Basket Option Pricing and the Mellin Transform

by

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A Thesis

Presented to

The Faculty of Graduate Studies

of

The University of Guelph

In partial fulfilment of requirements

for the degree of

Master of Science

in

Applied Mathematics

Guelph, Ontario, Canada

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Abstract

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Option pricing has been an increasingly popular area of study over the past four decades. The use of the Mellin transform in such a context, however, has not. In this work we present a general multi-asset option pricing formula in the context of Mellin transforms, extending previously known results. The analytic formula derived computes European, American, and basket options with $n$ underlying assets driven by geometric Brownian motion. Aside from the usual given parameters, the pricing formula requires three components to compute: (i) the Mellin basket payoff function, (ii) the characteristic function (or exponent) of a multivariate Brownian motion with drift, and (iii) the Mellin transform of the early exercise function. A fast discretization is solved, providing option prices at incremental values of initial asset prices. As an application, European put option prices are computed for Canadian bank stocks ($n = 1$) and foreign exchange rates ($n = 2$) with USD denomination.
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Acknowledgments

Foremost I would like to thank my advisor, Dr. Peter T. Kim, who has been a relentless source of support and inspiration for me. I am grateful to him for his life mentorship, his profound insight regarding research problems, and for providing me with the freedom to study engaging topics without rigid direction.

To the members of the examining committee: Dr. Tony Desmond, Dr. Hermann Eberl, and exam chair Dr. Rajesh Pereira. Thank you for taking the time to consume and decompose my work. I recognize the rarity of your time and appreciate the significance of your input.

To the members of the department whom had a profound impact on me: Dr. Jack Weiner for turning me on to the beauty of calculus, Dr. Dan Ashlock for keeping me turned on to the wonders of abstract math while providing entertaining venues to expand my learning, Dr. Rajesh Pereira for opening my awareness to the purity of math and the opportunity to study beneath him, Dr. Marcus Garvie for instilling the importance of numerical computation, Dr. Herb Kunze for demonstrating the applicability of analysis, Dr. Hermann Eberl for catalyzing my interest in partial differential equations, Dr. Monica Cojocaru for her hospitality, and Dr. Pal Fischer for being the coolest old person I know.

To my family and friends: for the companionship, the stability, and the humanism. Thank you for gracefully ignoring the 4n language that occasionally spews from my mouth.

To the funding sources during my tenure: the Department of Mathematics
and Statistics, the College of Physical and Engineering Sciences, the University of Guelph, Wilfred Laurier University, and Le Centre de recherches mathématiques.

For those who have been omitted here, but have knowingly impacted my life positively. Every interaction provides an opportunity to learn; thank you for being a source of encouragement, perspective, and wisdom.
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Chapter 1

Introduction

*It is not by augmenting the capital of the country, but by rendering a greater part of that capital active and productive than would otherwise be so, that the most judicious operations of banking can increase the industry of the country.* -Adam Smith

1.1 Outline of Work

The primary objective of this thesis is to review the recent literature relating to Mellin transforms in the context of option pricing where assets are driven by geometric Brownian motion. By doing so, we attempt to fill the gaps for the multi-asset basket option in an analytical and numerical framework. To achieve this, we present this thesis in the following manner. Chapter 1 introduces the necessary concepts of option pricing and provides financial motivation for the pursuit of obtaining accurate pricing methods. We also review stochastic calculus and formulate the partial differential equation attributed to Black, Scholes, and Merton. In chapter 2, we find the analytic solutions of European and American put/call options under a single asset driven by geometric Brownian motion. The solution method follows the conventional approach used in the literature; a change of variables followed by a reduction to a
diffusion equation. Chapter 3 introduces the Mellin transform in the multivariate case, providing properties on positive functions, as well as relations required thereafter. Chapter 4 demonstrates the applicability of the Mellin transform for solving European and American call/put options on a single asset, reviewing previous results from the literature. Chapter 5 extends these results to the multidimensional case, where we obtain the general analytic pricing formula, as well as expressions for the early exercise function, Mellin payoff function, and Greeks of put options. Chapter 6 considers the numerical pricing problem, where we discretize the general formula to obtain a numerically fast and accurate procedure. Additional numerical Mellin inversion techniques are reviewed. Chapter 7 concludes the results of this work, Chapter 8 presents possible areas of future work, And As It Is Such, So Also As Such Is It Unto You.

1.2 A Brief History of Options Trading

An option is a financial security that presents its holder with the right, but not the obligation, to purchase a given amount of underlying asset at some future date. In practice, the underlying asset is often the price of a stock, commodity, foreign exchange (FX) rate, index, or futures contract. With global issuance at over 4 billion a year, proper valuation is a significant concern among institutional and personal traders who use options to invest or mitigate risk by hedging.\footnote{Data queried from the Options Clearing Corporation (OCC) at www.theocc.com.} Despite their relatively recent market formalization by the Chicago Board of Options Exchange
(CBOE) in 1973, options have been in use for hundreds of years. Options written implicitly on contractual transactions were mentioned as early as the 4th century B.C. by Aristotle [49]. However, it was in the 16th century when free-standing exchangeable options became available at medieval market fairs, particularly in Antwerp, Belgium. Closely connected with the concentration of commercial activity and expansion of oceanic trade, arrival contracts could be issued for speculation on the quantity of goods entering ports. By the mid 17th century, the financial centres of London and Amsterdam had consumer accessible and liquid over-the-counter (OTC) options for trading. However it wasn’t until 1973 when the growth of options and derivatives trading exploded. Today, large volumes of options are traded daily on exchanges such as the CBOE, the London International Financial Futures and Options Exchange (LIFFE), Eurex Exchange (EE), the Osaka Securities Exchange (OSE), and the Montréal Exchange (MX). By far, the biggest players on derivatives markets are the investment and commercial banks of JPMorgan, Citibank, Bank of America, and Goldman Sachs with combined holdings of well over $200,000,000 [75]. Clearly, understanding how to efficiently price options and related derivatives is of supreme financial importance. As such, there is no doubt that a major objective of the option pricing field is to obtain models that accurately portray empirical features of option prices while subject to general assumptions that govern financial markets.

2Commodities such as grain, whale oil, herring, and salt were the more liquid contracts.
3The period coincided with Tulip mania, where over-speculation of futures and options contracts on tulips led to market collapse and widespread default. At the peak of the bull market, some tulip bulbs sold for as much as 10 times the annual salary of a skilled craftsman.
4Note that an option is a derivative security (i.e. a security which derives its value from an underlying asset). Despite the different derivative securities available for trade, many share similar mathematical formulations. For example, when an appropriate payoff function is specified, contingent claims are mathematically equivalent to European or plain vanilla options.
5As of quarter ending March 31st, 2013.
1.3 Market Assumptions

Prior to constructing financial models, certain assumptions must be made about the modelling and structure of markets. For instance, the absence of arbitrage eliminates the possibility of riskless returns (beyond what the risk-free rate of return \( r \) provides). This provides the scenario that markets are efficiently priced. For models in complete markets, this enables a portfolio to be perfectly hedged (through the construction of a self-financing portfolio), and gives rise to the relationship of put-call parity for European-style options. The risk-free rate of return \( r \) and volatility of the asset \( \sigma \) are assumed to be constant. Since asset price data is openly available to market participants and order execution is almost instantaneous (in highly liquid markets), we assume markets are frictionless. In other words, prices respond immediately to new information. Furthermore, keeping with the usual hypotheses, we assume that asset price history is fully reflective of the current price (i.e. the current price as a function of time is independent of the past).

1.4 Styles of Options

Recall that an option is a derivative security that presents its holder with the right to purchase a given amount of an underlying asset at some future date. For an option

---

6 The expression no free lunch is commonly used to describe this condition.

7 \( V_C - V_P = Se^{-q(T-t)} - Ke^{-r(T-t)} \), where \( S \) is the underlying asset price, \( K \) is the exercise price, \( r \) is the risk-free rate of return, \( q \) is the dividend rate, \( T \) is the time to maturity of the option, \( t \) is the current time, and \( V_C, V_P \) is the value of the call and put option, respectively. Similarly for American options, the notion of put-call symmetry applies (i.e. \( V_C^A(S,K,r,q,t) = V_P^A(K,S,q,r,t) \)).

8 In practice, short-term government bonds or interbank lending rates set by central banks are appropriate proxies for \( r \), while the average asset variance is sufficient for \( \sigma \). Rate \( r \) is discounted as a annual percentage, while volatility \( q \) is typically taken over the previous 10, 20, or 30 day period.
on a single asset, denoted by \( V = V(S, t; K; T; \sigma; r; q) \), the value is dependent on an underlying asset price \( S = S(t) \) at time \( t \geq 0 \), the exercise price \( K > 0 \), the maturity time \( T > 0 \), the volatility (or standard deviation) \( \sigma \) of the asset, the risk-free interest rate \( r > 0 \), and continuous dividend rate \( q > 0 \). The payoff of an option depends highly on its style and type. The type of option refers to whether it is a put or call. The conditions on when the option is exercisable and how much the option is worth at maturity specifies the style (or family) of option: (i) European options have fixed finite maturity time \( T \), (ii) American options have variable finite maturity time \( T \), and hence can be exercised at any time \( 0 \leq t \leq T \), (iii) Basket options have payoff functions that depend on a weighted sum of multiple underlying assets.

This thesis is concerned with the explicit valuation of European, American, and more generally, basket options. Consider the simplest example, the European option, whose value is given by the discounted expected payoff:

\[
V^E(S, t) = \mathbb{E}_Q[e^{-r(T-t)}\theta(S(T))|S(t) = S].
\]

The payoff at time \( T \) is \( \theta_C(S) = \max(S - K, 0) = (S - K)^+ \) for a call and \( \theta_P(S) = \max(K - S, 0) = (K - S)^+ \) for a put. Suppose we are the owner of a European option (put or call). If the future price of the underlying asset will be less (put) or greater (call) than the exercise price declared at issuance, we may sell the option for a positive return. Otherwise, the value of the option is zero (see Figure 1.1). Continuing with this approach, one can also value American options. Since American options are exercisable at any time up to \( T \), its value can be viewed as

---

9. This is analogous to specifying a short (put) or long (call) position on a stock, bond, FX rate, etcetera.

10. This list is not exhaustive; more complicated payoff structures and conditions exist in theory and practice. The class of available OTC and exchange-traded options continues to expand as investor and commercial demand grows.

11. A weighted sum on \( n \) assets \( A \) implies \( \sum_{i=1}^{n} A_i \omega_i \) for \( \sum_{i=1}^{n} \omega_i = 1 \) and \( 0 \leq \omega_i \leq 1 \).
Figure 1.1: The payoff for a European call and put option for different values of the asset price $S$, given exercise price $K = $100.

solving an optimal stopping problem: $V^A(S, t) = \sup_{t \leq T} \mathbb{E}_Q[e^{-r(T-t)}\theta(S(T))|S(t) = S]$.

An unknown variable known as the critical asset price $S^*(t)$ arises due to the free boundary, and divides the price space into two regions, separated by $S^*$. For a put option, when $S > S^*$ (known as the continuation region $\mathcal{C}$), it is optimal to hold the option, while when $S < S^*$ (known as the exercise region $\mathcal{E}$) it is optimal to exercise the option. In order for the transition at the boundary to be smooth, the option and its gradient must be continuous. The smooth pasting conditions supply this: \(^{12}\)

\[
(i) \quad \frac{\partial V(S^*, t)}{\partial S} = -1 \tag{1.1}
\]

and (ii) when $S = S^*$ the payoff function becomes $\theta_p(S) = (K - S^*)^+$. For a call option, $\mathcal{C}$ is realized when $S < S^*$ and $\mathcal{E}$ when $S > S^*$. The smooth pasting conditions

\(^{12}\)These conditions ensure the option and its first derivative with respect to the asset price is continuous.
are given by,

\[(i) \quad \frac{\partial V(S^*, t)}{\partial S} = 1 \quad (1.2)\]

and \((ii) \quad \theta_C(S) = (S^* - K)^+ \) when \( S = S^* \). In the exercise region \( E \), the payoff functions for puts and calls are equivalent to their European counterpart. Now suppose we construct a self-financing portfolio with an American option. If at any time the value of the option under-performs an investment earning the risk-free rate of return, it would be optimal to exercise early (converting risk yield to risk-free yield). This gives us the additional condition that \( V^A(S, t) \geq \theta(S) \) in \( C \). In the case of an American call option, it is never optimal to exercise the option before its maturity. To illustrate this, construct two portfolios \( A \) and \( B \). Let \( A \) be comprised of one American call option and \( Ke^{-r(T-t)} \) dollars in cash. Let \( B \) be comprised of one share of the asset paying no dividend. At any \( t < T \), the value of \( A \) when exercised is equal to \( S - K - Ke^{-r(T-t)} \). Since \( r > 0 \), \( S - K - Ke^{-r(T-t)} < S \). At maturity \( T \), \( A \) has a value of \((S, K)^+\). Clearly, \( S \leq (S, K)^+ \) and hence it is never optimal to exercise an American call before maturity. For a put or call option on a dividend-paying asset, this condition doesn’t hold, and the American option should be worth at least that of a European-style contract in \( C \).

Basket options can be viewed as multidimensional analogues of the single-asset European and American cases. Options issued on indices, mutual funds, or a basket of FX rates fall into this category. The increase in popularity is justified by their ability to minimize transaction costs while providing a sufficient hedge against risk. Since basket options are the primary focus of this work, do not discuss them
here but provide a thorough treatment in section 5.2

1.5 Option Pricing Theory

The introduction of modern mathematical finance and first published reference on option pricing is widely attributed to Louis Bachelier in 1900 [3]. Applying his theory to Parisian markets, he suggested that the underlying asset of an option follows a simple diffusion process \( \{ S(t) \}_{t \geq 0} \) where the asset price \( S(t) = S(0) + \sigma W(t) \), with volatility \( \sigma > 0 \), and \( \{ W(t) \}_{t \geq 0} \) standard Brownian motion. His results involving Brownian motion were published 5 years prior to Einstein’s famous paper on the subject [32].

Contrary to observed prices in markets, the proposed asset model may exhibit negative values. In 1959, Osborne considered that the distribution of log returns followed Brownian motion [76]. Conversely, in 1963 Mandelbrot assumed log returns mimicked non-normal Lévy motion in commodities markets [68]. He suggested that financial models be governed by \( \alpha \)-stable Lévy processes with index \( \alpha < 2 \). Two years later, economist Paul Samuelson and mathematician Henry McKean purported that asset prices be modelled by the normal case, \( S(t) = S(0)e^{(\mu - \sigma^2/2)t + \sigma W(t)} \), where \( \mu, \sigma \in \mathbb{R} \) and \( \sigma > 0 \) [69, 93]. This model enables asset prices to fluctuate randomly, while also preserving positivity. Despite this development in asset pricing, it would take 8 years and much rejection among journals before a quantitative method for pricing options was published. This method, attributed to physicist Fischer Black

\[ ^{13} \text{Bachelier also discovered the Chapman-Kolmogorov convolution probability integral (later accredited to C-K), a formula for at-the-money call options, and the reflection method for finding probability functions of diffusion processes with an absorbing boundary. Despite these accomplishments, the advisor of his dissertation Henri Poincaré assigned him a humbling grade of \textit{honorable} (rather than the arguably deserving \textit{très honorable}). } \]
and economist Myron Scholes, has been regarded as the most significant contribution to the field of mathematical finance \[7\]. Three years later in 1976, economist Robert C. Merton provided an alternate derivation by constructing a perfectly hedged portfolio under the no-arbitrage condition \[70\]. \textsuperscript{14} In 1979, Cox, Ross, and Rubenstein introduce a binomial tree lattice-based model which uses risk-neutral probabilities to price options \[18\]. Harrison, Kreps, and Pliska develop on the concept of risk-neutral probability by recognizing that discounted price processes are martingales \[50, 61\]. \textsuperscript{15} This recognition catalyzed the introduction of a general theory of martingales, which enabled the use of an equivalent martingale measure for risk-neutral pricing. The fundamental idea is to recast asset prices into martingales over a filtered probability space. \textsuperscript{16} However, as alluded to, the parametric form of the asset price is open to scrutiny. It is well known that the BSM model under geometric Brownian motion fails to account for certain empirical features of markets. Despite this, it remains the most popular model among practitioners for valuing options. As such, we assume the price of a risky asset $S$ at time $t$ is given by,

$$S_t = S_0 e^{X_t} \quad (1.3)$$

where $X_t$ is a Brownian motion with drift.

\textsuperscript{14}“For a new method to determine the value of derivatives”, Scholes and Merton were awarded the 1997 Nobel prize in Economics (Black passed away two years prior to the announcement).

\textsuperscript{15}The concept itself dates back to \[1\] and \[2\]. A martingale is a stochastic process whose expectation of future states is equivalent to the current state.

\textsuperscript{16}For a thorough treatment of the martingale approach, see \[26, 28, 72\].
1.6 Stochastic Processes

Definition 1 ([62] Measurable Space) Let $\Omega$ be a set and $2^\Omega$ denote its power set. Let $\mathcal{F}$ be a collection of subsets such that $\mathcal{F} \subset 2^\Omega$. The pair $(\Omega, \mathcal{F})$ is called a measurable space if the following three properties are satisfied:

(i) $\mathcal{F}$ is non-empty: At least one $A \subset \Omega / \emptyset$ is in $\mathcal{F}$.

(ii) $\mathcal{F}$ is closed under complementation: If $A \in \mathcal{F}$, so is $A^C$.

(iii) $\mathcal{F}$ is closed under countable unions: For any countable $A_i \in \mathcal{F}$, \( \bigcup_{i} A_i \in \mathcal{F} \).

Theorem 1 ([8] Fubini’s Theorem) Suppose $X$ and $Y$ are complete measurable spaces. Suppose $f(x,y)$ is $X \times Y$ measurable. If

\[
\int_{X \times Y} |f(x,y)| d(x,y) < \infty,
\]

where the integral is taken with respect to a product measure on the space over $X \times Y$, then

\[
\int_X \left( \int_Y f(x,y) dy \right) dx = \int_Y \left( \int_X f(x,y) dx \right) dy = \int_{X \times Y} f(x,y) d(x,y).
\]

In order to quantify sets of outcomes $A_i \in \mathcal{F}$, we establish a probability measure $P$ on $\mathcal{F}$.

Definition 2 ([62] Probability Measure) A probability measure $P$ is a real-valued function $2^\Omega \to [0,1]$ such that,

(i) $P(\emptyset) = 0$ and $P(\Omega) = 1$

(ii) $P(\bigcup_{i \in I} A_i) = \sum_{i \in I} P(A_i)$ where $A_i$ is a collection of pairwise disjoint sets

---

\(^{17}\)Every subset of every null set is measurable.
The triplet \((\Omega, \mathcal{F}, \mathbb{P})\) is called a probability space, where \(\Omega\) is a sample space (set of all possible outcomes), \(\mathcal{F}\) a set of events (containing zero or more outcomes), and \(\mathbb{P}\) is the measure that assigns probabilities to each event \([87]\). Let the probability space exhibit an underlying filtration of sub \(\sigma\)-algebras \(\mathcal{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}\) with \(\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}\) for \(s \leq t\). As usual, we assume \(\mathcal{F}\) satisfies the condition of right-continuity. \[18\] Under this framework, the filtration represents an increasing set of observables that becomes known to market participants as time progresses (e.g. tick-by-tick volatility, daily spot prices, intraday volume). To simulate the evolution of realized events over time, a random process is introduced. Under a random process one assumes events occur unpredictably, and can not be uniquely determined by a prescribed set of initial conditions.

**Definition 3 ([87] Stochastic Process)** A stochastic process \(X\) on \((\Omega, \mathcal{F}, \mathbb{P})\) is a collection of random variables \((X_t)_{0 \leq t < \infty}\).

If \(X_t \in \mathcal{F}_t\), the process \(X\) is said to be adapted to the filtration \(\mathcal{F}\), or equivalently, \(\mathcal{F}_t\)-measurable \([87]\). Stochastic processes have been studied extensively in economic, financial, actuarial, physical, biological, and chemical applications \([46, 47, 56, 62, 87]\). In finance, their primary function is to represent the price of an asset. Arguably, the most well-known stochastic process is a Wiener process which describes the movement of particles suspended in a fluid \([32, 56]\).

**Definition 4 ([87] Wiener Process)** A Wiener process \(W = (W_t)_{0 \leq t < \infty}\) is a stochastic process which satisfies the following three properties:

\[18\] No jump discontinuity occurs while approaching the limit from the right.
(i) \( \mathbb{W}_0 = 0 \)

(ii) \( W \) has independent increments: \( W_t - W_s \) is independent of \( \mathcal{F}_s \), \( 0 \leq s < t < \infty \)

(iii) \( W_t - W_s \) is a normal random variable: \( W_t - W_s \sim \mathcal{N}(0, t-s) \) \( \forall \ 0 \leq s < t < \infty \)

Property (ii) implies the Markov property (i.e. conditional probability distribution of future states depend only on the present state). Property (iii) indicates that knowing the distribution of \( W_t \) for \( t \leq \tau \) provides no predictive information about the process when \( t > \tau \). For property (iii), the density of a random variable from \( \mathcal{N}(\mu, \sigma^2) \) is given by,

\[
\eta(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{-(x-\mu)^2/2\sigma^2}. \quad (1.6)
\]

To model relationships involving a random time or state component, one can construct partial differential equations (PDEs) whose path structure is driven by a stochastic process. In order to solve these so-called stochastic differential equations, an alternative formulation of calculus is required to properly define probabilities of stochastic terms. Geometric Brownian motion can be constructed from definition 4 (see 1.8). This has a characteristic exponent of the form,

\[
\Psi(u) = \frac{1}{2}\sigma^2u^2 - iu\mu. \quad (1.7)
\]

for \( \mu \in \mathbb{R} \). One can simulate the asset over time by use of definition 4 and equation (1.3). For a daily timestep over 1000 days, figure 1.6 provides sample paths of prices which follow geometric Brownian motion.

\[\text{For an asset with functional form (1.3), log-returns are normally distributed.}\]

\[\text{For } \Psi(u) := -\frac{1}{t}\ln\mathbb{E}[e^{uX_t}]\]
Figure 1.2: Sample paths of an asset given by geometric Brownian motion with $\mu = 0.5$, $\sigma = 0.5$, $S_0 = 100$, and $r, q = 0$.

1.7 Itô’s Calculus

Consider the price process of an asset $X_t$ on $(\Omega, \mathcal{F}, \mathbb{P})$ given by,

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$  \hspace{1cm} (1.8)

where $\mu$ is the drift term and $\sigma > 0$ is the volatility. This stochastic differential equation (one-dimensional geometric Brownian motion) is an Itô drift-diffusion process given by the differential \[56\] \footnote{With the condition that $\int_0^t (\mu^2(X_s, s) + |\sigma(X_s, s)|)ds < \infty$.}.

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$  \hspace{1cm} (1.9)
or in integral form as,

$$X_{t+s} - X_t = \int_t^{t+s} \mu(X_u, u)du + \int_t^{t+s} \sigma(X_u, u)dW_u. \quad (1.10)$$

Since (1.8) can be represented as a sum of a Lebesgue and Itô integral, Itô’s lemma provides its solution.  

**Lemma 1 ([87] Itô’s lemma)** Let \( f(x, t) \in \mathbb{R}^2 \) be twice differentiable in \( x \) and once in \( t \). Then (1.9) becomes,

$$df(X_t, t) = \left( \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma \frac{\partial f}{\partial x} dW_t \quad (1.11)$$

in \( \mathbb{P} \), almost surely.  

For \( t \in [0, T] \) one-dimension Brownian motion becomes,

$$d \ln(X) = d \ln \left( \frac{X_t}{X_0} \right) = \sigma dW_t + (\mu - \frac{1}{2} \sigma^2) dt \quad (1.12)$$

and hence the solution of (1.8) is,

$$X_t = X_0 e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma W_t}. \quad (1.13)$$

The above lemma is crucial for solving stochastic differential equations in 1-dimensional space and time. Arguably the best known application of Itô’s lemma is for obtaining the solution to the Black-Scholes-Merton equation [7].

---

22Alternate forms of this theorem can be stated when the function is driven by a Lévy process or more general semimartingale of arbitrary dimension.

23Note the form of (1.11); the solution of an Itô drift-diffusion process is an Itô drift-diffusion process.
1.8 Black-Scholes-Merton Equation

Given the asset price \([1.3]\) and imposing general conditions on some function \(V(S, t)\), a PDE representing the option price can be obtained. Otherwise known as the Black-Scholes-Merton equation, it provides the price of European options when the appropriate boundary conditions are imposed. For GBM represented by \([1.8]\), a continuous dividend rate \(q\) is included in the model by setting \(\mu = r - q\). To derive the PDE, one may apply Itô’s lemma to a function \(V(S, t)\) representing the option value:

\[
dV = \frac{\partial V}{\partial S} dS + \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt \tag{1.14}
\]

\[
= \left( \frac{\partial V}{\partial t} + (r - q) S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t. \tag{1.15}
\]

By constructing a self-financing portfolio \(\Pi = V - S \frac{\partial V}{\partial S}\) consisting of an option \(V\) and stock \(S\),

\[
d\Pi = dV - S \frac{\partial V}{\partial S} = \left( \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - q S \frac{\partial V}{\partial S} \right) dt. \tag{1.16}
\]

Under the no-arbitrage condition, the portfolio must earn a risk-free rate of return such that \(d\Pi = r \Pi dt\). Hence,

\[
d\Pi = r \left( V - S \frac{\partial V}{\partial S} \right) dt. \tag{1.17}
\]

By combining (1.16) and (1.17) the Black-Scholes-Merton equation is obtained:

\[
\frac{\partial V}{\partial t} + (r - q) S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{1.18}
\]

In the following sections we derive European call and put option prices by the usual method, transforming the problem into a diffusion equation. For American options we omit the derivation.
Chapter 2

Analytic Option Pricing

Derivatives in and of themselves are not evil. There’s nothing evil about how they’re traded, how they’re accounted for, and how they’re financed, like any other financial instrument, if done properly. —James Chanos

2.1 European Call Option

By imposing the final time condition \( V(S,T) = (S - K)^+ \) and \( V(S,t) \to Se^{-q(T-t)} - Ke^{-r(T-t)} \) as \( S \to \infty, \) the solution of the PDE (1.18) gives the value of a European call option. To solve (1.18) set \( S = Ke^x, \) \( t = T - \frac{2\tau}{\sigma^2}, \) and \( V = KU(x,\tau). \) Then,

\[
\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} + (k_1 - 1) \frac{\partial U}{\partial x} - k_1 U
\]

(2.1)

where \( k_1 = \frac{2(r-q)}{\sigma^2} \) and \( U(x,0) = (e^x - 1,0)^+. \) With a change of variables \( U = e^{\alpha x + \beta \tau}C(x,\tau) \) (2.1) becomes,

\[
\beta C + \frac{\partial C}{\partial \tau} = \alpha^2 C + 2\alpha \frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2} + (k_1 - 1)(\alpha + \frac{\partial U}{\partial x}) - k_1 C.
\]

(2.2)

Choosing \( \beta = \alpha^2 + (k_1 - 1)\alpha - k_1 \) and \( 0 = 2\alpha + (k_1 - 1) \) yields \( \alpha = \frac{-1}{2}(k_1 - 1) \) and \( \beta = \frac{-1}{4}(k_1 + 1)^2. \) Furthermore, \( U = \exp[-\frac{1}{2}(k_1 - 1)x - \frac{1}{4}(k_1 + 1)^2\tau]\) \( C(x,\tau) \) and thus,

\[
\frac{\partial C}{\partial \tau} = \frac{\partial^2 C}{\partial x^2}
\]

(2.3)
where $-\infty < x < \infty$ and $\tau > 0$. The payoff function becomes

$$C(x, 0) = C_0(x) = \left(\exp\left[\frac{1}{2}(k_1 + 1)x\right] - \exp\left[\frac{1}{2}(k_1 - 1)x\right], 0\right)^+.$$ 

The solution to the diffusion equation is known. Hence,

$$C(x, \tau) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\infty} C_0(s) \exp\left(-\frac{(x - s)^2}{4\tau}\right) ds.$$ \hspace{1cm} (2.4)

To evaluate the integral above, make the change of variable $x' = \frac{x + s}{\sqrt{2\pi}}$ to obtain

$$C(x, \tau) = I_1 - I_2$$

where,

$$I_1 = \exp\left[-\frac{1}{2}(k_1 + 1)x + \frac{1}{4}(k_1 + 1)^2\tau\right] \Phi(d_1)$$ \hspace{1cm} (2.5)

$$I_2 = \exp\left[-\frac{1}{2}(k_1 - 1)x + \frac{1}{4}(k_1 - 1)^2\tau\right] \Phi(d_2).$$ \hspace{1cm} (2.6)

Above, $d_1 = \frac{1}{\sqrt{2\pi}} + \frac{1}{2}(k_1 + 1)\sqrt{2\pi}$, $d_2 = \frac{1}{\sqrt{2\pi}} + \frac{1}{2}(k_1 - 1)\sqrt{2\pi}$, and $\Phi$ is the cumulative distribution function of the normal distribution given by,

$$\Phi(d_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i} e^{-\frac{s^2}{2}} ds.$$ \hspace{1cm} (2.7)

Reverting back to $V$ and the original variables $S$ and $t$ yields the solution to the value of a European call option:

$$V_C^E(S, t) = S e^{-\phi(T-t)} \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2)$$ \hspace{1cm} (2.8)

with

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}$$ \hspace{1cm} (2.9)

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + (r - q - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} = d_1 - \sigma\sqrt{T - t}.$$ \hspace{1cm} (2.10)

---

Herein, subscripts $C$ and $P$ denote call and put, while $E$ and $A$ denote European and American, respectively.
2.2 European Put Option

Conversely, the final time condition for a put option is given by $V(S,T) = (K - S)^+$. The boundary conditions are $V(S,t) \to 0$ as $S \to \infty$ and $V(0,t) \to Ke^{-r(T-t)}$. Similar to the call option, the put can be reduced to diffusion equation (2.3) through the same change of variables $S = Ke^x$, $t = T - \frac{2\tau}{\sigma^2}$, $V = KU(x,\tau)$, and $U = e^{\alpha x + \beta \tau}P(x,\tau)$. The payoff function becomes $P(x,0) = P_0(x) = (\exp[\frac{1}{2}(k_1 - 1)x] - \exp[\frac{1}{2}(k_1 + 1)x],0)^+$. Again, make the change of variable $x' = \frac{x}{\sqrt{2\pi}}$ to obtain $P(x,\tau) = I_1 - I_2$ where,

\begin{align}
I_1 &= \exp\left[\frac{1}{2}(k_1 - 1)x + \frac{1}{4}(k_1 - 1)^2\tau\right]\Phi(-d_2) \\
I_2 &= \exp\left[\frac{1}{2}(k_1 + 1)x + \frac{1}{4}(k_1 + 1)^2\tau\right]\Phi(-d_1)
\end{align}
with \( d_1 = \frac{1}{\sqrt{2\pi}} + \frac{1}{2}(k_1+1)\sqrt{2\pi} \) and \( d_2 = \frac{1}{\sqrt{2\pi}} + \frac{1}{2}(k_1-1)\sqrt{2\pi} \). Returning to the original variables yields the solution to the value of a European put option:

\[
V_P^E(S, t) = Ke^{-r(T-t)}\Phi(-d_2) - Se^{-q(T-t)}\Phi(-d_1)
\]

(2.13) with \( d_1 \) and \( d_2 \) given by (2.9) and (2.10), respectively. Due to their closed form expressions, both the European call and put options can be computed numerically fast with high precision, and hence provide a sufficient benchmark for comparison.

**Remark 1** An application of put-call parity gives the price of a European put from a call (and vice-versa): \( V_C^E - V_P^E = Se^{-q(T-t)} - Ke^{-r(T-t)} \).

![European put option price](image)

Figure 2.2: The European put option price for exercise prices \( K \in [80, 120] \) of the asset price \( S = 100 \) with \( \sigma = 0.5, r = 0.05, q = 0.05, \) and \( T = 30/365 \).
2.3 American Call Option

Recall that American options differ from European in that they can be exercised at any time $t < T < \infty$. For a call option, one can solve the inhomogeneous Black-Scholes-Merton equation given by,

$$
\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = f
$$

where $0 < S < \infty$, $0 \leq t \leq T$, and the early exercise function,

$$
f(S, t) = \begin{cases} 
  rK - qS; & S^*(t) \leq S \leq \infty \\
  0; & 0 \leq S \leq S^*.
\end{cases}
$$

The final time condition is given by $V(S, T) = (S - K)^+$ and the critical asset price $S^*(t)$ is determined by the smooth pasting conditions,

$$
V(S^*, t) = S^* - K \quad \text{and} \quad \frac{\partial V}{\partial S}\bigg|_{S=S^*} = 1.
$$

Furthermore, we have the boundary conditions $V(S, t) \to \infty$ as $S \to \infty$ and $V(0, t) = 0$. There are numerous different formulations of the American option problem, and hence multiple different yet equivalent solutions. The integral solution in [52, 58] takes the form,

$$
V^A_C(S, \tau) = V^E_C(S, t) + \int_0^\tau qSe^{-q(\tau-s)}\Phi(d_1(S, \tau - s; S^*(s)))ds - \int_0^\tau rKe^{-r(\tau-s)}\Phi(d_2(S, \tau - s; S^*(s)))ds
$$

where $\tau = T - t$. Equations $d_i(S^*(\tau), \tau - s; S^*(s))$ are inherited from the European case with $S := S^*(\tau)$, $t := \tau - s$, and $K := S^*(s)$. The expression for an American
call option derived in [13] is,

\[ V_C^A(S, \tau) = (S - K)^+ + \frac{S\sigma^2}{2} \int_{0}^{\tau} e^{-q(\tau-s)} \Phi'(d_1(S, \tau - s; K)) \, ds \]

\[ + \int_{0}^{\tau} qSe^{-r(\tau-s)}[\Phi(d_1(S, \tau - s; S^*(s))) - \Phi(d_1(S, \tau - s; K))] \, ds \]

\[ - \int_{0}^{\tau} rKe^{-r(\tau-s)}[\Phi(d_2(S, \tau - s; S^*(s))) - \Phi(d_2(S, \tau - s; K))] \, ds \]  \quad (2.18)

where \( \Phi'(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \).

### 2.4 American Put Option

Similarly for a put option, the price is governed by the inhomogeneous Black-Scholes-Merton equation,

\[ \frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = f \]  \quad (2.19)

where \( 0 < S < \infty, 0 \leq t \leq T \), and

\[ f(S, t) = \begin{cases} 
-rK + qS; & 0 \leq S \leq S^* \\
0; & S^*(t) \leq S \leq \infty.
\end{cases} \]  \quad (2.20)

The final time condition is given by \( V(S, T) = (K - S)^+ \) and the free boundary \( S^*(t) \) is determined by the smooth pasting conditions,

\[ V(S^*, t) = K - S^* \quad \text{and} \quad \frac{\partial V}{\partial S} \bigg|_{S=S^*} = -1. \]  \quad (2.21)
Furthermore, we have the boundary conditions $V(S,t) \to 0$ as $S \to \infty$ and $V(0,t) = Ke^{-r(T-t)}$. The solution in \[52,58\] takes the form,

$$V_P^A(S,\tau) = V_P^E(S,t) - \int_0^\tau qSe^{-q(\tau-s)}\Phi(-d_1(S,\tau-s;S^*(s)))ds$$

$$+ \int_0^\tau rKe^{-r(\tau-s)}\Phi(-d_2(S,\tau-s;S^*(s)))ds. \quad (2.22)$$

The expression derived in \[13\] is given by,

$$V_P^A(S,\tau) = (K - S)^+ + \frac{S\sigma^2}{2} \int_0^\tau \frac{e^{-q(\tau-s)}}{\sigma \sqrt{\tau-s}}\Phi'(-d_1(S,\tau-s);K)ds$$

$$- \int_0^\tau qSe^{-r(\tau-s)}[\Phi(-d_1(S,\tau-s;S^*(s))) - \Phi(-d_1(S,\tau-s;K))]ds$$

$$+ \int_0^\tau rKe^{-r(\tau-s)}[\Phi(-d_2(S,\tau-s;S^*(s))) - \Phi(-d_2(S,\tau-s;K))]ds. \quad (2.23)$$

**Remark 2** An application of put-call symmetry gives the price of an American put from a call (and vice-versa): $V_C^A(S,K,r,q,t) = V_P^A(K,S,q,r,t)$. 
Chapter 3

The Mellin Transform

*The shortest path between two truths in the real domain passes through the complex domain.* - Jacques Hadamard

3.1 Motivation

Contrary to the role that integral transforms typically embody in the study of physical problems, Mellin transforms arose from a purely mathematical context. Without concern for a rigorous formalism of the transform, Riemann was the first to implement it in a memoir regarding the zeta function \[101\]. As the title implies, an extensive characterization and systematic treatment of the transform (and its inverse) was due to R.H. Mellin \[64\]. His primary concern was with the theory of special functions; particularly asymptotic expansions and solutions to hypergeometric differential equations. Since its initial formulation, the transform has found use primarily in the areas of digital data structures, probabilistic algorithms, communication/signal theory, and complex analysis \[21, 38, 39, 77, 105\]. Although its properties on functions are well developed, in relation to solving ordinary and partial differential equations it has

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1Robert Hjalmar Mellin (1854-1933) was a Finnish function-theorist who studied under Gösta Mittag-Leffler, Karl Weierstrass, and Leopold Kronecker. He was critically opposed to the theory of relativity, which he attempted to disprove through an assemblage of (philosophical) publications.
drawn much less attention than the Fourier and Laplace transforms \cite{9, 35, 79, 98}. Although options can be solved using these transforms, we may phrase the PDE in terms of market prices if we use the Mellin transform. For the non-technical trader, pricing options may be more accessible since their value can be determined directly from quotes on the exchange. The Mellin transform maps functions existing on some domain \(D \subseteq \mathbb{R}^+\) to \(\mathbb{C}\). For financial purposes this feature is particularly convenient, as most functions exhibit non-negativity. Furthermore, the transform enables option equations to be solved directly in terms of market prices rather than log-prices, providing a more natural setting to the problem of pricing. Despite this, the Mellin transform’s ascension into the realm of financial mathematics is only about a decade old \cite{20}.

### 3.2 Definition

**Definition 5 (\cite{38} Mellin Transform)** For a function \(f(x) \in \mathbb{R}^+\) for \(x \in (0, \infty)\), the Mellin transform is defined as the complex function,

\[
\mathcal{M}\{f(x); w\} := \hat{f}(w) = \int_0^\infty f(x)x^{w-1}dx.
\]

(3.1)

However, the Mellin transform of a function does not always exist.

**Lemma 2 (\cite{38} Existence)** Let \(f(x)\) be a continuous function such that,

\[
f(x) = \begin{cases} 
O(x^{a_1}), & x \to 0 \\
O(x^{a_2}), & x \to \infty
\end{cases}
\]

(3.2)

Then the Mellin transform \(\hat{f}(w)\) exists for any \(w \in \mathbb{C}\) on \(-a_1 < \Re(w) < -a_2\).
This interval, known as the **fundamental strip** and often denoted by $(-a_1, -a_2)$ is the domain of analyticity for $\hat{f}(w)$. To show this we consider the absolute bound of $\hat{f}(x)$,

$$
\left| \int_0^\infty f(x)x^{w-1}dx \right| \leq \int_0^1 |f(x)|x^{\Re(w)-1}dx + \int_1^\infty |f(x)|x^{\Re(w)-1}dx \quad (3.3)
$$

$$
\leq c_1 \int_0^1 x^{\Re(w)+a_1-1}dx + c_2 \int_1^\infty x^{\Re(w)+a_2-1}dx \quad (3.4)
$$

where $c_1, c_2 \in \mathbb{R}^+ \cup \{0\}$. Since the first integral in (3.4) converges for $\Re(w) > -a_1$ and the second integral converges for $\Re(w) < -a_2$, it follows that $\hat{f}(w)$ exists on $(-a_1, -a_2)$. Consider instead the inverse scenario; the Mellin transform of a function is known, and one wishes to recover the original function. For a function $\hat{f} : \mathbb{C} \to \mathbb{C}$ it can be shown under general conditions that an inverse $f(x) \in \mathbb{R}^+$ not only exists, but is also unique (for a given fundamental strip).

**Theorem 2 ([38] Mellin Inversion Theorem)** Suppose $f(x) \in \mathbb{R}^+$ is continuous and $\hat{f}(w) \in \mathbb{C}$ is analytic on $a < \Re(w) < b$. Then $f$ is recoverable via,

$$
f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{f}(w)x^{-w}dw \quad (3.5)
$$

where $c \in \Re(w)$.

**Remark 3** Furthermore, if $\hat{f}(w)$ satisfies the inequality $|\hat{f}(w)| \leq K|w|^{-2}$ for some constant $K$ then $f(x)$ is continuous on $x \in (0, \infty)$.

**Remark 4** If $f$ has a jump discontinuity at $x$, the function is equal to half of its limit values: $\frac{1}{2}[f(x^-) + f(x^+)]$. In this case, $\hat{f}$ must be absolutely convergent on $\Re(w) = c$ and of bounded variation in a neighbourhood of $x$. 
A famous example of (3.5) follows from considering $\Gamma(w)$ with real $c > 0$. By use of Stirling’s formula,

$$e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(w)x^{-w}dw.$$  \hfill (3.6)

Practical inversion can sometimes pose a challenge due to the complex nature of the integral. When possible, this is often achieved by direct contour integration, conversion to polar coordinates, recasting the problem as a product of gamma functions, exploiting properties of the transform in conjunction with the inversion theorem, or by use of previously solved tables of transforms [74, 85].

It should be noted that the Mellin transform bears a striking resemblance to other popular transforms, particularly Fourier and Laplace. Both can be obtained through a change of variables. By letting $x = e^{-t}$ and hence $dx = -e^{-t}dt$ the transform (3.1) becomes,

$$\mathcal{M}\{f(x); w\} = \int_{-\infty}^{\infty} f(e^{-t})e^{-wt}dt = L\{f(e^{-t}); w\} \hfill (3.7)$$

where $L\{\cdot\}$ denotes the two-sided Laplace transform. Similarly, let $a, b \in \mathbb{R}$ and set $w = a + 2\pi ib$ in (3.7). Then,

$$\mathcal{M}\{f(x); w\} = \int_{-\infty}^{\infty} f(e^{-t})e^{-at}e^{-2\pi ibt}dt = F\{f(e^{-t})e^{-at}; b\} \hfill (3.8)$$

where $F\{\cdot\}$ denotes the Fourier transform.

\footnote{$|\Gamma(a + ib)| \sim \sqrt{2\pi}|b|^{n-1/2}e^{-|b|\pi/2}$ when $|b| \to \infty$. See [85] for a derivation.}
3.3 Properties

The Mellin transform has the ability to reduce complicated functions by realization of its many properties. This subsection is dedicated to providing examples and general forms for Mellin-transformed functions. Arguably the most well-known property of the Mellin transform is how it manages functions with scaled domains. Suppose for some \( \alpha \in \mathbb{C} \) we wish to determine the transform of \( f(\alpha x) \). By the change of variable \( u = \alpha x \), it follows immediately that,

\[
\mathcal{M}\{f(\alpha x); w\} = \alpha^{-w} \int_0^\infty f(u) u^{w-1} du = \alpha^{-w} \hat{f}(w) \tag{3.9}
\]

Suppose instead that we multiply \( f(x) \) by \( x^\alpha \). It is clear that this operation is no different than changing \( w \) to \( w + \alpha \). Hence,

\[
\mathcal{M}\{x^\alpha f(x); w\} = \hat{f}(w + \alpha). \tag{3.10}
\]

Exponentiating the domain of \( f(x) \) by \( \alpha \) and solving for (3.1) requires a substitution of \( t = x^\alpha \). This yields,

\[
\mathcal{M}\{f(x^\alpha); w\} = \int_0^\infty f(t) t^{\frac{1-\alpha}{\alpha}} (t^{\frac{w-1}{\alpha}} dt) = \alpha^{-1} \int_0^\infty f(t) t^{\frac{w-1}{\alpha}} dt = \alpha^{-1} \hat{f}\left(\frac{w}{\alpha}\right) \tag{3.11}
\]

where \( \alpha \geq 0 \) is required for \( \hat{f}\left(\frac{w}{\alpha}\right) \) to be analytic. By a similar method to (3.10) and (3.11) we obtain the relation,

\[
\mathcal{M}\{x^{-1} f(x^{-1}); w\} = \hat{f}(1 - w). \tag{3.12}
\]

As we shall see in chapter 4, the ability to find alternate expressions for derivatives and integrals of functions allow us to reduce certain PDEs to linear combinations of
Mellin-transformed terms. It is helpful that derivatives of functions and derivatives of transformed functions have nice representations. The former case, given by (3.13), requires the fundamental theorem of calculus and induction to prove existence for all \( k \in \mathbb{N} \).

\[
\mathcal{M}\left\{ \frac{d^k}{dx^k} f(x); w \right\} = (-1)^k \frac{\Gamma(w)}{\Gamma(w-k)} \hat{f}(w-k) = (-1)^k (w)_k \hat{f}(w-k) \quad (3.13)
\]

The latter case, given by (3.14), is also derived inductively with use of the well-known formula \( \frac{d}{dx} x^{w-1} = \ln(x)x^{w-1} \).

\[
\mathcal{M}\{(\ln(x))^k f(x); w\} = \frac{d^k}{dw^k} \hat{f}(w) \quad (3.14)
\]

Suppose the Mellin transform of our function in (3.13) has a coefficient of \( x^k \). It follows that,

\[
\mathcal{M}\left\{ x^k \frac{d^k}{dx^k} f(x); w \right\} = (-1)^k \frac{\Gamma(w+k)}{\Gamma(w)} \hat{f}(w) = (-1)^k (w^{(k)}) \hat{f}(w) \quad (3.15)
\]

\[
= \mathcal{M}\left\{ \frac{d^k}{dx^k} f(x); w+k \right\}. \quad (3.16)
\]

Let \( r > 0 \) and suppose that a function \( f(r) \in \mathbb{R}^+ \) can be analytically continued into a function \( f(z) \in \mathbb{C} \) where \( z := re^{i\theta} \) for some \( |\theta| < M \). Applying (3.9) for complex coefficient \( e^{i\theta} \) yields,

\[
\mathcal{M}\{f(re^{i\theta})\} = e^{-i\theta w} \hat{f}(w). \quad (3.17)
\]

For the Mellin transform on a product of two univariate functions, a convolution formula exists.

---

3. The expression for the falling factorial \( \frac{\Gamma(w)}{\Gamma(w-k)} \) is often denoted by the Pochhammer symbol \((w)_k\).

4. The rising factorial \( \frac{\Gamma(w+k)}{\Gamma(w)} \) is typically denoted by \( w^{(k)} \).
**Theorem 3 (Convolution Theorem)** Let $f(x)$ and $g(x)$ satisfy lemma and define,

$$ (f * g)(x) = \int_0^\infty f(y)g\left(\frac{x}{y}\right) \frac{dy}{y}. $$

(3.18)

Then,

$$ M\{(f * g)(x)\} = M\{f(x)\}M\{g(x)\}. $$

(3.19)

Additionally, a Parseval-type formula can be defined in the following sense.

**Theorem 4 (Parseval-Mellin Formula)** Let $f(x)$ and $g(x)$ satisfy lemma.

If at least one of $\hat{f}(w), \hat{g}(w) \in L_1(\mathbb{R})$, then for $y > 0$

$$ \int_0^\infty f(x)g(yx)dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s}M\{f(x); 1-w\}M\{g(x); w\} dw. $$

(3.20)

If $\Re(w) = \frac{1}{2}$ then,

$$ \int_0^\infty f(x)^2dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |M\{f(x); \frac{1}{2} + ib\}|^2 db. $$

(3.21)

The properties presented here are merely a preview of the transform’s applicability on a function of one variable (see [37], [85], [97], [98], and [104] for a detailed approach).

### 3.4 Multidimensional Mellin Transform

A natural extension of the univariate transform exists for higher dimensions. The double Mellin transform was first introduced by Reed in 1944; he proved conditions for which the transform (and its inverse) exists [90].

\footnote{See [22], [35], and [73] for the double integral case.} In relation to option pricing,
it has been used to solve for the price of European and American basket options on two assets in [80], [82], and [102]. For the arbitrary multidimensional case, many distributional and functional properties are known [8, 57, 98]. Beyond two dimensions its use has been ignored in the derivatives pricing literature.

**Definition 6 ([8] Multidimensional Mellin Transform)** Let $x = (x_1, \ldots, x_n)'$ and $w = (w_1, \ldots, w_n)'$. For a function $f(x) \in \mathbb{R}^n_+$ the multidimensional Mellin transform is the complex function,

$$
\mathcal{M}\{f(x); w\} := \hat{f}(w) = \int_{\mathbb{R}^n_+} f(x)x^{-w}dx.
$$

Existence in the multidimensional case extends naturally from lemma [2]. Analogous to Fourier and Laplace, an inversion theorem in the multidimensional case holds under suitable conditions [8].

**Theorem 5 ([8] Multidimensional Mellin Inversion Theorem)** Let $w = (w_1, \ldots, w_n)'$, $x = (x_1, \ldots, x_n)'$, and $\hat{f}(w) \in \mathbb{C}^n$ be analytic on $\gamma = \times_{j=1}^n \gamma_j$ where $\gamma_j$ are strips in $\mathbb{C}^n$ defined by $\gamma_j = \{a_j + ib_j : a_j \in \mathbb{R}, b_j = \pm \infty\}$ with $a_j \in \Re(w_j)$. Suppose $f(x) \in \mathbb{R}^n_+$ is a continuous function. Then,

$$
f(x) = (2\pi i)^{-n} \int_\gamma \hat{f}(w)x^{-w}dw.
$$

If $f(x) = \prod_{j=1}^n f_j(x_j)$, then from theorem [1] it follows that $\hat{f}(x) = \prod_{j=1}^n \mathcal{M}\{f_j(x_j); w_j\}$. In this case, properties of the univariate Mellin transform can be used to obtain solutions of multidimensional Mellin transforms.

$\int x^{-w}dx$ is treated as $\int_1^\infty x^{-w}dx$. 

---

\[ \int \]
Lemma 3 Let $f(x) \in \mathbb{R}^{n+}$ be twice differentiable w.r.t $x_i$ and $x_j$. If $\prod_{i=1}^{n} x_i^{w_i} f(x)$ vanishes as $x_i \rightarrow 0$ and $x_i \rightarrow \infty$ then,

$$
\mathcal{M}\left\{ x_i x_j \frac{d^2}{dx_i dx_j} f(x); w \right\} = w_i (w_i - 1) \hat{f}(w) \quad i = j \tag{3.24}
$$

$$
\mathcal{M}\left\{ x_i x_j \frac{d^2}{dx_i dx_j} f(x); w \right\} = w_i w_j \hat{f}(w) \quad i \neq j. \tag{3.25}
$$

Proof: See Appendix.
Chapter 4

Analytic Option Pricing using the Mellin Transform

Despite the immense interest for using transforms to compute the value of options, the Mellin transform has received little attention. This may partly be because the PDE for pricing is often formulated in terms of log-prices. By far, the Fourier transform dominates the pricing literature. Although the introduction of the Mellin transform to derivatives pricing is relatively new, some progress has been made; the subsequent section reviews these results.

4.1 Previous Results

In 2002, Cruz-Báez and González-Rodríguez pioneer the method of using the Mellin transform to solve the associated PDE for a European call option [20]. In 2004, Panini and Srivastav use the Mellin transform method to solve for the European put, American put, European basket put \((n = 2\) underlying assets), and American basket put \((n = 2)\) [82]. In a separate publication, using Carr and Faguet’s result that the limit of a finite-lived option is perpetual, they obtain an expression for the

\[^{1}\text{In Carr’s presentation Option Pricing using Integral Transforms he lists over 75 references.}\]
free boundary and price of a perpetual American put option \[36, 81\]. In the same year Panini published his PhD dissertation, including the above results and adding the solution for perpetual basket put options \( (n = 2) \) \[80\]. For all of these cases, volatility was assumed constant and dividends were omitted. In 2005, Jódar, Sevilla-Peris, Cortés, and Sala directly solve the Black-Scholes equation to obtain the price of a European option \[59\]. No change of variables in the Black-Scholes equation is required, and hence the problem doesn’t involve solving a diffusion equation. \(^3\)

Cruz-Báez and González-Rodríguez extend their previous results and solve for the European call option where the underlying asset has a continuous dividend \[19\]. In 2006, Company, González, and Jódar solve for the discrete dividend case, providing numerical results using Gauss-Hermite quadrature \[88\]. Company, Jódar, Rubio, and Villanueva consider the case where the payoff is a generalized function, providing well-known solutions as particular cases (such as the Merton model) \[16\]. In 2007, Rodrigo and Mamon let the interest rate and dividend function be time dependent, and after solving via the Mellin convolution theorem, obtain a solution where the integration is over \( \mathbb{R}^+ \) rather than \( \mathbb{C} \) \[92\]. They also prove existence and uniqueness of the classical solution for two cases: \((i)\) when the payoff is bounded/continuous and \((ii)\) when the difference between the payoff and an arbitrary polynomial is bounded/-continuous. In his 2008 Masters thesis, Shen reproduces the results of \[88\] and adds a numerical scheme involving the composite Simpson’s rule \[90\]. Frontczak and Schöbel generalize the work of Panini and Srivastav by solving for European power options \(\text{Perpetual implies } T \to \infty.\) \(^2\)

\(^2\)Perpetual implies \( T \to \infty.\)

\(^3\)All of the authors mentioned above were seemingly unaware of each other’s results prior to publication.
and perpetual American put options with constant dividend [43]. In her 2009 Masters thesis, Vasilieva reproduces the results of [82] and introduces Newton’s method to calculate the American basket put option \( (n = 1, 2) \) [102]. By using the modified trapezoidal rule along with a truncation of the excess integration region, both computational error and CPU time is reduced relative to [82]. Using the discounted expectation approach, Dufresne, Garrido, and Morales provide formulas for \( \mathbb{E}_Q[\theta] \) with payoff \( \theta \) when options are European calls and puts [29]. In 2010, Frontczak and Schöbel deduce a new integral formulation for the price of a European and American call option (and its free boundary) with constant dividend. They provide numerical results using Gauss-Laguerre quadrature and compare the resultant prices among nine pre-existing numerical procedures [45]. In 2011, Frontczak obtains the solution for the Heston model with constant dividend; the first stochastic volatility model to be solved via the Mellin transform [40]. Similar to Carr’s formulation, the price of a European put only requires a single integration. A minor improvement is made on the European option model of [92] by Feng-lin who adds a transaction cost to the process [63]. Exotic options have rarely been considered using the Mellin approach. Elshegmani and Ahmed solve the PDE of the Rogers-Shi Asian option without dividend in [33]. They are able to circumvent existing numerical dimension reduction procedures by obtaining an analytic solution (both the Mellin and Fourier transform are used).

There are currently five working papers which utilize the Mellin transform as

\footnote{Stochastic volatility implies \( \sigma \) is driven by a separate stochastic process.}

\footnote{In 2007, Dewynne and Shaw provide a form of the Geman-Yor Asian option in an appendix to [24].}

\footnote{Published again with Zakaria in [34].}
part of a solution method to obtain prices for derivative securities: two by Frontczak, one by Kamdem, one by Shaw, and one by Chandra and Mukherjee. 7 Frontczak considers the solution for a general jump diffusion model and produces the explicit formula for the Merton model with constant dividend [42]. Additionally, he derives several analytic approximations for the critical stock price of an American option [43]. Kamdem considers the price of a European and American option when the underlying asset is driven by a general Lévy process [55]. Shaw uses the Mellin transform to focus more generally on the role of contour integration for pricing Asian options (in a numerical setting) [103]. He additionally proves asymptotic results for low volatility options. Chandra and Mukherjee numerically compute the price of Lookback options driven by Lévy processes and provide particular cases for the normal inverse Gaussian, CGMY, and Meixner process [15]. 8

Pricing options is important for two main reasons; speculation and hedging. When an investment portfolio has exposure to multiple foreign currencies, it is commonplace to hedge against currency risk by holding \( n \) options written on the FX rates of each currency. However, this is equivalent to holding a single basket option on \( n \) underlying FX rates, thus reducing the total transactual cost of obtaining a riskless hedge. Basket options also provide diversification, enabling one to decrease their risk. For basket options, since assets are modelled by GBM, prices are log-normally distributed, but the sum of log-normal variables is not. In fact, the sum has no closed-form distribution function, making basket option pricing a non-trivial

7 To the author’s knowledge.
8 See [?] for a thorough treatment of Lévy processes.
task. The majority of recent approaches rely on estimating analytical or numerical approximations via Monte-Carlo methods \cite{60}. By using transforms we can reduce the multidimensional PDE to an integral equation, allowing for numerical approximation via Mellin inversion.

**Remark 5** Throughout this work, call options are solved with the variable change $w \mapsto -w$. The change is made without declaration and the usual notation for the transform is preserved. This is to ensure the Mellin transform of the call payoff function exists for some fundamental strip.

### 4.2 European Call Option

Recall the problem formulation for a European call option in section \ref{section:problem_formulation}. Following \cite{19}, begin by rearranging the Black-Scholes-Merton PDE:

$$
\frac{\partial V(S, t)}{\partial t} + \frac{1}{2}\sigma^2(S \frac{\partial}{\partial S})^2 V(S, t) + (r - q - \frac{1}{2}\sigma^2)S \frac{\partial V(S, t)}{\partial S} - rV(S, t) = 0 \quad (4.1)
$$

Applying the Mellin transform \eqref{3.1}, using \eqref{3.15}, linearity, and independence of the time derivative yields,

$$
\frac{\partial \hat{V}(w, t)}{\partial t} + \left(\frac{1}{2}\sigma^2 w^2 - (r - q - \frac{1}{2}\sigma^2)w - r\right)\hat{V}(w, t) = 0 \quad (4.2)
$$

with the final time condition $\hat{V}(w, T) = \hat{\theta}(w)$. This first order ODE has a solution of the form,

$$
\hat{V}(w, t) = \hat{\theta}(w) \exp \left[ \left(\frac{1}{2}\sigma^2 w^2 - (r - q - \frac{1}{2}\sigma^2)w - r\right)(T - t) \right] \quad (4.3)
$$

$$
= \hat{\theta}(w) \exp \left[ \frac{1}{2}(T - t)((w + \alpha)^2 - \alpha^2 - \frac{2r}{\sigma^2}) \right] \quad (4.4)
$$
where \( \alpha = \frac{q}{\sigma^2} + \frac{1}{2} - \frac{r}{\sigma^2} \). The constant of integration is determined as \( \hat{\theta}(w) \) by setting \( t := T \). Applying the inverse Mellin transform (3.5) yields,

\[
V(S, t) = \theta(S) * M^{-1}\left\{ \exp \left[ \frac{1}{2}(T-t) \left( (w + \alpha)^2 - \alpha^2 - \frac{2r}{\sigma^2} \right) \right]; w \right\}
\]

\[
= \theta(S)e^{\beta(T-t)} * M^{-1}\left\{ \exp \left[ \frac{1}{2}(T-t)(w + \alpha)^2 \right]; w \right\}
\]

\[
:= \theta(S)e^{\beta(T-t)} * W(S, t)
\]

where \( \beta = -\frac{1}{2}\sigma^2(\alpha^2 + \frac{2r}{\sigma^2}) \). Substituting \( w + \alpha = x \) and using the scale property (3.9),

\[
W(S, t) = S^\alpha M^{-1}\left\{ \exp \left[ \frac{1}{2}\sigma^2(T-t)(w + \alpha)^2 \right]; x \right\}
\]

\[
= \frac{S^n}{2^{1/2}} \left( \frac{\pi}{2\sigma^2(T-t)} \right)^{-1/2} \exp \left[ -\frac{1}{4} \left( \frac{1}{2}\sigma^2(T-t) \right)^{-1} \ln(S)^2 \right].
\]

From (4.7), (4.9), and theorem (3) we obtain:

\[
V(S, t) = \theta(S) * M^{-1}\left\{ \exp \left[ \frac{1}{2}\sigma^2(T-t)(w + \alpha)^2 \right]; w \right\}
\]

\[
= \frac{e^{\beta(T-t)}}{\sigma \sqrt{2\pi(T-t)}} \int_{K}^{\infty} \frac{y - K}{y} \left( \frac{S}{y} \right)^{\alpha} \exp \left[ -\frac{1}{4} \left( \frac{1}{2}\sigma^2(T-t) \right)^{-1} \ln(y)^2 \right] dy.
\]

The equivalence of (4.11) and (2.8) is shown numerically in [19]. Alternatively, the European call option can be expressed as,

\[
V(S, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\theta}(w)e^{\frac{1}{2}\sigma^2w(T-t)} S^w dw
\]

using the following Mellin payoff function. For \( \Re(w) > 1 \),

\[
\hat{\theta}(w) = \int_{0}^{\infty} (S - K)^+ S^{-w-1} dS
\]

\[
= \int_{K}^{\infty} (S - K)S^{-w-1} dS
\]

\[
= \frac{K^{1-w}}{w(w-1)}
\]
by use of lemma 5.

**Proposition 1** The expressions (2.8), (4.11), and (4.12) for the European call option are analytically equivalent.

Proof: See Appendix.

### 4.3 European Put Option

Recall the problem formulation for a European put option in section 2.2. Following [82] but including a continuous dividend, apply the Mellin transform (3.1) to the Black-Scholes-Merton PDE:

$$\frac{\partial \hat{V}(w, t)}{\partial t} + \left( \frac{\sigma^2}{2} (w^2 + w) - (r - q)w - r \right) \hat{V}(w, t) = 0$$

(4.14)

which follows from (3.15), independence, and linearity of the Mellin transform. Following from the final time condition, this ODE has a solution given by,

$$\hat{V}(w, t) = \hat{\theta}(w) \exp \left[ -\frac{1}{2} \sigma^2 \left( w^2 + (1 - k_2)w - k_1 \right)(T - t) \right]$$

(4.15)

where \( k_1 = \frac{2r}{\sigma^2} \) and \( k_2 = \frac{2(r-q)}{\sigma^2} \). By setting,

$$\alpha(w) = w^2 + (1 - k_2)w - k_1$$

(4.16)

and applying the inverse Mellin transform (3.5),

$$V(S, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\theta}(w)e^{\frac{1}{2}\sigma^2\alpha(w)(T-t)}S^{-w}dw.$$  

(4.17)
The Mellin transform of the payoff function is given by,

$$\hat{\theta}(w) = \int_0^\infty (K - S)^+ S^{w-1} dS$$

$$= \int_0^K (K - S) S^{w-1} dS$$

$$= \left[ \frac{KS^w}{w} - \frac{S^{w+1}}{w + 1} \right]_0^K$$

$$= \frac{K^{w+1}}{w(w + 1)}$$ \hspace{1cm} (4.18)

which exists for $\Re(w) > 0$. When $q = 0$, the equivalence of (4.17) and (2.13) is shown analytically in [82]. By changing the final time condition in (4.10), we obtain the alternate expression for the put option:

$$V(S,t) = e^{\beta(T-t)} \frac{e^{\sigma \sqrt{T-t}}}{\sqrt{2\pi(T-t)}} \int_0^K K - y \left( \frac{S}{y} \right)^\alpha \exp \left[ - \frac{\ln(S/y)^2}{2\sigma^2(T-t)} \right] dy. \hspace{1cm} (4.19)$$

with $\alpha$ and $\beta$ as in section 4.2. In [19] the numerical equivalence to (2.13) is provided.

**Proposition 2** The expressions (2.13), (4.17), and (4.19) for the European put option are analytically equivalent.

Proof: Set $n = 1$ in the derivation for the European power put option in [44]. By noticing that (22) is (4.17), (25) is (4.19), and (28) is (2.13) the result follows. □
4.4 American Call Option

Recall the problem formulation for an American call option in section 2.3. Following [41], apply the Mellin transform to the inhomogeneous Black-Scholes-Merton PDE:

\[
\frac{\partial \hat{V}(w,t)}{\partial t} + \left( \frac{\sigma^2}{2} (w^2 + w) - (r - q)w - r \right) \hat{V}(w,t) = \hat{f}(w,t). \tag{4.20}
\]

The Mellin transform of the early exercise function (2.15) is,

\[
\hat{f}(w,t) = \int_0^\infty (rK - qS)S^{-w-1}dS = rK \frac{S^*(t)^{-w}}{w} - q \frac{S^*(t)^{1-w}}{w - 1} \tag{4.21}
\]

since \( f \) is non-zero on \( 0 \leq S \leq S^* \). The solution to the inhomogenous ODE is given by,

\[
\hat{V}(w,t) = C(w)e^{-\frac{1}{2}\sigma^2\alpha(-w)(T-t)} - \int_t^T \frac{rK}{w} S^*(s)^w e^{\frac{1}{2}\sigma^2\alpha(-w)(s-t)} ds \\
+ \int_t^T \frac{q}{w - 1} S^*(s)^{1-w} e^{\frac{1}{2}\sigma^2\alpha(-w)(s-t)} ds \tag{4.22}
\]

for constant of integration \( C(w) \) and \( \alpha(w) \) given by equation (4.16). By setting \( t := T \) and rearranging we obtain the expression,

\[
\hat{V}(w,t) = \hat{\theta}(w)e^{\frac{1}{2}\sigma^2\alpha(-w)(T-t)} - \int_t^T \frac{rK}{w} S^*(s)^w e^{\frac{1}{2}\sigma^2\alpha(-w)(s-t)} ds \\
+ \int_t^T \frac{q}{w - 1} S^*(s)^{1-w} e^{\frac{1}{2}\sigma^2\alpha(-w)(s-t)} ds \tag{4.23}
\]
where $\hat{\theta}(w)$ is the Mellin transform of the payoff function (4.13). Mellin inversion yields the price of the American call option:

$$V_{AC}(S,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\theta}(w)e^{\frac{1}{2}\sigma^2\alpha(-w)(T-t)}S^wdw$$

$$-\int_{c-i\infty}^{c+i\infty} \int_{t}^{T} \frac{rK}{w} \left( \frac{S}{S^*(s)} \right)^w e^{\frac{1}{2}\sigma^2\alpha(-w)(s-t)} ds dw$$

$$+\int_{c-i\infty}^{c+i\infty} \int_{t}^{T} \frac{qS^*(s)}{w-1} \left( \frac{S}{S^*(s)} \right)^w e^{\frac{1}{2}\sigma^2\alpha(-w)(s-t)} ds dw$$

$$= V_{EC}(S,t) + V_{EEP}(S,t). \hspace{1cm} (4.25)$$

The first term in (4.24) is equal to (4.12) i.e. the price of a European call option. The last two terms in represent the contribution of the early exercise premium, denoted $V_{EEP}(S,t)$.

**Proposition 3** The expressions (2.17), (2.18), and (4.24) for the American call option are analytically equivalent.

Proof: By proposition 4 and put-call symmetry for American options i.e. $V_{AC}(S,K,r,q,t) = V_{AP}(K,S,q,r,t)$, equivalence follows. □

### 4.5 American Put Option

Recall the problem formulation for an American put option in section 2.4. Following [44], apply the Mellin transform to the inhomogeneous Black-Scholes-Merton PDE:

$$\frac{\partial \hat{V}(w,t)}{\partial t} + \left( \frac{\sigma^2}{2}(w^2 + w) - (r - q)w - r \right) \hat{V}(w,t) = \hat{f}(w,t). \hspace{1cm} (4.26)$$
The Mellin transform of the early exercise function (2.20) is,
\[
\hat{f}(w,t) = \int_{0}^{\infty} (-rK + qS)S^{w-1}dS = -rK \frac{S^*(t)^w}{w} + q \frac{S^*(t)^{w+1}}{w+1} \quad (4.27)
\]
since \(f\) is non-zero on \(0 \leq S \leq S^*\). Similar to the call case, the solution to the inhomogeneous ODE is given by,
\[
\hat{V}(w,t) = C(w)e^{-\frac{1}{2}\sigma^2 \alpha(w)(T-t)} + \int_{t}^{T} \frac{rK}{w} S^*(s)^w e^{\frac{1}{2}\sigma^2 \alpha(w)(s-t)} ds - \int_{t}^{T} \frac{q}{w+1} S^*(s)^{w+1} e^{\frac{1}{2}\sigma^2 \alpha(w)(s-t)} ds \quad (4.28)
\]
for constant of integration \(C(w)\) and \(\alpha(w)\) given by equation (4.16). By setting \(t := T\) and rearranging we obtain the expression,
\[
\hat{V}(w,t) = \hat{\theta}(w)e^{\frac{1}{2}\sigma^2 \alpha(w)(T-t)} + \int_{t}^{T} \frac{rK}{w} S^*(s)^w e^{\frac{1}{2}\sigma^2 \alpha(w)(s-t)} ds - \int_{t}^{T} \frac{q}{w+1} S^*(s)^{w+1} e^{\frac{1}{2}\sigma^2 \alpha(w)(s-t)} ds \quad (4.29)
\]
where \(\hat{\theta}(w)\) is the Mellin transform of the payoff function (4.18). Mellin inversion yields the price of the American put option:
\[
V^E_A(S,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\theta}(w)e^{\frac{1}{2}\sigma^2 \alpha(w)(T-t)} S^{-w} dw + \int_{c-i\infty}^{c+i\infty} \int_{t}^{T} \frac{rK}{w} S^*(s) \left( \frac{S}{S^*(s)} \right)^{-w} e^{\frac{1}{2}\sigma^2 \alpha(w)(s-t)} ds dw - \int_{c-i\infty}^{c+i\infty} \int_{t}^{T} \frac{qS^*(s)}{w+1} \left( \frac{S}{S^*(s)} \right)^{-w} e^{\frac{1}{2}\sigma^2 \alpha(w)(s-t)} ds dw \quad (4.30)
\]
\[
= V^E_P(S,t) + V^{EEP}_P(S,t). \quad (4.31)
\]
The first term in (4.30) is equal to (4.17), the price of a European put option. The last two terms represent the contribution of the early exercise premium, denoted $V_{EP}^P(S,t)$.

**Proposition 4** *The expressions (2.22), (2.23), and (4.30) for the American put option are analytically equivalent.*

Proof: See Proposition 5.1.3 in [41].
Chapter 5

Analytic Basket Option Pricing using the Mellin Transform

To avoid having all your eggs in the wrong basket at the wrong time, every investor should diversify. -Sir John Marks Templeton

5.1 PDE for Multi-Asset Options

Consider the geometric Brownian motion model \( S_t = (S_{t1}, ..., S_{tn})' \) as the asset price process for a multi-asset option on \((\Omega, \mathcal{F}, P)\). For simplicity in notation, let the \( S_i \)'s be normalized so that each asset in the basket has an implicit weighting (i.e. \( S_i = \omega_i A_i \) for asset \( A_i \) and weight \( 0 \leq \omega_i \leq 1 \) where \( \sum \omega_i w_i = 1 \)). By the hypothesis of no-arbitrage, there exists a martingale measure \( Q \) equivalent to \( P \):

**Theorem 6 ([23] First Fundamental Theorem of Asset Pricing)** The market defined by the probability space \((\Omega, \mathcal{F}, P)\) is no-arbitrage iff there exists at least one risk-neutral probability measure \( Q \) that is equivalent to the original probability measure \( P \).
In particular, the asset \((S_t)_{t \in [0,T]}\) is a martingale under \(Q\). Hence \((e^{X_t})_{t \in [0,T]}\) is also a martingale. The asset model for \(1 \leq i \leq n\) is,

\[
S_{ti} = S_{0i} e^{X_{ti}} \quad \text{and} \quad X_{ti} = \mu_i t + \sigma_i W_{ti}.
\]

The martingale condition for \(S_t = S_0 e^{X_t}\) implies,

\[
\mu_i = r - q_i - \frac{\sigma_i^2}{2}.
\]

For multidimensional Brownian motion with drift, the characteristic exponent is given by the Lévy-Khintchine formula when the Lévy measure is zero \cite{62}:

\[
\Psi(u) = \frac{1}{2} u' \Sigma u - i \mu' u.
\]

This is equivalently defined by \(\Psi(u) := -\frac{1}{i} \ln(\mathbb{E}[\exp(iu' X_t)])\) where the characteristic function (Fourier transform) is \(\Phi(u) = \exp(-i\Psi(u))\). To incorporate correlation between the Brownian motions of the process, let \(\sigma \in \mathbb{R}^{n \times n}\) be a diagonal matrix of volatilities \(\sigma_i\) for \(1 \leq i \leq n\) and \(\rho \in \mathbb{R}^{n \times n}\) be a correlation matrix with \(\rho_{ij} = \text{corr}(dW_i, dW_j) \in [-1, 1]\) such that \(\Sigma = \sigma \rho \sigma\). For example, in the two-dimensional case,

\[
\Sigma = \begin{pmatrix}
\sigma_1 & 0 \\
0 & \sigma_2
\end{pmatrix}
\begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{pmatrix}
\begin{pmatrix}
\sigma_1 & 0 \\
0 & \sigma_2
\end{pmatrix}
= \begin{pmatrix}
\sigma_1^2 & \rho \sigma_1 \sigma_2 \\
\rho \sigma_2 \sigma_1 & \sigma_2^2
\end{pmatrix}
\]

where \(\rho = \rho_{12} = \rho_{21}\). It is known that when the asset of a European option \(V(S; \tau) \in C^{2,1}(\mathbb{R}^n \times (0, T)) \cap C^0(\mathbb{R}^n \cup \{0\} \times [0, T])\) is Markovian, the solution is given by

\footnote{This drift term is required to discount the option price and hence produce an equivalent martingale measure (EMM). This EMM is more commonly known as the mean-correcting martingale measure.}
Kolmogorov’s backward equation. One can construct this associated PDE problem using the generalized Black-Scholes-Merton equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_{i} \sigma_{j} S_{i} S_{j} \frac{\partial^{2} V}{\partial S_{i} \partial S_{j}} + \sum_{i=1}^{n} (r - q_{i}) S_{i} \frac{\partial V}{\partial S_{i}} - rV = 0.$$

(5.5)

where \(V(S, T) = \theta(S)\). From theorem 4.2 in [91], by letting the Lévy measure be zero and \(\theta(S)\) be Lipschitz continuous, \(V(S, t)\) is a classical solution of (5.5).

In much of the literature, especially in regards to Fourier pricing, the focus is on solving (5.5) in terms of log-prices \(x = \ln(S)\). With Fourier solution methods, the popularity with log-prices is handled with the shift theorem; \(\mathcal{F}\{f(x - \xi); t\} = e^{-2\pi i t \xi} \mathcal{F}\{f(x); t\}\). This allows functions with shifts in the original domain to be represented as an expression involving a coefficient. After the shift theorem has been applied, and manipulations in Fourier space are made, Fourier inversion reconstructs the original function (ideally as a simplified expression). With the Mellin transform, one can circumvent the change of variables in the asset price by instead applying the scale property (3.9). Similarly, working in Mellin space may prove to be fruitful. To solve the multi-asset option pricing problem, we opt for the Mellin approach with (5.5) in terms of \(S\). However, note that the above PDE formulation only considers the European case. It is known in the American case that a decomposition exists where the option value can be represented as a sum of a European option and an early exercise premium i.e. \(V_A(S, t) = V_E(S, t) + V_{EEP}(S, t)\). Consider the ansatz that the solution to an inhomogeneous PDE (5.5) solves the American case. Then for

\(^2\)See Appendix for derivation.
some \( f = f(S, t) \),

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} (r - q_i) S_i \frac{\partial V}{\partial S_i} - rV = f. \tag{5.6}
\]

If the solution of the European option can be separated from the solution of (5.6), an explicit expression for the early exercise premium can be obtained, and the ansatz will hold.

### 5.2 Boundary Conditions for Basket Options

Suppose \( V(S, t) \) is a multi-asset option with Lipschitz payoff function \( \theta(S) \). The boundary conditions imposed on (5.6) are reliant on the type of option. In general, we let \( 0 \leq t \leq T \) and \( 0 < S_1, ..., S_n < \infty \). To utilize the Mellin transform, we assume that \( V(S, t) \) is bounded of polynomial degree when \( S \to 0 \) and \( S \to \infty \). From the boundary conditions which follow, 4 cases for the basket option can be solved for: European basket call, European basket put, American basket call, and American basket put.

Consider an American option. Recall that when the option is granted exercise rights for any \( t \in [0, T] \), the problem divides the price space into two regions. This early exercise boundary will depend on the payoff function of the option under

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3Mathematically, this can be verified directly in the put case. In the call case, we obtain this condition by using the modified transform with \( w \to -w \). Financially, this claim is supported by the finiteness of option payouts.

4Not including the single asset European and American put/call options which are subcases of these.
consideration. The payoffs for an (arithmetic) basket option are:

\[ V_P(S, T) = \theta_P(S) = (K - \sum_{i=1}^{n} S_i)^+ \]  \hspace{1cm} (5.7)

\[ V_C(S, T) = \theta_C(S) = (\sum_{i=1}^{n} S_i - K)^+ \]  \hspace{1cm} (5.8)

In the put case, the continuation region \( C \) exists for \( \sum_{i=1}^{n} S_i > S^* \), while the exercise region \( E \) exists for \( \sum_{i=1}^{n} S_i < S^* \). Similar to the single asset option, the converse regions hold for the call case. The smooth pasting conditions can then be stated as:

\[ (i) \quad \left. \frac{\partial V_P(S, t)}{\partial S_i} \right|_{\sum_{i=1}^{n} S_i=S^*} = -1 \quad \text{for} \quad 1 \leq i \leq n \]  \hspace{1cm} (5.9)

\[ (ii) \quad \left. \frac{\partial V_C(S, t)}{\partial S_i} \right|_{\sum_{i=1}^{n} S_i=S^*} = 1 \quad \text{for} \quad 1 \leq i \leq n \]  \hspace{1cm} (5.10)

when \( \sum_{i=1}^{n} S_i = S^* \) the payoff becomes \( (iii) \ \theta_C(S) = S^* - K \) for a call, and \( (iv) \ \theta_P(S) = K - S^* \) for a put. Similar to the single asset option, the payoff functions are equivalent to (5.7) and (5.8) for the exercise region \( E \). As usual the option must satisfy \( V(S, t) \geq \theta(S, t) \) in \( C \). Due to the decomposition of American options, we can obtain the European option formula without directly solving for it.

\footnote{As usual, the subscripts \( P \) and \( C \) are used to distinguish between the terminal conditions of a put and call option.}
5.3 General Analytic Solution

Let the PDE formulation (5.6) be homogeneous. Splitting the $i = j$ terms in the double sum gives,

\[ V_t + \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^n \rho_{ij} \sigma_i \sigma_j S_i S_j V_{S_i S_j} + \frac{1}{2} \sum_{i=1}^n \sigma_i^2 S_i^2 V_{S_i S_i} + \sum_{i=1}^n (r - q_i) S_i V_{S_i} - rV = 0 \quad (5.11) \]

since $\text{corr}(dS_i, dS_i) = 1$. \(^6\) Now apply the multidimensional Mellin transform to (5.11). By theorem \(4\) lemma \(3\) independence of the time derivative, and linearity of the multidimensional Mellin transform we obtain,

\[ \hat{V}_t + \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^n \rho_{ij} \sigma_i \sigma_j w_i w_j \hat{V} \hat{V} + \frac{1}{2} \sum_{i=1}^n \sigma_i^2 w_i (w_i - 1) \hat{V} \hat{V} + (r - q_i) \sum_{i=1}^n w_i \hat{V} - r \hat{V} = 0 \quad (5.12) \]

where $\mathbf{w} = (w_1, ..., w_n)'$ are the Mellin variables. Equation (5.12) can be re-expressed as:

\[ \hat{V}_t(\mathbf{w}, t) = -Q(\mathbf{w}) \hat{V}(\mathbf{w}, t) \quad (5.13) \]

where

\[ Q(\mathbf{w}) = \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^n \rho_{ij} \sigma_i \sigma_j w_i w_j + \frac{1}{2} \sum_{i=1}^n \sigma_i^2 w_i (w_i - 1) + (r - q_i) \sum_{i=1}^n w_i - r \quad (5.14) \]

\[ = \frac{1}{2} \mathbf{w}' \Sigma \mathbf{w} + \mu' \mathbf{w} - r \quad (5.15) \]

\[ = -(\Psi(\mathbf{w}) + r) \quad (5.16) \]

\(^6\) To ease notation, subscripts in $V$ denote differentiation.
where (5.2) and (5.3) are used. Integrating both sides of (5.13) after separating for variables \( \hat{V} \) and \( t \) gives,

\[
\hat{V}(w, t) = C(w)e^{(\Psi(wi)+\tau)t}.
\]  

(5.17)

Setting \( t := T \) above and solving retrieves the constant of integration \( C = \hat{\theta}e^{QT} \). \(^7\)

After substitution the homogeneous solution becomes,

\[
\hat{V}(w, t) = \hat{\theta}(w)e^{-(\Psi(wi)+\tau)(T-t)}.
\]  

(5.18)

The solution for the inhomogeneous PDE can be obtained by considering the final time condition of the homogeneous problem.

**Theorem 7 ([54] Duhamel’s principle)** Let \( u \in C^{2,1}[\mathbb{R}^n \times [0, \infty)] \) and \( \mathcal{L} \) be a linear bounded operator. The solution of the inhomogeneous initial value problem (i) is given by,

\[
u(x, t) = \hat{u}(x, t) + \int_0^t \hat{w}(x, t; s)ds
\]  

(5.19)

where \( \hat{w}(x, t; s) \) is a family of functions solving the homogeneous initial value problem (ii).

(i) \( u_t = \mathcal{L}(u) + f(x, t), \quad x \in \mathbb{R}^n, \quad t > 0 \)

\( u(x, 0) = g(x), \quad x \in \mathbb{R}^n \)

(ii) \( \hat{w}_t = \mathcal{L}(\hat{w}), \quad x \in \mathbb{R}^n, \quad t > s \)

\( \hat{w}(x, s; s) = f(x, s), \quad x \in \mathbb{R}^n, \quad t = s \)

\(^7\)Since \( \hat{V}(w, T) = \hat{\theta}(w) \).
Duhamel’s principle reduces the problem of solving the American case to instead solving the European case under different boundary conditions. First we let $\tau = T - t$ so the use of Duhamel’s principle is justified. Then (5.18) becomes,

$$\hat{V}(w, \tau) = \hat{\theta}(w)e^{-(\Psi(w_i) + r)\tau}. \quad (5.20)$$

with initial time condition $\hat{V}(w, 0) = \hat{\theta}(w)$. After applying Duhamel’s principle,

$$\hat{V}(w, \tau) = \hat{\theta}(w)e^{-(\Psi(w_i) + r)\tau} - \int_0^\tau \hat{f}(w, s)(w, s)e^{-(\Psi(w_i) + r)(\tau - s)}ds. \quad (5.22)$$

Finally, by using Mellin inversion we arrive at,

$$\hat{V}(S, \tau) = (2\pi i)^{-n} \int_\gamma \hat{\theta}e^{-(\Psi(w_i) + r)\tau} S^{-w}dw \quad (5.21)$$

$$- (2\pi i)^{-n} \int_\gamma \int_0^\tau \hat{f}e^{-(\Psi(w_i) + r)(\tau - s)}S^{-w}dsdw. \quad (5.22)$$

Hence, the following main result holds.

**Theorem 8** Let $S = (S_1, ..., S_n)'$, $w = (w_1, ..., w_n)'$, $q = (q_1, ..., q_n)'$, $0 \leq \tau \leq T$, and $0 < K, T, S_j, r, q_j < \infty$ for all $1 \leq j \leq n$. For Lipschitz payoff $\theta(S)$, the value of a multi-asset American option $V(S, \tau)$ on $n$ geometric Brownian motion assets is given by,

$$V(S, \tau) = e^{-r\tau}\mathcal{M}^{-1}\left\{\hat{\theta}(w)\Phi(w_i, \tau)\right\} - \mathcal{M}^{-1}\left\{\int_0^\tau \hat{f}(wi, \tau - s)e^{-r(\tau - s)}ds\right\} \quad (5.23)$$

where $\Phi(\cdot)$ is the characteristic function of multivariate Brownian motion with drift.

**Remark 6** When $\hat{f} = 0$, $V(S, \tau)$ is the value of a multi-asset European option. When $\hat{f} \neq 0$, $V(S, \tau)$ is the value of a multi-asset American option. When $n = 1$ we obtain the single asset Black-Scholes-Merton cases.

\(^{8}\)The payoff function must occur at $\hat{V}(w, 0)$. This makes the backward in time model forward in time.
Remark 7  The fundamental strip $\mathcal{R}(\mathbf{w}) = (w_{\min}, w_{\max})$ of the integrand depends on the Mellin transform of the payoff function $\hat{\theta}$ and inhomogeneous term $\hat{f}$ since $e^z$ is holomorphic.

To implement this formula, one must know the following three components: (i) the Mellin payoff $\hat{\theta}(\mathbf{w})$, (ii) the characteristic function (or exponent) of a multivariate Brownian motion with drift, and (iii) the Mellin transform of the exercise function $\hat{f}(\mathbf{w})$. The parameters required are typically known or can be estimated prior to computation; these include $K, S_j, r, q_j$, and $\tau$. By recognition that (5.23) is in the form $V_A(S, \tau) = V_E(S, \tau) + V_{EEP}(S, \tau)$, the early exercise price is known.

Corollary 1  Let $S = (S_1, ..., S_n)^t$, $\mathbf{w} = (w_1, ..., w_n)^t$, $\mathbf{q} = (q_1, ..., q_n)^t$, $0 \leq \tau \leq T$, and $0 < K, T, S_j, r, q_j < \infty$ for all $1 \leq j \leq n$. The value of the early exercise premium $V_{EEP}(S, \tau)$ for a multi-asset option on $n$ geometric Brownian motion assets is given by,

$$V_{EEP}(S, \tau) = \mathcal{M}^{-1}\left\{-\int_0^\tau \hat{f}(\mathbf{w}, s)\Phi(\mathbf{w}_i, \tau - s)e^{-r(\tau-s)}ds\right\}$$

where $\Phi(\cdot)$ is the characteristic function of a multivariate Brownian motion with drift.

5.4 Basket Payoff Function

Generalized put-call parity and put-call symmetry enables European and American option prices to be calculated via the corresponding put or call. Hence an explicit expression for only one of the payoff functions (put or call) is needed to compute either case. Herein we consider the put option. Given an arithmetic basket payoff
according to (5.7), Hurd and Zhou have obtained an equivalent expression in terms of Fourier transforms when $K = 1$. We note that the authors only consider European basket options. With an appropriate change of variables ($u = wi$) their formula (Proposition 7.1) can be verified using the following Mellin formula.

**Proposition 5** Let $S = (S_1, ..., S_n)'$, $w = (w_1, ..., w_n)'$, $0 \leq \tau \leq T$, and $0 < K, T, S_j < \infty$ for all $1 \leq j \leq n$. For $\Re(w_j) > 0$, the put payoff value for a basket option is,

$$
\theta_P(S) = \mathcal{M}^{-1}\{\hat{\theta}_P(w)\} = \mathcal{M}^{-1}\left\{\frac{\beta_n(w)K^{1+\sum w}}{(\sum w)(\sum w + 1)}\right\}
$$

(5.25)

where $\beta_n(\cdot)$ denotes the multinomial Beta function.

Proof: See Appendix.

We may numerically implement this formula using methods discussed in chapter 6.

For two underlying assets, the value of the basket put payoff function in the above proposition and the corresponding call payoff are given by figure 5.1. It is trivial that when $n = 1$ proposition 5 becomes,

$$
\hat{\theta}_P(w) = \frac{K^{w+1}}{w(w + 1)}
$$

(5.26)

where the strip of analyticity is defined by $\Re(w) > 0$.

### 5.5 Early Exercise Boundary

An explicit form for the early exercise function $\hat{f}$ must be obtained for the practical computation of American options. It is responsible for contributing premium to the
Figure 5.1: The value of a basket call (left) and put (right) payoff function on $S_{1,2} \in [0,3]$. For the call and put payoff, $K = 4$ and $K = 2$ respectively.

price, as determined by the location of the free boundary. Recall that for non-jump models on one asset, non-zero $f = \pm rK \mp qS$, depending on whether the option is a call or put, respectively. Once again, consider a put option. For the multi-asset case, the early exercise function is given by,

$$f_P(S, t) = \begin{cases} 
-rK + \sum_{i=1}^{n} q_i S_i; & 0 < \sum_{i=1}^{n} S_i \leq S^*(t) \\
0; & S^*(t) < \sum_{i=1}^{n} S_i < \infty 
\end{cases} \quad (5.27)$$

which allows us to solve for the following expression.

**Proposition 6** Let $S = (S_1, ..., S_n)'$, $w = (w_1, ..., w_n)'$, $q = (q_1, ..., q_n)'$, $0 \leq t \leq T$, and $0 < K, T, S_j, q_j, r < \infty$ for all $1 \leq j \leq n$. For $\Re(w_j) > 0$, the early exercise
function for a multi-asset American put option is,

\[ f_P(S,t) = \mathcal{M}^{-1}\{ \tilde{f}_P(w,t) \} = \mathcal{M}^{-1}\left\{ \frac{\beta_n(w)(S^*)\sum w}{\sum w} \left[ \frac{q'wS^*}{\sum w + 1} - rK \right] \right\} \] (5.28)

where \( \beta_n(\cdot) \) is the multinomial beta function and \( S^*(t) \) is the critical asset price.

Proof: See Appendix.

As expected, when \( n = 1 \) we obtain (4.27).

### 5.6 Black-Scholes-Merton Models

Recall the single asset Black-Scholes-Merton models, where the asset price is driven by GBM according to (1.13). In the multi-asset case,

\[ S_{tj} = S_{0j} \exp\left[ (r - q_j - \sigma_j^2/2)t + \sigma_j W_{tj} \right] \] (5.29)

for \( 1 \leq j \leq n \). The joint characteristic function is given by,

\[ \Phi(w;T) = \exp[iw(rTe - qTe - \sigma^2T/2) - w\Sigma w'T/2] \] (5.30)

where \( e = (1,\ldots,1)' \). Consider the value of an American option. By using (5.30), proposition 5, and proposition 6 in theorem 8 we obtain,

**Corollary 2** Let \( S = (S_1, \ldots, S_n)', \; w = (w_1, \ldots, w_n)', \; q = (q_1, \ldots, q_n)', \; 0 \leq \tau \leq T, \) and \( 0 < K,T,S,j,q,r < \infty \) for all \( 1 \leq j \leq n \). For \( \Re(w) > 0 \), the value of a
multi-asset American put option on \( n \) GBM assets is given by,

\[
V^A_P(S, \tau) = e^{-r\tau} \mathcal{M}^{-1}\left\{ \frac{\beta_n(w)K^{1+\sum w}}{(\sum w)(\sum w + 1)} \Phi(w, \tau) \right\}
+ rK\mathcal{M}^{-1}\left\{ \frac{\beta_n(w)}{\sum w} \int_0^\tau (S^*(s))^{\sum w} \Phi(w, \tau - s)e^{-r(\tau-s)}ds \right\}
- \mathcal{M}^{-1}\left\{ \frac{q'w\beta_n(w)}{\sum w(\sum w + 1)} \int_0^\tau (S^*(s))^{1+\sum w} \Phi(w, \tau - s)e^{-r(\tau-s)}ds \right\}
\]

(5.31)

where \( \beta_n(\cdot) \) is the multinomial beta function, \( \Phi(\cdot) \) is the joint characteristic function of Brownian motion with drift, and \( S^* \) is the critical asset price.

By setting \( n = 2, t = T - \tau, \) and \( q = 0, \) corollary 2 reduces to the expression found in \([80, 82, 102]\):

\[
V^A_P(S_1, S_2, t) = V^E_P(S_1, S_2, t) - \int_t^T \left( \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} \hat{\theta}e^{Q(s-t)}S^{-w_1}S^{-w_2}ds \right) dw_1 dw_2
\]

T (5.32)

where the price of a European option on 2 assets is given by,

\[
V^E_P(S_1, S_2, t) = \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} \hat{\theta}e^{Q(T-t)}S^{-w_1}S^{-w_2}dw_1 dw_2
\]

(5.33)

which requires the following components for computation,

\[
\hat{\theta}(w_1, w_2) = \frac{\Gamma(w_1)\Gamma(w_2)}{\Gamma(w_1 + w_2 + 2)} K^{1+w_1+w_2}
\]

(5.34)

\[
Q(w_1, w_2) = -(r - \frac{\sigma^2}{2})w_1 - (r - \frac{\sigma^2}{2})w_2 - r
\]

(5.35)

\[
\hat{f}(w_1, w_2, s) = -rK\Gamma(w_1)\Gamma(w_2)\Gamma(w_1 + w_2 + 1)(S^*(s))^{-w_1-w_2}.
\]

(5.36)

If instead \( n = 1 \) and \( q > 0, \) we obtain the American Black-Scholes-Merton put formula \([4.30]\). Removing the second and third term retrieves the European Black-Scholes-Merton put formula \([4.17]\) with Mellin payoff \([4.18]\). The multi-asset call option
follows from put-call symmetry and corollary 2. When \( n = 1 \) and \( t = T - \tau \) we obtain the American Black-Scholes-Merton call formula (4.24). Removing the second and third term retrieves the European Black-Scholes-Merton call formula (4.12) with Mellin payoff (4.13).

5.7 Greeks

Option sensitivities or Greeks describe the relationship between the value of an option and changes in one of its underlying parameters. They play a vital role for risk management and portfolio optimization, since they have the ability to describe how vulnerable an option is to a particular risk factor (such as the asset price, time, volatility, or risk-free rate). In the multi-asset case, the presence of multiple first order derivatives leads to many possible computations. The focus below is on first partial derivatives; however, by direct substitution of derivative operators under the integral in theorem 8, many other formulas for Greeks may be obtained. As usual, let \( \mathbf{S} = (S_1, ..., S_n)' \), \( \mathbf{w} = (w_1, ..., w_n)' \), \( \mathbf{q} = (q_1, ..., q_n)' \), \( 0 \leq \tau \leq T \), and \( 0 < K, T, S_j, r, q_j < \infty \) for all \( 1 \leq j \leq n \). For \( \Re(w_j) > 0 \), Theta of a multi-asset put option \( V \) on \( n \) GBM assets is given by,

\[
\Theta := -\frac{\partial V(S, \tau)}{\partial \tau} = e^{-r\tau} \mathcal{M}^{-1}\{\hat{\theta} \hat{P}(r + \Psi(wi)) e^{-\Psi(wi)\tau} - \int_{0}^{\tau} \hat{f} \hat{P}(w_i)e^{(r-\Psi(wi))(\tau-s)} ds\}. \tag{5.37}
\]
Similarly, $\nu_i$ is given by,

$$
\nu := \frac{\partial V}{\partial \sigma_i} \\
= e^{-r\tau} \mathcal{M}^{-1} \left\{ \hat{\theta}_p \tau e^{-\Psi(w_i)\tau} + \int_0^\tau \hat{f}_p(\tau - s) e^{(r-\Psi(w_i))(\tau-s)} ds \right\} \nu_0
$$

(5.38)

where $\nu_0 = \frac{1}{2} \sum_{j=1}^n \rho_{ij} \sigma_j w_i \sigma_j w_j + \sigma_i w_i$. $\rho$ is given by,

$$
\rho := \frac{\partial V}{\partial r} \\
= e^{-r\tau} \mathcal{M}^{-1} \left\{ \hat{\rho}_p \tau (\rho_0 - \tau) e^{-\Psi(w_i)\tau} + \int_0^\tau \hat{f}_p s e^{(r-\Psi(w_i))(\tau-s)} ds \right\}
$$

(5.39)

where $\rho_0 = \sum_{j=1}^n w_j - 1$. Due to the form of the general multi-asset option, applying differential operators to the integrand can yield similar results for call options (which we omit here).
Chapter 6

Numerical Option Pricing

Valuing options on $n$ underlying assets is a difficult problem due to the curse of dimensionality. The issues stem from multiple integration, where the order of complexity does not scale linearly as $n$ increases. For the single asset case, numerical integration of European options was made popular by Carr and Madan who provide formulas in terms of the Fourier transform. This is particularly useful in numerical applications because of its relation to the fast Fourier transform (FFT); a computationally fast discretization \[13\]. Higher dimensional FFT pricing was considered in \[51\], where basket and spread option prices are computed in the European case. However, the Fourier transform is not the only transform that can achieve subquadratic computation speeds for $n = 1$. The Mellin transform (along with its inverse) can also be computed using the FFT (IFFT) after an exponential change of variables \[21\]. 

In fact, direct computation of the Mellin transform is not difficult, and can be implemented by truncating a Riemann sum (see \(6.1\)). However, the corresponding algorithm exhibits quadratic complexity which is not ideal for multiple integration procedures. \[2\] Theocaris and Chrysakis tackle the problem of numerical inversion by

\[\text{As one would expect from (3.8).}\]

\[\text{Quadratic complexity implies that the memory required to compute a problem of size } n \text{ squares for each increase in } n.\]
expanding the inverse as a series of Laguerre polynomials [100]. Following Dubner and Abate [27], Dishon and Weiss provide a series expansion in terms of sine and cosine functions (see (6.4)) [25]. Recently, Campos and Mejia obtained a quadrature formula for the Bromwich integral using Hermite polynomials [10]. Their expression enables the direct computation of the Mellin transform (and its inverse) in $n$ dimensions. By exploiting the link to other transforms, numerous algorithms to compute the Mellin transform (and inverse) are available. However, since computational speed is of supreme importance, we choose to provide expressions in terms of the FFT. In $n$ dimensions this can be achieved with a complexity of $O(N^n \log_2(N^n))$, where $N$ is the number of initial asset prices used to compute the panel of option prices. In this chapter we consider the numerical pricing of European options. This is primarily because no closed form solution exists for the American option, and hence no exact benchmark for measuring error. Furthermore, the critical asset price $S^*$ would require approximation, whose error would depend on the method of estimation or functional form chosen for $S^*$.

6.1 Numerical Mellin Inversion

To compute proposition 5, 6, and theorem 8 we require the inverse Mellin transform. In this section we present two methods for practical inversion in the univariate case. To facilitate the numerical procedures, consider the put payoff function in proposition
and make a change of variable \( w = a + ib \) so that \( dw = idb \). Thus,

\[
\theta(S) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \hat{\theta}(w)S^{-w} dw
\]

\[
= \lim_{M \to \infty} \frac{1}{2\pi} \int_{-M}^{+M} \hat{\theta}(a + ib)S^{-(a+ib)} db
\]

\[
\simeq \frac{\Delta}{2\pi} \sum_{i=1}^{N} \Re[\hat{\theta}(a + ib_i)S^{-(a+ib_i)}]
\]

(6.1)

where \( \Delta = 2M/N \) and \( b_i = -M + i\Delta \) for \( i = 1, \ldots, N \). Alternatively, following \[25\] we may expand the function into a series of cosine and sine functions by recalling (3.7):

\[
\mathcal{M}\{f(S); w\} = \int_{0}^{\infty} f(S)S^{w-1} dS = \int_{-\infty}^{\infty} f(e^{-t})e^{-tw} dt = \mathcal{L}\{f(e^{-t}); w\}.
\]

(6.3)

The Mellin connection to the two-sided Laplace transform enables numerical inversion via the method in \[27\]. With an appropriate truncation and letting \( S = e^{-t} \) the Mellin payoff function becomes,

\[
\theta(e^{-t}) \simeq \frac{e^{at}}{2M} \hat{\theta}(a) + \frac{e^{at}}{M} \sum_{r=1}^{N} \left\{ \Re[\hat{\theta}(a + \pi ir/M)] \cos \left( \frac{\pi rt}{M} \right) - \Im[\hat{\theta}(a + \pi ir/M)] \sin \left( \frac{\pi rt}{M} \right) \right\}
\]

(6.4)

where \( -M \leq t \leq M \). \[4\] To achieve a faster rate of convergence, it is understood that \( M \) should be chosen such that \( |t/M| \leq 1/2 \) and when the fundamental strip

\[
V_E(S,t) \simeq \frac{\Delta e^{-r(T-t)}}{2\pi} \sum_{i=1}^{N} \Re[\hat{\theta}(a + ib_i)\Phi(ai - b)S^{-(a+ib_i)}].
\]

(6.2)

\[
V_P(S,t) \simeq \frac{e^{ax}}{2M} \hat{g}(a) + \frac{e^{ax}}{M} \sum_{j=1}^{N} \left\{ \Re[\hat{g}(a + \pi ij/M)] \cos \left( \frac{\pi jx}{M} \right) - \Im[\hat{g}(a + \pi ij/M)] \sin \left( \frac{\pi jx}{M} \right) \right\}
\]

(6.5)

where \( \hat{g}(w) = \hat{\theta}(w)\Phi(wi) \) and \( S = e^{-x} \).
\( \langle w_{\min}, w_{\max} \rangle \) is finite, choose \( a = \Re(w_{\max} + w_{\min})/2 \). In both numerical methods presented, convergence of the sum will depend on the choice of \( a \), the truncation term \( M \) and the number of evaluation points \( N \). The algorithm associated with (6.1) converges to the true solution with less error than (6.1), when evaluated using the same parameters. To illustrate this, we fix all parameters except for \( N \) and compute the error (see figures 6.1 and 6.2). It should be noted that at-the-money payoff values produce a higher error. 

\[ \text{Figure 6.1: The absolute error between the approximated Mellin payoff and the original payoff function. On the left is the numerical procedure of (6.1) for } N = 2^8 \text{ (red), } N = 2^{10} \text{ (green), } N = 2^{12} \text{ (orange), and } N = 2^{14} \text{ (purple) points. On the right is the numerical procedure of (6.4) for } N = 2^{14}. \text{ In both cases, } K = 1.5, S \in [0, 3], a = 2, M = 10 \text{ and } 100 \text{ points are plotted in the line graphs.} \]

For \( K = S = 1.5 \) in (6.1), the integrand reduces to the function \( I(w_i) = \frac{1.5}{w_i(w_i+1)} \), whose numerator no longer varies with \( w_i \). So any term \( I(w_i) \) whose value is

\[ ^5\text{A similar phenomenon is observed in [14] when } \ln(K) = 0. \text{ To combat the increase in error, for general out-of-the-money options Carr and Madan divide the characteristic function by } \sinh(\alpha k) \text{ rather than multiply by } \exp(-\alpha k) \text{ in (6.10).} \]
over or under-estimated becomes magnified when taken as a sum over all $w_i$’s. The numerical method of (6.4) experiences at-the-money error to a lesser effect and can be efficiently overcome by evaluating more points in the sum. Regardless, by definition at-the-money payoff functions must be zero and hence no numerical computation is needed.

![Log Absolute error: Mellin payoff vs. original payoff](image)

**Figure 6.2:** The log absolute error between the approximated Mellin payoff and the original payoff function. On the left is the numerical procedure of (6.1) for $N = 2^8$ (red), $N = 2^{10}$ (green), $N = 2^{12}$ (orange), and $N = 2^{14}$ (purple) points. On the right is the numerical procedure of (6.4) for $N = 2^{14}$. In both cases, $K = 1.5$, $S \in [0, 3]$, $a = 2$, $M = 10$, and 100 points are plotted in the line graphs.

It was previously noted that these procedures do not scale well in higher dimensions. To combat this loss in computational speed, the following section implements the method of [14]. By utilizing the fast Fourier transform, the authors obtain a numerically accurate and fast procedure for evaluating European options.
6.2 Fast Fourier Transform Method

Option values can be calculated numerically by multiplying a payoff function with the transition density of an underlying asset, then taking its discounted expectation with respect to an equivalent martingale measure \[28, 72\]. This method of martingale pricing is often computed with respect to the space of the asset, despite often posing more challenges. To circumvent this, Carr and Madan proposed a method to value European options using the expectation of the asset in Fourier space \[14\]. In their approach, the only requirement is knowing the analytic characteristic function $\Phi$ for the log exercise price $k$ of some asset. By taking the Fourier transform of $\Phi$ and multiplying by the appropriate call payoff function, they obtain $\mathcal{F}_T^\alpha(u)$, the Fourier transform of the call option. Fourier inversion then retrieves the original value of the option $C_T(k)$:

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-i uk} \mathcal{F}_T^\alpha(u) du. \quad (6.5)$$

We note that to ensure $C_T(k) \in L^2(\mathbb{R})$, prior to transformation the payoff function is multiplied by a damping factor of $e^{\alpha k}$ for $\alpha > 0$. Given this general framework, assets do not need to have an explicit functional representation; only its distribution is needed. This is particularly useful for infinitely divisible distributions, which includes the general class of Lévy processes. We first present the derivation for a European call option \[6.5\]. As mentioned at the beginning of this section, the value of a call option with maturity at time $T$ is given by,

$$C_T(k) = \int_k^\infty e^{-rT}(e^s - e^k)q_T(s) ds. \quad (6.6)$$
However, the integrand may not be square integrable, so define \( c_T(k) := C_T(k)e^{\alpha k} \) with the condition that \( \alpha > 0 \). Taking the Fourier transform of \( c_T(k) \) yields,

\[
\mathcal{F}^\alpha_T(v) = \int_{-\infty}^{\infty} e^{-rT}e^{ivk}\int_k^{\infty} e^{\alpha k}(e^s - e^k)q_T(s)dsdk
\]  

(6.7)

\[
= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\int_s^{\infty} e^{ivk}(e^{s+\alpha k} - e^{k+\alpha k})dkds
\]  

(6.8)

\[
= \int_{-\infty}^{\infty} e^{-rT}q_T(s)\left(\frac{e^{(\alpha+1+iv)s}}{\alpha + iv} - \frac{e^{(\alpha+1+iv)k}}{\alpha + 1 + iv}\right)ds
\]  

(6.9)

\[
= \frac{e^{-rT}\phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}
\]  

(6.10)

where the characteristic function of the density is defined by,

\[
\phi_T(u) := \int_{-\infty}^{\infty} e^{ius}q_T(s)ds.
\]  

(6.11)

Multiplying by the inverse damping factor and taking the inverse transform retrieves the call option price,

\[
C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}^\alpha_T(v)e^{ivk}dv
\]  

(6.12)

\[
= \frac{e^{-\alpha k}}{\pi} \int_{-\infty}^{\infty} \mathcal{F}^\alpha_T(v)e^{ivk}dv
\]  

(6.13)

by recognizing that the call price is real (even in real part, odd in imaginary). Due to the condition on \( \alpha \), formula (6.13) is well defined. After discretizing, the above formula can be evaluated numerically by means of the fast Fourier transform:

\[
C_T(k_u) \simeq \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-2\pi i(j-1)(u-1)/N}e^{ibv_j}\mathcal{F}^\alpha_T(v_j)\frac{\eta}{3}[3 + (-1)^j - \delta_j]
\]  

(6.14)

where \( v_j = \eta(j - 1), k_u = -b + \lambda(u - 1), b = N\lambda/2, \lambda\eta = 2\pi/N, \) and \( \delta_n \) is the Kronecker delta function. As mentioned, the fast Fourier transform has profound

\[\delta_n = 1 \text{ for } n = 0 \text{ and zero otherwise. The Trapezoid and Simpson’s rule are used in the derivation.}\]
Figure 6.3: The European call (left) and put (right) option price using the FFT method and BSM formula for exercise prices $K \in [80, 120]$ of the asset price $S = 100$ with $\sigma = 0.5$, $r = 0.05$, $q = 0.05$, $T = 30/365$, $N = 2^{14}$, $\alpha = 2$ and $\eta = 0.05$. Points 'o' denote the BSM prices, while points 'x' denote the FFT prices.

computational advantages in terms of speed. However, with careful selection of the model parameters, a close approximation to the BSM formula is also attainable (see figure 6.3). Parameters $\eta$ and $N$ determine the fineness and size of the grid; thus defining the upper limit of integration. Despite error and runtime being inversely proportional, a high degree of accuracy can be obtained with less than a second of computation. In figure 6.4 we see that the absolute error between the BSM formula and FFT method is well within two decimal places governing market prices. It is this computational speed and accuracy of the FFT which motivates the construction of our algorithm in the subsequent section.

\footnote{The corresponding put option is easily computable via put-call parity.}

\footnote{0.01 seconds on Intel Core i3 CPU @ 2.53 GHz and 4 GB RAM}
Figure 6.4: The absolute and log absolute European option price error between the FFT method and BSM formula for exercise prices $K \in [80, 120]$ of the asset price $S = 100$ with $\sigma = 0.5$, $r = 0.05$, $q = 0$, $T = 30/365$, $N = 2^{14}$, $\alpha = 2$, and $\eta = 0.05$.

6.3 Mellin Transform Method

We present a method for numerical inversion of the Mellin transform to compute theorem 8. This procedure may similarly be applied to propositions 5 and 6. To derive the numerical formula, recall the value of a multi-asset option:

$$V(S, \tau) = e^{-r\tau} \lim_{b \to \infty} \frac{1}{(2\pi i)^n} \int_{a-ib}^{a+ib} \hat{\theta}(w)\Phi(wi, \tau)S^{-w}dw$$

$$+ \frac{1}{(2\pi i)^n} b^{-\tau} \lim_{b \to \infty} \int_{a-ib}^{a+ib} \int_f(w, s)\Phi(wi, \tau - s)e^{-r(\tau - s)}S^{-w}dsdw$$

The numerical methods in this section are motivated by the work of [14], [21], [51].
Make a change of variables by setting \( w = a + ib \) so that \( dw = idb \). Then,

\[
V(S, \tau) = \frac{e^{-\tau r}}{(2\pi)^n} \lim_{b \to \infty} \int_{-b}^{b} \hat{\theta}(a + ib) \Phi(ai - b, \tau) S^{-(a + ib)} db
\]

\[
+ \frac{1}{(2\pi)^n} \lim_{b \to \infty} \int_{-b}^{b} \tau \int_{0}^{\tau} \hat{f}(a + ib, s) \Phi(ai - b, \tau - s) e^{-r(\tau - s)} S^{-(a + ib)} ds db
\]

\[
= \frac{e^{-\tau r}}{(2\pi)^n} \lim_{b \to \infty} \int_{-b}^{b} \hat{\theta}(a + ib) \Phi(ai - b, \tau) e^{-(a + ib) \ln(S)} db
\]

\[
+ \frac{1}{(2\pi)^n} \lim_{b \to \infty} \int_{-b}^{b} \tau \int_{0}^{\tau} \hat{f}(a + ib, s) \Phi(ai - b, \tau - s) e^{-r(\tau - s)} e^{-(a + ib) \ln(S)} ds db
\]

Discretize the integrals over \( b \) and \( s \) by invoking the Trapezoid rule:

\[
V(S, \tau) \simeq \frac{\Delta b e^{-\tau r}}{(2\pi)^n} \sum_{j_1, \ldots, j_n=0}^{N-1} \hat{\theta}(a + ib_j) \Phi(ai - b_j, \tau) e^{-a' \ln(S)} e^{-ib'_j \ln(S)}
\]

\[
+ \frac{\Delta b \Delta \tau}{(2\pi)^n} \sum_{j_1, \ldots, j_n=0}^{N-1} \sum_{l=0}^{M-1} \hat{f}(a + ib_j, t_l) \Phi(ai - b_j, \tau - t_l) e^{-r(\tau - t_l) - a' \ln(S)} e^{-ib'_j \ln(S)}
\]

where \( t_l = 0, \ldots, M - 1 \) by step-size \( h = L/(M - 1) \), \( \Delta \tau = h/2 \), \( b_j = (b_{j_1}, \ldots, b_{j_n}) \), \( b_{j_i} := (j_i - N/2) \Delta_i \) for \( j_i = 0, \ldots, N - 1 \), and \( \Delta_b = \prod_{i=1}^{n} \Delta_i \). Note that we let the grid of each sum in \( j_i \) be bounded by \( N \). In order to apply the FFT, the reciprocal lattice for the log initial prices must be defined by,

\[
\ln(S) := s_k = (s_{k_1}, \ldots, s_{k_n}) \quad \text{where} \quad s_{k_i} := (k_i - N/2) \lambda_i. \quad (6.15)
\]

Hence, the multiple integral is approximated by a multiple sum over the lattice,

\[
B = \{ b_j = (b_{j_1}, \ldots, b_{j_n})| j = (j_1, \ldots, j_n) \in \{0, \ldots, N - 1\}^n \}. \quad (6.16)
\]

The lattice spacing \( \Delta_i \) and number of points on the lattice \( N = 2^m \) must be chosen so the FFT produces an acceptable error. The value of the exercise price \( K \) is fixed
and the reciprocal lattice $\mathbf{S}$ for computation are log-asset prices:

$$
\mathbf{S} = \{ \mathbf{s}_k = (s_{k_1}, \ldots, s_{k_n}) | k = (k_1, \ldots, k_n) \in \{0, \ldots, N-1\}^n \}. \quad (6.17)
$$

This enables a panel of option prices to be computed at varying asset prices, rather than exercise prices as in [14]. Choosing $\lambda_i \Delta_i = \frac{2\pi}{N}$ yields,

$$
V(\mathbf{S}, \tau) \simeq \frac{\Delta_b e^{-r\tau}}{(2\pi)^n} \sum_{j_1, \ldots, j_n=0}^{N-1} \hat{\theta}(a + ib_j) \Phi(a - b_j, \tau) e^{-a's_k e^{-\frac{2\pi i}{N}(j - \frac{k}{2})}}
+ \frac{\Delta_b \Delta_r}{(2\pi)^n} \sum_{j_1, \ldots, j_n=0}^{N-1} \sum_{l=0}^{M-1} \hat{f}(a + ib_j, t_l) \Phi(a - b_j, \tau - t_l) e^{-r(\tau - t_l) - a's_k e^{-\frac{2\pi i}{N}(j - \frac{k}{2})}}
= (-1) \sum_k \frac{\Delta_b e^{-r\tau}}{(2\pi)^n} \sum_{j_1, \ldots, j_n=0}^{N-1} \zeta^E e^{-a's_k e^{-\frac{2\pi i}{N}j'k}}
+ \frac{(-1) \sum k \Delta_b \Delta_r}{3(2\pi)^n} \sum_{j_1, \ldots, j_n=0}^{N-1} \sum_{l=0}^{M-1} \zeta^{EEP} e^{-r(\tau - t_l) - a's_k e^{-\frac{2\pi i}{N}j'k}}
$$

where $\zeta^E(j) = (-1) \sum \hat{\theta}(a + ib_j) \Phi(a - b_j, \tau)$ and $\zeta^{EEP}(j, t_l) = (-1) \sum \hat{f}(a + ib_j, \tau - t_l) \Phi(a - b_j, t_l)$. To compute the American option price, two FFT procedures must be computed with input arrays $\zeta^E(j)$ and $\zeta^{EEP}(j, t_l)$. Introducing the composite Simpson’s rule allows the integrand to be approximated using quadratic polynomials rather than line segments. This weighted smoothing implies,

$$
V(\mathbf{S}, \tau) \simeq \frac{(-1) \sum k \Delta_b e^{-r\tau}}{3(2\pi)^n} \sum_{j_1, \ldots, j_n=0}^{N-1} \zeta^E e^{-a's_k e^{-\frac{2\pi i}{N}j'k}} [3 + (-1) \sum j - \delta \sum j]
+ \frac{(-1) \sum k \Delta_b \Delta_r}{3(2\pi)^n} \sum_{j_1, \ldots, j_n=0}^{N-1} \sum_{l=0}^{M-1} \zeta^{EEP} e^{-r(\tau - t_l) - a's_k e^{-\frac{2\pi i}{N}j'k}} [3 + (-1) \sum j - \delta \sum j]
$$
where $\delta \sum_j = 1$ for $\sum_j = 0$ and zero otherwise. By defining $\gamma = (3 + (-1)^{1+\sum_j - \delta \sum_j})/3$ the above American option formula is succinctly,

$$
V^A(S, \tau) \simeq \frac{(-1)^{\sum_k \Delta b}}{(2\pi)^n} \mathcal{F}\mathcal{F}\mathcal{T} \{ \gamma \zeta^E \} e^{-r \tau - a's_k}
$$

$$
+ \frac{(-1)^{\sum_k \Delta b \Delta_c}}{(2\pi)^n} \mathcal{F}\mathcal{F}\mathcal{T} \left\{ \sum_{l=0}^{M-1} \gamma \zeta^{EEP} e^{-r(\tau-t_l)} \right\} e^{-a's_k}.
$$

(6.18)

This expression computes an $N \times \ldots \times N$ matrix of option prices at varying initial asset prices. The dimension of the matrix corresponds to the number of underlying assets of the option. Hence, for a European option price,

$$
V^E(S, t) \simeq \frac{(-1)^{\sum_k \Delta b}}{(2\pi)^n} \mathcal{F}\mathcal{F}\mathcal{T} \{ \gamma \zeta^E \} e^{-r \tau - a's_k}.
$$

(6.19)

To determine the option price in the solution matrix, one must find the index $k$ such

![Figure 6.5: The European put (left) and call (right) option price using the MT method and BSM formula for exercise prices $K \in [80, 120]$ of the asset price $S = 100$ with $\sigma = 0.5$, $r = 0.05$, $q = 0.05$, $T = 30/365$, $N = 2^{14}$, $a = 1$ and $\eta = 0.05$. Points ‘o’ denote the BSM prices, while points ‘x’ denote the MT prices.](image)
that $\exp(s_k) = S$. It follows that entry $k_1k_2\ldots k_n$ corresponds to the basket option price for $S$. To obtain an accurate value, parameters $\Delta_i$, $N$, and $a$ must be chosen carefully. They are responsible for solution convergence and grid fineness, which is necessary so that initial asset prices correspond to option prices within the solution matrix. For $n = 1$, equation (6.19) provides a close approximation to the BSM formula (see figure 6.5). For practical purposes, the error is well within tolerances required by industry practitioners and the speed of the procedure matches that of the FFT method (see figure 6.6). The missing data point in the log absolute graph indicates perfect convergence to the BSM formula (within precision defined by $R$).

Figure 6.6: The absolute and log absolute European option price error between the MT method and BSM formula for exercise prices $K \in [80, 120]$ of the asset price $S = 100$ with $\sigma = 0.5$, $r = 0.05$, $q = 0$, $T = 30/365$, $N = 2^{14}$, $a = 1$, and $\eta = 0.05$.

---

10. To obtain equivalence, a preliminary routine must be implemented to define $s_k$ appropriately.
11. 0.01 seconds on Intel Core i3 CPU @ 2.53 GHz and 4 GB RAM
6.4 Application to Equity Markets \((n = 1)\)

In this section we consider numerical pricing of European put options using historical equity price data. Although it is cumbersome to price European options using a method other than the BSM equation. Implied options are computed on five Canadian bank stocks: Canadian Imperial Bank of Commerce (CIBC), Bank of Montreal (BMO), Toronto-Dominion Bank (TD), Bank of Nova Scotia (Scotiabank), and Royal Bank of Canada (RBC). Each option under review has a maturity \(T \in [22, 45]\) days. A 1% risk-free rate of return is used, indicative of the overnight lending rate set by the Bank of Canada. Discrete dividends are not accounted for in our pricing formula so we omit them and set \(q = 0\). Each exercise price \(K\) and historical volatility \(\sigma\) are fixed, while stock prices \(S\) are taken as their Toronto Stock Exchange market prices over the period specified in the figures below. Volatility \(\sigma\) is calculated by taking the standard deviation of daily close price changes over the periods specified. The selection covers in-the-money, at-the-money, and out-of-the-money put options. \(^{12}\)

\(^{12}\) All data was obtained from the Thomson-Reuters DataStream, Data and Resource Centre, University of Guelph.
Figure 6.7: Implied option prices (green) and historical option prices (blue) are computed with $r = 0.01$, $\sigma = 0.18$, $\tau = 21/252$, and $K = 59$ for the period starting June 20, 2013.

Figure 6.8: Implied option prices (green) and historical option prices (blue) are computed with $r = 0.01$, $\sigma = 0.45$, $\tau = 37/252$, and $K = 100$ for the period starting December 22, 2012.
Figure 6.9: Implied option prices (green) and historical option prices (blue) are computed with $r = 0.01$, $\sigma = 0.32$, $\tau = 39/252$, and $K = 100$ starting January 22, 2013.

Figure 6.10: Implied option prices (green) and historical option prices (blue) are computed with $r = 0.01$, $\sigma = 0.18$, $\tau = 45/252$, and $K = 60$ for the period starting March 18, 2013.
Figure 6.11: Implied option prices (green) and historical option prices (blue) are computed with $r = 0.01$, $\sigma = 0.33$, $\tau = 42/252$, and $K = 72$ for the period starting June 20, 2013.

6.5 Application to Foreign Exchange Markets ($n = 2$)

In this section we consider numerical pricing of European basket put options using historical FX rate data. Implied options are computed on two underlying currencies of USD denomination. Each option is given a monthly maturity. The risk-free rate is set to the central bank interest rate of the base currency $r_b$, while the dividend term corresponds to the rate of the foreign currency $r_f$. Each exercise price $K = 1.5$ so that options exhibit in-the-money conditions. Historical volatility of each currency pair $\sigma_i$ is fixed, while prices $S$ are taken as their exchange prices over the period specified in the figures below. Volatility $\sigma$ is calculated by taking the standard deviation of daily
close price changes. Historical correlation is calculated by

$$\rho_{ij} = \frac{\sum_{k=1}^{m} (S_{1k} - \bar{S})(S_{2k} - \bar{S})}{\sqrt{\sum_{k=1}^{m} (S_{1k} - \bar{S})^2 \sum_{k=1}^{m} (S_{2k} - \bar{S})^2}}$$  (6.20)

for $m$ prices of $S_1$ and $S_2$. One-year trade correlation is used for the following computations.  

![European put option prices (n=2)](image1)

![Canada and Japan currency rates](image2)

Figure 6.12: Implied basket option prices (green) are computed on CAD/USD and YEN/USD with $r_{CAD} = 0.01$, $r_{USD} = 0.0025$, $r_{YEN} = 0$, $\sigma_{YEN/USD} = 0.1027$, $\sigma_{CAD/USD} = 0.0561$, $\rho_{12} = 0.81$, $\tau = 30/365$, and $K = 1.5$ for the period from January 1st, 2013 to October 1st, 2013. The contract size of the YEN position is $1/100^{th}$ the size of the EUR position.

All data was obtained from the Thomson-Reuters DataStream, Data and Resource Centre, University of Guelph.
Figure 6.13: Implied basket option prices (green) are computed on EUR/USD and YEN/USD with $r_{CAD} = 0.01$, $r_{USD} = 0.0025$, $r_{YEN} = 0$, $\sigma_{EUR/USD} = 0.0695$, $\sigma_{YEN/USD} = 0.1027$, $\rho_{12} = 0.06$, $\tau = 30/365$, and $K = 1.5$ for the period from January 1st, 2013 to October 1st, 2013. The contract size of the YEN position is $1/100^{th}$ the size of the EUR position.

Figure 6.14: Implied basket option prices (green) are computed on EUR/USD and CAD/USD with $r_{CAD} = 0.01$, $r_{USD} = 0.0025$, $r_{YEN} = 0$, $\sigma_{EUR/USD} = 0.0695$, $\sigma_{CAD/USD} = 0.0561$, $\rho_{12} = 0.21$, $\tau = 30/365$, and $K = 1.5$ for the period from January 1st, 2013 to October 1st, 2013.
Chapter 7

Conclusion

Failure saves lives. In the airline industry, every time a plane crashes the probability of the next crash is lowered by that. The Titanic saved lives because we’re building bigger and bigger ships. So these people died, but we have effectively improved the safety of the system, and nothing failed in vain. - Nassim Nicholas Taleb

In the wake of the 2008 financial crisis, media were prompt to assign blame, and synthesized the root of the problem to a lack of financial regulation, inadequate institutional disclosure, precarious lending practices, exaggerated rating claims of mortgage-backed financial derivatives, and even the derivatives themselves. Although when implemented properly they have the ability to provide a safer investment than the asset of its underlying, some argue there is no place for financial engineering in today’s markets.

Before humans developed the capacity to create structures that could support weighted transport, such as bridges, individuals would have to cross large bodies of water by boat. A bridge is efficient for travel because cargo does not need to be transferred between modes of transport. If a bridge fails to support the weight of the cargo, and it collapses, does one erect another structure? Or does one resort
to travelling by boat? It can be said that the long-term benefits of a bridge outweigh the risk of collapse, and when viable, are almost always the chosen option. The problem arises when the limits of the bridge are stressed beyond its bounds. The same could be said for financial models. Engineered models do not account for every implicit market feature because the complexity is profound and the dynamics are not well understood. The laws governing financial markets are known to change and adapt to influencing circumstances, while physical laws do not. If $F$ no longer equals $ma$ or $PV$ no longer equals $nRT$, it is our scientific judgement to abandon the model describing the law. The uncertainty inherent in the problem of pricing financial instruments is no reason to abandon progress or the pursuit of understanding. Phenomenon that are best modelled by stochastic processes are common in nature and as a result, warrant our attention.\footnote{One does not have to look farther than the movement of particles to discover the intrigue of Brownian motion.} The solution to financial crises should not be to eliminate the modelling and offering of financial derivatives, but to improve upon their regulation, understanding, and implementation. Thus is the motivation driving institutions, practitioners, academics, and the work presented within.

In conclusion, we review current literature regarding Mellin transforms for option pricing in the single-asset framework. We extend this work analytically by demonstrating the ability of the Mellin transform for obtaining multi-asset option prices. This thesis builds on the PhD results of [41, 82] by including continuous dividends and $n$ underlying assets. This enables options (i) to be solved without a change of variables or reduction to a diffusion equation, and (ii) to be represented in
terms of market asset prices, rather than the conventional logarithmic asset prices. For non-technical traders, pricing options directly from exchange prices may appear as a natural setting for the problem, however the choice is clearly preferential. As such, the Mellin method merely provides an alternative approach to the standard pricing problem. The unique formula discovered computes multi-asset European and American options with \( n \) assets driven by geometric Brownian motion. Hence, the generalized BSM formula is obtained as a special case and we present equivalence to previously known expressions for European and American options. Aside from the parameters known (and estimated) prior to pricing, the option formula only requires three components for computation: (i) the Mellin basket payoff function \( \hat{\theta}(w) \), (ii) the characteristic function (or exponent) of a multivariate Brownian motion with drift, and (iii) the Mellin transform of the early exercise function \( \hat{f}(w, t) \). As a result, new expressions for the basket payoff function \( \theta_p(S, t) \) and early exercise function \( f_p(S, t) \) are obtained for put options. Since no fundamental strips exist for the Mellin transform of the call payoff function, solving directly for the call option is not possible. The change of variable \( w \rightarrow -w \) is required. Alternatively, the corresponding call option can be obtained via generalized put-call parity for basket options. For practical computation of our main result, numerical Mellin inversion is required. We compare the error of two popular inversion techniques, which perform well in the single-asset case. However, since the algorithms do not scale linearly, the Trapezoid and Simpson’s rule are applied to obtain a numerically efficient discretization using the fast Fourier transform. For fixed exercise price \( K \), the algorithm provides option prices at incremental values of initial asset price \( S \). In the European case, computational error
is measured against the closed-form BSM formula, which we show is a comparable alternative to the FFT method of Carr and Madan [14]. Although the difference in speed is negligible, a smaller error is achieved with the Mellin method (especially for at-the-money options). As an application, European put option prices are computed for bank stocks \((n = 1)\) and foreign exchange rates \((n = 2)\) with USD denomination.
Chapter 8

Future Work

Over the past few decades analysis has shown that market data is inconsistent with some of these underlying model assumptions. For example, if asset price changes remain large while time periods shrink, one cannot assume that prices are continuous. Under this observation, Cox and Ross assume prices follow a pure jump process so that at any timestep the asset price results in a positive jump or negative drift \[17\]. Expanding on this theory, Merton introduced jumps superimposed on a continuous price process, whose parameters can be chosen to account for fat-tail returns observed in real markets \[71\]. Additionally, practitioners require models with accurate implied volatility surfaces for risk management. \[1\] While the surface can be explained without a jump model, the smile becomes augmented for short maturity options, which is better described by the presence of jumps \[99\]. \[2\] Moreover, distributions of returns exhibit skewness and leptokurtosis. Clearly, as more information about market structure is understood, improvements to parametric forms of asset prices are necessary. For the class of finite jump models, Kou extends Merton’s model by allowing jump sizes to follow an asymmetric double exponential distribution.

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1 Known as the smile effect of volatility, due to its graphical portrayal of happiness.
2 In non-jump models, as \(T \to 0\), \(\ln(S_t / S_{t-1}) \sim \text{Normal}(\mu, \sigma)\), when in reality this is not the case.
The model accounts for both the volatility smile and asymmetric leptokurtic returns. Most financial models fall into the class of infinite jump models, where many developments have been made. In 1987, Madan and Seneta suggest that increments of log-prices follow a (symmetric) variance gamma (VG) distribution, while providing statistical evidence using Australian stock market data [65, 67]. The VG distribution is a special case of the generalized hyperbolic (GH) distribution, which was initially proposed by Barndorff-Nielson to model the grain-size distribution of sand as it travels from a source to a deposit [4]. Other cases of the GH distribution have been considered for asset price modelling; Eberlein and Keller suggest that increments of log-prices follow a hyperbolic distribution [30]. Alternatively, Barndorff-Nielson propose the normal inverse Gaussian (NIG) distribution for log-prices in [5]. Finally, Eberlein and Prause extend their focus to the entire class of GH distributions, providing a thorough statistical treatment as well as applications to stochastic volatility [31, 86]. Carr, Geman, Madan, and Yor proposed the (generalized) tempered stable process. [12]. Coined the CGMY process, it coincides with the Lévy measure of a non-normal α-stable Lévy process (α ∈ (0, 2)) multiplied by an exponential factor. Another infinite jump model, which arose from the theory of orthogonal polynomials, is given by the Meixner process [94, 95]. All of the processes mentioned thus far share a common trait; they are Lévy processes. It is obvious that by modelling the asset dynamics with a general Lévy process, a large range of special cases are accessible for consideration. Not only does this general class of process sufficiently mimic the

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3 Later extended to the asymmetric variance gamma model in [66].
4 Which include (but are not limited to) the financial cases mentioned above.
implied volatility surface of real data, but as mentioned they can be parametrized
to exhibit skewness, kurtosis, an absence of autocorrelation in price increments, fi-
nite variance, aggregational normality, and have an ability to change discontinuously
[83, 84, 89].

With the Mellin transform, recall that Frontczak solves for a general jump
diffusion model and produces the explicit formula for the Merton model with constant
dividend [42], while the more general exponential Lévy processes were considered by
Kamdem in [55]. Pricing in the multi-asset framework is an area to consider in
the future. By subordinating a Lévy process with a stochastic time change, the asset
exhibits volatility clustering [11]. This observed empirical phenomenon is otherwise
known as the persistence of volatility. Representing the asset as the class of stochastic
volatility Lévy processes is a natural extension to this framework. Since basket options
were the only multi-asset option solved explicitly in this thesis, the opportunity is
available to study different payoff structures (e.g. geometric basket options, spread
options, log contract). The payoff structure alters the domain of optimality for the
option, which results in a different early exercise function. Hence, one must explicitly
solve for the Mellin transform of the payoff function and early exercise function to
price an option this way. From a numerical standpoint, further investigation into
higher dimensional error and estimation of parameters is an additional avenue for
exploration.

5The early exercise function is erroneously stated.
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Chapter 9

Appendix

Lemma 4 ([48, Eq.3.191.1]) For \( \Re(\mu), \Re(\nu) > 0 \),

\[
\int_0^u x^{\nu-1}(u - x)^{\mu-1}dx = u^{\mu+\nu-1}\beta(\mu, \nu) \tag{9.1}
\]

where \( \beta(\mu, \nu) \) denotes the Euler integral of the first kind (or beta function), \[ \text{[1]} \]

\[
\beta(\mu, \nu) = \int_0^1 t^{\nu-1}(1 - t)^{\mu-1}dt. \tag{9.2}
\]

Lemma 5 ([48, Eq.3.191.2]) For \( \Re(\mu), \Re(\nu) > 0 \),

\[
\int_u^\infty x^{-\nu}(x - u)^{\mu-1}dx = u^{\mu-\nu}\beta(\nu - \mu, \mu) \tag{9.3}
\]

where \( \beta(\mu, \nu) \) denotes the Euler integral of the first kind (or beta function).

Definition 7 ([48] Multinomial Beta function) The multinomial Beta function of \( \mathbf{w} = (w_1, \ldots, w_n) \) with \( w_i > 0 \) for \( 1 \leq i \leq n \) is defined by,

\[
\beta_n(w) := \frac{\prod_{j=1}^n \Gamma(w_j)}{\Gamma(\sum_{i=1}^n w_i)}. \tag{9.4}
\]

\[ ^1 \text{The Mellin transform of } f(t) = (1 - t)^{\mu-1} \text{ is (9.2).} \]
9.1 Proof of lemma

By use of theorem 1 we obtain,

\[
\mathcal{M}\left\{ x_i x_j \frac{d^2}{dx_i dx_j} f(x) ; w \right\} = \int_{\mathbb{R}_+^n} x_i x_j \frac{d^2}{dx_i dx_j} f(x) \prod_{k=1}^{n} x_k^{w_k-1} dx_k
\]

\[
= \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} x_i \frac{d}{dx_i} \int_{\mathbb{R}_+^n} x_j \frac{d}{dx_j} f(x) \prod_{k=1}^{n} x_k^{w_k-1} dx_k
\]

(9.5)

(9.6)

If \( i = j \), setting \( k = 2 \) in (3.15) yields (3.24). If \( i \neq j \), setting \( k = 1 \) in (3.15) yields (3.25). Existence of \( \hat{f}(w) \) follows from applying lemma 2. □

9.2 Proof of proposition

Proposition 5.2.1 in [41] gives the equivalence of (2.8) and (4.11), however for completeness we provide the derivation of all three. Starting from (4.11) let,

\[
\psi(S) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2} \alpha(w)(T-t)} S^{-w} dw
\]

(9.7)

which can be written alternatively with \( \alpha \) and \( \beta \) as,

\[
\psi(S) = e^{\beta(T-t)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\beta(w+\alpha)^2(T-t)} S^{-w} dw.
\]

(9.8)

Using equation 7.2.1 in [6] we obtain,

\[
\psi(S) = e^{\beta(T-t)} \frac{S^\alpha}{\sigma \sqrt{2\pi(T-t)}} \exp \left[ -\frac{1}{2} \left( \frac{\ln(S)}{\sigma \sqrt{T-t}} \right)^2 \right].
\]

(9.9)

Using this, applying theorem 3 to (4.11) yields,

\[
V(S, t) = \frac{e^{\beta(T-t)}}{\sigma \sqrt{2\pi(T-t)}} \int_{k}^{\infty} y - K \left( \frac{S}{y} \right)^\alpha \exp \left[ -\frac{\ln(S/y)^2}{2\sigma^2(T-t)} \right] dy
\]

(9.10)
which is (4.10). By defining,

\[ I_1 = \int_{K}^{\infty} \frac{1}{y^\alpha} \exp \left[ -\frac{\ln(S/y)^2}{2\sigma^2(T-t)} \right] dy \]  
\[ I_2 = \int_{K}^{\infty} \frac{1}{y^{\alpha+1}} \exp \left[ -\frac{\ln(S/y)^2}{2\sigma^2(T-t)} \right] dy \]  
\[ \gamma = \frac{e^{\beta(T-t)}}{\sigma \sqrt{2\pi(T-t)}} \]  

we can express (9.10) as,

\[ V(S,t) = S^{\alpha} \gamma I_1 - KS^{\alpha} \gamma I_2. \]  

By letting,

\[ x = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \left( \frac{S}{y} \right) - \sigma^2(T-t)(\alpha - 1) \right] \]  

for \( I_1 \) the expression becomes,

\[ I_1 = e^{-\beta(T-t)} \frac{\sigma \sqrt{2\pi(T-t)}}{S^{\alpha-1}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{x^2}{2}} dx \]  
\[ = \frac{e^{-\beta(T-t)}}{\gamma S^{\alpha-1}} \Phi(d_2) \]  

where \( d_2 = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \left( \frac{S}{K} \right) + (r - q - \frac{\sigma^2}{2})(T-t) \right] \). By letting,

\[ x = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \left( \frac{S}{y} \right) - \sigma^2(T-t)\alpha \right] \]  

for \( I_2 \) the expression becomes,

\[ I_2 = e^{-\beta(T-t)} \frac{\sigma \sqrt{2\pi(T-t)}}{S^{\alpha}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{x^2}{2}} dx \]  
\[ = \frac{e^{-\beta(T-t)}}{\gamma S^{\alpha}} \Phi(d_1) \]
where \( d_1 = d_2 + \sigma \sqrt{T-t} \). Combining (9.14) with (9.17) and (9.20) gives,

\[
V(S,t) = Se^{-q(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\Phi(d_1).
\]

\[\text{(9.21)}\]

\[\square\]

### 9.3 Proof of proposition 5

The \( n = 1, 2 \) cases have been solved in [80, 82, 102]. We first derive the \( n = 3 \) case with an approach similar to [102], then solve for the general case where \( n \in \mathbb{Z}^+ \). For \( n = 3 \), following [102] the simplifying assumption that \( K = 1 \) is made.

\[
\hat{\theta}(w_1, w_2, w_3) = \int_0^\infty \int_0^\infty \int_0^\infty (1 - S_1 - S_2 - S_3)^+ S_1^{w_1-1} S_2^{w_2-1} S_3^{w_3-1} dS_1 dS_2 dS_3 \quad (9.22)
\]

\[
= \int_D (1 - S_1 - S_2 - S_3) S_1^{w_1-1} S_2^{w_2-1} S_3^{w_3-1} dS_1 dS_2 dS_3 \quad (9.23)
\]

where \( D = \{ S_1 + S_2 + S_3 < 1; S_1, S_2, S_3 > 0 \} \). Let \( S_1 = a(1-b), S_2 = ab(1-c), S_3 = abc \) for \( a, b, c \in \mathbb{R} \). Solving for the new domain gives \( 1 > a > ab > abc > 0 \). The determinant of the Jacobian matrix is,

\[
|J| = \begin{vmatrix}
1-b & -a & 0 \\
b(1-c) & a(1-c) & -ab \\
bc & ac & ab
\end{vmatrix} = a^2b \quad (9.24)
\]
This transformation maps the tetrahedron \( D \) onto the unit cube. Hence (9.23) becomes,

\[
\begin{align*}
&= \int_0^1 \int_0^1 \int_0^1 (1 - a)[a(1 - b)]^{w_1 - 1}[ab(1 - c)]^{w_2 - 1}[abc]^{w_3 - 1}[a^2b]da \, db \, dc \quad (9.25) \\
&= \int_0^1 (1 - a)a^{w_1 + w_2 + w_3 - 1}da \, \int_0^1 (1 - b)b^{w_1 - 1}b^{w_2 + w_3 - 1}db \, \int_0^1 (1 - c)c^{w_2 - 1}c^{w_3 - 1}dc \quad (9.26) \\
&= \beta(2, w_1 + w_2 + w_3) \beta(w_2, w_3) \quad (9.27) \\
&= \frac{\Gamma(2)\Gamma(w_1 + w_2 + w_3)\Gamma(w_1)\Gamma(w_2 + w_3)\Gamma(w_2)\Gamma(w_3)}{\Gamma(2 + w_1 + w_2 + w_3)\Gamma(w_1 + w_2 + w_3)\Gamma(w_2 + w_3)} \quad (9.28) \\
&= \frac{\beta_3(w_1, w_2, w_3)}{(w_1 + w_2 + w_3)(w_1 + w_2 + w_3 + 1)} \quad (9.30)
\end{align*}
\]

by use of lemma 4, definition 7, \( \Gamma(z + 2) = z(z + 1)\Gamma(z) \), and \( \Gamma(2) = 1 \). The fundamental strip is \( \Re(w_j) > 0 \). By the Mellin inversion theorem (3.5), the payoff function is obtained. The general case is derived analogously.
For \( n \in \mathbb{Z}^+ \), let \( K \in \mathbb{R}^+ \). Apply the multidimensional Mellin transform to the payoff function. Since \( K - \sum_{i=1}^{n} S_i \geq 0 \),

\[
\hat{\theta}(w) = \int_{\mathbb{R}^n_+} (K - \sum_{i=1}^{n} S_i)^+ \prod_{i=1}^{n} S_i^{w_i-1} dS_i 
\]

(9.31)

\[
= \int_0^K \int_0^{K-S_1} \cdots \int_0^{K-\sum_{i=1}^{n-1} S_i} (K - \sum_{i=1}^{n-1} S_i) \prod_{i=1}^{n-1} S_i^{w_i-1} dS_i 
\]

(9.32)

\[
= \frac{\Gamma(w_n)}{\Gamma(w_n + 2)} \int_0^{K} \cdots \int_0^{K-\sum_{i=1}^{n-2} S_i} [(K - \sum_{i=1}^{n-2} S_i) - S_{n-1}] S_{n-1}^{w_n-1} \prod_{i=1}^{n-1} S_i^{w_i-1} dS_i 
\]

(9.33)

\[
= \frac{\Gamma(w_n)\Gamma(w_n-1)}{\Gamma(w_n + w_n-1 + 2)} \int_0^{K} \cdots \int_0^{K-\sum_{i=1}^{n-3} S_i} [(K - \sum_{i=1}^{n-3} S_i) - S_{n-2}] S_{n-2}^{w_n-1} \prod_{i=1}^{n-2} S_i^{w_i-1} dS_i 
\]

(9.34)

\[
= \frac{\prod_{j=0}^{n-1} \Gamma(w_n-j)}{\Gamma(2 + \sum_{j=0}^{n-1} w_n-j)} K^{1+\sum_{j=0}^{n-1} w_n-j} 
\]

(9.35)

\[
= \frac{\beta_n(w) K^{1+\sum w}}{\sum w(\sum w + 1)} 
\]

(9.36)

by use of lemma 4, definition 7, theorem 1, \( \Gamma(z+2) = z(z+1)\Gamma(z) \), and \( \Gamma(2) = 1 \). The fundamental strip is \( \Re(w_j) > 0 \).

Hence, Mellin inversion by theorem 5 returns the payoff function. This can alternatively be solved by induction. □
9.4 Proof of proposition

For \( n \in \mathbb{Z}^+ \), let \( K, S^* \in \mathbb{R}^+ \). Apply the multidimensional Mellin transform to the first term of the exercise function \( f(S, t) = f_1(S, t) + f_2(S, t) \). For non-zero \( f \) of a put option,

\[
\hat{f}_1(w, t) = \int_{\mathbb{R}^n_+} f S^{w-1} dS
\]

\[
= -rK \int_0^{S^*} \cdots \int_0^{S^* - \sum_{i=1}^{n-1} S_i} \frac{S_{n-1}^{w_n - 1} dS_n \prod_{i=1}^{n-1} S_i^{w_i - 1} dS_i}{w_n \int_0^{S^*} \cdots \int_0^{S^* - \sum_{i=1}^{n-2} S_i} (S^* - \sum_{i=1}^{n-2} S_i - S_{n-1})^{w_n} S_{n-1}^{w_{n-1}} dS_{n-1} \prod_{i=1}^{n-2} S_i^{w_i - 1} dS_i}
\]

using lemma \( \[6 \] \). This implies the \( k^{th} \) \( (1 \leq k \leq n) \) integration is,

\[
I_k = \frac{-rK \prod_{j=1}^{n} \Gamma(w_j) }{(\sum_{i=1}^{n} w_i) \Gamma(\sum_{i=1}^{n} w_i)} (S^* - \sum_{i=1}^{n-k} S_i)^{\sum_{i=1}^{n-k} w_i}
\]

Setting \( k := n \) and applying definition \( \[7 \] \) yields,

\[
\hat{f}_1(w, t) = \frac{-rK \prod \Gamma(w)}{(\sum w) \Gamma(\sum w)} (S^* \sum_{i=1}^{n} w_i)^n
\]

\[
= rK \beta_n(w) (S^*)^{\sum w}
\]

where \( S^*(t) \) is the critical stock price at time \( t \). \[9.38 \]
Similarly, apply the multidimensional Mellin transform to the \( n \)th term of \( f_2(S,t) \). Since \( S^* = \sum_{i=1}^{n} S_i \geq 0 \),

\[
\hat{f}_2^n(w,t) = q_n \int_{\mathbb{R}^{n+}} (S^* - \sum_{i=1}^{n} S_i) \left( \prod_{i=1}^{n} S_i^{w_i-1} \right) dS_i
\]

\[
= q_n \int_{0}^{S^*} \int_{0}^{S^*-S_i} \cdots \int_{0}^{S^*-\sum_{i=1}^{n-1} S_i} (S^* - \sum_{i=1}^{n} S_i) \left( \prod_{i=1}^{n} S_i^{w_i-1} \right) dS_i
\]

\[
= q_n \int_{0}^{S^*} \int_{0}^{S^*-S_i} \cdots \int_{0}^{S^*-\sum_{i=1}^{n-1} S_i} [(S^* - \sum_{i=1}^{n-2} S_i) - S_n] S_n^{w_n-1} \prod_{i=1}^{n-1} S_i^{w_i-1} dS_i
\]

\[
= q_n^{w_n} \frac{\Gamma(w_n)}{\Gamma(w_n+2)} \int_{0}^{S^*} \cdots \int_{0}^{S^*-\sum_{i=1}^{n-2} S_i} [(S^* - \sum_{i=1}^{n-2} S_i) - S_{n-1}]^{w_{n-1}} S_{n-1}^{w_{n-1}-1} \prod_{i=1}^{n-2} S_i^{w_i-1} dS_i
\]

\[
= q_n^{w_n} \frac{\Gamma(w_n)\Gamma(w_{n-1})}{\Gamma(w_n+w_{n-1}+2)} \int_{0}^{S^*} \cdots \int_{0}^{S^*-\sum_{i=1}^{n-3} S_i} [(S^* - \sum_{i=1}^{n-3} S_i) - S_{n-2}]^{w_{n-2}} S_{n-2}^{w_{n-2}-1} \prod_{i=1}^{n-3} S_i^{w_i-1} dS_i
\]

\[
= q_n^{w_n} \prod_{j=0}^{n-1} \frac{\Gamma(w_{n-j})}{\Gamma(2 + \sum_{j=0}^{n-1} w_{n-j})} (S^*)^{1+\sum_{j=0}^{n-1} w_{n-j}}
\]

\[
= q_n^{w_n} \beta_n(w)(S^*)^{1+\sum w}
\]

\[
= \sum_{k=1}^{n} q_k w_k \beta_n(w)(S^*)^{1+\sum w} / \sum w(\sum w + 1)
\]

by use of lemma 8, definition 7, theorem 1, \( \Gamma(z+2) = z(z+1)\Gamma(z) \), and \( \Gamma(2) = 1 \). The fundamental strip is \( \Re(w_j) > 0 \).

Use theorem 1 to change the integration for all \( k \) in \( \hat{f}_2(w,t) = \sum_{k=1}^{n} \hat{f}_2^k(w,t) \). This yields,

\[
\hat{f}_2(w,t) = \sum_{k=1}^{n} q_k w_k \beta_n(w)(S^*)^{1+\sum w} / \sum w(\sum w + 1)
\]
Combining (9.38) and (9.46) yields,

\[
\hat{f}(w, t) = -rK\beta_n(w)\sum w + \beta_n(w)(S^*)^{\frac{1}{2}}\sum q_kw_k \tag{9.47}
\]

\[
= \beta_n(w)(S^*)^{\frac{1}{2}}\sum \left[ \frac{S^*}{w + 1} \sum q_kw_k - rK \right] \tag{9.48}
\]

Mellin inversion by theorem 5 returns an alternate expression for the early exercise function \(f(S, t)\). This can alternatively be solved by induction. \(\square\)

### 9.5 Generalized Black-Scholes-Merton Equation

For higher dimensional functions, Itô’s lemma extends naturally. For \(1 \leq i \leq n\), consider a multidimensional Itô process of the form,

\[
dX_{it} = \mu_i(X_{it}, t)dt + \sigma_i(X_{it}, t)dW_{it} \tag{9.49}
\]

where \(\rho_{ij} = \text{corr}(W_i, W_j) \in [-1, 1]\).

**Lemma 6 (Multidimensional Itô’s lemma)** Let \(f(x, t) \in \mathbb{R}^{n+1}\) be twice differentiable in \(x\) and once in \(t\). Then (9.49) is equal to,

\[
df(X_{t}, t) = \left( \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dt + \sum_{i=1}^{n} \sigma_i \frac{\partial f}{\partial x_i} dW_i \tag{9.50}
\]

in \(\mathbb{P}\), almost surely.

To derive the PDE of a multi-asset option price, we begin by constructing a portfolio consisting of an option \(V(S, t)\) and \(\frac{\partial V}{\partial S_i}\) amount of assets \(S_i\). Let the assets \(S_i\) be
driven by correlated geometric Brownian motions. Applying the multidimensional Itô’s formula to some function \( V(S, t) \) yields

\[
dV = \left( \frac{\partial V}{\partial t} + \sum_{i=1}^{n} (r - q_i) \frac{\partial V}{\partial S_i} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right) dt + \sum_{i=1}^{n} \sigma_i \frac{\partial V}{\partial S_i} dW_i \tag{9.51}
\]

Then the value of the self-financing portfolio is given by,

\[
\Pi = V(S, t) - \sum_{i=1}^{n} S_i \frac{\partial V}{\partial S_i}. \tag{9.52}
\]

The change in value of the portfolio over time is,

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} - \sum_{i=1}^{n} q_i S_i \frac{\partial V}{\partial S_i} \right) dt. \tag{9.53}
\]

Similar to the univariate case, the change in value over time must earn the risk-free rate of return. Hence,

\[
d\Pi = r\Pi dt = r \left( V - \sum_{i=1}^{n} S_i \frac{\partial V}{\partial S_i} \right) dt \tag{9.54}
\]

By combining (9.53) and (9.54) the generalized Black-Scholes-Merton equation is obtained:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} (r - q_i) S_i \frac{\partial V}{\partial S_i} - rV = 0. \tag{9.55}
\]

\(\square\)