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010

AN EXACT SOLUTION OF STIKKER'S NONLINEAR HEAT EQUATION*

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Abstract. Exact solutions are derived for a nonlinear heat equation where the conductivity is a linear fractional function of (i) the temperature gradient or (ii) the product of the radial distance and the radial component of the temperature gradient for problems expressed in cylindrical coordinates. It is shown that equations of this form satisfy the same maximum principle as the linear heat equation, and a uniqueness theorem for an associated boundary value problem is given. The exact solutions are additively separable, isolating the nonlinear component from the remaining independent variables. The asymptotic behaviour of these solutions is studied, and a boundary value problem that is satisfied by these solutions is presented.

Key words. nonlinear heat conduction, exact solution, diffusion

AMS subject classifications. 35C05, 80A20, 35K05

1. Introduction. Considerable work has been done in the analysis of certain types of nonlinear heat conduction (diffusion) equations where the conductivity (diffusivity) is a function of the temperature (concentration) [2], [6]. The porous media equation, a nonlinear diffusion equation of this type, is an example [1], [9]. A review of some nonlinear equations that admit exact solutions has been completed recently by Rogers and Ames [10]. Some analysis has also been done for equations where the conductivity is a function of the temperature gradient or its magnitude [3].

In this paper we shall develop exact solutions for the following two nonlinear heat conduction equations in $\mathbb{R}^3$:

$$u_t = \left(\frac{a + bu_x}{c + du_x} u_x\right)_x + u_{yy} + u_{zz}, \tag{1}$$

$$u_t = \frac{1}{r}\left(\frac{a + bru_r}{c + dru_r} ru_r\right)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}, \tag{2}$$

where (x, y, z) and (r, θ, z) are Cartesian and cylindrical coordinates, respectively; subscripts denote partial differentiation; all variables are nondimensional; and the nonzero constants a, b, c, and d satisfy certain conditions given below. The constructed solution to (2) will be radially symmetric. An equation of type (2) was derived by Stikker for the problem of heat conduction in steel coils during the batch annealing process [11].

The remainder of this paper is organized as follows. In § 2 we shall give Stikker's physical derivation of the nonlinear radial heat conductivity, consider some of the physical constraints on the constants, and simplify (1) and (2). Some properties of solutions to these equations are given in § 3, exact solutions for both equations will be derived in § 4, and § 5 will discuss the types of boundary value problems that these solutions will satisfy.

2. Derivation and simplification. Since Stikker's work is not generally available, we shall summarize his derivation of the heat conduction equation in § 2.1 below.

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2.1. Stikker's derivation. Stikker was concerned with the heat conduction equation relevant to the tight-coil batch annealing process. Batch annealing is a stage in the production of steel whose purpose is to restore the steel's ductility after cold rolling. In this process, steel coils are heated to a temperature of about 700° C and then, after some time, allowed to cool.

To simplify the mathematics, the steel coil is viewed geometrically as a collection of concentric rings rather than a spiral, thus achieving axial symmetry. This type of approximation has also been employed in the numerical work by Harvey [5] and Jaluria [7], [8]. Forte [4] gives some discussion of how the conductivity in the radial direction for Stikker's model may be modified to account for its failure to reflect the direct connection between the coil windings.

Since the gaps between the windings of the coil have only a very slight influence on the specific heat of the coils, for all practical purposes, the specific heat of the coils may be taken as that of the steel, c, which has been found experimentally to increase between 20° and 700° C by a factor of 1.7. The heat conductivity in the axial direction, or z-direction, is the heat conductivity of the steel, λ_s, which decreases between 20° and 700° C by a factor of 1.5. The heat conductivity of the coils in the radial direction is, however, highly dependent on the width of the gaps between the windings. The heat conduction equation then reads

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\lambda_{eq} r \frac{\partial T}{\partial r}\right] + \frac{\partial}{\partial z}\left[\lambda_s \frac{\partial T}{\partial z}\right], \tag{3}$$

where ρc is the specific heat of the steel per unit volume, λ_s is the heat conductivity of the steel, and λ_{eq} is the equivalent heat conductivity of the coil in the radial direction.

Figure 1 illustrates the situation in the coils. The resistance for the heat flow between the planes r_1 and r_2 is given by

$$R = \frac{d_1}{\lambda_s} + \frac{d_2}{\lambda_g},$$

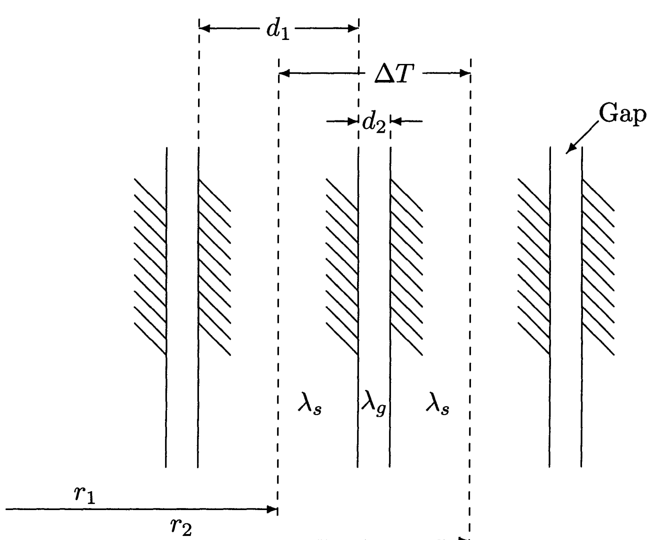

FIG. 1. *Coil cross section.*

where λ_g is the heat conductivity of the gas surrounding the coils. The heat resistance per unit length is

$$R_L = \frac{(d_1/\lambda_s) + (d_2/\lambda_g)}{d_1 + d_2},$$

and the equivalent heat conductivity is $\lambda_{eq} = 1/R_L$. The quantities d_1 and d_2 are functions of the temperature, which can be approximated by

$$d_1 \approx d_{10}, \qquad d_2 \approx d_{20} + r_2 A \Delta T, \tag{4}$$

where A is the expansion coefficient of steel. Although Stikker does not elucidate the above claim, we shall provide some justification.

The expansion coefficient of steel is of the order $10^{-5}\mathrm{C}^{-1}$. A temperature difference of ΔT between two rings of the steel would be reflected by a somewhat smaller difference across the thickness of one ring since the gap would certainly account for a significant portion of the temperature gradient. Consequently, the change in thickness of the steel due to the (assumed linear) temperature difference would be less than

$$\int_0^{d_{10}} A\Delta T \left(\frac{r}{d_{10}}\right) dr = \frac{1}{2} A \Delta T d_{10},$$

which would require an unphysically high value of ΔT ($\approx 200°$ C) to obtain even a 0.1% increase in the steel thickness. The steel thickness is thus virtually unaffected by an inter-ring temperature gradient.

In contrast, the width of the gap is significantly altered by a temperature difference between two successive rings. Suppose the two rings are at the same temperature. We then have the relation

$$r_2 - r_1 = d_{10} + d_{20}. \tag{5}$$

If the temperature of the outer ring is subsequently increased by ΔT, then, due to thermal expansion of the steel in the circumferential direction, the new radius, r_2', of the outer ring is given by

$$\begin{aligned} 2\pi r_2' &= 2\pi r_2 + 2\pi r_2 A \Delta T, \\ \Rightarrow r_2' &= r_2 + r_2 A \Delta T. \end{aligned} \tag{6}$$

Thus the new gap thickness, d_2, satisfies $r_2' - r_1 = d_{10} + d_2$, which from (5) and (6) gives

$$d_2 = r_2' - r_1 - d_{10} = r_2 + r_2 A\Delta T - r_1 - d_{10} = d_{20} + r_2 A \Delta T.$$

Since the gap thickness is roughly 10^{-3} to 10^{-2} cm and the radius is three to four orders of magnitude larger, the correction term, $r_2 A \Delta T$, represents a significant change (1%–10%) in the gap thickness when the temperature difference is of the order of unity (a physically realistic range).

Absolute temperature, T, would also affect the values of d_1, d_2, and A, but over moderate temperature ranges these effects may be ignored and over the relatively large ranges of the annealing process may be approximated by using piecewise constant functions for d_{10}, d_{20}, and A.

Substituting (4) into the expression for λ_{eq} gives

$$\lambda_{eq} = \lambda_s \frac{1 + (d_{20}/d_{10}) + r_2 A(\Delta T/d_{10})}{1 + (\lambda_s/\lambda_g)\left((d_{20}/d_{10}) + r_2 A(\Delta T/d_{10})\right)}.$$

Since the dimensions of the gaps are small compared with the thickness of the steel, we write $d_{10} \approx \Delta r$, and

$$\lambda_{eq} = \lambda_s \frac{1 + (d_{20}/d_{10}) + r_2 A(\Delta T/\Delta r)}{1 + (\lambda_s/\lambda_g)\left((d_{20}/d_{10}) + r_2 A(\Delta T/\Delta r)\right)}. \tag{7}$$

Note that as $\Delta T/\Delta r$ decreases from zero, since $\lambda_s > \lambda_g$, the denominator of the expression on the right side of (7) reaches zero before the numerator does, hence $\lambda_{eq} \to +\infty$ as $r_2 A\Delta T/\Delta r \to -(d_{20}/d_{10} + \lambda_g/\lambda_s)^+$. To avoid this situation Stikker imposes the physical constraint that the gap between the successive windings, d_2, is bounded from below by a minimum distance, $d_{\min}$, determined by the roughness of the steel ($\approx 10^{-4}$ cm). Consequently, (7) holds for

$$d_2 = d_{20} + rA\Delta T \geq d_{\min}$$

or

$$\frac{d_{20}}{d_{10}} + rA\frac{\Delta T}{\Delta r} \geq \frac{d_{\min}}{d_{10}}, \tag{8}$$

while for smaller values λ_{eq} is given by

$$\lambda_{eq} = \lambda_s \frac{1 + (d_{\min}/d_{10})}{1 + (\lambda_s/\lambda_g)(d_{\min}/d_{10})}. \tag{9}$$

Note then that the domain of validity of (7) is exclusive of the singularity that occurs at

$$\frac{d_{20}}{d_{10}} + rA\frac{\Delta T}{\Delta r} = -\frac{\lambda_g}{\lambda_s} < 0,$$

and includes points where $\Delta T/\Delta r = 0$. Also, when λ_{eq} is given by (9), (3) becomes linear for constant ρ, c, λ_s, and λ_g.

Using the above expressions for λ_{eq}, Stikker discretized (3) and solved it numerically to achieve his results. Harvey also utilized Stikker's derivation for the radial conductivity in a more complex model of the annealing process [5].

2.2. Constraints and simplification. If we now employ the approximation $\Delta T/\Delta r \approx \partial T/\partial r$, we obtain what we shall designate as Stikker's heat conduction differential equation:

$$\rho c\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda_s \frac{1 + d_{20}/d_{10} + rA\frac{\partial T}{\partial r}}{1 + (\lambda_s/\lambda_g)(d_{20}/d_{10} + rA\frac{\partial T}{\partial r})}\frac{\partial T}{\partial r}\right] + \frac{\partial(\lambda_s \frac{\partial T}{\partial z})}{\partial z}. \tag{10}$$

Taking λ_s, λ_g, A, and ρc as constants (which is valid over moderate temperature ranges) and introducing the nondimensional variables

$$u = AT, \qquad \hat{t} = \frac{\lambda_s t}{\rho c R_0^2}, \qquad \hat{r} = \frac{r}{R_0}, \qquad \text{and } \hat{z} = \frac{z}{R_0},$$

where R_0 is some suitable reference length (such as $(\lambda_s/\rho)\sqrt{A/c^3}$ or some other length that may arise from the problem geometry), (10) becomes

$$u_{\hat{t}} = \frac{1}{\hat{r}}\left[\frac{1 + d_{20}/d_{10} + \hat{r}u_{\hat{r}}}{1 + (\lambda_s/\lambda_g)(d_{20}/d_{10} + \hat{r}u_{\hat{r}})}\hat{r}u_{\hat{r}}\right]_{\hat{r}} + u_{\hat{z}\hat{z}}, \tag{11}$$

which is of the form (2) with $a = 1 + d_{20}/d_{10}$, $b = 1$, $c = 1 + (\lambda_s/\lambda_g)(d_{20}/d_{10})$, $d = \lambda_s/\lambda_g$, and radial symmetry imposed.

We now shift our attention to the general form of Stikker's differential equation, namely (2), and a similar equation, (1), in Cartesian coordinates. It is likely that (1) will be applicable for a problem in one spatial dimension ($u_{yy} = u_{zz} = 0$) rather than three; however, we shall consider the three-dimensional case in the following discussion.

Set $\gamma = u_x$ or ru_r corresponding to equations (1) and (2), respectively, and let

$$k = \frac{a + b\gamma}{c + d\gamma}.$$

Note that the outward flux, $f = -k\gamma$, has zeros at $\gamma = 0$, $-a/b$, and that at $\gamma = -c/d$ a singularity occurs. Both an infinite flux and a simultaneous occurrence of a zero flux and a nonzero temperature gradient are nonphysical situations; therefore, we shall restrict the domain of validity of (1) and (2) by applying the following two physical conditions.

First, since the domain of physical interest includes the region where the temperature gradient, and hence γ, is close to zero, near $\gamma = 0$ we shall require that the flux behave like the linear flux, $-k_0\gamma$, $k_0 > 0$, i.e.,

$$\frac{df(0)}{d\gamma} = -\frac{a}{c} < 0. \tag{12}$$

Second, we impose the restriction

$$\begin{cases} -\dfrac{a}{b} < -\dfrac{c}{d} < \gamma & \text{if } -\dfrac{c}{d} < 0, \\ \gamma < -\dfrac{c}{d} < -\dfrac{a}{b} & \text{if } -\dfrac{c}{d} > 0. \end{cases} \tag{13}$$

The above condition says that the region of interest is located on the same side of the singularity as the origin and that the position where a zero flux and a nonzero gradient occur simultaneously is located on the opposite side of the singularity, hence outside the region of interest. Conditions (12) and (13) are equivalent to

$$\frac{ad}{bc} > 1, \qquad \frac{b}{d} > 0. \tag{14}$$

We remark that Stikker's differential equation, (11), satisfies (14) while the limitation on the domain of validity of (11) is given by (8) and can be expressed in the form

$$-\frac{c}{d} = -\frac{\lambda_g}{\lambda_s} - \frac{d_{20}}{d_{10}} < -\frac{d_{20}}{d_{10}} + \frac{d_{min}}{d_{10}} \le ru_r.$$

With the restrictions (14), we may write

$$k = \frac{a + b\gamma}{c + d\gamma} = \frac{b}{d}\left(1 + \frac{\frac{ad}{bc} - 1}{1 + \frac{d}{c}\gamma}\right),$$

and hence, with the respective rescalings

$$\hat{u} = \frac{d}{c}\sqrt{\frac{d}{b}}\,u, \quad \hat{x} = \sqrt{\frac{d}{b}}\,x, \qquad \text{and} \qquad \hat{u} = \frac{d}{c}u, \quad \hat{r} = \sqrt{\frac{d}{b}}\,r, \quad \hat{\theta} = \sqrt{\frac{b}{d}}\,\theta,$$

equations (1) and (2) become, after dropping the hats,

$$u_t = \left[\left(1 + \frac{\beta}{1+u_x}\right)u_x\right]_x + u_{yy} + u_{zz} \tag{15}$$

and

$$u_t = \frac{1}{r}\left[\left(1 + \frac{\beta}{1+ru_r}\right)ru_r\right]_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}, \tag{16}$$

where $\beta = ad/bc - 1 > 0$, by (14). For Stikker's equation, (11), the parameter β is given by $(\lambda_s/\lambda_g - 1)/(1 + (\lambda_s/\lambda_g)(d_{20}/d_{10}))$.

Owing to the particular form of the nonlinearity, the first term of (15) may be written as

$$\left[\left(1 + \frac{\beta}{1+u_x}\right)u_{xx} - \frac{\beta u_{xx}}{(1+u_x)^2}u_x\right] = \left(1 + \frac{\beta}{(1+u_x)^2}\right)u_{xx};$$

hence (15) becomes

$$\begin{aligned} u_t &= \left(1 + \frac{\beta}{(1+u_x)^2}\right)u_{xx} + u_{yy} + u_{zz} \\ &= \triangle u - \left(\frac{\beta}{1+u_x}\right)_x. \end{aligned} \tag{17}$$

Similarly, (16) is equivalent to

$$\begin{aligned} u_t &= \frac{1}{r}\left(1 + \frac{\beta}{(1+ru_r)^2}\right)(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} \\ &= \triangle u - \frac{1}{r}\left(\frac{\beta}{1+ru_r}\right)_r. \end{aligned} \tag{18}$$

Note from (17) and (18) that the effect of the nonlinearity is the introduction of a "perturbation" term that vanishes as $\beta \to 0$.

The translation into our new variables of condition (13) with regard to the domain of validity of γ is given by

$$-1 < u_x, \qquad\qquad -1 < ru_r, \tag{19}$$

for equations (17) and (18), respectively.

3. Properties of solutions. Solutions to (17) and (18) satisfy the same maximum principle as the standard linear heat equation. Let Ω be an open set in $\mathbb{R}^3$, and define the sets D, D_T, and C_T (the parabolic boundary of D_T) by

$$D = \Omega \times (0,\infty), \qquad D_T = \Omega \times (0,T),\ T > 0, \qquad C_T = \partial D_T \setminus \{(\mathbf{x},T) \mid \mathbf{x} \in \Omega\}.$$

THEOREM 3.1. *If* $u \in C(\overline{D}) \cap C^2(D)$ *and satisfies* (17) *or* (18), *then*

$$u(\mathbf{x},t) \le \max_{C_T} u, \qquad\qquad \forall\, (\mathbf{x},t) \in D_T,\ T > 0.$$

Proof. Note that if a maximum occurs at a point p on the interior of D_T or on $\{(\mathbf{x},T) \mid \mathbf{x} \in \Omega\}$, then we have $u_r(p) = u_x(p) = 0$, and (17) and (18) evaluated at p are given by

$$u_t = (1+\beta)u_{xx} + u_{yy} + u_{zz}$$

and

$$u_t = \frac{1}{r}(1+\beta)(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz},$$

respectively. The result then follows in the same manner as it does for the linear heat equation. □

A uniqueness result may also be given.

THEOREM 3.2. *For a bounded region* D *in* $\mathbb{R}^3$, *and constant* $\beta > 0$, *there exists at most one solution* $u \in C(\overline{D}) \cap C^2(D)$ *to each of the problems*

$$\begin{cases} u \text{ satisfies } (17) \text{ or } (18) & \text{in } D \times \mathbb{R}_+, \\ u(\mathbf{x},0) = f(\mathbf{x}) & \text{in } D, \\ u(\mathbf{x},t) = g(\mathbf{x},t) & \text{on } \partial D \times \mathbb{R}_+, \\ \gamma > -1, \ \text{or } \gamma < -1 & \text{in } D \times \mathbb{R}_+, \end{cases}$$

where $\gamma = u_x$ *or* ru_r *corresponding to* u *satisfying* (17) *or* (18).

Proof. Set

$$\begin{array}{lll} \alpha = 1, & (\xi,\eta,\zeta) = (x,y,z) & \text{if } u \text{ satisfies (17) or} \\ \alpha = r, & (\xi,\eta,\zeta) = (r,\theta,z) & \text{if } u \text{ satisfies (18).} \end{array}$$

Suppose there are two solutions u_1, u_2. Set $v = u_1 - u_2$ and $\gamma_i = \alpha u_{i_\xi}$, $i = 1,2$; then v satisfies

$$\begin{cases} v_t = \Delta v + \dfrac{\beta}{\alpha}\left(\dfrac{\alpha v_\xi}{(1+\gamma_1)(1+\gamma_2)}\right)_\xi & \text{in } D \times \mathbb{R}_+, \\ v(\xi,\eta,\zeta,0) = 0 & \text{in } D, \\ v(\xi,\eta,\zeta,t) = 0 & \text{on } \partial D \times \mathbb{R}_+. \end{cases}$$

Consider the function

$$I(t) = \frac{1}{2}\int_D v^2\, dV. \tag{20}$$

Clearly, $I(t) \ge 0$ and $I(0) = 0$. Differentiating (20) yields

$$I'(t) = \int_D vv_t\, dV = \int_D v\Delta v\, dV + \int_D \frac{v\beta}{\alpha}\left(\frac{\alpha v_\xi}{(1+\gamma_1)(1+\gamma_2)}\right)_\xi dV. \tag{21}$$

Using Green's first identity,

$$\int_D (f\Delta g + \nabla f \cdot \nabla g)\, dV = \int_{\partial D} f\frac{\partial g}{\partial n}\, ds,$$

and noting that $dV = \alpha\, d\xi\, d\eta\, d\zeta$, (21) becomes

$$
\begin{aligned}
I'(t) &= -\int_D |\nabla v|^2\, dV + \int_{\partial D} v \frac{\partial v}{\partial n}\, ds \\
&\quad + \beta \iint_{\zeta\ \eta} \left[\int_{\xi_1(\eta,\zeta)}^{\xi_2(\eta,\zeta)} v \left(\frac{\alpha v_\xi}{(1+\gamma_1)(1+\gamma_2)} \right)_\xi d\xi \right] d\eta\, d\zeta \\
&= -\int_D |\nabla v|^2\, dV + \int_{\partial D} v \frac{\partial v}{\partial n}\, ds \\
&\quad + \beta \iint_{\zeta\ \eta} \left[\frac{v \alpha v_\xi}{(1+\gamma_1)(1+\gamma_2)} \Bigg|_{\xi_1(\eta,\zeta)}^{\xi_2(\eta,\zeta)} - \int_{\xi_1(\eta,\zeta)}^{\xi_2(\eta,\zeta)} \frac{\alpha v_\xi^2}{(1+\gamma_1)(1+\gamma_2)} d\xi \right] d\eta\, d\zeta.
\end{aligned} \tag{22}
$$

Since $v \equiv 0$ on ∂D, and therefore zero on $\xi_1(\eta,\zeta)$ and $\xi_2(\eta,\zeta)$, (22) collapses to

$$I'(t) = -\int_D \left(|\nabla v|^2 + \frac{\beta v_\xi^2}{(1+\gamma_1)(1+\gamma_2)} \right) dV.$$

Finally, by the theorem assumptions, since $\beta > 0$ and both $1+\gamma_1$ and $1+\gamma_2$ have the same sign, it follows that $I'(t) \leq 0$, implying that $I(t) \equiv 0$ and therefore $v \equiv 0$, hence $u_1 = u_2$. $\square$

Remark. The Dirichlet boundary condition in Theorem 3.2 may be replaced by a Neumann condition provided that $n = \pm\hat{\xi}$ at all points $(\xi_1(\eta,\zeta),\eta,\zeta)$, $(\xi_2(\eta,\zeta),\eta,\zeta)$, that is, if the boundary of D is composed of surfaces $\xi =$ constant and surfaces tangent to the $\hat{\xi}$-direction.

4. Exact solutions. In this section we shall construct analytical solutions of (17) and radially symmetric analytical solutions of (18) by assuming separability of the $x\,/\,r$ dependence from the remaining independent variables.

4.1. Solutions of (17). Suppose $u(x,y,z,t) = X(x)V(y,z,t)$; then substitution into equation (17) yields

$$XV_t = VX'' \left(1 + \frac{\beta}{(1+VX')^2} \right) + X(V_{yy} + V_{zz}),$$

$$\frac{V_t - V_{yy} - V_{zz}}{V} = \frac{X''}{X} \left(1 + \frac{\beta}{(1+VX')^2} \right).$$

Note that the V, X dependence does not completely separate, but one possible solution is

$$X'' = 0 = V_t - V_{yy} - V_{zz}, \tag{23}$$

implying

$$X(x) = Ax + B.$$

Solutions of (23) are, however, not very interesting, since they are also solutions of the linear heat equation, $\beta = 0$.

Solutions to (17) that are not also solutions to the linear heat equation may be found by assuming an additive separability of the form $u(x,y,z,t) = X(x)+V(y,z,t)$. In this case, substitution into (17) yields

$$V_t = X'' + V_{yy} + V_{zz} - \beta \frac{\partial}{\partial x}\left(\frac{1}{1+X'}\right),$$

which may be written as

$$V_t - V_{yy} - V_{zz} = \frac{d}{dx}\left(1 + X' - \frac{\beta}{1+X'}\right), \tag{24}$$

achieving complete separation of the x dependence. Setting both sides of (24) equal to a constant, 4λ, and integrating the X equation once gives

$$V_t - V_{yy} - V_{zz} = 4\lambda, \tag{25}$$

$$1 + X' - \frac{\beta}{1+X'} = 4(\lambda x + A), \tag{26}$$

where A is a constant.

Equation (25) is an inhomogeneous linear heat equation in two spatial dimensions, for which there are standard methods of solving, given a set of boundary and initial conditions.

We now turn our attention to solving the first-order nonlinear ordinary differential equation (ODE) (26). Multiplying by $1 + X'$ gives

$$(1+X')^2 - 4(\lambda x + A)(1+X') - \beta = 0;$$

hence,

$$1 + X'_{\pm} = 2\lambda x + 2A \pm \sqrt{4(\lambda x + A)^2 + \beta}. \tag{27}$$

Integration yields

$$X_{\pm}(x) = \lambda x^2 + (2A-1)x \pm \int \sqrt{4(\lambda x + A)^2 + \beta}\, dx + B, \tag{28}$$

where B is a constant. Note from (27) that since $\beta > 0$, $1 + X'$ is either always positive or always negative so that there are no singularities of (17). Furthermore, regarding the domain of validity of our equation, given by (19), we see that X_+ is the valid solution while X_- is a nonphysical solution. Note also that if $\lambda = 0$, then equations (25) and (27) collapse to the linear equation (23); we shall therefore assume that $\lambda \neq 0$.

Completing the integral in (28) gives

$$\begin{aligned} X_{\pm}(x) = \lambda x^2 + (2A-1)x + B \pm \frac{1}{2\lambda}\Big[&(\lambda x + A)\sqrt{4(\lambda x + A)^2 + \beta} \\ &+ \frac{\beta}{2}\ln\left|2(\lambda x + A) + \sqrt{4(\lambda x + A)^2 + \beta}\right|\Big]. \end{aligned} \tag{29}$$

Two classes of exact solutions to (17) are therefore given by

$$u_{E\pm}(x,y,z,t) = X_{\pm}(x) + V(y,z,t), \tag{30}$$

where $V(y,z,t)$ is any solution to (25).

4.2. Solutions of (18). We shall restrict ourselves to radially symmetric solutions of (18); hence, the $u_{\theta\theta}$ term vanishes. As in the previous section, a multiplicative solution of the form $u(r,z,t) = R(r)V(z,t)$ does not completely separate, yielding

$$RV_t = V\left(R'' + \frac{R'}{r}\right)\left(1 + \frac{\beta}{(1+VrR')^2}\right) + RV_{zz},$$
$$\frac{V_t - V_{zz}}{V} = \frac{1}{R}\left(R'' + \frac{R'}{r}\right)\left(1 + \frac{\beta}{(1+VrR')^2}\right),$$

which admits solutions to

(31) $$R'' + \frac{R'}{r} = 0 = V_t - V_{zz},$$

implying

$$R(r) = A\ln r + B.$$

Again, these solutions are also solutions to the linear problem.

Assuming an additive separability of the form $u(r,z,t) = R(r) + V(z,t)$ gives, on substitution into (18) and manipulation identical to that in the previous section, the equation

(32) $$V_t - V_{zz} = \frac{1}{r}\frac{d}{dr}\left(1 + rR' - \frac{\beta}{1+rR'}\right).$$

Setting both sides of (32) equal to 4λ and integrating the R equation once yields

(33) $$V_t - V_{zz} = 4\lambda,$$

(34) $$1 + rR' - \frac{\beta}{1+rR'} = 2(\lambda r^2 + A).$$

As before, (33) is an inhomogeneous linear heat equation. Multiplying (34) by $1+rR'$ gives

$$(1+rR')^2 - 2(\lambda r^2 + A)(1+rR') - \beta = 0;$$

hence,

(35) $$1 + rR'_{\pm} = \lambda r^2 + A \pm \sqrt{(\lambda r^2 + A)^2 + \beta},$$

and therefore

(36) $$R'_{\pm} = \lambda r + \frac{A-1}{r} \pm \frac{1}{r}\sqrt{(\lambda r^2 + A)^2 + \beta}.$$

Integration yields

(37) $$R_{\pm}(r) = (A-1)\ln r + \frac{\lambda}{2}r^2 \pm \int \frac{1}{r}\sqrt{(\lambda r^2 + A)^2 + \beta}\, dr + B.$$

Again, (35) shows that since $\beta > 0$, $1+rR'$ is either always positive or always negative; hence, there are no singularities of (18). From (19) we see that R_+ represents the valid solution while R_- is a nonphysical solution. Also, as before, $\lambda = 0$ yields only solutions of the linear problem (31) so that we assume $\lambda \neq 0$.

Using the transformation $y = \lambda r^2$, the integral in (37) becomes

$$I = \int \frac{1}{r}\sqrt{(\lambda r^2 + A)^2 + \beta}\, dr = \frac{1}{2}\int \frac{1}{y}\sqrt{y^2 + 2Ay + A^2 + \beta}\, dy.$$

The above integral can be found in most published tables of indefinite integrals, from which, after transforming back to r, we get

$$\begin{aligned} R_\pm(r) = \frac{\lambda}{2}r^2 + \frac{A-1}{2}\ln r^2 + B \pm \frac{1}{2}\Bigg\{ &\sqrt{(\lambda r^2 + A)^2 + \beta} \\ &- \sqrt{A^2+\beta}\,\ln\left|\frac{1}{\lambda r^2}\left(A^2 + \beta + A\lambda r^2 + \sqrt{(A^2+\beta)((\lambda r^2 + A)^2 + \beta)}\right)\right| \\ &+ A\ln\left[A + \lambda r^2 + \sqrt{(\lambda r^2 + A)^2 + \beta}\right]\Bigg\}. \end{aligned} \tag{38}$$

Two classes of exact radially symmetric solutions to (18) are therefore given by

$$u_{E\pm}(r,z,t) = R_\pm(r) + V(z,t), \tag{39}$$

where $V(z,t)$ is any solution to (33).

4.3. Asymptotic behaviour of $X_\pm$ and $R_\pm$. It is immediately clear that $X_\pm$ given by (29) is well behaved as $x \to 0$. By expressing (29) in the form

$$\begin{aligned} X_\pm(x) = \frac{(\lambda x + A)^2}{\lambda} - x - \frac{A^2}{\lambda} + B \pm \frac{1}{\lambda}\Bigg[&(\lambda x + A)\sqrt{(\lambda x + A)^2 + \frac{\beta}{4}} \\ &+ \frac{\beta}{4}\ln\left|2\left(\lambda x + A + \sqrt{(\lambda x + A)^2 + \frac{\beta}{4}}\right)\right|\Bigg], \end{aligned}$$

with some analysis it can be shown that the solutions $X_\pm$ diverge to infinity as $x \to \pm\infty$.

From (27) we may also establish the asymptotic behaviour of $1 + X'_\pm$.

$$\begin{aligned} \lim_{x\to 0}(1 + X'_\pm) &= 2A \pm \sqrt{4A^2 + \beta}, \\ \lim_{x\to\infty}(1 + X'_+) &= \begin{cases} \infty & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda < 0, \end{cases} \\ \lim_{x\to\infty}(1 + X'_-) &= \begin{cases} 0 & \text{if } \lambda > 0, \\ -\infty & \text{if } \lambda < 0, \end{cases} \\ \lim_{x\to -\infty}(1 + X'_+) &= \begin{cases} 0 & \text{if } \lambda > 0, \\ \infty & \text{if } \lambda < 0, \end{cases} \\ \lim_{x\to -\infty}(1 + X'_-) &= \begin{cases} -\infty & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda < 0. \end{cases} \end{aligned}$$

Analysis of the asymptotic behaviour of $R_\pm$, (38), as $r \to 0$ and as $r \to \infty$, is slightly more complex.

As $r \to 0$, the solutions $R_\pm$ approach

$$\frac{A-1}{2}\ln r^2 + B \pm \frac{1}{2}\left\{\sqrt{K} + A\ln\left[A + \sqrt{K}\right] - \sqrt{K}\,\ln\left|\frac{2K}{\lambda r^2}\right|\right\}, \tag{40}$$

where $K = A^2 + \beta$. The first and last terms of (40) are unbounded as $r \to 0$ and must be combined to cancel each other out. For this we require

$$A - 1 = \pm\left(-\sqrt{A^2+\beta}\right), \tag{41}$$

$$\Rightarrow A^2 - 2A + 1 = A^2 + \beta,$$

$$\Rightarrow A = \frac{1-\beta}{2}. \tag{42}$$

Substitution of (42) in (41) shows that (42) is valid only for the "plus" solution, since $A - 1 < \sqrt{A^2+\beta}$, $\forall\, A$, and $\forall\, \beta > 0$. Hence, $R_- \to \pm\infty$ as $r \to 0$, while for A given by (42) we have

$$A^2 + \beta = \left(\frac{1+\beta}{2}\right)^2 \quad \text{and } A + \sqrt{A^2+\beta} = 1$$

so that

$$\begin{aligned}\lim_{r\to 0} R_+ &= \frac{1}{2}\left(\frac{1+\beta}{2}\right) - \frac{1}{2}\left(\frac{1+\beta}{2}\right)\ln\left|\frac{2}{\lambda}\left(\frac{1+\beta}{2}\right)^2\right| + B \\ &= \frac{1+\beta}{4}\left(1 - \ln\left|\frac{(1+\beta)^2}{2\lambda}\right|\right) + B.\end{aligned}$$

To control the behaviour of $R_\pm$ as $r \to \infty$, we must control both the logarithmic terms and the algebraic terms separately. For the algebraic terms of (38) we have

$$\frac{\lambda}{2}r^2 \pm \frac{1}{2}\sqrt{(\lambda r^2 + A)^2 + \beta} \to \begin{cases} +\infty & \text{for } R_+ \text{ if } \lambda > 0, \\ -\infty & \text{for } R_- \text{ if } \lambda < 0, \\ -A/2 & \text{otherwise.}\end{cases}$$

For the logarithmic terms of (38) we have

$$\ln\left|\frac{1}{\lambda r^2}\left(A^2 + \beta + A\lambda r^2 + \sqrt{(A^2+\beta)\left((\lambda r^2 + A)^2 + \beta\right)}\right)\right| \to \ln\left[A + \sqrt{A^2+\beta}\right],$$

while the remaining two terms,

$$\frac{A-1}{2}\ln r^2 \pm \frac{A}{2}\ln\left[A + \lambda r^2 + \sqrt{(\lambda r^2 + A)^2 + \beta}\right], \tag{43}$$

are ill behaved. If $\lambda < 0$, then the argument of the logarithm of the second term of (43) approaches zero, and it can be shown that

$$\frac{A + \lambda r^2 + \sqrt{(\lambda r^2 + A)^2 + \beta}}{r^b}$$

approaches either ∞ or 0 for any b; hence, the case $\lambda < 0$ is unbounded as $r \to \infty$. If $\lambda > 0$, then we need $b = 2$ to control these terms; i.e., we require

$$-(A-1) = \pm A. \tag{44}$$

For R_+ then we need $A = 1/2$, while for R_-, equation (44) cannot be satisfied. We see, therefore, that it is impossible to control both the algebraic and the logarithmic terms simultaneously; the former are bounded for R_+ when $\lambda < 0$ and for R_- when $\lambda > 0$, while the latter are bounded only for R_+ with $\lambda > 0$ and $A = 1/2$. Hence, $R_\pm$ can be bounded as $r \to 0$ but are unbounded as $r \to \infty$.

The asymptotic behaviour of $1 + rR'_\pm$ is easily determined from (35).

$$\begin{aligned}
\lim_{r\to 0}(1 + rR'_\pm) &= A \pm \sqrt{A^2 + \beta}\,, \\
\lim_{r\to\infty}(1 + rR'_+) &= \begin{cases} \infty & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda < 0, \end{cases} \\
\lim_{r\to\infty}\left(1 + rR'_-\right) &= \begin{cases} 0 & \text{if } \lambda > 0, \\ -\infty & \text{if } \lambda < 0. \end{cases}
\end{aligned}$$

5. Boundary conditions. We shall now analyze the types of boundary and initial conditions that can be satisfied by the exact solution (39). The following observations, with straightforward modification from cylindrical to Cartesian coordinates and removal of the radial symmetry requirement, also apply to (30).

Consider the following initial boundary value problem (with a general mixed-type boundary condition) for radially symmetric u on a bounded radially symmetric region D in $\mathbb{R}^3$.

$$\text{(P1)} \qquad \begin{cases} u_t = \triangle u - \dfrac{1}{r}\left(\dfrac{\beta}{1 + ru_r}\right)_r & \text{in } D \times \mathbb{R}_+, \\ u(r, z, 0) = f(r, z) & \text{in } D, \\ Ku(r, z, t) + L\dfrac{\partial u}{\partial n}(r, z, t) = g(r, z, t) & \text{on } \partial D \times \mathbb{R}_+, \end{cases}$$

where f and g are arbitrary continuous functions and K and L are constants. The additive form of the solution $u_{E\pm}$, (39), places restrictions on the type of boundary and initial conditions that it can satisfy. In particular, the separation of the radial dependence implies that the initial r profile persists, up to a translation with time; i.e., for a fixed height z_0, we have

$$u_{E\pm}(r, z_0, t) = u_{E\pm}(r, z_0, 0) + h_{z_0}(t),$$

where

$$h_{z_0}(t) = V(z_0, t) - V(z_0, 0).$$

It follows that for $u_{E\pm}$ to satisfy (P1), the initial condition must be of the form

$$f(r, z) = u_{E\pm}(r, z, 0) = R_\pm(r) + f_1(z), \tag{45}$$

where $f_1(z)$ is an arbitrary function.

Whereas it is necessary that the initial condition, f, coincide with $R_\pm$ through (45) for $u_{E\pm}$ to satisfy (P1), the boundary condition, g, may take on a more standard form when $K = 0$, i.e., when a Neumann boundary condition is imposed. Suppose D is the region between two concentric cylinders and consider the problem

$$\text{(P2)} \qquad \begin{cases} u_t = \triangle u - \dfrac{1}{r}\left(\dfrac{\beta}{1 + ru_r}\right)_r, & r \in (r_1, r_2),\ \ z \in (z_1, z_2),\ \ t > 0, \\ u(r, z, 0) = R_\pm(r) + f_1(z), & r \in (r_1, r_2),\ \ z \in (z_1, z_2), \\ u_z(r, z_1, t) = g_1(t), & r \in (r_1, r_2),\ \ t > 0, \\ u_z(r, z_2, t) = g_2(t), & r \in (r_1, r_2),\ \ t > 0, \\ u_r(r_1, z, t) = c_1, & z \in (z_1, z_2),\ \ t > 0, \\ u_r(r_2, z, t) = c_2, & z \in (z_1, z_2),\ \ t > 0, \end{cases}$$

where c_1 and c_2 are constants, and f_1, g_1, and g_2 are arbitrary continuous functions. The exact solution $u_{E\pm}$ will satisfy (P2) provided that

$$R'_{\pm}(r_1) = c_1, \qquad R'_{\pm}(r_2) = c_2, \tag{46}$$

and V satisfies

$$\begin{cases} V_t - V_{zz} = 4\lambda, & z \in (z_1, z_2), \quad t > 0, \\ V(z,0) = f_1(z), & z \in (z_1, z_2), \\ V_z(z_1,t) = g_1(t), & t > 0, \\ V_z(z_2,t) = g_2(t), & t > 0. \end{cases}$$

It can be shown that (46) completely determines the constants λ and A through equation (36), provided that $c_1 r_1, c_2 r_2 \neq -1$, i.e., provided that the boundary is not a singularity of (18). The constant B in $R_{\pm}$ is completely arbitrary and may be absorbed into the function f_1. It remains then to solve the above initial boundary value problem for V. Having completed this, the function $u_{E\pm}$ will be a solution of (P2).

If we consider a solid cylinder, $r_1 = 0$, and replace the condition $u_r(r_1, z, t) = c_1$ in problem (P2) with *u bounded at the origin*, then by the asymptotic analysis of § 4.3 we must use u_{E+} and choose A according to (42). The second condition of (46) then serves to establish the value of λ, and we may proceed as in the previous case.

Unfortunately, the additive nature of our exact solution, (39), precludes it from satisfying the boundary conditions relevant to Stikker's problem. The boundary conditions on the outer surface of the steel coil for the annealing process are of the form

$$\lambda_R \frac{\partial T}{\partial r}\bigg|_{r=R} = \alpha_R(T_g - T_{cl}(R)) + c_1(T_c^4 - T_{cl}^4(R)) + c_2(T_g^4 - T_{cl}^4(R)),$$

involving a conduction term with the surrounding gas and radiation terms between the coil and gas and between the coil and the cover separating the steel from the furnace. Consequently the boundary conditions are highly temperature- and therefore time-dependent, whereas $\partial u_{E\pm}/\partial r$ is independent of time.

6. Conclusion. We have considered two nonlinear heat conduction equations where the conductivity was a function of the temperature gradient. An equation of this type was derived by Stikker for the problem of heat conduction inside a steel coil during the batch annealing process, but, to our knowledge, such equations have not received much attention in the literature. After the physical implications of these equations were discussed and the domain of validity for them was established, a uniqueness theorem and a maximum principle were given. Exact solutions to both of these equations were derived by assuming an additive separability of the nonlinear component from the remaining variables. Some asymptotic analysis of these solutions was performed, and the types of boundary value problems that these solutions could satisfy was discussed. Although these solutions do not satisfy the boundary conditions of Stikker's problem, it was shown that a problem with a constant Neumann boundary condition could be satisfied by these solutions. It is possible that these equations will also have application in the realm of diffusion research, where a constant flux condition at the boundary is a common requirement.

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