The Static Self-Force in Schwarzschild-de Sitter and
Schwarzschild-Anti-de Sitter Spacetimes

by

Joseph Kuchar

A Thesis
presented to
The University of Guelph

In partial fulfilment of requirements
for the degree of
Master of Science
in
Physics

Guelph, Ontario, Canada
©Joseph Kuchar, August, 2013
I investigate the self-force acting on static scalar and electric charges in Schwarzschild-de Sitter and Schwarzschild-Anti-de Sitter spacetimes. The self-force occurs when a charged particle’s field interacts with the curvature of spacetime so that the particle interacts with its own field. Because the field of a point particle is singular at the location of the particle, it is necessary to decompose the field into a regular part responsible for the self-force and a singular part that does not contribute to the self-force. To do this, I use the mode-sum regularization scheme introduced by Barack and Ori [2], in which the field is decomposed into a sum over modes, and the singular part is removed from each mode using so-called regularization parameters. I find that the electrostatic self-force in Schwarzschild-de Sitter and Schwarzschild-Anti-de Sitter behaves similarly to Schwarzschild self-force near the black hole, but can deviate strongly at larger distances. This is especially true in Schwarzschild-Anti-de Sitter, where the self-force is seen to increase linearly with distance. I provide an explanation for this behaviour using conformal transformations. A particular feature evident in Schwarzschild-Anti-de Sitter is that the self-force can become negative (attractive) at small distances when the Schwarzschild radius and the cosmological length scale are of a similar order. I find that the scalar self-force in Schwarzschild-de Sitter can not actually be computed, and in Schwarzschild-Anti-de Sitter the asymptotic behaviour
is similar to its electrostatic counterpart.
ACKNOWLEDGEMENTS

I am thankful to my advisor Eric Poisson for his help and support. I am also grateful to Ian Vega for his help with this project. I would also like to thank the members of my advisory committee, Luis Lehner and Rob Wickham. I would also like to thank my fiancée, Hilary Paulin, whose support I am especially grateful for.
# Contents

1 Introduction ............................................. 1

2 Background .............................................. 4
   2.1 The Regularization of the Bare Field ................. 4
      2.1.1 Electromagnetic Fields in Flat Spacetime ....... 4
      2.1.2 The Green’s Function Approach .................. 6
   2.2 Mode-Sum Regularization: History and a Simple Example . 10
   2.3 The Spacetimes Under Consideration ................. 15
      2.3.1 The Schwarzschild Spacetime .................... 16
      2.3.2 The de Sitter Spacetime ....................... 17
      2.3.3 The Anti-de Sitter Spacetime .................. 19
      2.3.4 The Schwarzschild-(Anti-)de Sitter Spacetime .... 21
   2.4 The Field Equations ............................... 22
      2.4.1 Electromagnetic Case ......................... 22
      2.4.2 Scalar Case .................................. 25

3 Computing the Self-Force .............................. 29
   3.1 Computational Method ................................ 29
   3.2 Asymptotic Behaviour of the Mode Sum ............... 32

4 Results ................................................. 35
   4.1 The Self-Force in the Schwarzschild-de-Sitter Spacetime .. 35
      4.1.1 Electromagnetic Self-Force ....................... 35
4.2 The Self-Force in Schwarzschild-Anti-de Sitter .......................... 38
  4.2.1 The Electromagnetic Self-Force ..................................... 38
  4.2.2 The Self-Force in AdS and Flat Spacetimes ....................... 43
  4.2.3 The Scalar Case .......................................................... 49

5 Conclusion ................................. 54
  5.1 Summary of Research ...................................................... 54
  5.2 Self-Force Intuition ...................................................... 55
  5.3 Future Research Directions ............................................. 56

6 Appendix ................................................. 58
  6.1 Static Assumption in the Anti-de Sitter Spacetime ................. 58
  6.2 The Electromagnetic Self-Force in AdS ............................... 62
List of Figures

2.1 Retarded and advanced potentials .......................... 8

3.1 Characteristic mode decay example .......................... 33

4.1 Self-force on electric charge in Schwarzschild-de-Sitter ........ 36
4.2 Normalized self-force on electric charge in Schwarzschild-de-Sitter ... 37
4.3 Characteristic mode behaviour near and far from the cosmological horizon ........................................ 39
4.4 Self-force on electric charge in Schwarzschild-Anti-de Sitter .......... 41
4.5 Normalized self-force on electric charge in Schwarzschild-Anti-de Sitter 42
4.6 Asymptotic behaviour of normalized self-force in SAdS .......... 43
4.7 Asymptotic behaviour of normalized self-force in AdS .......... 49
4.8 Asymptotic behaviour of self-force in terms of cosmological constant 50
4.9 Self-force on scalar charge in Schwarzschild-Anti-de Sitter and Anti-de Sitter ........................................ 51
4.10 Normalized self-force on scalar charge in Schwarzschild-de Sitter and pure de Sitter ........................................ 52
4.11 Asymptotic behaviour of normalized scalar self-force in AdS ........ 53
4.12 Asymptotic behaviour of normalized scalar self-force in SAdS ........ 53

6.1 Conformal coordinates $\eta, x$ as functions of $t$ ................. 61
6.2 Self-force and normalized self-force on electric charge in Anti-de Sitter 63
Chapter 1

Introduction

Consider a particle in motion outside of a black hole. If the particle is a test particle, then its trajectory will be that of a geodesic of the background spacetime of the black hole. If that particle has a mass, however, then its motion will deviate from a geodesic in the background spacetime, and this deviation is indicative of a force acting on the particle. This force is the \textit{self-force}, and it arises as a consequence of the particle interacting with its own gravitational field, in the case of a particle with mass, but a similar effect arises if the particle has an electric or scalar charge.

There has been much focus recently in gravitational self-force research, particularly in the context of extreme-mass-ratio inspirals (EMRIs) [3]. An EMRI describes a system composed of a small compact object inspiralling into a larger compact object, a common astronomical example of which is a small or stellar mass black hole plunging into a supermassive black hole. In this sort of scenario the spacetime would be dominated by the supermassive black hole, but there would be self-force effects on the smaller black hole arising from its self-interactions. This type of scenario is expected to produce gravitational waves detectable by space-based gravitational wave detectors [10], and so the self-force is something that needs to be well understood if accurate models of gravitational waves from EMRIs are to be developed. The concern of this research, however, is purely in the static electromagnetic and scalar self-force, and our aim is simply to improve our understanding of the often non-intuitive behaviour of the self-force by looking at simple yet interesting scenarios.
That an electric charge should be acted upon by its own field is not a new concept, and in fact it has been known since the early 1900s that an accelerating charge is subject to a radiation reaction (Abraham-Lorentz) force. It turns out that a non-accelerating charge can also be the subject of a self-force, provided spacetime is not flat. In 1980, Smith and Will [18] calculated the external force required to keep an electric charge $q$ static outside a Schwarzschild black hole of mass $M$. They found that there was a radially acting self-force, given by

$$F_r = \frac{q^2 M \sqrt{f_0}}{r_0^3}, \quad (1.1)$$

where $r_0$ denotes the position of the charge, and $f_0 = 1 - 2M/r_0$ is the Schwarzschild metric function evaluated at the location of the particle. This is a positive self-force, which means that it is repulsive. It is not obvious why the self-force should be repulsive, and there is in fact an argument to be made that it should be attractive instead. According to the membrane paradigm [19], we can consider the event horizon of the black hole to be an equipotential surface. To gain some intuition, then, we can try replacing our black hole with a spherical conductor. If we do this, and place a positive charge $q$ outside the conducting surface, then we would expect a build-up of negative charge density on the near face of the conductor, and a build-up of positive charge density on the opposite face, resulting in a net attractive force between the conductor and charge. If we replace our black hole with a spherical conductor of the same radius, then we can use techniques of electrostatics to determine what force we would expect to act on our charge. If we use the method of images and assume the surface of the conductor is held at zero potential, then we can replace the conductor with a charge $Q = -aq/r_0$ at a position $r = a^2/r_0$, where $a$ is the radius of the conductor. The expected force is then

$$F_r = -\frac{q^2 a}{r_0^3(1 - a^2/r_0^2)^2}, \quad (1.2)$$

This yields the correct $1/r_0^3$ dependence, but the sign is the opposite of what was found by Smith and Will. In a sense, our intuition yields the precisely wrong
The scalar analog to the study of Smith and Will was carried out by Wiseman in 2000 [20], who studied the static scalar self-force in the Schwarzschild spacetime, and found it to vanish. The work of Smith and Will and Wiseman was focused on static particles in simple black hole spacetimes. A Schwarzschild black hole is entirely characterized by its mass, and the spacetime is asymptotically flat. More recently, work was done by Isoyama and Poisson [13], who calculated the static self-force on both electric and scalar charges outside of a spherical star in an otherwise flat background, in order to determine the effects of internal structure on the self-force. The focus of the research of this thesis, on the other hand, is to investigate what effects changing the asymptotic boundary conditions of the spacetime has on the self-force. We do this by investigating the self-force on a static charge outside a Schwarzschild black-hole, but in spacetimes where that black-hole is in either a de Sitter or Anti-de Sitter background. Specifically, I investigate the electric and scalar self-force in Schwarzschild-de Sitter (SdS) and Schwarzschild-Anti-de Sitter (SAdS) spacetimes, which correspond to a Schwarzschild black hole in an expanding and contracting universe, respectively. There are several reasons for choosing these spacetimes to investigate. It is believed that the universe is expanding (and accelerating), and so SdS more closely represents the universe in which we live than does a pure Schwarzschild black hole. Anti-de Sitter space is a natural extension of the research in part because it differs from de Sitter only by a sign in the metric. Anti-de Sitter is also a popular spacetime to investigate thanks to the AdS-CFT conjecture, which states that given some phenomenon in the bulk spacetime (AdS), there is a mapping to another phenomenon in a conformal field theory on the boundary of the spacetime. A final interesting feature is that while in Schwarzschild it is known that the electrostatic self-force is non-zero and the scalar self-force vanishes, in the scalar case, the self-force is known to vanish in Schwarzschild. Curiously, we will see that the scalar self-force can not be computed in the de Sitter and Schwarzschild-de Sitter spacetimes, but it can be computed in the Anti-de Sitter and Schwarzschild-Anti-de Sitter spacetimes.
Chapter 2

Background

2.1 The Regularization of the Bare Field

We will go into detail in following sections about how the self force is computed in practice, but first we will discuss how one can decompose a singular field into its singular and regular components. Most of what follows is similar to what can be found in the self-force review of Poisson, Pound, and Vega [16]. Our present goal is not to provide a full, rigorous, derivation of the self-force as much as it is to provide a brief, pedagogical overview, to illustrate how decomposing a field in curved spacetimes into regular and singular parts is a sensible and well-defined thing to do, and to motivate the mode-sum method of field decomposition that we use to compute the self-force.

2.1.1 Electromagnetic Fields in Flat Spacetime

The simplest picture in which a self-force can be calculated is that of an accelerating charged particle in a flat spacetime, and this is the scenario considered by Dirac in 1938 [8]. Consider a charged particle $q$ orbiting a fixed point (for example, a fixed charge at the origin). The orbiting particle will have a vector potential $A^\mu$ that satisfies the wave equation,

$$\Box A^\mu = -4\pi j^\mu, \quad (2.1)$$
where \( j^\alpha \) is the vector current density. As this charge orbits the centre it emits outgoing radiation, and as a result we expect it will undergo a radiation reaction that will drive it towards the centre charge. If we wish to determine the self-force then we must solve the equations of motion, but there is a problem: the field and potential of a point particle diverge at the position of the particle. This makes it difficult to determine the self-force, but there is a clever trick that we can employ. First, notice that of all possible solutions, we have chosen that of the charge emitting outgoing radiation and plunging towards the centre (the retarded solution, \( A^\alpha_{\text{ret}} \)). Alternatively, we could have chosen the advanced solution, \( A^\alpha_{\text{adv}} \) corresponding to the particle absorbing ingoing radiation and spiralling outwards. Because \( A^\alpha_{\text{ret}} \) and \( A^\alpha_{\text{adv}} \) are both solutions to the wave equation, so is the linear superposition

\[
A^\alpha_S = \frac{1}{2} (A^\alpha_{\text{ret}} + A^\alpha_{\text{adv}}),
\]  

(2.2)

where the S can be taken to stand for singular, since this solution is just as singular as the solutions that make it up. The important thing to notice about the singular potential is that it has equal parts ingoing and outgoing radiation, which means that this solution cannot be responsible for the motion of the charge. Because this potential is just as singular as the retarded potential, we can subtract it from the retarded solution and obtain a potential that is well behaved on the world line of the charge:

\[
A^\alpha_R = A^\alpha_{\text{ret}} - A^\alpha_S = \frac{1}{2} (A^\alpha_{\text{ret}} - A^\alpha_{\text{adv}}).
\]  

(2.3)

\( A^\alpha_R \) is a solution to the homogeneous wave equation. More importantly, because the singular solution was not at all responsible for the motion of the particle, \( A^\alpha_R \) (where the R stands for regular) must be entirely responsible for the motion of the particle, and so it is the potential of interest when we wish to compute the self-force. We can use this regular potential to construct a corresponding regular electromagnetic field tensor, which we can then use to calculate the self-force.

We will show that the same sort of decomposition can still be done in curved spacetime, but it will take a certain amount of mathematical abstraction.
2.1.2 The Green’s Function Approach

Staying in flat spacetime for the time being, we look once more to our retarded and advanced potentials. The retarded solution to (2.1) can be written as a convolution of a Green’s function and the source term,

\[ A_{\text{ret}}^\alpha(x) = \int G_{+\beta'}^\alpha(x, x') j_\beta'(x')dV', \] (2.4)

where \(dV' = d^4x'\) and \(G_{+\beta'}^\alpha(x, x')\) is the retarded Green’s function associated with the retarded potential, and is defined as

\[ G_{+\beta'}^\alpha(x, x') = \frac{\delta_\beta' \delta(t - t' - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}. \] (2.5)

The position vector \(x\) is defined as \(x = (t, \vec{x})\). Unprimed coordinates denote an arbitrary field point, and primed coordinates denote the source point. We can similarly define the advanced potential with an advanced Green’s function as

\[ A_{\text{adv}}^\alpha(x) = \int G_{-\beta'}^\alpha(x, x') j_\beta'(x')dV', \] (2.6)

with

\[ G_{-\beta'}^\alpha(x, x') = \frac{\delta_\beta' \delta(t - t' + |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}. \] (2.7)

The retarded Green’s function is zero whenever \(x\) lies outside of the future lightcone of \(x'\), and \(G_{+\beta'}^\alpha(x, x')\) is infinite on the future light cone. The advanced Green’s function behaves similarly, but with respect to the past light cone of \(x'\). The retarded and advanced Green’s functions satisfy the reciprocity relation,

\[ G_{\beta'\alpha}(x', x) = G_{\alpha\beta'}^+(x, x'). \] (2.8)

We can now define a singular Green’s function as

\[ G_{S\beta'}^\alpha(x, x') = \frac{1}{2} \left[ G_{+\beta'}^\alpha(x, x') + G_{-\beta'}^\alpha(x, x') \right], \] (2.9)
and a regular Green’s function as $G^\alpha_{R\beta'} = G^\alpha_{+\beta'} - G^\alpha_{S\beta'}$. Because of the reciprocity relation, the singular Green’s function is symmetric in both indices and arguments. The regular Green’s function, on the other hand, is antisymmetric. We can use these Green’s functions to define singular and regular potentials as follows:

\[ A^\alpha_S(x) = \int G^\alpha_{S\beta'}(x,x') j^{\beta'}(x') dV' \]  \hspace{1cm} (2.10)

\[ A^\alpha_R(x) = \int G^\alpha_{R\beta'}(x,x') j^{\beta'}(x') dV'. \]  \hspace{1cm} (2.11)

It is easy to see that the singular and regular potentials defined this way are consistent with our definitions from the previous section. The retarded potential in flat spacetime is generated by a single point (the intersection of the world line and the past light cone of $x$). We call this point $z(u)$, where $u$ is the retarded time. Similarly, the advanced potential is generated by the point that is the intersection of the world line and the future light cone of $x$. This point is denoted $z(v)$, where $v$ is the advanced time. An illustration of the dependences of the retarded and advanced potentials is given in figure 2.1. In curved spacetime the dependence is a bit more complicated.

In curved spacetime, the wave equation becomes

\[ \Box A^\mu - R^\alpha_{\beta\gamma} A^\beta = -4\pi j^\mu, \]  \hspace{1cm} (2.12)

where $R^\alpha_{\beta\gamma}$ is the Ricci tensor. We can once again define retarded and advanced Green’s functions for this equation, and they take the same form as in the flat spacetime case, with the exception that now $dV' = \sqrt{-g(x')} d^4x'$.

A significant departure from flat spacetime, however, is in the causal structure of the problem. We mentioned previously that the advanced and retarded potential in flat spacetime depended only on two single points ($z(u)$ and $z(v)$). This is no longer the case. In curved spacetime, the retarded Green’s function is non-zero not only on the future-pointing light cone of $x'$, but at all points inside it as well. Similarly, the advanced Green’s function has support at all points within the past light-cone of $x'$ as well as on the cone itself. This dependence exists because in curved space a photon
Figure 2.1: In flat spacetime, the retarded potential at $x$ is generated by the point that is the intersection of the past light-cone of $x$ and the world line of the source ($z(u)$), where the world line $\gamma$ is described by $z(\tau)$ with $\tau$ the proper time. The advanced potential is generated by the intersection of the future light-cone of $x$ and the world line of the source ($z(v)$). In curved spacetime, the retarded or advanced potential is also generated by all points on the world line that fall within the past or future light-cone of $x$, respectively.
can travel at all speeds up to and including the speed of light. A consequence of this is that the retarded potential at \( x \) now depends on the entire past history of the particle, for all times \( \tau \leq u \). The advanced potential now depends on the entire future history of the particle, so it is necessary to know its state of motion for all times \( \tau \geq v \). These new features make it necessary to change our definitions of the singular and regular Green’s functions. If we did not, then the singular Green’s function would depend on the entire past and future history of the particle’s state of motion. So, too, would the regular Green’s function. Any regular potential constructed from such a Green’s function would be highly non-causal, and so must be rejected.

The proper definition of the singular Green’s function was provided by Detweiler and Whiting [7], who proposed the following correction:

\[
G_{\alpha \beta'}^\alpha(x, x') = \frac{1}{2} \left[ G_{\alpha \beta'}^\alpha(x, x') + G_{\alpha \beta'}^\alpha(x, x') - H_{\alpha \beta'}^\alpha(x, x') \right]. \tag{2.13}
\]

The \( H \) function is designed to remove the complete history dependence of the particle from the singular Green’s function. It is symmetric in its indices and arguments (and so too is the singular Green’s function), and it is a solution to the homogeneous wave equation. It is also defined to agree with the advanced Green’s function when \( x \) is in the chronological future of \( x' \). The potential constructed from the singular Green’s function depends only on the particle’s motion from \( u \leq \tau \leq v \), and can be shown to exert no force on the particle. We can now construct the Detweiler-Whiting regular Green’s function as

\[
G_{\alpha \beta'}^\alpha(x, x') = G_{\alpha \beta'}^\alpha(x, x') - G_{\alpha \beta'}^\alpha(x, x') = \frac{1}{2} \left[ G_{\alpha \beta'}^\alpha(x, x') - G_{\alpha \beta'}^\alpha(x, x') + H_{\alpha \beta'}^\alpha(x, x') \right]. \tag{2.14}
\]

The potential constructed from this regular Green’s function depends on the entire history of the particle up to the advanced time, \( \tau \leq v \), but not on the particle’s future history. It is a solution to the homogeneous wave equation, and so is well behaved without singularities. This is the potential that is responsible for the self-force. We
can construct a regular electromagnetic field tensor, \( F^R_{\alpha\beta} = \nabla_\alpha A^R_\beta - \nabla_\beta A^R_\alpha \), and define the self-force using the Lorentz force equation,

\[
F^\text{self}_\alpha = qF^R_{\alpha\beta} U^\beta,
\]

(2.16)

where \( U^\beta \) is the particle’s four-velocity.

### 2.2 Mode-Sum Regularization: History and a Simple Example

All we have really done so far is lay down some groundwork to show that it is possible to obtain a regular self-force by decomposing the retarded field into regular and singular components. What we have yet to do is demonstrate how it is done in practice. In our research, we employ the Mode Sum Regularization Procedure (MSRP). The idea of using a mode decomposition to obtain the self-force was first had by Lousto in 2000 [15], in the context of binary black hole systems. The MSRP was presented in the same year by Barack and Ori [2], where they developed the MSRP framework and used it to examine the case of a scalar charge falling radially into a Schwarzschild black hole. Burko calculated the static self-force for electromagnetic and scalar charges outside of a Schwarzschild black hole with MSRP, also in 2000 [5]. The MSRP was further developed and improved by Barack, Ori, Mino, Nakano, and Sasaki in 2002 [1], who investigated the gravitational self-force in the Schwarzschild spacetime. Since then, many more papers have been published which have further justified the MSRP as a valid and (relatively) straight-forward way to compute the self-force.

The problem that we have is that we would like to solve Maxwell’s equations for the retarded potential and the electromagnetic field tensor at the position of a point particle, but both the particle’s potential and field are singular there. The key insight of the MSRP is that if we decompose the potential into a sum over modes, and if our spacetime has spherical symmetry, then when the potential and the current
density are decomposed into spherical harmonics, the full field equations decouple into equations for the radial modes of the potential. While the potential is singular, the contribution of any particular mode is finite. The modes of the singular field can also be computed and removed from the modes of the retarded field, and then finally summed over to obtain the regular field, from which we can calculate the self-force. Explicitly, the self-force is proportional to the gradient of the regular potential,

\[ F_r = \frac{e}{\sqrt{f}} A_{t,r}^R, \quad (2.17) \]

where \( f \) is the metric function which is determined by the spacetime. The force can be written explicitly in terms of a mode-sum:

\[ F_r = \frac{e}{\sqrt{f}} (A_{t,r} - A_{t,r}^S) \]

\[ = e^2 \lim_{x \to z} \sum_l \left[ e^{-1} f_0^{-1/2} (\partial_r A_t)_l + S_l \right], \quad (2.19) \]

where \( S_l \) is the decomposition of the singular field. The modes of the singular field are given by the regularization parameters \( A, B, C \) and \( D \):

\[ S_l := A(l + 1/2) + B + \frac{C}{l + 1/2} + \frac{D}{(l - 1/2)(l + 3/2)} + O(l^{-4}). \quad (2.20) \]

The singular field is therefore defined as a sum in powers of \( l^{-1} \), where the coefficients (the regularization parameters) are determined by the spacetime geometry and are independent of \( l \). The form of equation (2.20) reveals the singularity structure of the retarded field (linear with \( l \)) as well as the expected behaviour of the regularized field with respect to \( l \). More will be said about this in the following chapter.

The simplest way to see how we can use mode-sum regularization to obtain a finite self-force is by going through the somewhat silly steps of calculating the self-force on a static charged particle in a flat spacetime at a position \( r = r_0 \). We can determine the bare modes of the force by solving Maxwell’s equations,

\[ \nabla_\nu F^{\mu\nu} = 4\pi j^\mu, \quad (2.21) \]
where the electromagnetic field tensor $F_{\mu\nu}$ is defined in terms of the vector potential, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, and the current density $j^\mu = \rho u^\mu$. For a point charge, $j^\alpha = q \int u^\alpha \delta_4(x, z(\tau))d\tau$. Because of the static nature of the problem, the only non-vanishing component is $\mu = t$. In terms of the vector potential, we have

$$\left( r^2 A_t, r \right)_r + \left[ \frac{1}{\sin \theta} (\sin \theta A_t, \theta) + \frac{1}{\sin^2 \theta} A_t, \phi \phi \right] = 4\pi r^2 j^t.$$

(2.22)

We now perform our mode decomposition of the potential, and for concreteness we place the charge along the $z$-axis. We also make use of the completeness relation of spherical harmonics to express the current density as a mode sum:

$$A_t(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_l(r) Y^{lm}(\theta, \phi) \quad (2.23)$$

$$j^t(r, \theta, \phi) = \frac{q \delta(r - r_0)}{r_0^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^{lm*}(0, 0) Y^{lm}(\theta, \phi). \quad (2.24)$$

Substituting these into our differential equation gives us the equation for the radial potential:

$$r^2 \frac{d^2 R_l}{dr^2} + 2r \frac{dR_l}{dr} - l(l+1)R_l = 4\pi q \sqrt{\frac{2l+1}{4\pi}} \delta(r - r_0). \quad (2.25)$$

The solutions to the homogeneous ODE are $R_{l-} = c_1 r^l$ and $R_{l+} = c_2 r^{-(l+1)}$. We can obtain the general solutions in the $r < r_0$ and $r > r_0$ regions by enforcing continuity at the particle, $R_1(r_0) = R_2(r_0)$, and by integrating the ODE immediately around the particle in order to obtain the jump condition:

$$\lim_{\epsilon \to 0} \left[ \int_{r_0-\epsilon}^{r_0+\epsilon} r^2 \frac{d^2 R_l}{dr^2} dr + \int_{r_0-\epsilon}^{r_0+\epsilon} 2 \frac{dR_l}{dr} dr - l(l+1) \int_{r_0-\epsilon}^{r_0+\epsilon} \frac{R_l}{r^2} dr \right] = 4\pi q \sqrt{\frac{2l+1}{4\pi}}.$$

(2.26)
This gives us the jump condition for the first derivative,
\[
\left. \frac{dR}{dr} \right|_{r_0^+} - \left. \frac{dR}{dr} \right|_{r_0^-} = \frac{q \sqrt{4\pi (2l + 1)}}{r_0^2}.
\] (2.27)

This condition, together with continuity, allows us to determine \( c_1 \) and \( c_2 \). We will be considering the potential from the right hand side (that is, \( r > r_0 \)), so all we need right now is \( c_2 \). We find that \( c_2 = -q \sqrt{\frac{4\pi}{2l + 1}} l_0 \), so that the specific solution is given by
\[
R_{l+} = -\frac{q}{r_0} \sqrt{\frac{4\pi}{2l + 1}} \left( \frac{r_0}{r} \right)^{l+1}.
\] (2.28)

We can now finally compute the modes of the force:
\[
F^r_l = q g^{rr} A_{l+} u^t \\
= q (\partial_r R_{l+}) Y^{lm}(\theta, \phi) \\
= q^2 (l + 1) \frac{r_0^l}{r^{l+2}},
\] (2.29-2.31)

where in the last line we have evaluated the spherical harmonics at the point \( (\theta = 0, \phi = 0) \) since we are interested in the field at the point of the particle. We now let \( r \to r_0 \), and we obtain
\[
F^r_l = \frac{q^2 (l + 1/2)}{r_0^2} - \frac{q^2}{2r_0^2}.
\] (2.32)

We have determined that the force that we would compute from the bare field by summing the individual modes diverges linearly with \( l \), but in flat spacetime we know that the self force vanishes. This means that the regularization parameters have to be such that they exactly cancel the modes of the bare force shown above. Hence, our regularization takes the form \( (l + 1/2) \frac{q^2}{r_0^2} - \frac{q^2}{2r_0^2} \). The form we have written it in is chosen to highlight the sum in powers of \( l^{-1} \) that was mentioned previously. In the flat spacetime case, we see that the only non-zero terms are \( A \) and \( B \). It’s a general feature of the static, spherically symmetric spacetimes we are studying that \( C = 0 \). For the rest of the spacetimes we consider in this thesis, however, we will see that \( D \) is in general non-vanishing. The regularized electromagnetic self-force is calculated
We also compute the self-force caused by a scalar charge. In this case, the regularized self-force is calculated slightly differently, and is given by

$$F^r = q^2 f_0 \lim_{x \to z} \sum_l \left[ q^{-1} f_0^{-1/2} (\partial_r A_t)_l + S_l \right],$$

where

$$(\partial_r A_t)_l = \sum_m \partial_r R_l(r) Y^{lm}(\theta, \phi).$$

$q$ is now a scalar charge, and $\Phi$ is a scalar rather than an electromagnetic potential. Scalar charges and fields are a bit of a physical abstraction. The scalar potential is a solution to the wave equation,

$$\Box \Phi = -4\pi \mu,$$

where $\mu$ is the scalar charge density. The force on a scalar charge in a scalar field is simply $F = q \nabla \Phi$. An familiar example of a scalar potential is the Newtonian gravitational potential, satisfying $\nabla^2 U = -4\pi \rho$ and $F = m \nabla U$. Unlike the electric self-force, for which we can straight-forwardly imagine an electron or proton held outside a black hole by some mechanical system, we have no physically relevant examples of scalar charges to imagine being held outside of black holes. We compute the scalar self-force all the same, however, because the situation is analogous to the electric self-force problem, and the potential is simpler.

The regularization parameters, incorporated in the $S_l$ term, also take different forms in the scalar and electrostatic cases. The regularization parameters for static electric and scalar charges in spherically symmetric spacetimes have been derived by...
Casals, Vega, and Poisson [6], and they are, for the electric case,

\[ A = \frac{-1}{r^2} f^{-1/2} \text{sign}(\Delta) \]  
\[ B = \frac{-1}{2r^2}(1 - r\psi') \]  
\[ C = 0 \]  
\[ D = \frac{-1}{16r^2}[(1 - r\psi') - (1 - r\psi' + 3r^2\psi'^2 - r^3\psi'^3 + 6r^2\psi'' + 2r^3\psi''')f + (1 - 4r\psi' - 3r^2\psi'')rf' + (1 - r\psi')r^2f''] \]

and for the scalar case,

\[ A = \frac{-1}{r^2} f^{-1/2} \text{sign}(\Delta) \]  
\[ B = \frac{-1}{2r^2}(1 + r\psi') \]  
\[ C = 0 \]  
\[ D = \frac{-1}{16r^2}[(1 + r\psi') - (1 + r\psi' + 3r^2\psi'^2 + r^3\psi'^3 - 6r^2\psi'' - 2r^3\psi''')f + (1 + 4r\psi' + 3r^2\psi'')rf' + (1 + r\psi')r^2f''] \]

where \( \Delta = r - r_0 \), a prime denotes differentiation with respect to \( r \), and \( f \) and \( \psi \) are defined by the form of the metric

\[ ds^2 = -e^{2\psi} dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \].

For all the spacetimes we are considering, in the forms that we have written the metric, we have that \( \psi = \frac{1}{2} \log(f) \).

### 2.3 The Spacetimes Under Consideration

Our goal is to compute the self-force in Schwarzschild-de Sitter and Schwarzschild-Anti-de Sitter spacetimes, so it seems prudent to give an overview of the Schwarzschild, de Sitter, and Anti-de Sitter spacetimes.
2.3.1 The Schwarzschild Spacetime

The Schwarzschild spacetime is the solution to Einstein’s equations corresponding to the space around a static and spherically symmetric mass (for example, a non-rotating black hole). In spherical coordinates, the Schwarzschild metric is given by

\[ ds^2 = -(1 - \frac{2M}{r}) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.41} \]

where \( M \) is the mass parameter. There are a few things to notice. First of all, we can see that the metric has two singularities, one at \( r = 0 \) and the other at \( r = 2M \). The \( r = 0 \) singularity is the real singularity at the centre of the black hole. The \( r = 2M \) singularity is a coordinate singularity and denotes the location of the event horizon of the black hole. The real singularity is a place where the curvature of the spacetime diverges. A coordinate singularity, on the other hand, does not have any associated curvature singularity, and can be done away with by a change of coordinates. We will always be examining situations where a charged particle is safely outside the event horizon of the black hole, so the spherical coordinate system is fine for our purposes.

The second thing to notice about this metric is that it is of the form

\[ ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.42} \]

In fact, this is not a special feature of Schwarzschild - it is a feature of many spherically symmetric and static spacetimes, and all of the spacetimes that we will be considering can be written in this form, with just the form of \( f(r) \) varying.

When \( M \to 0 \), the metric collapses to the Minkowski metric. A final thing to notice about the Schwarzschild spacetime is that it is asymptotically flat, and it will reduce to the Minkowski spacetime in the limit that \( r \to \infty \). This particular feature is not evident in the other spacetimes we will be examining. Effectively, by looking at the self-force in the SdS and SAdS spacetimes, we are asking “what happens to the self-force when spacetime is not asymptotically flat?”
2.3.2 The de Sitter Spacetime

The de Sitter spacetime is a vacuum solution of Einstein’s equations with a cosmological constant $\Lambda$. It can be visualized as a 4-dimensional hyperboloid,

\[- Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = a^2, \quad (2.43)\]

where $a = \sqrt{3/\Lambda}$, embedded in a 5-dimensional Minkowski spacetime [12]:

\[ds^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + dZ_4^2. \quad (2.44)\]

The spacetime can be parameterized by spherical polar coordinates $(t, r, \theta, \phi)$ by making the coordinate transformation

\[
\begin{align*}
Z_0 &= \sqrt{a^2 - r^2} \sinh t/a \quad (2.45a) \\
Z_1 &= \sqrt{a^2 - r^2} \cosh t/a \quad (2.45b) \\
Z_2 &= r \cos \theta \quad (2.45c) \\
Z_3 &= r \sin \theta \cos \phi \quad (2.45d) \\
Z_4 &= r \sin \theta \sin \phi. \quad (2.45e)
\end{align*}
\]

In these coordinates, the metric has the form

\[
ds^2 = -\left(1 - \frac{r^2}{a^2}\right) dt^2 + \left(1 - \frac{r^2}{a^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (2.46)
\]

Anti-de Sitter space has nearly the same metric, with the difference of a sign and a redefinition of $a$. In de Sitter, $\Lambda > 0$, while in Anti-de Sitter, $\Lambda < 0$. As we might expect, when the cosmological constant is taken to be zero (or equivalently, when $a \to \infty$), the metric reduces to the flat Minkowski metric. Unlike Schwarzschild, we do not obtain the Minkowski metric when $r \to \infty$. More interestingly, when $\Lambda > 0$, $f(r) = 1 - \frac{\Lambda}{3} r^2$ has a real positive root at $r = a$, which means that this metric has a coordinate singularity, similar to the $r = 2M$ singularity of Schwarzschild that
denotes the location of the event horizon. This singularity denotes what is called the *cosmological horizon*, represented as \( r_c \). This horizon exists because the cosmological constant causes an accelerated expansion of spacetime. This means that, for an observer at \( r = 0 \) equipped with a light source, there is some distance that the photons sent by the observer can never reach, because space is expanding too quickly for the photons to ever get there. This distance is the cosmological horizon, and it’s a horizon because the observer at \( r = 0 \) is causally disconnected from events beyond it.

A final note to make about de Sitter is that its conformal structure means that the self-force vanishes in de Sitter space. We can write the de Sitter metric in a conformally flat form by making the coordinate transformation [12]

\[
\begin{align*}
Z_0 &= \frac{a^2 + s^2}{2\eta} \quad (2.47a) \\
Z_1 &= \frac{a^2 - s^2}{2\eta} \quad (2.47b) \\
Z_2 &= \frac{ax}{\eta} \quad (2.47c) \\
Z_3 &= \frac{ay}{\eta} \quad (2.47d) \\
Z_4 &= \frac{az}{\eta} \quad (2.47e)
\end{align*}
\]

where \( s^2 = -\eta^2 + x^2 + y^2 + z^2 \). In these coordinates, the metric is

\[
ds^2 = \frac{a^2}{\eta^2} (-d\eta^2 + dx^2 + dy^2 + dz^2) .
\]  

(2.48)

This is just the Minkowski metric with a conformal factor \( \Omega^2 = a^2/\eta^2 \). That is, \( g_{\alpha\beta}^{\text{dS}} = \Omega^2 \eta_{\alpha\beta} \). The conformal invariance of Maxwell’s equations then means that we have \( F_{\mu\nu}^{\text{dS}} = F_{\mu\nu}^{\text{flat}} \). For a static particle in flat spacetime, the self-force is obtained from the electromagnetic field tensor as

\[
F^\alpha = qF^\alpha_{\eta} U^\eta ,
\]  

(2.49)
and because the self-force vanishes in flat space, $F_{\alpha\eta}^{\text{flat}} = 0$, and so $F_{\alpha\eta}^{\text{dS}} = 0$, and so the self-force vanishes in the de Sitter spacetime as well.

### 2.3.3 The Anti-de Sitter Spacetime

Similarly to de Sitter, we can picture Anti-de Sitter space as a 4-dimensional hyperboloid,

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 - Z_4^2 = -a^2,$$

where $a = \sqrt{-3/\Lambda}$, embedded in a flat 5-dimensional space:

\begin{equation}
    ds^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 - dZ_4^2.
\end{equation}

The difference is that now that 5-dimensional space is not Minkowski, and it has two time directions [12]. We can transform to spherical polar coordinates via a similar transformation to the one we had for de Sitter space:

\begin{align}
    Z_0 &= \sqrt{a^2 + r^2 \sin t/a} \\
    Z_1 &= r \cos \theta \\
    Z_2 &= r \sin \theta \cos \phi \\
    Z_3 &= r \sin \theta \sin \phi \\
    Z_4 &= \sqrt{a^2 + r^2 \cos t/a}.
\end{align}

The metric in these coordinates is

\begin{equation}
    ds^2 = -\left(1 + \frac{r^2}{a^2}\right)dt^2 + \left(1 + \frac{r^2}{a^2}\right)^{-1}dr^2 + r^2d\Omega^2.
\end{equation}

Because $a^2 \geq 0$, $1 + \frac{r^2}{a^2}$ has no real roots, and so Anti-de Sitter has no cosmological horizon. It does, however, have another interesting feature. The negative cosmological constant in the metric induces an accelerated negative expansion of space. The result
of this is that a photon can reach spatial infinity in a finite amount of coordinate time (since spatial infinity is accelerating towards the photon). It is therefore possible to send a beam of light to infinity, and have it come back in a finite amount of time. Consider a radially travelling null ray describing a light beam sent from the origin to spatial infinity. Because it is null, $ds^2 = 0$, and because it is radial, $d\phi = d\theta = 0$. This gives us

$$dt^2 = \frac{dr^2}{(1 + r^2/a^2)^2}. \quad (2.54)$$

We can take the square root and integrate to obtain

$$t = \int_0^R \frac{dr}{1 + r^2/a^2} = a \arctan(R/a), \quad (2.55)$$

and in the limit that $R \to \infty$ we have that $t \to \frac{a\pi}{2}$. This means that when the cosmological constant approaches zero, and the spacetime approaches Minkowski, the time taken by a photon to reach spatial infinity will approach infinity, as it should. However, if the cosmological constant becomes large, and so $a$ becomes small, then the time taken for a photon to reach spatial infinity also becomes small. Of the spacetimes we are considering, this feature is unique to Anti-de Sitter and Schwarzschild-Anti-de Sitter, and it makes it necessary to impose special boundary conditions at $r = \infty$, since there is no guarantee the fields will have vanished before they get there. The common constraint to impose is energy conservation: if we do not wish energy to leak out of the universe, it is necessary to impose reflective boundary conditions at spatial infinity. An implication of doing this is that, unlike de Sitter space, where the self-force vanishes, in Anti-de Sitter space it does not vanish. This is because Anti-de Sitter is conformal not to Minkowski space, but to Minkowski space with special boundary conditions. Just what this means in practice will be looked into in detail once we begin the discussion of the computed self-force results in Schwarzschild-Anti-de Sitter space.
2.3.4 The Schwarzschild-(Anti-)de Sitter Spacetime

The SdS and SAdS metric is a simple combination of the Schwarzschild and dS/AdS metric, and is given by [17]

\[
 ds^2 = - \left( 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) dt^2 + \left( 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega^2. 
\] (2.56)

This metric satisfies the obvious requirements: when the black hole mass \( M = 0 \), it collapses to the de Sitter or Anti-de Sitter metric. When the cosmological constant \( \Lambda = 0 \) it collapses to the Schwarzschild metric. It also approaches the de Sitter or Anti-de Sitter metric asymptotically for large values of \( r \), and behaves like Schwarzschild for small values of \( r \).

When \( \Lambda \) is positive, the metric function \( f(r) \) admits two positive roots, corresponding to the black hole event horizon \( r_e \) and to the de Sitter cosmological horizon \( r_c \). In this work we parameterize spacetimes using \( r_e \) and \( r_c \) rather than \( M \) and \( \Lambda \), so it is instructive to explicitly write the metric function in terms of those variables. In SdS we have

\[
 f(r) = \frac{1}{(r_e r_c - s^2)r} (r - r_e)(r - r_c)(r - s). 
\] (2.57)

Here \( s \) is a constant given by \( s = -(r_e + r_c) \). The relations between \( M, \Lambda \) and \( r_e, r_c \) are given by

\[
 r_e r_c s = -\frac{6M}{\Lambda} 
\] (2.58)

\[
 -s^2 + r_e r_c = -\frac{3}{\Lambda}. 
\] (2.59)

When \( \Lambda \) is negative and the metric is that of SAdS, then the metric function only has one positive root. In this case the metric function can be written as \( f(r) = -\frac{\Lambda}{3r}(r - r_e)(r^2 + ar + b) \). Consistency gives the condition on \( a \), which is \( a = r_e \), and we define \( b := r_c^2 \), where \( r_c \) here defines not a cosmological horizon, but what we refer
to as a cosmological length scale. The metric function is

$$f(r) = \frac{1}{(r^2 - r_e^2)r}(r - r_e)(r^2 + r_e r + r_e^2),$$

(2.60)

and the relations with $M, \Lambda$ are given by

$$r_e r_c^2 = -\frac{6M}{\Lambda},$$

(2.61)

$$r_c^2 - r_e^2 = -\frac{3}{\Lambda}.$$  

(2.62)

Because the SAdS spacetime is asymptotically Anti-de Sitter, our previous discussion of the asymptotic boundary conditions in Anti-de Sitter space applies here as well.

2.4 The Field Equations

2.4.1 Electromagnetic Case

We consider a charged particle held a fixed distance from a Schwarzschild black hole. Maxwell’s equations in an arbitrary background spacetime are given by

$$\nabla_\nu F^{\mu\nu} = 4\pi j^\mu,$$

(2.63)

where the electromagnetic field tensor $F^{\mu\nu}$ is defined by the vector potential, $F^{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, and the current density is defined by $j^\mu = \rho u^\mu = q \int u^\nu \delta_4(x,z(\tau))d\tau$. Because of the static nature of the problem, the only non-vanishing component is $\mu = t$, and we have

$$F^{t\nu} = g^{\nu\alpha} g^{tt} (-A_{t,\alpha}),$$

(2.64)

so we have

$$(g^{\nu\alpha} g^{tt} A_{t,\alpha})_{;\nu} = -4\pi j^t.$$  

(2.65)
We can avoid having to compute the covariant derivative by incorporating the metric determinant, so that we have

$$
\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\nu\alpha}g^{t\alpha}A_{t,\nu}) = -4\pi j^t.
$$

(2.66)

All of the spacetimes that we are considering are spherically symmetric and static, and so we can write the metric as

$$
ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
$$

(2.67)

where $f$ is a function that only depends on the radial coordinate. With the metric known, we have

$$
(r^2 A_{t,r})_r + f^{-1} \left[ \frac{1}{\sin \theta} (\sin \theta A_{t,\theta})_\theta + \frac{1}{\sin^2 \theta} A_{t,\phi\phi} \right] = 4\pi r^2 j^t.
$$

(2.68)

We now take the first step of our mode-sum procedure by assuming a mode expansion for $A_t$ of the form

$$
A_t(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_l(r) Y^{lm}(\theta, \phi),
$$

(2.69)

and by making a variable substitution, the time component of the current density can be written as

$$
j^t(r, \theta, \phi) = \frac{q \delta(r - r_0)}{r_0^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^{lm*}(0,0) Y^{lm}(\theta, \phi).
$$

(2.70)

Substituting these into our differential equation gives us, after a bit of work,

$$
r^2 \frac{d^2 R_l}{dr^2} + 2r \frac{dR_l}{dr} - \frac{l(l+1)}{f} R_l = 4\pi q Y^{lm}(0,0) \delta(r - r_0).
$$

(2.71)

Equation (2.71) is the differential equation for the $l$'th mode of the radial potential, so this is what we must solve for many values of $l$. We have also placed the particle
along the $z$-axis, so that $\theta = \phi = 0$. This makes the sum over $m$ trivial, and the full differential equation is finally given by

$$r^2 \frac{d^2 R_l}{dr^2} + 2r \frac{dR_l}{dr} - \frac{l(l+1)}{f(r)} R_l = q \sqrt{4\pi (2l+1)} \delta (r - r_0) \tag{2.72}$$

We will be computing the self force for Schwarzschild-de Sitter and Schwarzschild-Anti-de Sitter spacetimes, as well as in pure de Sitter and Anti-de Sitter, and we can see that at this point the only difference that arises between them is in the function $f(r)$ coupled to the $l$-dependent term. This means that the $l = 0$ solution is the same for all the spacetimes we are considering, so we may as well derive it now. The homogeneous differential equation is

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} = 0, \tag{2.73}$$

which has the solutions $R_1 = c_1$ and $R_2 = \frac{q}{r}$. The condition that $R$ vanish at infinity indicates that the constant solution is the one that applies to the region from the black hole to the particle, while the second solution is the one that extends from the particle to infinity. We can determine the particular solution by integrating the differential equation around $r_0$:

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} = q \sqrt{4\pi} \delta (r - r_0) \tag{2.74}$$

$$\lim_{\epsilon \to 0} \left[ \int_{r_0 - \epsilon}^{r_0 + \epsilon} \frac{d^2 R}{dr^2} \, dr + \int_{r_0 - \epsilon}^{r_0 + \epsilon} \frac{2}{r} \frac{dR}{dr} \, dr \right] = q \sqrt{4\pi} \frac{1}{r_0^2} \tag{2.75}$$

The integral of $\frac{dR}{dr}$ will vanish because $R$ is continuous at the point of the particle. We therefore have

$$\left. \frac{dR}{dr} \right|_{r_0^+} - \left. \frac{dR}{dr} \right|_{r_0^-} = q \frac{\sqrt{4\pi}}{r_0^2}. \tag{2.76}$$
Because $R_1$ is constant its derivative is zero, and so we have $\frac{dR_2}{dr} \big|_{r_0} = \frac{q \sqrt{4\pi}}{r_0}$. Hence,

$$R_1^{l=0}(r) = -\frac{q \sqrt{4\pi}}{r_0} \tag{2.77}$$

$$R_2^{l=0}(r) = -\frac{q \sqrt{4\pi}}{r} \tag{2.78}$$

All modes after the $l = 0$ mode must be solved numerically.

### 2.4.2 Scalar Case

The field equation for a scalar charge is given by

$$\Box \Phi = -4\pi \mu, \quad \tag{2.79}$$

where the box operator is the D’Alembertian, defined by $\Box = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$. We will once again make use of the divergence formula, $A_\alpha = \frac{1}{\sqrt{-g}}(\sqrt{-g} A^\alpha)_\alpha$ for our covariant derivative:

$$\nabla_\alpha \nabla^\alpha \Phi = \frac{1}{\sqrt{-g}}(\sqrt{-g} \nabla^\alpha \Phi)_\alpha \tag{2.80}$$

$$= \frac{1}{\sqrt{-g}}(\sqrt{-g} g^{\alpha\beta} \Phi_{,\beta})_\alpha. \tag{2.81}$$

In all the spacetimes we are considering, $\sqrt{-g} = r^2 \sin \theta$, and the expression becomes

$$\Box \Phi = \frac{1}{r^2 \sin \theta} \left[ (r^2 \sin \theta f(r) \Phi_{,r})_{,r} + (\sin \theta \Phi_{,\theta})_{,\theta} + \frac{\Phi_{,\phi\phi}}{\sin \theta} \right]. \tag{2.82}$$

We now decompose the scalar field into a mode sum:

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_l(r) Y_l^m(\theta, \phi). \tag{2.83}$$
The scalar charge density is defined as \( \mu = g \int \delta_4(x, z(\tau)) d\tau \), and by making a variable substitution can be written as

\[
\mu(r, \theta, \phi) = \frac{g f_0^{1/2} \delta(r - r_0)}{r_0^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^{lm*}(0, 0) Y^{lm}(\theta, \phi),
\]

(2.84)

where we have once again placed the particle along the \( z \)-axis in order to make the sum over \( m \) as simple as possible. Making these substitutions, we obtain

\[
\sum_{l} \sum_{m} \frac{1}{r_0^2} Y^{lm}(\theta, \phi) \left[ r^2 f \psi_{l,rr} + (r^2 f)_{,r} \psi_{l,r} - l(l + 1) \psi \right] = -4\pi \sqrt{f_0} \delta(r - r_0) \sum_{l} \sum_{m} Y^{*lm}(0, 0) Y^{lm}(\theta, \phi).
\]

(2.85)

This gives us the final equation for the radial component of the potential:

\[
r^2 \frac{d^2 \psi}{dr^2} + \frac{2r f + rf' d\psi}{f} - \frac{l(l + 1) \psi}{f} = -4\pi q \sqrt{\frac{2l + 1}{4\pi}} f_0^{-1/2} \delta(r - r_0),
\]

(2.86)

where \( f' = \frac{df}{dr} \). We can immediately see that equation (2.86) is more complicated than what we had in the electromagnetic case. In particular, the coupling to the metric function is no longer isolated to the \( l \)-dependent term, which means that our \( l = 0 \) solution will not be independent of the spacetime. We can obtain a general solution in terms of \( f(r) \), however. The homegenous differential equation for the \( l = 0 \) mode,

\[
\frac{d^2 \psi}{dr^2} + \frac{2f + rf' d\psi}{rf} = 0,
\]

(2.87)

has the solutions \( \frac{d\psi_1}{dr} = 0 \) and \( \frac{d\psi_2}{dr} = \frac{c_2}{r^2 f} \). We can determine \( c_2 \) by integrating our differential equation in the immediate neighborhood of the charge, as we did in the electrostatic case.
\[
\frac{d^2 \psi}{dr^2} + \frac{2f + rf'}{rf} \frac{d\psi}{dr} = -q\sqrt{4\pi f_0^{-1/2}} \frac{\delta(r - r_0)}{r_0^2} \quad (2.88)
\]
\[
\lim_{\epsilon \to 0} \left[ \int_{r_0 - \epsilon}^{r_0 + \epsilon} \frac{d^2 \psi}{dr^2} \, dr + \int_{r_0 - \epsilon}^{r_0 + \epsilon} \frac{2f + rf'}{rf} \frac{d\psi}{dr} \, dr \right] = -q\sqrt{4\pi f_0^{-1/2}} \quad (2.89)
\]

This gives us
\[
\left. \frac{d\psi}{dr} \right|_{r_0^+} - \left. \frac{d\psi}{dr} \right|_{r_0^-} = -q\sqrt{4\pi f_0^{-1/2}} \quad (2.90)
\]
Because \( \psi \) is constant from the black hole to the particle, its derivative on the left side of the particle vanishes, and we are left with
\[
\frac{c_2}{r_0^2 f_0} = -q\sqrt{4\pi f_0^{-1/2}} \quad (2.91)
\]
so that \( c_2 \) is given by
\[
c_2 = -q\sqrt{4\pi f_0} \quad (2.92)
\]
so that we have the general solution
\[
\frac{d\psi}{dr} = -q\sqrt{4\pi f_0} \frac{1}{r^2 f(r)} \quad (2.93)
\]
We do not provide a general expression for the radial potential itself, since it will be an integral of the above expression, but it is not actually important because the self-force is proportional to the gradient of the potential and not the potential itself. However, there is something noteworthy here. The proper boundary conditions on the scalar field in the Anti-de Sitter and Schwarzschild-Anti-de Sitter spacetimes are that the field vanish at \( r = \infty \). The boundary conditions in the case of the de Sitter and Schwarzschild-de Sitter spacetimes is that the field be regular at the cosmological horizon. The solution above satisfies the boundary conditions in the Anti-de Sitter and Schwarzschild-Anti-de Sitter spacetimes, but not in the de Sitter and Schwarzschild-de Sitter spacetimes. In particular, the metric function \( f(r) \) has a root at \( r = r_c \) in both the de Sitter and Schwarzschild-de Sitter spacetimes, so the field...
will diverge at $r = r_c$. In fact, we can see there is no way to combine the two general solutions such that they can be sensibly matched at the position of the particle, and so that the field vanishes at the cosmological horizon. We will therefore only be presenting results for the scalar field in the Schwarzschild-Anti-de Sitter spacetime.
Chapter 3

Computing the Self-Force

3.1 Computational Method

The self-force was computed via a mode-sum decomposition scheme. The code to do so was written in Python, using the Numerical Python (NumPy) and Scientific Python (SciPy) packages.

The code is given an event horizon distance $r_e$, as well as a cosmological length scale or horizon $r_c$, and an array of particle positions. It then loops through the array and calls the function Self_force(), which depends on particle position, $M$, $\Lambda$, and $q$. In practice the only important parameters are $M$ and $\Lambda$ because $q$ is set so that $q = 1$. Self_force() computes the regularization parameters $A, B, C$, and $D$, and includes a loop from $l = 0$ to $l_{\text{max}}$ (usually set to around 80), and during this loop it calls the function potential(), which computes the modes of the bare field at $r_0$, $\frac{dR_k}{dr}|_{r_0}$. Self_force() then computes the regularized field, and then $F_r, F^r$, and $F_{\text{norm}}$, which will be defined in the following chapter.

The potential() function computes the $l = 0$ mode of the bare field using the expressions analytically derived in the previous section. For all modes $l \neq 0$, the homogeneous field equation is written as a system of two first-order differential equations and is solved numerically using the SciPy function ODEint. ODEint has user defined absolute and relative error tolerances, and can be given the Jacobian of the system for greater stability and accuracy. In our computations, the relative tolerance was
generally specified as $\sim 10^{-11}$ and the absolute tolerance as $\sim 10^{-10}$.

The homogeneous field equation needs to be solved first in the region from the event horizon up to the particle’s position, and then again in the region from the particle’s position to infinity or the cosmological horizon. To do this, position arrays are set up that span from $r_e + \epsilon$ to $r_0$ and from $r_c - \epsilon$ to $r_0$ in Schwarzschild-de Sitter, and from $r_{\infty}$ to $r_0$ for Schwarzschild-Anti-de Sitter, where $r_{\infty}$ is simply a large value of $r$, which we took to be about $r_{\infty} = 200$. The value of $\epsilon$ used was $\epsilon = 10^{-5}$ in SdS and $10^{-8}$ in SAdS. We start much closer to the event horizon because we computed fewer terms in the recursion relation used to determine the boundary conditions (see below). Boundary conditions at $r_e + \epsilon$, $r_c - \epsilon$, and $r_{\infty}$ must be provided to the ODE solver, and they are generally computed by assuming a Frobenius-type series solution of the homogenous field equations, such as

$$R_{r_e}(r) = \sum_n \alpha_n (r - r_e)^{n+p}$$  \hspace{1cm} (3.1a)

$$R_{r_c}(r) = \sum_n \beta_n (r - r_c)^{n+p}$$  \hspace{1cm} (3.1b)

$$R_{\infty}(r) = \sum_n \gamma_n r^{-n-p},$$  \hspace{1cm} (3.1c)

where $p$ is determined by the indicial equations. By substituting these forms into the differential equation, recursion relations can be derived for $\alpha_n$, $\beta_n$, and $\gamma_n$, which are used to compute the boundary conditions in the code. The exception to this rule is the electromagnetic self-force in AdS and SAdS, where a closed-form analytic expression for the asymptotic boundary conditions at $r_{\infty}$ was derived. For SAdS, the field equation is

$$r^2 \frac{d^2 R_l}{dr^2} + 2r \frac{dR_l}{dr} - \frac{l(l+1)}{1-2M/r - \Lambda r^2/3} R_l = 4\pi q \delta(r - r_0) \sqrt{\frac{2l + 1}{4\pi}}. \quad (3.2)$$

We can perform a change of variables to $w = 1/r$, which gives us the new field equation

$$w^2 \frac{d^2 R_l}{dw^2} - \frac{l(l+1)}{1-2mw - \Lambda/(3w^2)} R_l = 0 \quad (3.3)$$
Asymptotically, this gives us

\[ \frac{d^2 R}{dw^2} = \frac{l(l + 1)}{-\Lambda/3} R \]

This has the general solution

\[ R_{\infty}(w) = A e^{\sqrt{-\frac{3l(l+1)}{\Lambda}} w} + B e^{-\sqrt{-\frac{3l(l+1)}{\Lambda}} w} \] (3.5)

or,

\[ R_{\infty}(r) = A e^{\sqrt{-\frac{3l(l+1)}{\Lambda}}/r} + B e^{-\sqrt{-\frac{3l(l+1)}{\Lambda}}/r} \] (3.6)

The constants \( A \) and \( B \) are determined by the boundary conditions. As discussed in the introduction to AdS, because infinity is rushing towards the origin, it is necessary to impose boundary conditions there. If we demand that energy is conserved within the universe, then we must impose reflective boundary conditions. With these conditions in place, we have \( R'_{\infty} = 0 \) and \( \frac{dR_{\infty}}{dr} = 0 \). This gives us the condition that \( A = -B \).

Once the homogeneous solutions in both regions are computed, the inhomogeneous solution is computed by matching the two solutions at the position of the particle using the method of variation of parameters [4]. The solution to the non-homogeneous equation is

\[ R(r) = c_1(r)R_1(r) + c_2(r)R_2(r), \] (3.7)

where the functions \( c_1(r) \) and \( c_2(r) \) are defined, up to a constant of integration, by

\[ c'_1 = -\frac{R_2(r)F(r)}{W(r)}, \quad c'_2 = \frac{R_1(r)F(r)}{W(r)}, \] (3.8)

where \( F(r) \) is the source term of the non-homogenous differential equation, and \( W(r) \) is the Wronskian, \( W(r) = R_1 \frac{dR_2}{dr} - R_2 \frac{dR_1}{dr} \). In our case, \( F(r) \) is proportional to a delta function, \( F(r) = f(r)\delta(r - r_0) \), and because \( R_1 \) should only apply up to the particle, and \( R_2 \) should apply from the particle onwards, we know that \( c_1(r) \) must include a
Heaviside $\Theta(r_0 - r)$ and $c_2(r)$ must include $\Theta(r - r_0)$:

$$c_1(r) = \int \frac{R_2(r)f(r)\delta(r - r_0)}{W(r)}dr$$

$$= \frac{R_1(r_0)f(r_0)}{W(r_0)} \left[ \Theta(r - r_0) + d_1 \right]$$

$$c_2(r) = \int \frac{R_1(r)f(r)\delta(r - r_0)}{W(r)}dr$$

$$= \frac{R_1(r_0)f(r_0)}{W(r_0)} \left[ \Theta(r - r_0) + d_2 \right],$$

Where $d_1$ and $d_2$ are constants of integration. Because $c_2(r)$ should be proportional to $\Theta(r - r_0)$, $d_2 = 0$. On the other hand, $c_1(r)$ should be proportional to $\Theta(r_0 - r)$, which means that $d_1 = -1$, as $\Theta(r - r_0) - 1 = -\Theta(r_0 - r)$, and in the end we have

$$R(r) = \frac{R_2(r_0)f(r_0)}{W(r_0)} \Theta(r_0 - r)R_1(r) + \frac{R_1(r_0)f(r_0)}{W(r_0)} \Theta(r - r_0)R_2(r),$$

and so the two solutions agree at $r = r_0$.

The potential() function then returns $R(r_0)$ and $\frac{dR}{dr}(r_0)$ as computed from the 2nd region. We then subtract the regularization term, $(\partial_r \Phi^S)_l$, from each mode, and sum the remainder to determine the self-force.

### 3.2 Asymptotic Behaviour of the Mode Sum

The regularization of the retarded field takes the form

$$F_{tr} = \sum_l \left( F_{tr}^{ret} - \left( (l + 1/2)A + B + \frac{C}{l + 1/2} + \frac{D}{(l + 1/2)(l - 3/2)} + O(l^{-4}) \right) \right).$$

There are characteristic behaviours of the retarded field before and during the regularization process which can be seen from the form of equation (3.11). As an example, see figure 3.1. The retarded field diverges, and when expressed as a sum over modes, it diverges linearly in $l$. When the $(l + 1/2)A$ term is removed, we obtain a constant curve. When the $B$ term is removed, we get something that behaves like $l^{-2}$, and
with the removal of $\frac{D}{(l+1/2)(l-3/2)}$, we get a curve that decreases like $l^{-4}$. Ensuring we get the proper asymptotic behaviour of the mode fall-off curves is a strong test of numerical accuracy for the regularized field.

Figure 3.1: A typical example of the behaviour of the modes of the field as it is being regularized. This plot shows the mode behaviour for a particle at position $r_0 = 3.0$ in an SdS spacetime with $r_e = 2.0$ and $r_c = 5.0$. The black dash-dotted line is the retarded field. The flat dashed line is the retarded field with the $A$ term removed. The red dashed curve includes the removal of the $B$ term, and the dotted curve includes the removal of the $D$ term.

The regularization parameters $A, B, C,$ and $D$ are displayed below in terms of $M, \Lambda$. This representation is useful because the metric function looks the same in SdS and SAdS, which means that these are also the regularization parameters for SAdS.

\begin{align*}
A &= -\frac{\text{sign}(\Delta)}{r^2 \sqrt{1 - 2M - \Lambda r^2/3}} \\
B &= \frac{3M - r}{2r^3(1 - 2M - \Lambda r^2/3)} \\
C &= 0 \\
D &= \frac{4M^2r^6 - 3\Lambda r^5 + 9M\Lambda r^4 - 24M^2\Lambda r^3 + 27M^2r - 45M^3}{48r^6(1 - 2M/r - \Lambda r^2/3)^2}
\end{align*}

A problem that arises when regularizing $r$ near $r_c$ in SdS is that $r_c$ is a root of the metric function, and $D$ is inversely proportional to the square of the metric function.
If the metric function is factored out to eliminate this divergence, then the numerator grows more quickly than the denominator, and $D$ is a larger number for larger values of $r$, making it necessary to sum to higher $l$ values for proper convergence.

The regularization parameters for the static scalar self-force in SAdS (as discussed, we do not compute the scalar self-force in SdS) are

\[
\begin{align*}
A &= -\frac{\text{sign}(\Delta)}{r^2 \sqrt{1 - 2M - \Lambda r^2/3}} \\
B &= -\frac{(1 - M/r - 2\Lambda r^2/3)}{2r^2(1 - 2M - \Lambda r^2/3)} \\
C &= 0 \\
D &= -\frac{2\Lambda^2 r^7 + 4M\Lambda^2 r^6 + 9\Lambda r^5 - 45M\Lambda r^4 + 66M^2 \Lambda r - 9M^2 r + 9M^3}{48r^5(1 - 2M/r - \Lambda r^2/3)^2}. 
\end{align*}
\] (3.13)
Chapter 4

Results

4.1 The Self-Force in the Schwarzschild-de-Sitter Spacetime

4.1.1 Electromagnetic Self-Force

The static self-force on an electric charge in SdS was calculated for spacetimes with $r_e = 2.0$ and $r_e = 3.0, 5.0, 10.0$, and $20.0$. It is instructive to compare the computed results with the self-force in a pure Schwarzschild spacetime. The Schwarzschild static self-force was calculated by Smith and Will in 1980 [18], and they found it to be

$$F_r = \frac{q M \sqrt{f_0}}{r_0^3}. \quad (4.1)$$

This gives us a point of reference for what follows, and allows us to be sure that we recover Schwarzschild behaviour in the limit that the cosmological constant is small compared to the mass.

The SdS electromagnetic self-force is displayed in figure (4.1), alongside the self-force felt by a particle outside a Schwarzschild black hole with $M = 1$, so that the event horizon radius is the same for the Schwarzschild black hole and the SdS black hole. What we observe is that the SdS self-force has the same overall behaviour as the Schwarzschild self-force, but with a smaller magnitude. This is especially noticeable
for small values of $r_c$. As $r_c$ increases, the SdS self-force starts to coincide with the Schwarzschild self-force. This agrees with intuition, since a smaller cosmological horizon corresponds to a larger cosmological constant. The electromagnetic self-force vanishes in the pure de Sitter spacetime, and as we increase the cosmological constant we are entering a regime where the de Sitter behaviour plays a larger and larger role. As we push the cosmological horizon further and further away, its influence decreases, and we are left with behaviour that is nearly identical to Schwarzschild. As $r_c \to \infty$ we expect the SdS self-force to collapse to the Schwarzschild self-force.

![Diagram of self-force as a function of particle position for different values of $r_c$.](image)

Figure 4.1: The self-force ($F^r$) as a function of particle position is shown for an event horizon of $r_e = 2$ and a cosmological horizon of $r_c = 3.0, 5.0, 10.0, 20.0$. The Schwarzschild self-force of a black hole with $M = 1$ is shown with a dashed line.

As a way to measure the departure of the SdS self-force from the Schwarzschild self-force, we also show what we call the normalized self-force, which is not to be confused with the norm of the self-force vector (though sometimes they overlap). This is defined

36
in SdS as
\[
F_{\text{norm}} := \frac{F^r r_0^3}{qM\sqrt{f_0}},
\]  
(4.2)

where \(f_0\) is the SdS metric function evaluated at the particle’s position. Essentially, we have taken the form we know the self-force takes in Schwarzschild and we have divided it out of our computed self-force (the difference, of course, that \(f_0\) in SdS is not the same as \(f_0\) in Schwarzschild). We display some examples of the normalized self-force in figure (4.2).

\[\begin{array}{ccc}
\text{Figure 4.2: The normed self-force is shown for an event horizon of } r_e = 2 \text{ and a cosmological horizon of } r_c = 3.0, 5.0, 10.0, \text{ and } 20.0. \\
\end{array}\]

The normalised self force plots reinforce the previous observations that our results conform to the Schwarzschild results when the cosmological horizon is large. A more interesting observation is that in all cases it seems that the curves are approaching a value of 1 at \(r = r_c\), so that at \(r = r_c\) the self-force would be given by the Schwarzschild
self-force expression with the SdS metric function replacing the Schwarzschild one. We also observe a fairly linear decrease in the normalized self force with particle position for small values of $r_c$, which departs to something very non-linear as $r_c$ increases. One thing we can take away from the normalised force plots is that unless you are in the regime of the cosmological horizon being very near the black hole horizon, the Schwarzschild self-force (with the appropriate redefinition of $f(r)$) can actually be a reasonable approximation to the actual self-force.

A small turning point can be seen near the cosmological horizon in the $r_c = 10$, and a larger turning point in the $r_c = 20$, normalized self-force plot. This is not a physical feature, and is an artifact stemming from the fact that proper regularization in SdS becomes much more demanding near the cosmological horizon. This can be seen in figure (4.3), where the $D$-term starts out higher than the bare field at small values of $l$ for the particle position near $r_c$, which makes it necessary to sum to higher values of $l_{max}$ near the cosmological horizon for accurate results there.

4.2 The Self-Force in Schwarzschild-Anti-de Sitter

4.2.1 The Electromagnetic Self-Force

The SAdS electromagnetic self-force is displayed in figure (4.4). The first thing to notice in the SAdS self-force results is that we actually see an approximately linear increase in the self-force with particle position. This should seem very strange, because it means that the further away a charged particle is from the black hole, the more strongly it will be repelled from it. This picture is not as strange as it seems, however, because the linear repulsive force has very little to do with the black hole, and very much to do with the Anti-de Sitter background.

Referring to figure (4.4), we can see that the self-force is generally largest for small values of the cosmological length scale $r_c$, corresponding to large values of the cosmological constant $\Lambda$. Like de Sitter, there is no known analytic expression for the Anti-de Sitter static self-force. However, we have computed it numerically (see
Figure 4.3: The mode-sum fall off for two particle positions, one far from and one close to the cosmological horizon. The black dash-dotted line is the bare force contribution. The flat dashed line is the bare force with the A regularization term removed. The red dashed line also removes the B regularization term, and the the black dotted line includes the removal of the D term. Near the cosmological horizon, while we still have the correct asymptotic behaviour, the curve starts off quite a bit higher than the bare field. The result of this is that it becomes necessary to sum to higher modes to remove enough of the field that it converges to the proper self-force.
Appendix) and the results seen here are consistent with those of Anti-de Sitter. In particular, the linear behaviour is a strong feature of the AdS self-force. Unlike in the SdS case, \( r_c \) is not a horizon in SAdS, which makes it possible to compute the self-force for particle positions well past \( r = r_c \).

The fact that the self-force has the same linear behaviour in pure Anti-de Sitter also seems strange, since the self-force is known to vanish in pure de Sitter space. We show in the next section how this linear self-force behaviour arises in pure Anti-de Sitter. In short, it has to do with the fact that while de Sitter is conformally related to Minkowski space, Anti-de Sitter is conformally related to Minkowski space with a boundary. The boundary conditions imposed there result in the linearly increasing self-force. The Anti-de Sitter results confirm that far from the black hole, the self-force in SAdS is dominated by the cosmological constant.

In figure (4.4b) we again show the self-force \( F^r \), but this time for several values of the cosmological length \( r_c \) that are close to the event horizon radius. In this regime, we see an interesting phenomenon, which is that the self-force actually becomes negative near the event horizon. This is easiest to see in figure (4.5b), where we show the normalized self-force for small values of \( r_c \). There is no obvious reason that the self-force should be negative in this regime, and we observed no negative self-force in the Anti-de Sitter spacetime for small values of \( r_c \). These results are indicative of the non-intuitive behaviour of the self-force.

The normalized self-force is defined differently in SAdS than in SdS. We define \( F_{\text{norm}} \) by

\[
F_{\text{norm}} := \frac{F^r}{q^2 M \sqrt{f}}, \tag{4.3}
\]

we omit the factor of \( r^3 \) that is present in the SdS \( F_{\text{norm}} \) definition, since including it would make the self-force diverge very quickly.

Because the leading behaviour of \( \sqrt{f} \) is linear, when we remove it from \( F^r \) in \( F_{\text{norm}} \), we obtain curves that are asymptotically flat. Something that we can observe from the normalized self-force curves is that there is a change in the behaviour that occurs between \( r_c = 3.0 \) and \( r_c = 5 \), where the self-force asymptotically approaches
Figure 4.4: The self-force $F^r$ is shown for an event horizon of $r_e = 2$ for various values of $r_c$ by the solid curves. The self-force for a Schwarzschild black hole of mass $M = 1$ is represented by the dashed curves.
Figure 4.5: The normalized self-force is shown for an event horizon of $r_e = 2$ and various values of $r_c$. The normalized self-force is defined as $F_{\text{norm}} := \frac{F}{q^2 M \sqrt{f}}$. 

(a) $r_c = 3.0, 5.0, 10.0$, and $20.0$.

(b) $r_c = 2.5, 2.6, 2.7$, and $3.0$. 

42
some value from below for the small values of $r_c$, but it switches to an approach from above for larger values of $r_c$.

![Graph](image)

Figure 4.6: Here we show the asymptotic (at a particle position of $r = 8r_c$) values of the normalized self-force versus $r_c$. With the red dashed line we show $1/(4r_c^2)$ vs. $r_c$, which fits the trend quite well. The reason for this particular guess is given in section 4.2.2.

In figure (4.6) we show asymptotic normalized self-force as a function of $r_c$. We’ve determined that the normalized self-force scales approximately as $r_c^{-2}$. If we were to define the normalized self-force without dividing out the black hole mass, that is, as $F^r/\sqrt{f}$, then we would find instead that the asymptotic self-force scales with $-\Lambda$, and has no dependence on $M$. This will be explained in the following section, after we’ve discussed the self-force in AdS for context.

### 4.2.2 The Self-Force in AdS and Flat Spacetimes

We’ve seen that the self-force in SAdS has a tendency to increase linearly with particle position. In fact, this is not particular to SAdS, and the same behaviour is evident in Anti-de Sitter as well. This seems strange, considering we’ve made the argument that the de Sitter spacetime is conformally flat and therefore has no self-force (because Maxwell’s equations are conformally invariant and flat spacetime has no self-force).
Shouldn’t the same be true for the Anti-de Sitter spacetime, then, since the only difference between dS and AdS is a sign? The answer to this is no, and the reason is somewhat subtle. In the Anti-de Sitter spacetime, the negative cosmological constant provides a negative acceleration of space, which insures that if a photon is sent towards spatial infinity, it can actually arrive there in a finite amount of time. This is a feature not evident in the de Sitter spacetime (in fact, in the de Sitter spacetime we have much the opposite case, where the presence of the cosmological horizon insures that a photon is essentially disconnected from spatial infinity forever). This makes the boundary conditions at \( r = \infty \) in the Anti-de Sitter spacetime very important. A common way to deal with spatial infinity in Anti-de Sitter is to impose reflective boundary conditions, the justification being that it is the only way to ensure that the universe conserves energy. With these boundary conditions in place, then, we should not expect Anti-de Sitter to be conformal to flat space, but to flat space with a boundary. In this case, we know more or less what the force should be.

AdS space, like dS space, can be pictured as a hyperboloid,

\[
-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 - Z_4^2 = -a^2,
\]

where \( a = \sqrt{-3/\Lambda} \), embedded in flat 5-dimensional space:

\[
ds^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 - dZ_4^2.
\]

The transformation to the spherical coordinate representation is given by [12]:

\[
Z_0 = \sqrt{a^2 + r^2} \sin t/a \\
Z_1 = r \cos \theta \\
Z_2 = r \sin \theta \cos \phi \\
Z_3 = r \sin \theta \sin \phi \\
Z_4 = \sqrt{r^2 + a^2} \cos t/a,
\]

and the transformation to the manifestly conformally flat coordinate system is given
by

\begin{align}
Z_0 &= \frac{1}{2x}(a^2 + s) \\
Z_1 &= \frac{1}{2x}(a^2 - s) \\
Z_2 &= \frac{ay}{x} \\
Z_3 &= \frac{az}{x} \\
Z_4 &= \frac{a\eta}{x},
\end{align}

where \( s = -\eta^2 + x^2 + y^2 + z^2 \) and the metric is given by

\[ ds^2 = \frac{a^2}{x^2}(-d\eta^2 + dx^2 + dy^2 + dz^2). \]

We see that \( x = 0 \) corresponds to the boundary. We need a relationship between \( x \) and \( r \). Note that \( r^2 = Z_1^2 + Z_2^2 + Z_3^2 \). This gives us

\begin{align}
\frac{1}{4x^2} (a^2 - s)^2 + \frac{a^2 y^2}{x^2} + \frac{a^2 z^2}{x^2} \\
\frac{1}{4x^2} (a^2 + \eta^2 - x^2)^2,
\end{align}

where in the second line we’ve taken the liberty of setting \( y = z = 0 \). If we stay in the regime where we are close to the boundary, and let \( x \to 0 \) (corresponding to the large \( r \) limit), we have

\[ r^2 = \frac{1}{4x^2}(a^2 + \eta^2)^2. \]

There is something important to notice here. We have assumed that we have a static particle in the spherical representation, that is, \( r = r_0 = \text{constant} \). In \( x, \eta \) coordinates, \( r = \text{constant} \) describes a hyperbola. This means that our static particle in spherical coordinates is not static in the conformally flat coordinates. The particle is, however, quasi-static in the limit that \( x \to 0 \). This is evident from the hyperbolic equation, and can also be seen by computing the 4-velocity of the particle. The static particle in spherical coordinates has a 4-velocity with only a time component, given
by $U^t = \frac{1}{\sqrt{1+r^2/a^2}}$. It is shown in the appendix that when the 4-velocity is transformed to the conformally flat coordinates $(\eta, x, y, z)$ that it takes the form

$$
U^\alpha = -\frac{x[(a^2 + \eta^2 + x^2), 2\eta x, 0, 0]}{a\sqrt{(a^4 + \eta^4 + x^4 + 2a^2\eta^2 + 2a^2x^2 - 2x^2\eta^2)}},
$$

(4.12)

which reduces to $U^\alpha = \frac{-x}{a}[1, 0, 0, 0]$ in the small $x$ limit. This corresponds to the large $r$ limit, so the self-force we determine will only be valid in this regime.

The picture is that we have a charged particle held static at some large $r$ in the spherical representation of the Anti-de Sitter spacetime, which has a reflective boundary at $r = \infty$. In the conformally flat representation, our particle is moving slowly at some small $x$, and the $x = 0$ surface is seen to take the place of the $r = \infty$ boundary. To determine the force felt by the particle, we consider an analogous situation in flat Minkowski spacetime with $(\eta, x, y, z)$ coordinates. For clarity, we have the metric of the Minkowski spacetime,

$$
ds^2 = -d\eta^2 + dx^2 + dy^2 + dz^2,
$$

(4.13)

which is conformally related to the Anti-de Sitter metric in equation (4.8),

$$
\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}
$$

(4.14)

$$
= \eta_{\alpha\beta}.
$$

(4.15)

where tildes are used for quantities in the Minkowski spacetime, and the conformal factor is given by $\Omega^2 = \frac{x^2}{a^2}$. Furthermore, the conformal invariance of Maxwell’s equations can be stated as

$$
\tilde{F}_{\alpha\beta} = F_{\alpha\beta}.
$$

(4.16)

This means that we can determine the relevant components of the electromagnetic field tensor in the Anti-de Sitter spacetime by considering an analogous situation in the Minkowski spacetime. Therefore, we imagine we have a particle slowly tracing out a hyperbolic path in Minkowski space at some small $x$, and where we have also
placed a conducting surface at $x = 0$ to mimic the reflective boundary conditions of the Anti-de Sitter spacetime. The electric field of a static charge $q$ a distance $x$ from a conducting sheet can be determined from the method of images. In order for the potential to vanish everywhere on the sheet, a mirror charge $-q$ can be placed at the position $-x$, and the electric field at the position of $q$ caused by the image charge is $E = -\frac{q}{4x^2}$. Strictly speaking, our charge $q$ is not stationary, and it has an acceleration. We should therefore expect there to also be radiative effects present. We make the assumption, however, that the image charge field will dominate over any radiative effects. This is reasonable because in the limit that $x \to 0$ the image charge field will diverge as $\frac{1}{x^2}$, and as $x \to 0$ the particle becomes approximately static, in which case radiative effects are likely to be small.

With the assumption above, we have that $\tilde{F}_{\eta x} = -\frac{q}{4x^2}$ (the sign is a consequence of the 4-velocity being negative). Because Maxwell’s equations are conformally invariant, we also have that $F_{\eta x} = -\frac{q}{4x^2}$. The tensor transformation law is given by

$$T_{\alpha\beta} = T'_{\alpha'\beta'} \frac{\partial x'^\alpha}{\partial x^\alpha} \frac{\partial x'^\beta}{\partial x^\beta}.$$  \hspace{1cm} (4.17)

This means that $F_{tr}$ can be computed by

$$F_{tr} = F_{\eta x} \frac{\partial \eta}{\partial t} \frac{\partial x}{\partial r} + F_{x\eta} \frac{\partial x}{\partial t} \frac{\partial \eta}{\partial r}. \hspace{1cm} (4.18)$$

Equations 4.6 and 4.7 can be inverted to obtain $\eta$ and $x$ as functions of $t$ and $r$, which are then given by (assuming $y = z = 0$)

$$\eta = \frac{aZ_4}{Z_0 + Z_1} = \frac{a \sqrt{a^2 + r^2} \cos(t/a)}{\sqrt{a^2 + r^2} \sin(t/a) + r} \hspace{1cm} (4.19)$$

$$x = \frac{a^2}{Z_0 + Z_1} = \frac{a^2 \sqrt{a^2 + r^2} \sin(t/a)}{\sqrt{a^2 + r^2} \sin(t/a) + r}. \hspace{1cm} (4.20)$$

This gives the result that $F_{tr} = -\frac{q}{4a^2}$. The radial self-force is then given by

$$F^r = qg^{rt} F_{rt} U^t. \hspace{1cm} (4.21)$$
In the spherical coordinates, \( g^{rr} = \sqrt{f(r)} \), \( U^t = \frac{1}{\sqrt{f(r)}} \), and \( F_{rt} = -F_{tr} \), giving us

\[
F^r = \frac{q^2 \sqrt{f(r)}}{4a^2} = \frac{q^2 \sqrt{1 + \frac{r^2}{a^2}}}{4a^2},
\]

which asymptotically gives us \( F^r \sim \frac{q^2 r}{4a^2} \).

We see that the self force in the spherical coordinates, and in the regime where \( r \) is large, should be proportional to \( r \), which is what we observe in practice. We can also normalise this by dividing it by \( \sqrt{f} \). If we do that, then we get \( F_{\text{norm}} = \frac{q^2}{4a^2} \).

In AdS, \( a \) is exactly the cosmological length scale, so we have also found that our simple argument has recovered the approximate asymptotic \( 1/r_c^2 \) behaviour for the normalised self-force that we saw in SAdS.

We also computed self-force results in AdS numerically, and display \( F_{\text{norm}} \) vs \( r_c \) in figure (4.7), where we confirm this behaviour.

There is something to note here, however. \( F_{\text{norm}} \) is defined differently in the Anti-de Sitter spacetime and the Schwarzschild-Anti-de Sitter spacetime. In SAdS we defined \( F_{\text{norm}} = \frac{F^r M}{\sqrt{f}} \), but in AdS there is no mass. What this tells us is that in SAdS, the asymptotic behaviour of the self-force is similar to that of the AdS self-force, but it is re-scaled by the black hole mass \( M \). This seems strange, because asymptotically we expect the self-force to depend on the AdS features of the spacetime rather than the Schwarzschild features. In fact, this is still the case, and the confusion here comes from the variables we are using.

In Ads, we have \( r_c^2 = -\frac{3}{\Lambda} \). We can therefore equivalently write our \( \frac{q^2}{4r_c^2} \) force as \( -\frac{q^2 \Lambda}{12} \). In SAdS, the parameters \( r_e \) and \( r_c \) introduce a mixing between \( M \) and \( \Lambda \), and this is what makes it seem like the asymptotic self-force depends on the mass. In fact it only depends on \( \Lambda \), but it seems to depend on the mass when expressed in terms of \( r_c \). This is because we have \( \Lambda = \frac{M}{2r_c^2} \). When \( r_e = 2 \), this gives us the \( \frac{M}{4r_c^2} \) behaviour that we saw in the previous section.

We display asymptotic values of \( \frac{F^r}{\sqrt{f}} \) in SAdS in figure (4.8), and show that we recover the same asymptotic behaviour in SAdS as in AdS, provided we properly define things in terms of the cosmological constant, and not the cosmological horizon \( r_c \). This is reassuring, because when we are in the
asymptotic regime it seems natural the self-force should be dominated by the cosmological component of the spacetime rather than the black hole component.

Figure 4.7: Here we show the asymptotic (at a particle position of $r = 8r_c$) values of the normalized self-force versus $r_c$ in AdS. With the red dashed line we show $1/(4r_c^2)$ (or equivalently, $-\frac{\Lambda}{12}$) vs. $r_c$.

4.2.3 The Scalar Case

Asymptotically, the scalar self-force (displayed in figure (4.9) alongside the self-force in the Anti-de Sitter spacetime) in SAdS behaves very similarly to the electric self-force, that is, we see the same linear behaviour. Once again, this asymptotic behaviour can be anticipated, because the same behaviour is evident in the Anti-de Sitter spacetime. The SAdS behaviour departs from AdS behaviour near the black hole, however, where we see the self-force plunges quickly towards zero as the particle approaches the event horizon of the black hole.

The normalized self-force is pictured in figure (4.10), along with the normalized Anti-de Sitter self-force. Two interesting features are evident. The first is that while both seem to be asymptotically approaching a constant, the Anti-de Sitter self-force does so universally from below (i.e., it is uniformly increasing), whereas in the SAdS case, it is uniformly decreasing for $r_c = 3.0$, but for higher values of $r_c$ it features a
Figure 4.8: Here we show the asymptotic (at a particle position of \( r = 8r_c \)) values \( \frac{F_r}{\sqrt{f}} \) (this is the same as the normalized SAdS self-force, but without the factor of \( M \) divided out) versus \( r_c \) in AdS. With the red dashed line we show \( -\frac{\Lambda}{12} \) vs. \( r_c \).

minima and is neither uniformly increasing or decreasing. It also seems that when the SAdS self-force is normalized by the mass, that the SAdS self-force and the Anti-de Sitter self-force may be approaching the same (or nearly the same) asymptotic value, which is something that was also seen in the electrostatic self-force.

Unfortunately, we do not have conformal invariance for the scalar field equations, so we can not make the same sort of argument as we made in the electromagnetic case to derive an expression for the asymptotic value of the self-force. However, it does seem reasonable that we obtain the same sort of linear behaviour if we think of our scalar charge as an electrostatic charge, and consider the dominant feature of the spacetime to be the reflective boundary. We can also plot the asymptotic values of \( \frac{F_r}{\sqrt{f}} \), as we did in the electromagnetic case, for both the Anti-de Sitter and SAdS scalar self-force, and these are displayed in figures (4.11) and (4.12). Once again we see that both trends can be fit by the same curve (in this case, by \( -\frac{A}{4} \) rather than \( -\frac{\Lambda}{12} \) as in the electrostatic case).
Figure 4.9: The scalar self-force for Schwarzschild-Anti-de Sitter and Anti-de Sitter spacetimes.
Figure 4.10: The normalized scalar self-force for Schwarzschild-Anti-de Sitter and Anti-de Sitter spacetimes. We define the normalized self-force to be $F_{\text{norm}} = F(r)/(M\sqrt{f})$ for Schwarzschild-Anti-de Sitter, and $F_{\text{norm}} = F(r)/\sqrt{f}$ for Anti-de Sitter.
Figure 4.11: Here we show the asymptotic (at a particle position of $r = 8r_c$) values of the normalized self-force ($F_r/\sqrt{f}$) versus $r_c$ in AdS. With the red dashed line we show $3/(4r_c^2)$ (or equivalently, $-\Lambda/4$) vs. $r_c$.

Figure 4.12: Here we show the asymptotic (at a particle position of $r = 8r_c$) values of $F_r/\sqrt{f}$ versus $r_c$ in SAdS. With the red dashed line we show $-\Lambda/4$ vs. $r_c$. 
Chapter 5

Conclusion

5.1 Summary of Research

The static electromagnetic self-force has been computed for the Schwarzschild-de Sitter and Schwarzschild-Anti-de Sitter spacetimes (as well as pure de Sitter and pure Anti-de Sitter spacetimes). The static scalar self-force has been computed in the SAdS and Anti-de Sitter spacetimes, and was found to be not computable in the de Sitter and SdS spacetimes. Additionally, an approximation for the asymptotic electromagnetic self-force $F^r$ in Anti-de Sitter has been derived, and was found to be linear with a slope of $1/(4r_c^3)$. The magnitude of the force, $F^r / r_c$, was found to approach the constant $-q^2 \Lambda_{12}$, and this seems to hold as well for SAdS. It was seen in SdS that the electromagnetic self-force has a similar form to Schwarzschild, but is generally weaker, and approaches the Schwarzschild self-force as $r_c \to \infty$. It was also seen through the normalised self-force that the EM self-force approaches $F^r = q^2 M \sqrt{\gamma} / r_c$ as $r \to r_c$. The scalar self-force in AdS and SAdS also were found to be asymptotically linear, and though we have no theoretical derivation of the asymptotic behaviour of the self-force in the scalar case, numerical results suggest that the magnitude of the self-force, $F^r / r_c$, asymptotically approaches $-q^2 \Lambda / 4$.

We’ve found that the SAdS electromagnetic self-force has an asymptotic linear behaviour, as well as a dependence on $1/r_c^3$, and we’ve confirmed that this behaviour is to be expected by considering the conformal transformation from Anti-de Sitter
to Minkowski with a boundary, as well as by computing the Anti-de Sitter self-force numerically. Even though the Schwarzschild self-force is positive, and the asymptotic Anti-de Sitter self-force that was derived is positive, we observed a negative self-force in SAdS in the regime where $r_c$ is close to $r_e$. This demonstrates the non-intuitive nature of the self-force, particularly in regimes where there are two competing scales. There was no similar negative behaviour seen in the scalar SAdS self-force.

### 5.2 Self-Force Intuition

In our introductory statements, we named the non-intuitive nature of the self-force as a motivation for this research. In particular, the repulsive behaviour of the self-force in the Schwarzschild spacetime is the opposite of what one would expect from electrostatic considerations after replacing the Schwarzschild black hole with a spherical conductor. A reasonable question to ask, then, is whether or not this work has succeeded in improving our intuition of the self-force. The answer to this is both yes and no.

Why does the spherical conductor argument fail for the Schwarzschild self-force? One possible reason is that while the event horizon is an equipotential surface, it is not a conductor, and it is not a home to free charges that can redistribute to create a net force on the outside charge. Another reason is that the self-force in Schwarzschild is a product of the curvature of the spacetime, and not of any particular features of the event horizon. The self-force arises because the charge interacts with its own field in the presence of the curvature of the black hole. This is a picture that does not change in the Schwarzschild-de Sitter spacetime. This is in sharp contrast to the anti-de Sitter spacetime, however, where we did successfully use electrostatic arguments to predict not only the sign of the asymptotic self-force, but the magnitude as well.

Why does this type of argument succeed in the anti-de Sitter (and asymptotic region of SAdS) spacetime while it failed in the Schwarzschild spacetime? The answer is that unlike the Schwarzschild spacetime where the self-force is a product of curvature, it seems that in the anti-de Sitter spacetime, the self-force is a product of
the reflective boundary conditions. In the de Sitter spacetime, the self-force vanishes, so we can conclude that the cosmological constant does not introduce any type of curvature that induces a self-force in the charge. When we change the sign of the cosmological constant this behaviour should not change (the spacetime is still conformally flat), but the effect that it does have is to contract the spacetime so that a boundary must appear at spatial infinity. In that sense, the curvature indirectly produces the self-force through the boundary conditions. The intuition we have obtained into the self-force, then, is dependent on the origin of that force.

5.3 Future Research Directions

Two natural extensions of this work are i) to compute the static self-force in spherically symmetric spacetimes in higher dimensions, and ii) to repeat this study, but to remove the condition that the charge be static.

Black holes in higher dimensions are an active area of research, partly because string theory involves gravity in more than 4 dimensions, and partly because the AdS-CFT correspondence maps \(d\)-dimensional black holes to \(d - 1\)-dimensional conformal field theories (so if one is interested in a 4-\(d\) field theory, they must consider a 5-\(d\) black hole)\cite{9}. The work would require generalizing the mode-sum regularization scheme to higher dimensions. An expectation is that fundamentally different behaviours would exist in even and odd dimensions. The largest complication that would likely arise is in the renormalization in higher dimensions. In 3 + 1 dimensions, the retarded field is decomposed into a regular component responsible for the self-force, and a singular component that is absorbed into the effective mass of the particle. In 2002 Gal’tsov investigated the radiation-reaction force in flat spacetime in higher dimensions, and found that in even dimensions higher than 4 that the divergences could not be removed by mass renormalization, and in odd dimensions greater than 4, the Lorentz-Dirac force depends on the entire past history of the charge \cite{11}. Also in 2002, Kazinski, Lyakhovich, and Sharapov investigated radiation reactions in higher dimensions and found that the divergences could be regularized not by mass renormalization, but by
introducing additional terms to the action [14]. This suggests that generalizing the mode-sum procedure to higher dimensions will not be trivial. Some of this will not be a problem for the static case (there is no radiation reaction component to the self-force on a static charge), but it is indicative of a more complicated Green’s function structure in higher dimensions.

The second possibility, that of considering moving charges in SdS and SAdS, would allow us to see how the trajectories of the charges would evolve in time. This would be particularly interesting in SAdS, where the static self-force increases with distance. In the case of a non-static charge, it would be expected that it would feel a similar force to the static case, but it would also feel an additional radiative force opposing its motion. It is likely that special care would have to be taken in defining the initial conditions of the problem because of the added complication of the reflective boundary. Calculations of the motion of charges subject to a self-force are sensitive to initial conditions, and noise (“junk radiation”) at early times in the simulations can arise when the initial conditions are unphysical. It seems possible that the reflective boundary property of SAdS will allow the junk radiation to interfere with the charge at later times in the simulation as well. If this ends up being the case, care will have to be taken to either begin the simulation with physical initial conditions (which may be difficult, since the self-force depends on the entire past history of motion), or else to carefully remove the noise after the simulation begins.
Chapter 6

Appendix

6.1 Static Assumption in the Anti-de Sitter Spacetime

In spherical coordinates we have the metric

\[ ds^2 = -(1 + r^2/a^2)dt^2 + (1 + r^2/a^2)^{-1}dr^2 + r^2d\Omega^2, \]  

(6.1)

where \( a = \sqrt{-3/\Lambda} \). In these coordinates, because the 4-velocity of the particle only has a time component (because the particle is static), normalization gives us

\[ U_t = \frac{1}{\sqrt{1+\frac{r^2}{a^2}}} = \frac{a}{\sqrt{a^2+r^2}}. \]

Now we can transform to the general AdS coordinates \((Z_0, Z_1, ... )\), and from there we can transform to the manifestly conformally flat coordinates. The coordinate transformation rules are given by [12]

\[ Z_0 = \sqrt{a^2+r^2} \sin t/a \]  

(6.2a)

\[ Z_1 = r \cos \theta \]  

(6.2b)

\[ Z_2 = r \sin \theta \cos \phi \]  

(6.2c)

\[ Z_3 = r \sin \theta \sin \phi \]  

(6.2d)

\[ Z_4 = \sqrt{r^2+a^2} \cos t/a. \]  

(6.2e)
Note that \( \sqrt{Z_0^2 + Z_4^2} = \sqrt{a^2 + r^2} \), so that in terms of the \( Z \) coordinates, \( U^t = \frac{a}{\sqrt{Z_0^2 + Z_4^2}} \).

The vector transformation law is \( U^\alpha' = \partial x'^\mu \partial x^\alpha U^\mu \), which gives us

\[
U^Z_0 = \frac{\partial Z_0}{\partial t} U^t = \frac{\sqrt{a^2 + r^2}}{a} \cos\left(\frac{t}{a}\right) \frac{a}{\sqrt{Z_0^2 + Z_4^2}} = \frac{Z_4}{\sqrt{Z_0^2 + Z_4^2}} \quad (6.3)
\]

\[
U^Z_4 = \frac{\partial Z_4}{\partial t} U^t = -\frac{\sqrt{a^2 + r^2}}{a} \sin\left(\frac{t}{a}\right) \frac{a}{\sqrt{Z_0^2 + Z_4^2}} = -\frac{Z_0}{\sqrt{Z_0^2 + Z_4^2}}. \quad (6.4)
\]

So in this coordinate system, our 4-velocity is given by \( \frac{1}{\sqrt{Z_0^2 + Z_4^2}} (Z_4, 0, 0, -Z_0) \). Now we can transform to the manifestly flat coordinates, and the coordinate transformation is given by

\[
Z_0 = \frac{1}{2x} (a^2 + s) \quad (6.5a)
\]

\[
Z_1 = \frac{1}{2x} (a^2 - s) \quad (6.5b)
\]

\[
Z_2 = \frac{ay}{x} \quad (6.5c)
\]

\[
Z_3 = \frac{az}{x} \quad (6.5d)
\]

\[
Z_4 = \frac{a\eta}{x} \quad (6.5e)
\]

where \( s = -\eta^2 + x^2 + y^2 + z^2 \). Note that \( Z_0 + Z_1 = a^2/x \), so we have \( x = \frac{a^2}{Z_0 + Z_1} \). We can now invert the relations to express \( \eta, x, y, z \) in terms of \( Z_0, Z_1, \ldots \). We have

\[
\eta = \frac{aZ_4}{Z_0 + Z_1} \quad (6.6a)
\]

\[
x = \frac{a^2}{Z_0 + Z_1} \quad (6.6b)
\]

\[
y = \frac{aZ_2}{Z_0 + Z_1} \quad (6.6c)
\]

\[
z = \frac{aZ_3}{Z_0 + Z_1}. \quad (6.6d)
\]

The \( \eta \) component of the four velocity, then, is given by
\[ U^\eta = \frac{\partial \eta}{\partial Z_0} U^{Z_0} + \frac{\partial \eta}{\partial Z_4} U^{Z_4} \]  
\[ = -\frac{a Z_4}{(Z_0 + Z_1)^2 \sqrt{Z_0^2 + Z_1^2}} Z_4 - \frac{a}{(Z_0 + Z_1)^2 \sqrt{Z_0^2 + Z_1^2}} Z_0 \]  
\[ = -\frac{a}{(Z_0 + Z_1)^2 \sqrt{Z_0^2 + Z_1^2}} Z_4 \]  
\[ \]  
(6.7)  
(6.8)  
(6.9)  

The other components of \( U^\alpha \) are

\[ U^x = -\frac{a^2 Z_4}{(Z_0 + Z_1)^2 \sqrt{Z_0^2 + Z_1^2}} \]  
\[ U^y = -\frac{a Z_2 Z_4}{(Z_0 + Z_1)^2 \sqrt{Z_0^2 + Z_1^2}} \]  
\[ U^z = -\frac{a Z_3 Z_4}{(Z_0 + Z_1)^2 \sqrt{Z_0^2 + Z_1^2}} \]  

(6.10a)  
(6.10b)  
(6.10c)  
(6.10d)  

We can make this slightly simpler by letting \( y = z = 0 \), which is equivalent to setting \( Z_2 = Z_3 = 0 \). In this case the only non-zero components of \( U^\alpha \) are \( U^\eta \) and \( U^x \). We can write \( U^\alpha \) terms of \( \eta \) and \( x \) by substituting in, and after a good deal of simplification, we wind up with

\[ U^\alpha = -\frac{x[(a^2 + \eta^2 + x^2), 2\eta x, 0, 0]}{a \sqrt{(a^4 + \eta^4 + x^4 + 2a^2\eta^2 + 2a^2x^2 - 2x^2\eta^2}}} \]  

(6.11)  

This is plainly much different from the 4-velocity we would obtain if we assumed the particle was static in these coordinates. However, in the limit that \( x \to 0 \), the \( U^x \) component vanishes, and the rest simplifies enormously:

\[ \lim_{x \to 0} U^\alpha = \frac{-x}{a} [1, 0, 0, 0]. \]  

(6.12)  

So we do have a static particle in the conformally flat representation, provided we stay in the large \( r/\text{small } x \) limit. We have an apparent problem, however. The \( x \) was factored from a square root, and therefore it is necessarily a positive quantity. This
makes $U^\eta$ necessarily negative, and thus, our particle exists backwards in time. More specifically, it is travelling backwards in time as described by the time coordinate $\eta$. In terms of our spherical coordinates, $x$ and $\eta$ are given by

$$\eta = \frac{a Z_4}{Z_0 + Z_1} = \frac{a \sqrt{a^2 + r^2 \cos(t/a)}}{\sqrt{a^2 + r^2 \sin(t/a) + r}}$$

(6.13)

$$x = \frac{a^2 Z_0 + Z_1}{Z_0 + Z_1} = \frac{a^2}{\sqrt{a^2 + r^2 \sin(t/a) + r}}$$

(6.14)

Note that we have substituted $r$ and not $r \cos(\theta)$ for $Z_1$ because the condition $y = z = 0$ is equivalent to the condition $\theta = 0$. What we can do now is ask what $\eta$ is doing when we impose the condition that $x$ be positive. We choose $a = 0.5$ and $r = 1.0$, and plot $x$ and $\eta$ as functions of $t$ below:

![Figure 6.1: The $\eta$ and $x$ coordinates as functions of $t$. Note that $\eta$ is actually a decreasing function of time $t$.](image)

What we notice is that when $x$ is positive, $\eta$ is decreasing with respect to the time coordinate $t$. In other words, as $t$-time moves forwards, $\eta$-time moves backwards. The negative we have in our 4-velocity ensures that our particle moves backwards in backwards time, and so it must actually be moving forwards in time. We conclude that the negative sign is correct, and the 4-velocity of our semi-static particle is $U^\alpha = -\frac{Z}{a}[1, 0, 0, 0]$. 

61
6.2 The Electromagnetic Self-Force in AdS

Below we display results for the numerically computed electromagnetic self-force $F^r$ and $F_{\text{norm}}$ in the Anti-de Sitter spacetime. What’s observed for $F^r$ is some early non-linear behaviour when $r$ is small, which then gives way to the expected linear behaviour.
Figure 6.2: The self-force and normalized self-force in Anti-de Sitter. We define the normalized self-force to be $F_{\text{norm}} = F^r / \sqrt{f}$. 

(a) $F^r$

(b) $F_{\text{norm}}$
Bibliography


