\epsilon \in S$ which satisfies $\epsilon^2 = \mathbf{1}$. One candidate for ϵ that exists in every semiring is the multiplicative identity $\mathbf{1}$ If $\epsilon = \mathbf{1}$, we get $det_\epsilon(A) = per(A)$.

Tan has shown that the ϵ-determinant satisfies two very important identities analogous to the Binet-Cauchy theorem and the Laplace expansion of the ordinary determinant. In order to state them, we introduce the following notation. Let $\alpha = \{\alpha_1 < \alpha_2 < ... < \alpha_k\}$ and $\beta = \{\beta_1 < \beta_2 < ... < \beta_k\}$ be two subsets of $\{1, 2, ..., n\}$ of equal cardinality. Then $A[\alpha, \beta]$ is the k by k submatrix of A whose (i, j)th entry is $a_{\alpha_i \beta_j}$. We define $\pi(\alpha) = \sum_{j=1}^{k} \alpha_j$.

Theorem 7.0.24. [**85**, Theorem 3.5] (*Cauchy-Binet Theorem*) *Let* $A \in M_{m,n}(S)$,$B \in M_{n,p}(S)$, $k \leq \min(m, n, p)$. *Let* α *and* β *be two subsets of* $\{1, 2, ..., n\}$ *of cardinality* k. *Then there exists an* $\delta \in S$ *such that* $det_\epsilon(AB[\alpha, \beta]) = (\bigoplus_{\gamma \subseteq \{1,2,...,n\}; |\gamma|=k} det_\epsilon(A[\alpha, \gamma]) \otimes det_\epsilon(B[\gamma, \beta])) \oplus [\delta \otimes (\mathbf{1} \oplus \epsilon)]$.

Theorem 7.0.25. [**85**, Theorem 3.3] (*Laplace Expansion*) *Let* $A \in M_n(S)$, $\alpha \subseteq \{1, 2, ..., n\}$, *then* $det_\epsilon(A) = \bigoplus_{\beta \subseteq \{1,2,...,n\}; |\beta|=|\alpha|} (\epsilon)^{\pi(\alpha)+\pi(\beta)} \otimes det_\epsilon(A[\alpha|\beta]) \otimes det_\epsilon(A[\alpha^c|\beta^c])$.

In the case where S is a commutative ring and $\epsilon = -1$, the ϵ-determinant reduces to the regular determinant and the two theorems above reduce to the usual Binet-Cauchy theorem and the Laplace expansion.

7.1 The ϵ-determinant and ϵ-rank

We can use Tan's generalization of the Laplace expansion to obtain a relationship between the ϵ-determinant of $A + B$ and those of various submatrices of A and B.

Corollary 7.1.1. *Let $A, B \in M_n(S)$, Then*

$$det_\epsilon(A+B) = \bigoplus_{\alpha,\beta\subseteq\{1,2,\ldots,n\};|\beta|=|\alpha|} (\epsilon)^{\pi(\alpha)+\pi(\beta)} \otimes det_\epsilon(A[\alpha|\beta]) \otimes det_\epsilon(B[\alpha^c|\beta^c]).$$

Proof. Let C_α be the n by n matrix whose ith row is equal to the ith row of A if $i \in \alpha$ and whose ith row is equal to the ith row of B if $i \notin \alpha$. Then using the multilinearity of the ϵ-determinant, $det_\epsilon(A+B) = \bigoplus_{\alpha\subseteq\{1,2,\ldots,n\}} det_\epsilon(C_\alpha)$. Now we use the Laplace expansion on $det_\epsilon(C_\alpha)$, for any fixed α, to get

$$det_\epsilon(C_\alpha) = \bigoplus_{\beta\subseteq\{1,2,\ldots,n\};|\beta|=|\alpha|} (\epsilon)^{\pi(\alpha)+\pi(\beta))} \otimes det_\epsilon(A[\alpha|\beta]) \otimes det_\epsilon(B[\alpha^c|\beta^c]).$$

Hence

$$det_\epsilon(A+B) = \bigoplus_{\alpha,\beta\subseteq\{1,2,\ldots,n\};|\beta|=|\alpha|} (\epsilon)^{\pi(\alpha)+\pi(\beta)} \otimes det_\epsilon(A[\alpha|\beta]) \otimes det_\epsilon(B[\alpha^c|\beta^c]).$$

∎

Definition 7.1.2. *Let S be a commutative semiring and let A be an m by n matrix over S. If $1 \le k \le \min(m,n)$, let $I_k^\epsilon(A)$ be the ideal in S generated by the set of all the k by k ϵ-minors of A. We define $I_0^\epsilon(A) = S$ and $I_k^\epsilon(A) = \{\mathbf{0}\}$ when $k > \min(m,n)$.*

It follows immediately from the Laplace expansion that if $I_k^\epsilon(A) \subseteq I_{k-1}^\epsilon(A)$ for all $k \in \mathbb{N}$.

Definition 7.1.3. *Let S be a semiring and let ϵ be an element of S such that $\epsilon^2 = \mathbf{1}$ and that $\mathbf{1} \oplus \epsilon$ is not a unit. Let I be any proper ideal of S which contains $\mathbf{1} \oplus \epsilon$. Then the (ϵ, I)-rank of an m by n matrix A (denoted by $rank_{det}^{\epsilon,I}(A)$) is the largest nonnegative integer k such that $I_k^\epsilon(A)$ is not contained in I.*

For certain semirings, it may be possible that there is no choice of ϵ for which $\mathbf{1} \oplus \epsilon$ is not a unit. An example of this is the semiring of nonnegative real numbers $\mathbb{R}^+$ with the usual addition and multiplication. In this case the only $\epsilon \in \mathbb{R}^+$ satisfying $\epsilon^2 = 1$ is 1 itself and $2 = 1 + 1$ is a unit. The max-plus and max-min semirings are other examples of this. For semirings such as these, one cannot immediately define an (ϵ, I)-rank. We will examine how to handle cases like this in section 7.4 and 7.3.

The principal ideal generated by $\mathbf{1} \oplus \epsilon$ is a natural choice for our ideal I.

Definition 7.1.4. *Let S be a semiring and let ϵ be an element of S such that $\epsilon^2 = \mathbf{1}$ and that $\mathbf{1} \oplus \epsilon$ is not a unit. Let $I_{\mathbf{1}\oplus\epsilon}$ be the principal ideal generated by $\mathbf{1} \oplus \epsilon$. Then ϵ-rank of an m by n matrix A (denoted by $rank^{\epsilon}_{det}(A)$) is the largest nonnegative integer k such that $I^{\epsilon}_k(A)$ is not contained in $I_{\mathbf{1}\oplus\epsilon}$.*

It is clear that any I containing $\mathbf{1} \oplus \epsilon$ contains $I_{\mathbf{1}\oplus\epsilon}$ and therefore $rank^{\epsilon,I}_{det}(A) \leq rank^{\epsilon}_{det}(A)$. If S is a ring, then the (ϵ, I)-rank of a matrix A over S is equal to the standard ring-theoretic rank (the size of the largest submatrix with nonzero determinant) of the matrix $\phi(A)$ over the quotient ring S/I where ϕ is the natural entrywise quotient map.

We now examine some inequalities satisfied by these ranks that are implied by the Binet-Cauchy theorem, the Laplace expansion and our determinant sum identity. The first is the relationship between this rank and the factor rank.

Proposition 7.1.5. *Let S be a commutative semiring and $A \in M_{m,n}(S)$. Then $rank^{\epsilon}_{det}(A) \leq f(A)$.*

Proof. Let $r = f(A)$, then there exist matrices $B \in M_{m,r}(S)$ and $C \in M_{r,n}(S)$ such that $A = BC$. Let $\hat{B} \in M_{m,r+1}(S)$ be the matrix obtained by adding a zero column to the right of B and $\hat{C} \in M_{r+1,n}(S)$ be the matrix obtained by adding a zero row to the bottom of C. Clearly $A = \hat{B}\hat{C}$. Now we compute the ϵ-minors of $A(= \hat{B}\hat{C})$ of order (r+1)×(r+1), using the Cauchy-Binet theorem, i.e, $det_\epsilon(\hat{B}\hat{C}[\alpha,\beta]) = det_\epsilon(\hat{B}[\alpha,\{1,2,...r,,r+1\}]) \otimes det_\epsilon(\hat{C}[\{1,2,...r,,r+1\},\beta]) \oplus \delta \otimes (\mathbf{1} \oplus \epsilon)$. Since the first summand is $\mathbf{0}$, I^ϵ_{r+1} is contained in the ideal generated by $\mathbf{1} \oplus \epsilon$. This implies that $rank^\epsilon_{det}(A) \leq r$. ■

We can also prove a version of Sylvester's inequality for the (ϵ, I)-rank.

Proposition 7.1.6. *Let S be a semiring and let ϵ be an element of S such that $\epsilon^2 = \mathbf{1}$ and that $\mathbf{1} \oplus \epsilon$ is not a unit. Let I be an proper ideal of S which contains $\mathbf{1} \oplus \epsilon$. Let $A \in M_{m,n}(S)$ and $B \in M_{n,p}(S)$. Then $rank^{\epsilon,I}_{det}(AB) \leq \min(rank^{\epsilon,I}_{det}(A), rank^{\epsilon,I}_{det}(B))$.*

Proof. Let $r = \min(rank^{\epsilon,I}_{det}(A), rank^{\epsilon,I}_{det}(B))$. If $r \geq \min(m,n,p)$ we are done so suppose $r < \min(m,n,p)$ and let α and β be arbitrary subsets of $\{1,2,...,n\}$ both of cardinality $r+1$. Then either $I^\epsilon_{r+1}(A)$ or $I^\epsilon_{r+1}(B)$ is contained in I. It follows from the Binet-Cauchy theorem that $det_\epsilon(AB[\alpha,\beta]) \in I$. ■

It should be noted that the condition $\mathbf{1} \oplus \epsilon \in I$ is required for our version of Sylvester's inequality to hold; this is largely our motivation for insisting on this condition.

We also have the following rank-sum inequality.

Proposition 7.1.7. *Let S be a semiring and let ϵ be an element of S such that $\epsilon^2 = \mathbf{1}$ and that $\mathbf{1} \oplus \epsilon$ is not a unit. Let I be an proper ideal of S which contains $\mathbf{1} \oplus \epsilon$. Let $A, B \in M_{m,n}(S)$, then $rank_{det}^{\epsilon,I}(A+B) \leq rank_{det}^{\epsilon,I}(A) + rank_{det}^{\epsilon,I}(B)$.*

Proof. We begin by proving the inequality in the special case where $m = n$ and $rank_{det}^{\epsilon,I}(A) + rank_{det}^{\epsilon,I}(B) = n-1$. Hence $A, B \in M_n(S)$. Let $r = rank_{det}^{\epsilon,I}(A)$ and then $n - r - 1 = rank_{det}^{\epsilon,I}(B)$. We can use Corollary 7.1.1 to show that $det_\epsilon(A+B) \in I$. Note that every term in the expansion of $det_\epsilon(A+B)$ is of a power of ϵ times $det_\epsilon(A[\alpha|\beta]) \otimes det_\epsilon(B[\alpha^c|\beta^c])$ where α and β are subsets of $\{1, 2, ..., n\}$ satisfying $|\alpha| = |\beta|$. Let $k = |\alpha| = |\beta|$. If $k \leq r$, then $n - k \geq n - r > rank_{det}^{\epsilon,I}(B)$ and since $|\alpha^c| = |\beta^c| = n - k$ we must have $det_\epsilon(B[\alpha^c|\beta^c]) \in I$. Similarly, if $k > r$, then $det_\epsilon(A[\alpha|\beta]) \in I$. Therefore every term in the expansion of $det_\epsilon(A+B)$ is in I and hence $rank_{det}^{\epsilon,I}(A+B) \leq n - 1 = rank_{det}^{\epsilon,I}(A) + rank_{det}^{\epsilon,I}(B)$.

Now we prove the general case. Let $r = rank_{det}^{\epsilon,I}(A) + rank_{det}^{\epsilon,I}(B)$. If $r \geq \min(m,n)$ then we are done so suppose $r < \min(m,n)$. Now let α and β be subsets of $\{1, 2, ..., m\}$ and $\{1, 2, ..., n\}$ respectively both of cardinality $r+1$. Then $rank_{det}^{\epsilon,I}((A+B)[\alpha|\beta]) \leq rank_{det}^{\epsilon,I}(A[\alpha|\beta]) + rank_{det}^{\epsilon,I}(B[\alpha|\beta]) \leq rank_{det}^{\epsilon,I}(A) + rank_{det}^{\epsilon,I}(B) = r$. Hence $det_\epsilon((A+B)[\alpha|\beta]) \in I$ and since $(A+B)[\alpha|\beta]$ is an arbitrary $r+1$ by $r+1$ submatrix of $A+B$, we have $rank_{det}^{\epsilon,I}(A+B) \leq r = rank_{det}^{\epsilon,I}(A) + rank_{det}^{\epsilon,I}(B)$. ∎

7.2 Bijective Linear ϵ-rank Preservers

In this section, we classify those bijective linear operators which preserve ϵ-rank of matrices over antinegative commutative semirings with no zero divisors.

Definition 7.2.1. [**11**] *Let S be a semirings and A be an m by n matrix over S. The term rank of A is the minimum number of lines (rows and columns) needed to include all nonzero entries of A. The term rank of a matrix A is denoted by $t(A)$.*

Proposition 7.2.2. [**11**] *Let S be a semirings and A be an m by n matrix over S, then $f(A) \leq t(A)$.*

Remark 7.2.3. *If S is commutative a semirings and A be an m by n matrix over S, then by Proposition 7.1.5, we have $rank^{\epsilon}_{det}(A) \leq f(A)$ and by Proposition 7.2.2 we have $f(A) \leq t(A)$. This implies that $rank^{\epsilon}_{det}(A) \leq t(A)$.*

Let S be a semirings and $A, B \in M_{m,n}(S)$. We write $A \leq B$ if there exists $C \in M_{m,n}(S)$ such that $A \oplus C = B$. We note that the relation ($\leq$) is a reflexive and transitive relation but not antisymmetric in general. Therefore it is a *pre-order*. It is easy to check that any linear operator $T : M_{m,n}(S) \longrightarrow M_{m,n}(S)$ preserves this pre-order. Further, if S is an antinegative semirings then the term rank is a monotone function, i.e., if $A \leq B$ then $t(A) \leq t(B)$.

Definition 7.2.4. *Let S be a semirings and $A, B \in M_{m,n}(S)$. The Schur product of A and B denoted as $A \circ B$, is an m by n matrix whose $(i, j)^{th}$ entry is $a_{ij} \otimes b_{ij}$.*

Definition 7.2.5. *Let S be a semirings and a matrix $A \in M_{m,n}(S)$ is called a submonomial matrix if every line (row or column) of A contains at most one non-zero entry. A matrix $A \in M_n(S)$ is called a monomial matrix if every line (row or column) of A contains exactly one non-zero entry.*

Lemma 7.2.6. [**15**, Lemma 2] *Let S be an antinegative semiring and $A \in M_{m,n}(S)$, then there exists a submonomial matrix $M \leq A$ such that $t(M) = t(A)$.*

Remark 7.2.7. *The term rank of a submonomial matrix $A \in M_{m,n}(S)$ is equal to the number of non-zero entries of A. It has been observed in* [**76**, Proposition 2.4] *that if S is a semiring with no zero divisors, then the factor rank and determinantal rank of a submonomial matrix A are equal to the term rank of A. We also note that if S is a semiring with no zero divisors, then the ϵ-rank of a submonomial matrix A is equal to the term rank of A.*

Skornyakov [**79**] characterized invertible matrices over an antinegative commutative semiring with no zero divisors.

Proposition 7.2.8. [**79**, Theorem 1] *Let S be an antinegative commutative semiring with no zero divisors, then $A \in M_n(S)$ is invertible if and only if it is a monomial matrix all of whose non-zero entries are units.*

This result is also a special case of Dolžan and Oblak's [**31**, theorem 1] result about invertible matrices over antinegative semirings.

The concept of a (P, Q, B) operator is a fundamental concept in the theory of linear preservers over semirings. We will use weak (P, Q, B) operator and strong (P, Q, B) operator to distinguish between the two different definitions in the literature.

Definition 7.2.9. [**15**] *Let T be a linear operator from $M_{m,n}(S)$ to itself. Then we say that T is a weak (P, Q, B) operator if there exist $P \in M_m(S)$, $Q \in M_n(S)$, and $B \in M_{m,n}(S)$ such that P and Q are permutation matrices, B has no entries which are zeros and zero divisors, and either $T(X) = P(X \circ B)Q$ or $m = n$ and $T(X) = P(X^T \circ B)Q$.*

Definition 7.2.10. [**76**] *Let T be a linear operator from $M_{m,n}(S)$ to itself. Then we say that T is a strong (P, Q, B) operator if there exist $P \in M_m(S)$, $Q \in M_n(S)$, and $B \in M_{m,n}(S)$ such that P and Q are permutation matrices, all of the entries of B are units and either $T(X) = P(X \circ B)Q$ or $m = n$ and $T(X) = P(X^T \circ B)Q$.*

Lemma 7.2.11. [**15**, Theorem 1] *Let S be a commutative semiring and let T be a linear operator which maps $M_{m,n}(S)$ to itself. Then T preserves term rank if and only if it is a weak (P, Q, B) operator.*

We now prove our main theorem which characterizes bijective linear preservers of ϵ-rank functions of matrices over antinegative commutative semirings.

Theorem 7.2.12. *Let S be an antinegative commutative semiring with no zero divisors and let T be a linear operator on $M_{m,n}(S)$. If T is bijective and preserves ϵ-rank then T is a strong (P, Q, B) operator, where the ϵ-rank of B is equal to one.*

Proof. Suppose the linear operator T on $M_{m,n}(S)$ is bijective and preserves ϵ-rank. To prove this theorem we will first show that T also preserves the term tank. For any $A \in M_{m,n}(S)$, there exists a submonomial matrix $B \leq A$ such that ϵ-rank$(B) = t(B) = t(A)$. Therefore $t(A) = \epsilon$-rank$(B) = \epsilon$-rank$(T(B)) \leq t(T(B)) \leq t(T(A))$. Now we will show that $t(T(A)) \leq t(A)$. There exists a submonomial matrix $C \leq T(A)$ such that ϵ-rank$(C) = t(C) = t(T(A))$. Let $D \in M_n(S)$ be such that $C + D = T(A)$. Since T is surjective there exists $E, F \in M_n(S)$ such that $T(E) = C$ and $T(F) = D$. Since T is injective, $E + F = A$ and $t(T(A)) = \epsilon$-rank$(C) = rank(E) \leq t(E) \leq t(A)$. Therefore we have $t(T(A)) = t(A)$ for all $A \in M_n(S)$, which by Lemma 7.2.11 means that T is a weak (P, Q, B) operator. Since T is surjective there must be $G \in M_{m,n}(S)$ such that $T(G) = J_{m,n}$ which means that all the entries of B must be units and hence T is a strong (P, Q, B) operator. Further, $rank^{\epsilon}_{det}(B) = rank^{\epsilon}_{det}(PBQ) = rank^{\epsilon}_{det}(T(J_{m,n})) = 1$.

■

7.3 Symmetrized Semirings

The ranks introduced in the previous sections all require an element ϵ satisfying the condition that $\epsilon^2 = \mathbf{1}$ and $\mathbf{1} \oplus \epsilon$ is not a unit. Such an element may not exist in a given semiring; the max-min and max-plus semirings are examples of semiring which lack an ϵ. Fortunately, there is a known construction which allows us to append such an element. This construction is from [**71**], in which it was applied to the max-plus semiring. In this section we explore applications of this construction both to general semirings as well as to the specific examples such as the Boolean semiring and the sign pattern semiring.

If $(S, \oplus, \otimes)$ is a commutative semiring then $S^2 = \{(a, b)|a, b \in S\}$ is also a commutative semiring with addition and multiplication defined at the end of the section 2.2.1 and it is called a symmetrized semiring.

Essentially this construction allows us to append an element $\epsilon = (\mathbf{0}, \mathbf{1})$ with the property $\epsilon^2 = \mathbf{1}$ to the semiring S in a natural way giving us a way to apply the ϵ-determinant theory to semirings which do not have non-trivial self inversive elements. The ideal in S^2 generated by $(\mathbf{1}, \mathbf{1}) = (\mathbf{1}, \mathbf{0}) + \epsilon$ is $\Delta = \{(x, x) : x \in S\}$ which we will call the diagonal ideal. The ϵ-determinant in this case is the standard bideterminant and the ϵ-rank is the standard determinantal rank.

Remark 7.3.1. *Let $\beta = \{0, 1\}$ be a Boolean semiring. Then $\beta^2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ is also a semiring with the addition and the multiplication defined above for S^2. Moreover it is isomorphic to the sign pattern semiring as $(0, 0) = 0$, $(1, 0) = +1$, (0,1) = -1, (1,1) = # .*

7.4 The ϵ-rank and SNS Submatrix

In this section, we explore connections between the sign pattern matrices and ϵ-rank.

For matrices over the sign pattern semiring, we can give a more concrete interpretation of the ϵ-rank. The sign pattern semiring has only two elements whose square is the identity are 1 and -1. The ideal generated by $1 = 1 + 1$ is the entire semiring but $\# = 1 + -1$ generates the unique proper ideal $\{\#, 0\}$. Therefore -1 is the only available choice for ϵ and we have a unique ϵ-rank. Hence $det_\epsilon(A) = det_+(A) \oplus (-1 \otimes det_-(A))$. It is easy to show that an n by n sign pattern matrix has ϵ-rank n if and only if it is an SNS matrix. Hence the ϵ-rank of a sign pattern matrix A is the largest integer k for which there exists a k by k SNS submatrix of A.

7.5 The CP-rank and The Nonnegative rank

In this section, we define the nonnegative rank for matrices over semirings and compare it with the CP-rank of completely positive matrices over semirings. Our definition of the nonnegative rank for matrices over semirings is a natural generalization of the nonnegative rank for real nonnegative matrices. The nonnegative rank for nonnegative real matrices has been studied in [**8**]. A real matrix A is said to have a nonnegative rank factorization if A can be written as $A = BC$, where B and C are nonnegative real matrices. The *nonnegative rank* of an $m \times n$ real matrix A is the smallest number r such that A can be decomposed as $A = BC$, where B are C are nonnegative real matrices of order $m \times r$ and $r \times n$, respectively. One can easily check that a real matrix has a nonnegative rank factorization if and only if it is a nonnegative matrix, since $A = AI$ or $A = IA$, where I is an identity matrix. We generalize this rank function to matrices over semirings as follows:

Definition 7.5.1. (*The Nonnegative Rank*) *Let S be a commutative semiring and $P(S)$ be the positive subsemiring of S which consists of all the finite sums of perfect squares in S. The nonnegative rank of an $m \times n$ matrix A over S, denoted as $rank_n(A)$, is the smallest number k such that $A = BC$, where B is an $m \times k$ matrix over $P(S)$ and C is an $k \times n$ matrix over $P(S)$.*

A factorization $A = BC$ of a matrix A over a semiring S such that B and C are matrices over $P(S)$ is called a nonnegative rank factorization of the matrix A over S. It is clear from the definition that the nonnegative rank of matrices over semirings exist if and only if all the entries of the matrices are the elements of $P(S)$. Moreover, if the nonnegative rank of an $m \times n$ matrix A over a semiring S exists then it is always less that or equal to $min\{m, n\}$, since $A = AI$ or $A = IA$, where I is an identity matrix over S (we will call this a trivial nonnegative rank factorization of A).

The nonnegative rank of matrices over semirings is different from the factor rank of matrices over semiring. The factor rank of a matrix A over a semiring S always exists, since we can write $A = AI$. However, if all the entries of the matrix A are not the elements of $P(S)$, where $P(S)$ is the positive subsemiring of S, then the nonnegative rank of A does not exist. We also note that if the nonnegative rank of a matrix over a semiring exists then it is always greater than or equal to the factor rank.

Proposition 7.5.2. *Let A be an $m \times n$ matrix over a semiring S and all the entries of A are the elements $P(S)$, where $P(S)$ is the positive subsemiring of S. Then $f(A) \leq rank_n(A) \leq min\{m,n\}$.*

Proof. The nonnegative rank of A is less than or equal to $min\{m,n\}$, because of a trivial nonnegative rank factorization ($A = AI$ or $A = IA$) of A. Let the nonnegative rank of A be k, where $k \leq min\{m,n\}$. This implies that there exist $B \in M_{m,k}(P(S))$ and $C \in M_{k,n}(P(S))$ such that $A = BC$. Therefore the factor rank of A is less than or equal to k. ■

We note that if $A \in M_{n,n}(S)$ is a symmetric matrix and all the entries of A are the elements of $P(S)$, where $P(S)$ is a subsemiring of S which consists of all the finite sums of perfect squares in S, then the nonnegative rank of A always exists. On the other hand, the CP-rank of A may not exist, i.e., the matrix A may not be a completely positive matrix over S. Here we have an example of a square symmetric matrix A over the Boolean semiring from [**11**] which is not completely positive but we find a non-trivial nonnegative rank factorization of A.

Example 7.5.3. *Let $A = J_7 - I_7$ be a derangement matrix over the Boolean semiring. Clearly*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

is a symmetric Boolean matrix. Over the Boolean semiring the nonnegative rank of A is less than or equal to 5 due to the nonnegative rank factorization

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Since A is not diagonally dominant matrix over the Boolean semiring, by proposition 4.1.2, A is not a completely positive matrix over the Boolean semiring.

Proposition 7.5.4. *Let A be an $n \times n$ completely positive matrix over a semiring S. Then*

$$CP\text{-}rank(A) \geq rank_n(A)$$

Proof. Given that A is an $n \times n$ completely positive matrix over a semiring S. Let us suppose that the CP-rank of A is k, this implies that there exists an $n \times k$ matrix B over the subsemiring $P(S)$ such that $A = BB^T$. Clearly $B^T = C$ is an $k \times n$ matrix over the subsemiring $P(S)$ such that $A = BC$. This implies that the nonnegative rank of A is less than or equal to k. ■

We have shown in previous chapters that the upper bound of completely positive $n \times n$ matrices over special semirings is bounded above my the $max\{n, [n^2/4]\}$. There also exist some semirings, such as the sign pattern semiring, in which a completely positive matrix of order n may have the CP-rank greater than $max\{n, [n^2/4]\}$. In other words, if an $n \times n$ matrix over a semiring is completely positive then the CP-rank of that matrix can be larger than n. An example of the matrix given in remark 4.2.4, shows the strict inequality between the CP-rank and the nonnegative rank of a completely positive matrix over the Boolean semiring.

Equality also holds in the CP-rank and the nonnegative rank of completely positive matrices over semirings in some special cases.

Proposition 7.5.5. *Let A be an $n \times n$ completely positive matrix over a semiring S and $P(S)$ be the positive subsemiring of S. If the positive subsemiring $P(S)$ has the unique square root property, then the CP-rank of A is equal to one if and only if the nonnegative rank of A is one.*

Proof. If the CP-rank of an $n \times n$ completely positive matrix over a semiring S is one then it is obvious that the nonnegative rank of A is also one. Conversely, suppose that the nonnegative rank of A over the semiring S is one. This implies that $A = bc^T$ where $b \in M_{n,1}(P(S))$ and $c \in M_{n,1}(P(S))$. It is given that A is completely positive matrix over S, this implies that all the entries of A are the elements of $P(S)$. We also have that every element of $P(S)$ has a unique square root in $P(S)$. Thus we get

$$\sqrt{a_{ii}}^2 = a_{ii} = b_i c_i. \tag{7.1}$$

Since we have $A = bc^T$ where $b \in M_{n,1}(P(S))$ and $c \in M_{n,1}(P(S))$, the factor rank of A is equal to one. By proposition 7.0.18, we get that the determinantal rank of A is equal to one and hence every two by two submatrix of A has the positive determinant equal to the negative determinant. Thus we get for all i, j

$$a_{ii}a_{jj} = a_{ij}a_{ji} = a_{ij}^2 \qquad \text{(since } A \text{ is symmetric)}$$

This implies that

$$\sqrt{a_{ii}}\sqrt{a_{jj}} = a_{ij}, \tag{7.2}$$

(using the fact that the unique square root property implies that the square root function is multiplicative). From equation (7.1) and (7.2), we get that

$$A = \begin{bmatrix} \sqrt{a_{11}} \\ \sqrt{a_{22}} \\ . \\ . \\ \sqrt{a_{nn}} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & \sqrt{a_{11}} & . & . & \sqrt{a_{11}} \end{bmatrix}$$

and this implies that the CP-rank of A is one. ■

Note that in proposition 7.5.5, if we remove the condition that the subsemiring $P(S)$ has the unique square root property then the CP-rank of a completely positive matrix A over a semiring S may not be equal to one when the nonnegative rank of A is equal to one. Here we have an example:

Example 7.5.6. *Let $S = \mathbb{N}$, where $\mathbb{N}$ be the set of all natural numbers including zero. It is evident that $\mathbb{N} = P(\mathbb{N})$ and $P(\mathbb{N})$ does not have the unique square root property. Let*

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

be a completely positive matrix over $\mathbb{N}$. The nonnegative rank of A is equal to one, since

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \end{bmatrix}$$

However, the CP-rank of A is equal to two by example 3.6.3.

Proposition 7.5.7. *Let A be an $n \times n$ completely positive matrix over a semiring S and $P(S)$ be the positive subsemiring of S. If the subsemiring $P(S)$ has the unique square root property then the CP-rank of A is two implies the nonnegative rank of A equal to two.*

Proof. Let us suppose that the CP-rank of A is equal to two. Then by proposition 7.5.4, the nonnegative rank of A is less than or equal two. Let us suppose that the nonnegative rank of A is one then the CP-rank of A is also one by proposition 7.5.5, which is a contradiction. Hence the nonnegative rank of A is equal to two. ■

Note that the converse of proposition 7.5.7 may not be true. There exist completely positive matrices over special semirings with the nonnegative rank equal to two and the CP-rank strictly greater than two.

Example 7.5.8. *Let S be the sign pattern semiring and $A = \begin{bmatrix} \# & +1 \\ +1 & \# \end{bmatrix}$ be a completely positive matrix over S. Clearly A can be written as:*

$$A = \begin{bmatrix} \# & +1 \\ +1 & \# \end{bmatrix} \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix} = AI.$$

This implies that the nonnegative rank of A is less than or equal to two, but there does not exist any nonnegative rank factorization ($A = BC$) of A, where $B \in M_{2,1}(P(S))$ and $C \in M_{1,2}(P(S))$. Therefore, the nonnegative rank of A is equal to two. However, by example 5.4.4, the CP-rank of A is equal to three.

Chapter 8

Conclusion

In this chapter we summarize the novel results of the thesis.

In this thesis, we extended the notion of complete positivity for matrices over real numbers to matrices over general semirings. For completely positive matrices over real numbers there is no efficient algorithm to decide if a given real matrix is completely positive. We found simple necessary and sufficient conditions for matrices over certain semirings to be completely positive. We showed that diagonal dominance is a necessary and sufficient condition for symmetric matrices over special semirings to be completely positive (theorem 3.3.3). One direction of our result generalizes the Kaykobad's result for completely positive matrices over real numbers. We also showed that symmetric matrices over certain special types of inclines are completely positive if and only if they are diagonally similar to matrices with all diagonal entries equal to **1** (theorem 3.3.1).

In 1994, Drew, Johnson and Loewy conjectured that the CP-rank of real completely positive matrices of order n is always less than or equal the maximum of n and $[n^2/4]$. This conjecture was open before being disproved by Bomze et al. in 2014. We generalized the Drew-Johnson-Loewy conjecture to completely positive matrices over semirings and proved the truth of this conjecture for completely positive matrices over certain types of semirings, which include the Boolean semiring, the negative interval subsemiring of the max-plus semiring $([-\infty, 0], max(x, y), x + y)$, the max-min fuzzy algebra $([\mathbf{0}, \mathbf{1}], max(x, y), min(x, y))$, Max-min semirings also called chain semirings, the max-times semiring $([\mathbf{0}, \mathbf{1}], max(x, y), xy)$ where xy is the ordinary real multiplication and Boolean algebras (theorem 5.1.2, and theorem 4.4.6). Although the original Drew-Johnson-Loewy conjecture was disproved, the generalized Drew-Johnson-Loewy conjecture is still open for many other semirings (Question 1.0.1). Our work gives researchers, who were working on the original Drew-Johnson-Loewy conjecture, a new direction to look at this conjecture. In addition, we showed that the Drew-Johnson-Loewy conjecture for completely positive matrices over the Boolean semiring is equivalent to a well-known result of Erdős, Goodman and Pósa in graph theory. We also proved the Erdős, Goodman and Pósa result for fuzzy graphs: every fuzzy graph on a set S of $n \geq 2$ points can be covered by at most $max\{n, [n^2/4]\}$ complete fuzzy graphs and in covering with complete fuzzy graphs we need only single edges and triangles (theorem 5.2.6).

We showed that the notion of complete positivity for matrices over special semirings has some important similarities with the standard notion of complete positivity of real matrices. We proved a semiring version of Markham's theorems which give sufficient conditions for a completely positive matrix over special types of semirings to have a triangular factorization (section 3.4). We generalized an algorithm given by Kaykobad to find a completely positive factorization of symmetric diagonally dominant matrices over certain semirings. We examined that $n(n+1)/2 - N$ is a sharp upper bound on the CP-rank of completely positive matrices over special semirings (theorem 3.5.3 and remark 5.4.5). We also showed that in many cases the matrices of interest in graph theory are completely positive matrices over special types of semirings. This enables us to use completely positive matrices over special semirings to formulate a wide variety of path finding problems. In addition, we formulated various CP-rank inequalities of completely positive matrices over special semirings.

Lastly, we used the ϵ-determinant introduced by Ya-Jia Tan to define a family of ranks of matrices over certain semirings. We showed that these ranks generalize some known rank functions over semirings such as the determinantal rank. We also showed that this family of ranks satisfies the rank-sum and Sylvester inequalities. Moreover, we classified all bijective linear preservers of these ranks.

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