

A Collection of Results on Simonyi's Conjecture

by

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ABSTRACT

A COLLECTION OF RESULTS OF SIMONYI'S CONJECTURE

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Given two set systems \mathcal{A} and \mathcal{B} over an n -element set, we say that $(\mathcal{A}, \mathcal{B})$ forms a recovering pair if the following conditions hold:

$$\forall A, A' \in \mathcal{A} \text{ and } \forall B, B' \in \mathcal{B}, A \setminus B = A' \setminus B' \Rightarrow A = A'$$

$$\forall A, A' \in \mathcal{A} \text{ and } \forall B, B' \in \mathcal{B}, B \setminus A = B' \setminus A' \Rightarrow B = B'$$

In 1989, Gábor Simonyi conjectured that if $(\mathcal{A}, \mathcal{B})$ forms a recovering pair, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$. This conjecture is the focus of this thesis.

This thesis contains a collection of proofs of special cases that together form a complete proof that the conjecture holds for all values of n up to 8. Many of these special cases also verify the conjecture for certain recovering pairs when $n > 8$. We also present a result describing the nature of the set of numbers over which the conjecture in fact holds. Lastly, we present a new problem in graph theory, and discuss a few cases of this problem.

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Chapter 1

Definitions

1.1 Definitions, Notation, Useful Facts

Consider the n -element set $\{1, 2, \dots, n\}$, we will denote this by $[n]$. The collection of all subsets of $[n]$ is called the *power set* of $[n]$ and is denoted by $\mathcal{P}([n])$. A subset of $\mathcal{P}([n])$ is called a *set system* over $[n]$. Note that a set system may also be referred to as a set family, or a family or collection of subsets of $[n]$.

Definition 1.1.1 [8] *Let \mathcal{A} and \mathcal{B} be set systems over an n -element set such that the following conditions hold:*

$$\forall A, A' \in \mathcal{A} \text{ and } \forall B, B' \in \mathcal{B}, A \setminus B = A' \setminus B' \Rightarrow A = A'$$

$$\forall A, A' \in \mathcal{A} \text{ and } \forall B, B' \in \mathcal{B}, B \setminus A = B' \setminus A' \Rightarrow B = B'$$

Then $(\mathcal{A}, \mathcal{B})$ is said to be a recovering pair.

Given set systems \mathcal{A} and \mathcal{B} , we define $|\mathcal{A}|$ and $|\mathcal{B}|$ to be the cardinalities of \mathcal{A} and \mathcal{B} respectively.

As examples, consider the following pairs of set systems over $[n]$ where $n = 5$:

1. $\mathcal{A} = \mathcal{P}(\{1, 2\})$ and $\mathcal{B} = \mathcal{P}(\{3, 4, 5\})$. Then $(\mathcal{A}, \mathcal{B})$ is a recovering pair.
2. $\mathcal{A} = \{\{1, 2, 3, 4\}, \{1, 2, 5\}\}$ and $\mathcal{B} = \{\{2, 3, 4\}, \{2, 3, 5\}\}$. Then $(\mathcal{A}, \mathcal{B})$ is not a

recovering pair due to the fact that $\{1, 2, 3, 4\} \setminus \{2, 3, 4\} = \{1, 2, 5\} \setminus \{2, 3, 5\} = \{1\}$.

Conjecture 1.1.2 (Simonyi) [1] *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.*

Unless otherwise stated, we will assume throughout this thesis that the letters \mathcal{A} and \mathcal{B} refer to set systems that form a recovering pair.

We will now consider some definitions regarding lattices.

Definition 1.1.3 [5] *Let P be a set that satisfies the following conditions with respect to the ordering relation \leq for all $a, b, c \in P$:*

1. $a \leq a$
2. If $a \leq b$ and $b \leq a$, then $a = b$
3. If $a \leq b$ and $b \leq c$, then $a \leq c$

Then P is called a partially ordered set.

Definition 1.1.4 [5] *Let $C \subseteq P$ where P is a partially ordered set. Then C is called a chain if for any $a, b \in C$, either $a \leq b$ or $b \leq a$.*

Definition 1.1.5 [1] *Let C be a chain. Then the length of C is the number of elements in C .*

It should be noted that there is more than one definition for the length of a chain. We follow the choice of [1].

Definition 1.1.6 [6] *Let C_1, C_2, \dots, C_k be chains. Then the product of these chains is the partially ordered set $P = C_1 \times C_2 \times \dots \times C_k = \{(c_1, c_2, \dots, c_n) : c_i \in C_i\}$ such that $(c_1, c_2, \dots, c_k) \leq (c'_1, c'_2, \dots, c'_k)$ iff $c_i \leq c'_i$ for all i .*

Definition 1.1.7 [5] Let P be a partially ordered set and let $H \subseteq P$. If $a \in P$ and $h \leq a$ for all $h \in H$, then a is an upper bound of H . Similarly, if $b \in P$ and $b \leq h$ for all $h \in H$, then b is a lower bound of H .

Definition 1.1.8 [5] Let P be a partially ordered set and let $H \subseteq P$. If A is an upper bound of H such that $A \leq a$ for any upper bound a of H , then A is called the least upper bound or supremum of H . Similarly, if B is a lower bound of H such that $B \geq b$ for any lower bound b of H , then B is called the greatest lower bound or infimum of H .

Definition 1.1.9 [5] Let L be a partially ordered set such that for any $H \subseteq L$ where H is finite and non-empty, H has both a supremum and an infimum. Then L is called a lattice.

The following are two examples of lattices:

1. Consider the set $\{1, 2, 3, 4, 5\}$ and let L be the collection of all subsets of this set where for any two elements $a, b \in L$, $a \leq b$ iff $a \subseteq b$. Then L is a lattice. For any $H \subseteq L$, the supremum of H is the union of all elements of H , and the infimum of H is the intersection of all elements of H .

2. Let $L = \{1, 2, 3, 6, 9, 18\}$ with the property that for any $a, b \in L$, $a \leq b$ iff $a|b$. Then L is a lattice. For any $H \subseteq L$, the supremum of H is the least common multiple of all elements of H , and the infimum of H is the greatest common divisor of all elements of H .

Definition 1.1.10 (Notation) Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair, let $A \in \mathcal{A}$ and let

$B^* \subseteq [n] \setminus A$. Then we denote by B^{*+} the unique element $B \in \mathcal{B}$ which satisfies $B \setminus A = B^*$.

In this definition we will always assume that we are referring to a specific $A \in \mathcal{A}$. The rational behind this definition is that in some cases we will only need to consider the value of $B \setminus A$, therefore we only need to consider the elements of B that are not in A . Because we must have that $B \setminus A \neq B' \setminus A$ whenever $B \neq B'$, we can define any $B \in \mathcal{B}$ by its elements that aren't in A . For example, suppose that we have a recovering pair over $[n]$ where $n = 6$ and $A = \{1, 2\}$, then if $\{2, 4, 5\} \in \mathcal{B}$, we denote this set by $\{4, 5\}^+$.

Definition 1.1.11 *Let $G(V, E)$ be a graph over $V = [n]$. We say that G is edge-wise triangle complete if for every edge $\{a, b\} \in E$, $\exists c \in V$ such that $\{a, c\} \in E$ and $\{b, c\} \in E$.*

This definition says that an *edge-wise triangle complete* graph is a graph such that every edge is part of a triangle. We will use it to talk about the 2-element sets in a set family.

Note that for any triangle $\{\{a, b\}, \{a, c\}, \{b, c\}\}$, if $\{a, b\} \in \mathcal{A}$ and $\{b, c\} \in \mathcal{A}$, then $\{a, c\} \notin \mathcal{B}$ whenever $(\mathcal{A}, \mathcal{B})$ forms a recovering pair.

These graphs have also appeared elsewhere in literature, most notably in [7], where they were referred to as *T-graphs*.

Definition 1.1.12 *(notation) Consider a recovering pair $(\mathcal{A}, \mathcal{B})$, then we denote the sets of k -element members of \mathcal{A} and \mathcal{B} by \mathcal{A}_k and \mathcal{B}_k respectively.*

Note that in the proof of one theorem in the literature review, \mathcal{A}_i and \mathcal{B}_i are used to describe the sets $\{A \in \mathcal{A} : i \notin A\}$ and $\{B \in \mathcal{B} : i \notin B\}$ respectively. Otherwise, such notation is used according to the definition above.

Definition 1.1.13 [13] *Let $G(V, E)$ be a graph. Then the total degree of G is the sum of the degrees of all its vertices.*

Lemma 1.1.14 [9] *The total degree of a graph G is equal to $2|E|$.*

Definition 1.1.15 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$. Then we call $(\mathcal{A}, \mathcal{B})$ a total recovering pair if for every $x \in [n]$, there exists an $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $x \in A$ and $x \in B$.*

Definition 1.1.16 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$ and let $A \in \mathcal{A}$. Then we define the set $A \setminus \mathcal{B}$ to be $A \setminus \mathcal{B} = \{A \setminus B : B \in \mathcal{B}\}$.*

Chapter 2

Introduction

2.1 Problem and Origin

Consider an n -element set, and assume for simplicity that this set is $[n] = \{1, 2, \dots, n\}$. A *set system* over $[n]$ is defined as a collection of subsets of $[n]$. When we restrict one or more set systems to a certain set of criteria, we can then ask how large these set systems can be on their own or in relation to one another.

Definition 2.1.1 *Let \mathcal{A} and \mathcal{B} be two set systems, then $(\mathcal{A}, \mathcal{B})$ forms a recovering pair if the following conditions hold:*

$$\forall A, A' \in \mathcal{A}, \forall B, B' \in \mathcal{B}, A \setminus B = A' \setminus B' \rightarrow A = A'$$

$$\forall A, A' \in \mathcal{A}, \forall B, B' \in \mathcal{B}, B \setminus A = B' \setminus A' \rightarrow B = B'$$

The term recovering pair is used because if we are given a value of $A \setminus B$ for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$, we can recover our set A . Similarly, if we are given a value of $B \setminus A$ for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$, we can recover our set B . [1]

After studying write-unidirectional memories in [11], Gábor Simonyi first proposed the following conjecture at the Oberwolfach Conference in 1989:

Conjecture 2.1.2 *Let \mathcal{A} and \mathcal{B} be set systems over an n -element set such that*

$(\mathcal{A}, \mathcal{B})$ forms a recovering pair, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

This conjecture is now known as Simonyi's Conjecture. If the conjecture in fact holds, then the upper bound of 2^n is best possible by considering a subset S of our n -element set and letting \mathcal{A} be every subset of S and \mathcal{B} be every subset of our n -element set that is disjoint from S . Then $(\mathcal{A}, \mathcal{B})$ is a recovering pair and $|\mathcal{A}||\mathcal{B}| = 2^{|S|}2^{n-|S|} = 2^n$.

As mentioned above, this conjecture arose from the study of write-unidirectional memories from information theory. More specifically, "write-unidirectional memories are binary storage media for multiple uses in which n binary symbols can be stored at every single usage." [11] They require the use of both an encoder and a decoder, both of which can be informed or uninformed about the previous state of the memory. Simonyi's Conjecture originates from the study of the case where both the encoder and the decoder are uninformed.

2.2 An Alternate Form

Simonyi's Conjecture was first published in the above form by Ron Holzman and János Körner in 1995 [8]. Before this, the conjecture had the same statement, although an alternate, yet equivalent, definition of a recovering pairs was given [1]. A pair of set systems $(\mathcal{A}, \mathcal{B})$ was said to form a recovering pair if the following conditions hold:

$$\forall A, A' \in \mathcal{A}, \forall B, B' \in \mathcal{B}, A \cup B = A' \cup B' \rightarrow A = A'$$

$$\forall A, A' \in \mathcal{A}, \forall B, B' \in \mathcal{B}, B \cap A = B' \cap A' \rightarrow B = B'$$

To understand that the two sets of criteria for recovering pairs are in fact equivalent, note that one set of criteria can be obtained by taking the complement of every set A in the other.

2.3 Two Generalizations

The first generalization of Simonyi's Conjecture is the Sandglass Conjecture, which was proposed by Rudolf Ahlswede and Gábor Simonyi, and was published at the same time that Simonyi's Conjecture was first published. Before stating the conjecture, we must discuss some definitions, including the definition of a recovering pair in a lattice. All definitions are from [1].

Definition 2.3.1 *Let \mathcal{L} be a lattice. An ordered pair of subsets of \mathcal{L} , $(\mathcal{A}, \mathcal{B})$, is a recovering pair if for every $a, a', c, c' \in \mathcal{A}$ and $b, b', d, d' \in \mathcal{B}$, the following conditions hold:*

$$a \vee b = a' \vee b' \rightarrow a = a'$$

$$c \wedge d = c' \wedge d' \rightarrow d = d'$$

For any lattice \mathcal{L} , let $r(\mathcal{L})$ denote the largest possible value of $|\mathcal{A}||\mathcal{B}|$ among all recovering pairs in \mathcal{L} .

Definition 2.3.2 *A pair $(\mathcal{A}, \mathcal{B})$ of subsets of a lattice \mathcal{L} is said to form a sandglass if there exists an element c of \mathcal{L} such that $a \geq c$ for all $a \in \mathcal{A}$ and $b \leq c$ for all $b \in \mathcal{B}$. Furthermore, a sandglass is full if by adding any new element to \mathcal{A} or \mathcal{B} the new pair will no longer be a sandglass.*

Conjecture 2.3.3 *Let a lattice \mathcal{L} be the product of k finite length chains. Then there exists a (full) sandglass $(\mathcal{A}, \mathcal{B})$, $\mathcal{A}, \mathcal{B} \subset \mathcal{L}$ such that $|\mathcal{A}||\mathcal{B}| = r(\mathcal{L})$.*

To demonstrate that the Sandglass Conjecture implies Simonyi's conjecture, consider the lattice consisting of all subsets of $[n]$ and the property that for all $x, y \subseteq [n]$, $x \leq y$ in the lattice if and only if $x \subseteq y$. This lattice is a product of n chains of length two, namely the chains $\{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}, \dots, \{\emptyset, \{n\}\}$. Then for any $C \in \mathcal{L}$, we form the full sandglass $(\mathcal{A}', \mathcal{B}')$ such that $B \in \mathcal{B}'$ if and only if $B \leq C$ and $A \in \mathcal{A}'$ if and only if $A \geq C$. According to the Sandglass Conjecture, this recovering pair will achieve the value $r(\mathcal{L})$. Using this sandglass, we can then form a pair $(\mathcal{A}, \mathcal{B})$ that satisfies the first definition given for a recovering pair. This is the recovering pair $(\mathcal{A}, \mathcal{B})$ such that $B \in \mathcal{B}$ if and only if $B \in \mathcal{B}'$ and $A \in \mathcal{A}$ if and only if $[n] \setminus A \in \mathcal{A}'$. Then $|\mathcal{A}||\mathcal{B}| = 2^{n-|C|} * 2^{|C|} = 2^n$, where 2^n is proposed by Simonyi's Conjecture to be the largest possible value of $|\mathcal{A}||\mathcal{B}|$.

A second generalization of Simonyi's Conjecture using recovering pairs as defined using set subtraction was proposed by Ron Aharoni in [12] as follows:

Conjecture 2.3.4 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair, then $\sum_{A \in \mathcal{A}, B \in \mathcal{B}} 2^{|A \cap B|} \leq 2^n$.*

Again, if this conjecture holds then the upper bound of 2^n is best possible, as shown by the same construction as before where we take a subset S of our n -element set and let \mathcal{A} be every subset of S and let \mathcal{B} be every subset of the n -element set that is disjoint from S . Then $(\mathcal{A}, \mathcal{B})$ is a recovering pair and there are 2^n pairs (A, B) ,

each of which meet the condition that $A \cap B = \emptyset$ and therefore $2^{|A \cap B|} = 1$, thus giving equality.

Simonyi's conjecture follows immediately from this because $2^{|A \cap B|} \geq 1$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

2.4 My Contributions

When beginning my studies of recovering pairs $(\mathcal{A}, \mathcal{B})$, my first question was how could restricting elements of \mathcal{A} via certain criteria impose restrictions on the elements of $[n]$ that could be in \mathcal{B} ? My main approach to answering this was to consider and impose restrictions on the largest elements of \mathcal{A} . In some cases this directly gave the desired result of $|\mathcal{A}||\mathcal{B}| \leq 2^n$, while for other cases it provided enough information about the elements of \mathcal{B} that I was then able to obtain further restrictions on the elements of \mathcal{A} to obtain that $|\mathcal{A}||\mathcal{B}| \leq 2^n$. My main goal in this approach was to methodically relax these restrictions and arrive at an inductive proof, but finding a pattern proved difficult. These results hold for any n , but they were particularly useful for small values of n because they were able to provide a complete subset of cases needed for the proof.

With the data obtained from the previously described method, it was possible to obtain a proof that Simonyi's Conjecture holds whenever $n \leq 5$. The results also proved many cases for $n = 6$, so the natural question was to ask how to prove the remaining cases. A proof was found by considering the elements that are within both \mathcal{A} and \mathcal{B} and observing how the presence of a sufficient number of elements

forced most of them to be contained within one set system or the other and hence restrict the number of elements that could be in the other set system. This was done by considering the notion of an edge-wise triangle complete graph (see definitions), and applying this concept to set systems.

The next section of results begins with a theorem that justifies the use of recovering pairs where $(\mathcal{A}, \mathcal{B})$ is a total recovering pair (see definitions section) in proving the conjecture holds for higher values of n . Following this will be a collection of results for such recovering pairs over general values of n , as well as proofs that the conjecture holds for $n = 7$, some more general results, and finally for $n = 8$.

The results for various values of n are broken up into four sections, each one for $n \leq 5$, $n = 6$, $n = 7$, and $n = 8$, where each section follows the same general structure: First a collection of general results needed to prove the next value of n , followed by the proof for that n using those results as well as some results from previous sections.

Following these results that verify Simonyi's Conjecture for $n \leq 8$, there will be one section discussing a theorem that describes the set of numbers over which Simonyi's Conjecture holds. Lastly, we will present a new problem in graph theory and discuss some related results.

A listing of all results and the results they call upon in their proof is given in the Appendix.

Chapter 3

Literature Review

3.1 Previous Research

It appears that there have been two key approaches in attempting to solve the conjecture. One approach is to work on proving slightly weaker upper bounds. The other is to set restrictions on the nature of \mathcal{A} and \mathcal{B} and prove that the conjecture holds under these conditions. This type of proof would rely on finding a complete subset of cases and proving all of them to be true, or to then eliminate the true cases and be left with a counterexample.

The first upper bound to be proven for $|\mathcal{A}||\mathcal{B}|$ was 3^n , and was found by Cohen in 1988. The proof of this was given as a remark by Ahlswede and Simonyi in [1]. Using the definition of recovering pairs using set subtraction, the proof of this is as follows: Let $t = \max(|A| : A \in \mathcal{A})$, then each set $B \in \mathcal{B}$ must give a different result when we consider $B \setminus A$, which implies that $|\mathcal{B}| \leq 2^{n-t}$. Also, we know that $|\mathcal{A}| \leq \sum_{i=0}^t \binom{n}{i}$. Then $|\mathcal{A}||\mathcal{B}| \leq \sum_{i=0}^t \binom{n}{i} 2^{n-t} = \sum_{i=n-t}^n \binom{n}{i} 2^{n-t} \leq \sum_{i=0}^n \binom{n}{i} 2^i = 3^n$.

Currently the best upper bound known for $|\mathcal{A}||\mathcal{B}|$ is $\approx 2.3264^n$, as found in 1995 by Ron Holzman and János Körner in [8]. Before giving the proof of this, we must first review some definitions and facts.

Definition 3.1.1 [8] Let \mathcal{A} and \mathcal{B} be two set systems such that $\forall A, A' \in \mathcal{A}, \forall B, B' \in \mathcal{B}, A \setminus B = A' \setminus B \rightarrow A = A'$ and $\forall A, A' \in \mathcal{A}, \forall B, B' \in \mathcal{B}, B \setminus A = B' \setminus A \rightarrow B = B'$. Then $(\mathcal{A}, \mathcal{B})$ is called a cancellative pair.

It can be noted that any recovering pair will also be a cancellative pair, and therefore any bound on $|\mathcal{A}||\mathcal{B}|$ for cancellative pairs will also hold for recovering pairs.

Definition 3.1.2 [8] Let ξ be a random variable which assumes k distinct values with respective probabilities p_1, \dots, p_k $\left(p_j > 0, \sum_{j=1}^k p_j = 1\right)$. Then the entropy of ξ is defined as $H(\xi) = -\sum_{j=1}^k p_j \log(p_j)$. In the special case where $k = 2$ and the probabilities are p and $1-p$, we define the entropy function to be $h(p) = -p \log(p) - (1-p) \log(1-p)$.

We will also use the fact that if $\xi = (\xi_1, \dots, \xi_n)$ is an n -dimensional random variable, then $H(\xi) \leq \sum_{i=1}^n H(\xi_i)$. [8]

The concept of entropy was originally introduced by Shannon as a means of quantifying the level of uncertainty associated with a random variable that assumes k distinct variables as defined above. The function H was found by requiring the following [10]:

1. H should be continuous over the values p_i .
2. If all values of p_i are equal, i.e., $p_i = 1/n$, then H should be a monotonic increasing function of n . This is to ensure that H reflects the greater level of uncertainty as the value of n increases.
3. The value of H should only depend on the probabilities of each possible out-

come occurring, and not the way in which each outcome is chosen. For example, $H(1/2, 1/3, 1/6) = H(1/2, 1/2) + (1/2)H(2/3, 1/3)$.

We now define the function $f(p, q) = qh(p) + ph(q)$ and consider the following two lemmata that will be required in the proof of the next theorem.

Lemma 3.1.3 [8] *On each hyperbola of the form $pq = C$, the maximum of $f(p, q)$ is attained when $p = q$.*

Lemma 3.1.4 [8] *$f(p, p)$ is increasing for $0 \leq p \leq \sqrt{1/\theta}$ and assumes the value $\log(\theta)$ at $p = \sqrt{1/\theta}$.*

The proof of these lemmata follow from calculus, and the reader is directed to [8] for the details.

Theorem 3.1.5 (Holzman and Körner) [8] *Let $(\mathcal{A}, \mathcal{B})$ be a cancellative pair over $[n]$ and let θ be the largest solution of the equation $\sqrt{1/\theta}h(\sqrt{1/\theta}) = \log \sqrt{1/\theta}$. Then $\theta \approx 2.3264$ and $|\mathcal{A}||\mathcal{B}| \leq \theta^n$.*

Proof. (By induction) It is easy to verify the case when $n = 1$

Now let $(\mathcal{A}, \mathcal{B})$ be a cancellative pair over an n -element set, and WLOG assume this set is $[n]$. Then for all $i \in [n]$, we consider the following sets and values:

$$\mathcal{A}_i = \{A \in \mathcal{A} : i \notin A\}, p_i = |\mathcal{A}_i|/|\mathcal{A}|$$

$$\mathcal{B}_i = \{B \in \mathcal{B} : i \notin B\}, q_i = |\mathcal{B}_i|/|\mathcal{B}|$$

Note that $(\mathcal{A}, \mathcal{B})$ is a cancellative pair over $[n] \setminus \{i\}$, and by the induction hypothesis we know that $|\mathcal{A}_i||\mathcal{B}_i| < \theta^{n-1}$. If $p_i q_i \geq 1/\theta$, then we obtain $|\mathcal{A}||\mathcal{B}| < \theta^n$, as desired. So we will assume that $p_i q_i < 1/\theta$ for all $i \in [n]$.

Now for any set $B \in \mathcal{B}$, consider the random variable $\xi^B = A \setminus B$, where $A \in \mathcal{A}$ is chosen according to the uniform distribution on \mathcal{A} , i.e., the probability of any certain set $A \in \mathcal{A}$ being chosen is $1/|\mathcal{A}|$. Because $(\mathcal{A}, \mathcal{B})$ is a cancellative pair, ξ^B assumes distinct values for distinct sets A , and therefore the entropy of ξ^B is

$$H(\xi^B) = - \sum_{A \in \mathcal{A}} (1/|\mathcal{A}|) \log(1/|\mathcal{A}|) = -\log(1/|\mathcal{A}|) = \log(|\mathcal{A}|).$$

Now we can consider ξ^B as an n -dimensional random variable with components ξ_1^B, \dots, ξ_n^B , where $\xi_i^B = 1$ if $i \in A \setminus B$ and $\xi_i^B = 0$ otherwise. Then for $i \in B$, we have $H(\xi_i^B) = 0$ and for $i \notin B$ we have $H(\xi_i^B) = h(p_i)$. We also have that

$$\log(|\mathcal{A}|) = H(\xi^B) \leq \sum_{i \in X \setminus B} h(p_i).$$

But then we have such an inequality for every $B \in \mathcal{B}$, so we can take an average of all these values to obtain

$$\log(|\mathcal{A}|) \leq 1/|\mathcal{B}| \sum_{B \in \mathcal{B}} \sum_{i \in X \setminus B} h(p_i) = 1/|\mathcal{B}| \sum_{i=1}^n |\mathcal{B}_i| h(p_i) = \sum_{i=1}^n q_i h(p_i).$$

Similarly, we can obtain that

$$\log(|\mathcal{B}|) \leq \sum_{i=1}^n p_i h(q_i).$$

This then gives that

$$\log(|\mathcal{A}||\mathcal{B}|) \leq \sum_{i=1}^n [q_i h(p_i) + p_i h(q_i)].$$

From lemmata 1.3 and 1.4 we know that

$$\log(|\mathcal{A}||\mathcal{B}|) \leq \sum_{i=1}^n [q_i h(p_i) + p_i h(q_i)] \leq \sum_{i=1}^n \log(\theta) = \log(\theta^n)$$

Therefore $|\mathcal{A}||\mathcal{B}| \leq \theta^n$. ■

These are the only published results that give a general upper bound for $|\mathcal{A}||\mathcal{B}|$.

The remainder of the results found on Simonyi's Conjecture are special cases where the conjecture holds. One of these results focuses on the conjecture itself, while the remainder prove specific cases of the Sandglass Conjecture. The proven cases of the Sandglass Conjecture are as follows:

Theorem 3.1.6 [1] *Let \mathcal{L} be the lattice obtained as the product of two finite length chains. Then $r(\mathcal{L})$ can be achieved by a Sandglass.*

Theorem 3.1.7 [4] *Let \mathcal{L} be the lattice obtained as the product of three finite length chains. Then $r(\mathcal{L})$ can be achieved by a Sandglass.*

Theorem 3.1.8 [4] *For each n there exists $K_0(n)$ such that if $\mathcal{L} = \prod_{i=1}^n C_i$, where C_i are finite length chains with $|C_i| \geq K_0(n)$ for $i = 1, \dots, n$, then $r(\mathcal{L})$ is realized by a Sandglass.*

Theorem 1.4 was published by Ron Ahlswede and Gábor Simonyi in 1994 in [1] and was proven via a series of lemmata. The reader is directed to that paper if they are interested. Theorems 1.5 and 1.6 were published by Rita Csákány in 2000 in [4], however no proof was given for either theorem.

Although these theorems deal with the more general Sandglass Conjecture, they also prove Simonyi's Conjecture when $n \leq 3$.

The last result we will discuss in one found by Attila Sali and Gábor Simonyi

in 1997, as presented in [3]. Before presenting the theorem, we must state the following definition and lemma.

Definition 3.1.9 *Let $[n] = X \cup Y \cup Z$ be a partition of $[n]$ where $|Z| = 1$. Let \mathcal{A}_k be the set of all k -element subsets of $X \cup Z$ and let \mathcal{B}_k be the set of all k -element subsets of $Y \cup Z$. Then the pair $(\mathcal{A}_k, \mathcal{B}_k)$ is called the quasi-disjoint pair with parameter $\max\{x, y\}$ where $x = |X|$ and $y = |Y|$.*

Lemma 3.1.10 *[3] The quasi-disjoint pair $(\mathcal{A}_k, \mathcal{B}_k)$ defined above is always a recovering pair.*

Proof. Consider the quasi-disjoint pair $(\mathcal{A}_k, \mathcal{B}_k)$ and assume WLOG that $X = \{1, \dots, z-1\}$, $Z = \{z\}$, and $Y = \{z+1, \dots, n\}$. Now WLOG consider any element $A \in \mathcal{A}_k$. If $z \notin A$, then $A \setminus B = A$ for any $B \in \mathcal{B}$. If $z \in A$, then A is composed of z as well as $k-1$ elements of X . Then $A \setminus B \in \{A, A \setminus \{z\}\}$ for any $B \in \mathcal{B}$. From here it is easy to see that $(\mathcal{A}_k, \mathcal{B}_k)$ is a recovering pair. ■

In our quasi-disjoint pair defined above, we have that $|\mathcal{A}_k||\mathcal{B}_k| = \binom{x+1}{k} \binom{y+1}{k} \leq 2^n$. It is easy to see that $|\mathcal{A}_k||\mathcal{B}_k|$ will be maximal when, WLOG, $x = \lceil (n-1)/2 \rceil$ and $y = \lfloor (n-1)/2 \rfloor = \lceil (n-2)/2 \rceil$. Hence $|\mathcal{A}_k||\mathcal{B}_k| \leq \binom{\lceil (n+1)/2 \rceil}{k} \binom{\lceil n/2 \rceil}{k}$.

Theorem 3.1.11 *For every $k \geq 1$ there is an integer $n_0(k)$ such that if $(\mathcal{A}_k, \mathcal{B}_k)$ is a recovering pair of set systems of k -subsets of an n -set with $n \geq n_0(k)$, then $|\mathcal{A}_k||\mathcal{B}_k| \leq \binom{\lceil (n+1)/2 \rceil}{k} \binom{\lceil n/2 \rceil}{k}$. Furthermore, equality holds if and only if $(\mathcal{A}_k, \mathcal{B}_k)$ is a quasi-disjoint pair with parameter $\lceil (n-1)/2 \rceil$.*

3.2 A Connection to Graph Theory

The following definitions, first published by Attila Sali and Gábor Simonyi, provide a connection between graphs and recovering pairs:

Let G be a simple graph with vertex set $\{1, 2, \dots, m\}$. A family of set systems $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ over $[n]$ forms a *recovering family with respect to G* if $(\mathcal{A}_i, \mathcal{A}_j)$ is a recovering pair whenever $\{i, j\} \in E(G)$. For any graph G , we want to find the largest possible value of $\prod_{1 \leq i \leq m} |\mathcal{A}_i|$. In the general case, this is denoted by $M(G, n)$, and in the case that all set systems \mathcal{A}_i are *k -uniform* (i.e., all elements of \mathcal{A} have size k), we denote the value by $M_k(G, n)$. The (*general*) *recovering number* of a graph G is the quantity $RC(G) = \limsup_{n \rightarrow \infty} [M(G, n)]^{1/n}$ and the *k -uniform recovering number* of G is the quantity $RC_k(G) = \limsup_{n \rightarrow \infty} \frac{M_k(G, n)}{\binom{n}{k}^m}$.

In this paper, the authors present and prove a direct relation between the k -uniform recovering number of a graph G and graph entropy, as well as prove several facts on the topic. As these topics are not directly related to proving Simonyi's Conjecture, the reader is directed to [3] if they wish to read more on this.

Chapter 4

Results

4.1 Proof for $n \leq 5$

Conjecture 4.1.1 (*Simonyi*) Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Throughout this thesis, the letters \mathcal{A} and \mathcal{B} will be used to denote families of sets in a recovering pair.

Lemma 4.1.2 Let $a = \max\{|A| : A \in \mathcal{A}\}$. Then $|\mathcal{B}| \leq 2^{n-a}$.

Proof. Let $A \in \mathcal{A}$ such that $|A| = a$. Then for any $B^* \subseteq [n] \setminus A$, there exists at most one $B \in \mathcal{B}$ such that $B \setminus A = B^*$, because if two such sets existed, subtracting A from both would give the same answer and not meet the required conditions for a recovering pair. Therefore $|\mathcal{B}| \leq 2^{|[n] \setminus A|} = 2^{n-a}$.

■

Corollary 4.1.3 Let $a = \max\{|A| : A \in \mathcal{A}\}$ and let $b = \max\{|B| : B \in \mathcal{B}\}$. If $a + b \geq n$, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Proof. This result follows directly from Lemma 1.2 because $|\mathcal{A}| \leq 2^{n-b}$ and $|\mathcal{B}| \leq 2^{n-a}$, so $|\mathcal{A}||\mathcal{B}| \leq 2^{2n-(a+b)} \leq 2^n$ whenever $a + b \geq n$. ■

Lemma 4.1.4 *If either \mathcal{A} or \mathcal{B} only contains sets of size at most one, then the conjecture holds.*

Proof. WLOG assume that \mathcal{A} has the property that $|A| \leq 1$ for all $A \in \mathcal{A}$.

Case 1. ($\emptyset \in \mathcal{A}$) It is easy to see that for any singleton A , the number contained in A cannot be in any element B of \mathcal{B} because we would then have $A \setminus B = \emptyset \setminus B = \emptyset$ and our pair $(\mathcal{A}, \mathcal{B})$ would no longer be recovering. Let k be the number of singletons in \mathcal{A} , then $|\mathcal{B}| \leq 2^{n-k}$. We know that $|\mathcal{A}| = k+1$, so $|\mathcal{A}||\mathcal{B}| \leq (k+1)2^{n-k}$. But $k+1 \leq 2^k \forall k \geq 0$, so $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Case 2. ($\emptyset \notin \mathcal{A}$) i.e., \mathcal{A} is a set of singletons. Assume that $|\mathcal{A}| = k+1$. Now assume for a contradiction that there are two elements of \mathcal{A} , $\{a_1\}$ and $\{a_2\}$, such that $\{a_1\} \subseteq B_1$ and $\{a_2\} \subseteq B_2$ where $B_1, B_2 \in \mathcal{B}$. Then $\{a_1\} \setminus B_1 = \{a_2\} \setminus B_2 = \emptyset$, a contradiction. So we know that there exists at most one element of \mathcal{A} such that its element is contained in a member of \mathcal{B} , so $|\mathcal{B}| \leq 2^{n-k}$, and therefore $|\mathcal{A}||\mathcal{B}| \leq (k+1)2^{n-k} \leq 2^n$. ■

Lemma 4.1.5 *Suppose that either \mathcal{A} or \mathcal{B} contains $[n]$ as an element. Then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.*

Proof. WLOG assume that $[n] \in \mathcal{A}$, then $|\mathcal{B}| \leq 1$ because $B \setminus [n] = \emptyset$ for all $B \in \mathcal{B}$, and hence $|\mathcal{A}||\mathcal{B}| \leq 2^n$. ■

Lemma 4.1.6 *Suppose that either \mathcal{A} or \mathcal{B} contains a set of cardinality $n - 1$. Then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.*

Proof. WLOG assume there exists some $A \in \mathcal{A}$ such that $|A| = n - 1$. By Lemma 1.2, $|\mathcal{B}| \leq 2^{n-(n-1)} = 2$. If $|\mathcal{B}| \leq 1$, the result follows trivially. If \mathcal{B} contains two elements, then at least one of these must be non-empty and by Lemma 1.2, $|\mathcal{A}| \leq 2^{n-1}$. Therefore, by Corollary 1.3, $|\mathcal{A}||\mathcal{B}| \leq 2^n$. ■

Lemma 4.1.7 *Suppose that either \mathcal{A} or \mathcal{B} contains a set of cardinality $n - 2$. Then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.*

Proof. WLOG assume there exists some $A \in \mathcal{A}$ such that $|A| = n - 2$. By Lemma 1.2 we know that $|\mathcal{B}| \leq 2^{n-(n-2)} = 4$. If any set in \mathcal{B} has size ≥ 2 , then we will obtain, again by Lemma 1.2, that $|\mathcal{A}| \leq 2^{n-2}$, and we have $|\mathcal{A}||\mathcal{B}| \leq 2^n$. We also know from Lemma 1.4 that if all elements of \mathcal{B} have size ≤ 1 , then $|\mathcal{A}||\mathcal{B}| \leq 2^n$, thus completing the proof. ■

Lemma 4.1.8 *Suppose that either \mathcal{A} or \mathcal{B} contains a set of cardinality $n - 3$. Then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.*

Proof. WLOG assume there exists some $A \in \mathcal{A}$ such that $|A| = n - 3$. For the sake of simplicity in our proof, assume that our n -element set is $[n]$ and that $A = \{4, 5, \dots, n\}$.

Because $|A| = n - 3$, we know that by Lemma 1.2 $|\mathcal{B}| \leq 2^3 = 8$. We also know that if any $B \in \mathcal{B}$ has size ≥ 3 , then $|\mathcal{A}|$ will be $\leq 2^{n-3}$ and hence $|\mathcal{A}||\mathcal{B}| \leq 2^n$. Similarly, we know from Lemma 1.4 that if all members of \mathcal{B} have size ≤ 1 , then $|\mathcal{A}||\mathcal{B}| \leq 2^n$. When the maximum size of any element in $\mathcal{B} = 2$, we obtain that $|\mathcal{A}| \leq 2^{n-2}$, so if $|\mathcal{B}| \leq 4$, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$. So we are left to prove the case where the largest element of \mathcal{B} has size 2 and $|\mathcal{B}| \geq 5$. We will assume that these conditions hold for the remainder of the proof. Because $|\mathcal{B}| \leq 2^3$, it will suffice to show that whenever $|\mathcal{B}| \geq 5$, we have that $|\mathcal{A}| \leq 2^{n-3}$.

We will refer to all elements of \mathcal{B} in the form B^{*+} where $B^* \in [n] \setminus A = \{1, 2, 3\}$. Because $|B| \leq 2$ for all $B \in \mathcal{B}$ by assumption, we know that $\{1, 2, 3\}^+ \notin \mathcal{B}$ and also that if $|B^*| = 2$, then $B^{*+} = B^*$. This means that \mathcal{B} will be a subset of $\{\emptyset^+, \{1\}^+, \{2\}^+, \{3\}^+, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Because we assumed that $|\mathcal{B}| \geq 5$, it is clear that we must have at least one element of \mathcal{B} such that $|B^*| = 2$. The proof will now be split into cases based on the size of the set $\{B : |B^*| = 2\}$.

Case 1. ($|\{B : |B^*| = 2\}| = 1$) WLOG assume that $\{1, 2\} \in \mathcal{B}$. Because $|\mathcal{B}| \geq 5$ and $\{1, 2\}$ is the only set in \mathcal{B} such that $|B^*| = 2$, we know that $\mathcal{B} = \{\emptyset^+, \{1\}^+, \{2\}^+, \{3\}^+, \{1, 2\}\}$. We now claim that 1 cannot be an element of any set in \mathcal{A} . To see this, recall our set $A = \{4, 5, \dots, n\}$ and assume there exists an element $A_1 \in \mathcal{A}$ such that $1 \in A_1$. Then we have either that $\{1, 2\} \setminus A_1 = \{2\}$ or $\{1, 2\} \setminus A_1 = \emptyset$. But we also know that $\{1\}^+ \setminus A = \{1\}$ and $\emptyset^+ \setminus A = \emptyset$. Hence if

$A_1 \in \mathcal{A}$, we obtain a contradiction to our assumption that $(\mathcal{A}, \mathcal{B})$ forms a recovering pair, therefore 1 is not an element of any set in \mathcal{A} . In the same way, we can see that 2 is not an element of any set in \mathcal{A} . So \mathcal{A} must be a set system over the set $\{3, 4, \dots, n\}$ and hence we know that $|\mathcal{A}| \leq 2^{n-2}$.

Now to see that $|\mathcal{A}| \leq 2^{n-3}$, assume for a contradiction that $|\mathcal{A}| > 2^{n-3}$. Create 2^{n-3} pigeonholes P_S corresponding to subsets $S \subseteq \{4, 5, \dots, n\}$ such that $A_i \in P_S$ if and only if $A_i \cap \{4, 5, \dots, n\} = S$. Because we assumed that $|\mathcal{A}| > 2^{n-3}$, we know there must be some pigeonhole containing at least two elements. When this happens, we know that we have two elements $A_3, A_{\setminus 3}$ in \mathcal{A} such that $3 \notin A_{\setminus 3}$ and $A_{\setminus 3} \cup \{3\} = A_3$. However, we then get that $A_3 \setminus \{3\}^+ = A_{\setminus 3} \setminus \{3\}^+$, contradicting the assumption that \mathcal{A}, \mathcal{B} is a recovering pair. Therefore $|\mathcal{A}| \leq 2^{n-3}$ and $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Case 2. ($|\{B : |B^*| = 2\}| = 2$) WLOG assume that $\{1, 2\} \in \mathcal{B}$ and $\{2, 3\} \in \mathcal{B}$. Because $|\mathcal{B}| \geq 5$, we know that ($|\{B : |B^*| = 1\}| \geq 2$) and hence that either $\{1\}^+ \in \mathcal{B}$ or $\{3\}^+ \in \mathcal{B}$. Assume WLOG that $\{1\}^+ \in \mathcal{B}$.

Subcase 2.1. ($\emptyset^+ \in \mathcal{B}$) We claim that there cannot be any $A_2 \in \mathcal{A}$ such that $2 \in A_2$, as if such an A_2 were to exist, then we would either have that $\{1, 2\} \setminus A_2 = \{1\} = \{1\}^+ \setminus A$, or we would have that $\{1, 2\} \setminus A_2 = \emptyset = \emptyset^+ \setminus A$. Now we cannot have two elements $A_1, A_3 \in \mathcal{A}$ such that $1 \in A_1$ and $3 \in A_3$, because we would then have $\{1, 2\} \setminus A_1 = \{2, 3\} \setminus A_3 = \{2\}$ and $(\mathcal{A}, \mathcal{B})$ would no longer be recovering. Assume WLOG that 1 is not contained in any element of \mathcal{A} . Then we know that \mathcal{A} is a set system over the set $\{3, 4, \dots, n\}$ and we have that $|\mathcal{A}| \leq 2^{n-2}$. Because $3 \in \{2, 3\} \in \mathcal{B}$, we can use the pigeonhole argument from case 1 to see that $|\mathcal{A}| \leq 2^{n-3}$, and hence $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Subcase 2.2. ($\emptyset^+ \notin \mathcal{B}$) In this case, we have that $\mathcal{B} = \{\{1\}^+, \{2\}^+, \{3\}^+, \{1, 2\}, \{2, 3\}\}$. We claim that no element of $\{1, 2, 3\}$ can be in any element of \mathcal{A} . To prove this, assume for a contradiction that at least one element of $\{1, 2, 3\}$ is found in some element of \mathcal{A} . But we already know that $\{1\}^+ \setminus A = \{1\}$, $\{2\}^+ \setminus A = \{2\}$, and $\{3\}^+ \setminus A = \{3\}$. We also know by assumption that either $\{1, 2\} \cap A' \neq \emptyset$ or $\{2, 3\} \cap A' \neq \emptyset$ for some $A' \in \mathcal{A}$. WLOG assume that $\{1, 2\} \cap A' \neq \emptyset$. However, because $(\mathcal{A}, \mathcal{B})$ is a recovering pair, we are forced to have that $\{1, 2\} \setminus A' = \emptyset$. But we also can see now that $\{2, 3\} \setminus A'$ is equal to either $\{3\}$ or \emptyset , providing a contradiction to $(\mathcal{A}, \mathcal{B})$ being a recovering pair. Therefore no element of $\{1, 2, 3\}$ is contained in any element of \mathcal{A} , and hence $|\mathcal{A}| \leq 2^{n-3}$, and hence $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Case 3. ($|\{B : |B^*| = 2\}| = 3$) In this case, we have that $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ are all elements of \mathcal{B} . Clearly we cannot have $\{1, 2, 3\} \subseteq A' \in \mathcal{A}$ because we would then have $\{1, 2\} \setminus A' = \{2, 3\} \setminus A' = \emptyset$. We also cannot have any $A' \in \mathcal{A}$ such that $|A' \cap \{1, 2, 3\}| = 2$. To demonstrate this, assume WLOG that $\{1, 2\} \subseteq A'$ for some $A' \in \mathcal{A}$. Then $\{1, 3\} \setminus A' = \{2, 3\} \setminus A' = \{3\}$, a contradiction. So we have that $|\{1, 2, 3\} \cap A'| \leq 1$ for all $A' \in \mathcal{A}$.

Now assume WLOG that $1 \in A_1$ for some $A_1 \in \mathcal{A}$. We cannot have any $A_2 \in \mathcal{A}$ such that $2 \in A_2$, because otherwise we would have $\{1, 3\} \setminus A_1 = \{2, 3\} \setminus A_2 = \{3\}$, and similar we cannot have any $A_3 \in \mathcal{A}$ such that $3 \in A_3$. Hence $|\mathcal{A}| \leq 2^{n-2}$. Because we know that $\{1, 2\} \in \mathcal{B}$, we can use the pigeonhole argument from case 1

to prove that $|\mathcal{A}| \leq 2^{n-3}$, and hence $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

■

Lemmata 1.5 through 1.8 can be summarized by the following theorem:

Theorem 4.1.9 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$ such that either \mathcal{A} or \mathcal{B} contains a set of size at least $n - 3$, then Simonyi's Conjecture holds.*

Theorem 4.1.10 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over an n -element set where $n \leq 5$. Then Simonyi's Conjecture holds.*

Proof. Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$ where $n \leq 5$. We know by Lemma 1.4 that if \mathcal{A} or \mathcal{B} contain elements of size at most one, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$, and by Theorem 1.9 we know the same inequality holds if either \mathcal{A} or \mathcal{B} contain an element of size ≥ 2 , thus completing the proof.

■

4.2 Proof for $n = 6$

Lemma 4.2.1 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$. Then there exists at most one $A \in \mathcal{A}$ such that $A \subseteq B$ for some $B \in \mathcal{B}$.*

Proof. Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair and assume for a contradiction that there exists $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$ such that $A \subseteq B$ and $A' \subseteq B'$. Then $A \setminus B =$

$A' \setminus B' = \emptyset$, contradicting our assumption that $(\mathcal{A}, \mathcal{B})$ is a recovering pair. ■

The following corollaries follow directly from Lemma 2.1 and will be used in later proofs.

Corollary 4.2.2 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$. Then there exists at most one set $p \subseteq [n]$ such that $p \in \mathcal{A}$ and $p \in \mathcal{B}$.*

Corollary 4.2.3 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$. If $\emptyset \in \mathcal{A}$ or $\emptyset \in \mathcal{B}$, then there can be no non-empty set that is an element of both \mathcal{A} and \mathcal{B} .*

Proof. WLOG assume for a contradiction that $\emptyset \in \mathcal{A}$ and there exists a set $p \subseteq [n]$ such that $p \in \mathcal{A}$ and $p \in \mathcal{B}$. Then $\emptyset \setminus p = p \setminus p = \emptyset$, thus contradicting the assumption that $(\mathcal{A}, \mathcal{B})$ is a recovering pair. ■

Lemma 4.2.4 *Any simple graph with 5 vertices with at least 8 edges will be edge-wise triangle complete.*

Proof. Consider a graph $G(V, E)$, over the vertex set $V = \{1, 2, 3, 4, 5\}$ where $|E| \geq 8$ and assume WLOG that $\{1, 2\} \in E$. Of the remaining 9 possible edges in G , six edges have an endpoint in common with the edge $\{1, 2\}$ and three have no endpoint in common. Because $|E| \geq 8$, we must have at least seven edges other than $\{1, 2\}$, and hence at least four that share an endpoint with the edge $\{1, 2\}$. This

means there are at least four edges between an element of the vertex set $\{1, 2\}$ and the vertex set $\{3, 4, 5\}$. By the pigeonhole principle, there exists at least one $v \in \{3, 4, 5\}$ such that $\{v, 1\} \in E$ and $\{v, 2\} \in E$, and hence the edge $\{1, 2\}$ is part of a triangle. Therefore, G is edge-wise triangle complete. ■

In the above lemma, the requirement that the graph has at least 8 edges is best possible, as shown by the following graph with 5 vertices and 7 edges that is not edge-wise triangle complete: $G(V, E)$ where $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$.

Lemma 4.2.5 *Any graph on 6 vertices with at least 10 edges will have an induced subgraph with 5 vertices that is edge-wise triangle complete.*

Proof. Let $G(V, E)$ be a graph over 6 vertices and assume that G has at least 10 edges. By Lemma 2.4, if we can show that G has an induced subgraph over 5 vertices with at least 8 edges, then the lemma will be proven.

If $|E| \geq 11$, we can use the averaging principle on the degree of each vertex. When $|E| = 11$, we know that the total degree of G is equal to 22, but $22/6 < 4$, so there must be at least one vertex of degree ≤ 3 , so by removing this vertex we will obtain an induced subgraph over 5 vertices with at least 8 edges, and by Lemma 2.4 we are done. A similar proof holds for all graphs with greater than 11 edges.

To consider the case when $|E| = 10$, first note that if any vertex in G has degree ≤ 2 , we can remove that vertex and have an induced subgraph with 5 vertices

and at least 8 edges, and we are done. What remains to be proven are the cases where $|E| = 10$ and every vertex has degree ≥ 3 . Because $|E| = 10$, we know that the total degree of G is 20, so the only possibilities for the degree set of our graph are $\{3, 3, 3, 3, 4, 4\}$ and $\{3, 3, 3, 3, 3, 5\}$. If the degree set of G is $\{3, 3, 3, 3, 4, 4\}$, then pick one vertex of degree 4, label it X , and remove the vertex that isn't adjacent to it. If the degree set of G is $\{3, 3, 3, 3, 3, 5\}$, then label the vertex of degree 5 by X and remove one of the vertices of degree 3. In either case, we are left with a graph G' over 5 vertices, including X which is adjacent to every other vertex, and every other vertex has degree at least two. To show that this graph is edge-wise triangle complete, label the vertices by X, v_1, v_2, v_3, v_4 . Because the edge $\{X, v_i\} \in G'$ for all v_i , we know that each edge $\{v_i, v_j\}$ in G' is part of a triangle. Note that each v_i has degree at least 2 and therefore is adjacent to some v_j , and therefore v_i is part of the triangle $\{\{v_i, v_j\}, \{X, v_i\}, \{X, v_j\}\}$, hence G' is edge-wise triangle complete, thus completing the proof. ■

It can be noted that in the above proof, every edge-wise triangle complete subgraph we found had no vertex that has degree zero. This will be used in a later proof.

The assumption that G had at least 10 edges is best possible, as shown by the fact that the complete bipartite graph $K_{3,3}$ has 9 edges and no induced subgraph over 5 vertices that is edge-wise triangle complete.

Lemma 4.2.6 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over a 6-element set such that $\max(|A| :$*

$A \in \mathcal{A}) = \max(|B| : B \in \mathcal{B}) = 2$, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Proof. Assume WLOG we are working over $\{1, \dots, 6\}$. Then we know that $|\mathcal{A}_1 \cup \mathcal{B}_1| \leq 6$ because there are only 6 possible singletons and $|\mathcal{A}_0 \cup \mathcal{B}_0| \leq 1$. By Corollary 2.2, $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_2 \cup \mathcal{B}_2| + |\mathcal{A}_1 \cup \mathcal{B}_1| + |\mathcal{A}_0 \cup \mathcal{B}_0| + 1 \leq |\mathcal{A}_2 \cup \mathcal{B}_2| + 8$ because only one element can occur in both \mathcal{A} and \mathcal{B} . Furthermore, if we have any non-empty element occurring in both \mathcal{A} and \mathcal{B} , then $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_2 \cup \mathcal{B}_2| + |\mathcal{A}_1 \cup \mathcal{B}_1| + 1 \leq |\mathcal{A}_2 \cup \mathcal{B}_2| + 7$.

Case 1. If $|\mathcal{A}_2 \cup \mathcal{B}_2| \leq 8$, then $|\mathcal{A}| + |\mathcal{B}| \leq 16$. Any two real numbers that sum to 16 or less will have a product no greater than 64, so $|\mathcal{A}||\mathcal{B}| \leq 2^6$.

Case 2. If $|\mathcal{A}_2 \cup \mathcal{B}_2| = 9$, then $|\mathcal{A}| + |\mathcal{B}| \leq 17$. The only time anything needs to be proven is when then $|\mathcal{A}| + |\mathcal{B}| = 17$, and by Corollaries 2.2 and 2.3 this only occurs when $|\mathcal{A}_2 \cup \mathcal{B}_2| = 9$, $\mathcal{A}_2 \cap \mathcal{B}_2 = \emptyset$, every singleton occurs once in either \mathcal{A} or \mathcal{B} but not both, and $\emptyset \in \mathcal{A}$ and $\emptyset \in \mathcal{B}$. Now assume these conditions hold. Because \emptyset is in both \mathcal{A} and \mathcal{B} , then WLOG if $\{a\} \in \mathcal{A}$, then $a \notin B$ for any $B \in \mathcal{B}$. Consider the following, WLOG:

If \mathcal{A} contains no singletons, then $|\mathcal{A}| = 1$ and $|\mathcal{B}| = 16$, and $|\mathcal{A}||\mathcal{B}| = 16 \leq 2^6$.

If \mathcal{A} contains one singleton, then $|\mathcal{A}| = 2$ and $|\mathcal{B}| = 15$, and $|\mathcal{A}||\mathcal{B}| = 30 \leq 2^6$.

If \mathcal{A} contains two singletons, then $|\mathcal{A}| \leq \sum_{i=0}^2 \binom{2}{i} = 4$ and $|\mathcal{B}| \leq \sum_{i=0}^2 \binom{4}{i} = 11$, which implies that $|\mathcal{A}| + |\mathcal{B}| \leq 15$, thus providing a contradiction.

If \mathcal{A} contains three singletons, then $|\mathcal{A}| \leq \sum_{i=0}^2 \binom{3}{i} = 7$ and

$|\mathcal{B}| \leq \sum_{i=0}^2 \binom{3}{i} = 7$, which implies that $|\mathcal{A}| + |\mathcal{B}| \leq 14$, thus providing a contradiction.

Case 3. If $|\mathcal{A}_2 \cup \mathcal{B}_2| \geq 10$, then by Lemma 2.5 there exists a 5-element subset of our 6-element set such that the collection of pairs of $\mathcal{A}_2 \cup \mathcal{B}_2$ taken from the 5-element subset is edge-wise triangle complete. Hence all pairs from this 5-element subset must be in \mathcal{A} or all be in \mathcal{B} , but not both. WLOG assume they are all in \mathcal{A} and our 5-element subset is $\{1,2,3,4,5\}$. We know from the note after the proof of Lemma 2.5 that each element of $\{1, \dots, 5\}$ is used in some pair $\{a, b\}$. So any $B \in \mathcal{B}$ must either be the empty set, a singleton, or a pair that has 6 as one of its elements. So for any $B \in \mathcal{B}$, $\exists A \in \mathcal{A}$ such that either $B \setminus A = \emptyset$ or $B \setminus A = \{6\}$. Therefore $|\mathcal{B}| \leq 2$. If $|\mathcal{B}| = 2$, then at least one of its elements must be non-empty, and we know by Lemma 1.2 that $|\mathcal{A}||\mathcal{B}| \leq 2^6$.

■

Theorem 4.2.7 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over a 6-element set, then Simonyi's conjecture holds.*

Proof. By Lemma 1.4 we know this is true if either \mathcal{A} or \mathcal{B} is composed of sets of size at most 1. By Lemma 2.6, we know it is true whenever the largest elements of both \mathcal{A} and \mathcal{B} have size 2. Then by Theorem 1.9, we know it is true when either \mathcal{A} or \mathcal{B} contains a set of size at least 3, thus completing the proof.

■

4.3 Proof for $n = 7$

Lemma 4.3.1 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$. If there exists two elements $A_1, A_2 \in \mathcal{A}$ such that $|A_1| = |A_2| = k$ and $|A_1 \cap A_2| = k - 1$, then $|\mathcal{B}| \leq 2^{n-(k+1)}$.*

Proof. Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair of $[n]$ and assume WLOG that $A_1 = \{1, 2, \dots, k - 1, k\}$, $A_2 = \{1, 2, \dots, k - 1, k + 1\} \in \mathcal{A}$. Then for any $B \in \mathcal{B}$, we know that $\{k, k + 1\}$ is not a subset of B because then we would have $A_1 \setminus B = A_2 \setminus B$. So then we know that for any $B \in \mathcal{B}$, by subtracting either A_1 or A_2 from B , we can obtain a subset of $\{k + 2, \dots, n\}$. Because no two elements of B can produce the same subset, we know that $|\mathcal{B}| \leq 2^{n-(k+1)}$. ■

Theorem 4.3.2 *Suppose that $(\mathcal{A}, \mathcal{B})$ is a recovering pair over $[n]$ such that only k elements of $[n]$ appear in both some element of \mathcal{A} and some element of \mathcal{B} , where $k < n$. If Simonyi's Conjecture holds for any recovering pair over $[k]$, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.*

Proof. Let $k < n$ and assume that Simonyi's Conjecture holds for any recovering pair over $[k]$. Now consider a recovering pair $(\mathcal{A}, \mathcal{B})$ over $[k + 1]$ and assume WLOG that $k + 1$ appears in at least one element of \mathcal{A} , but does not occur in any element of \mathcal{B} . Let \mathcal{A}_{k+1} be the set of elements of \mathcal{A} that contain $k + 1$, and let $\mathcal{A}_{\setminus k+1}$ be the set of elements of \mathcal{A} that do not contain $k + 1$. Because $(\mathcal{A}, \mathcal{B})$ is a recovering pair, we know that $(\mathcal{A}_{\setminus k+1}, \mathcal{B})$ is a recovering pair over $[k]$ and hence $|\mathcal{A}_{\setminus k+1}||\mathcal{B}| \leq 2^k$.

Now consider the recovering pair $(\mathcal{A}_{k+1}, \mathcal{B})$. Because $k + 1$ occurs in every

element of \mathcal{A}_{k+1} and does not occur in any element of \mathcal{B} , it is clear that $A \setminus B = A' \setminus B'$ if and only if $(A \setminus \{k+1\}) \setminus B = (A' \setminus \{k+1\}) \setminus B'$, and similarly, $B \setminus A = B' \setminus A'$ if and only if $B \setminus (A \setminus \{k+1\}) = B' \setminus (A' \setminus \{k+1\})$. Therefore, by removing $k+1$ from every element of \mathcal{A}_{k+1} , we would obtain a recovering pair over $[k]$, and hence $|\mathcal{A}_{k+1}||\mathcal{B}| \leq 2^k$.

$$\text{So } |\mathcal{A}||\mathcal{B}| \leq (|\mathcal{A}_{k+1}| + |\mathcal{A}_{k+1}|)|\mathcal{B}| \leq 2 * 2^k = 2^{k+1}.$$

By repeating this process, we can obtain that $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

■

Theorem 3.2 is a very useful result. If we know that if Simonyi's Conjecture holds for some value of n , then we know that when considering recovering pairs over $[n+1]$, we only need to prove the conjecture holds for total recovering pairs $(\mathcal{A}, \mathcal{B})$ over $[n+1]$. We recall that a total recovering pair over $[n]$ is a recovering pair such that for every $x \in [n]$, there exists an $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $x \in A$ and $x \in B$. This not only reduces the number of cases that remain to be proven, but also allows us to make stronger assumptions and claims in our proofs.

Additionally, this theorem gives us insight to the nature of a counterexample in the case that the conjecture is in fact false. To demonstrate this, assume that the conjecture is false, and consider the smallest n for which a counterexample exists. Then this counterexample must be a total recovering pair over $[n]$.

For the remainder of this section, as well as the following section, we will assume that $(\mathcal{A}, \mathcal{B})$ is a total recovering pair.

Lemma 4.3.3 *Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over $[n]$. If $|A \setminus \mathcal{B}| = 1$ for*

some $A \in \mathcal{A}$, then $A \setminus \mathcal{B} = \{\emptyset\}$.

Proof. Assume that $|A \setminus \mathcal{B}| = 1$ for some $A \in \mathcal{A}$. Because $(\mathcal{A}, \mathcal{B})$ is total, we know that for every $a \in A$, there exists $A^a \in \mathcal{A} \setminus \mathcal{B}$ such that $a \notin A^a$. But we know that $A \setminus \mathcal{B}$ only contains one set, so this means that this set cannot contain any elements, and hence $A \setminus \mathcal{B} = \{\emptyset\}$. ■

Corollary 4.3.4 *If $(\mathcal{A}, \mathcal{B})$ forms a total recovering pair, then there can only be at most one set $A \in \mathcal{A}$ such that $|A \setminus \mathcal{B}| = 1$. Furthermore, there can only be at most one set $A \in \mathcal{A}$ such that $A \setminus \mathcal{B}$ contains only one set that is not A .*

By the definition of a recovering pair, we know that for all $A_1, A_2 \in \mathcal{A}$,
 $(A_1 \setminus \mathcal{B}) \cap (A_2 \setminus \mathcal{B}) = \emptyset$.

Lemma 4.3.5 *Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair, then each of \mathcal{A} and \mathcal{B} can have at most one element that has size ≤ 1 .*

Proof. Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair and WLOG assume for a contradiction that there exists $A_1, A_2 \in \mathcal{A}$ such that $|A_1| \leq 1$ and $|A_2| \leq 1$. Then there exists $B_1, B_2 \in \mathcal{B}$ such that $A_1 \setminus B_1 = A_2 \setminus B_2 = \emptyset$, thus arriving at our contradiction. ■

Lemma 4.3.6 *Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over $[n]$. Then $|\{A \in \mathcal{A} : |A| \leq 2\}| \leq \lfloor n/2 \rfloor + 1$ and similarly $|\{B \in \mathcal{B} : |B| \leq 2\}| \leq \lfloor n/2 \rfloor + 1$.*

Proof. WLOG consider the elements $A \in \mathcal{A}$ such that $|A| \leq 2$. Then for each $A \in \mathcal{A}$, the set $(A \setminus \mathcal{B}) \setminus \{A\}$ consists of sets that are either singletons or the empty set. Hence the number of elements contained in the union of all such sets is at most $n + 1$. We know that each set $(A \setminus \mathcal{B}) \setminus \{A\}$ is disjoint from any other, and we also know by Corollary 3.4 that at most one of these sets contains only one element. So then we can have at most $1 + \lfloor n/2 \rfloor$ elements $A \in \mathcal{A}$ such that $|A| \leq 2$. ■

This result is in fact best possible, as shown by the following construction:

If n is even, then let $\mathcal{A} = \{\emptyset, \{1, 2\}, \dots, \{n - 1, n\}\}$ and let $\mathcal{B} = \{\{1, 3, \dots, n - 1\}, \{2, 4, \dots, n\}\}$. If n is odd, then let $\mathcal{A} = \{\{1\}, \{2, 3\}, \dots, \{n - 1, n\}\}$ and let $\mathcal{B} = \{1, 3, \dots, n\}, \{2, 4, \dots, n - 1\}\}$. In either case \mathcal{A}, \mathcal{B} forms a total recovering pair and $|\mathcal{A}| = \lfloor n/2 \rfloor + 1$.

Corollary 4.3.7 *Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over $[n]$ where $n \leq 7$. If either \mathcal{A} or \mathcal{B} only contains elements of size at most 2, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.*

Proof. Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over $\{1, \dots, 7\}$ and assume WLOG that $|A| \leq 2$ for all $A \in \mathcal{A}$. Then we know by Lemma 3.6 that $|\mathcal{A}| \leq 1 + \lfloor n/2 \rfloor = 4 = 2^2$ and we know from an earlier lemma that $|\mathcal{B}| \leq 2^{n-2}$, therefore $|\mathcal{A}||\mathcal{B}| \leq 2^n$. ■

It can be noted that by assuming that $(\mathcal{A}, \mathcal{B})$ is a total recovering pair and that Simonyi's Conjecture holds for $n = 5$, a similar statement to Corollary 3.7 could be used to prove Simonyi's Conjecture for $n = 6$ in the case that either \mathcal{A} or \mathcal{B} only

contains elements of size at most 2.

Lemma 4.3.8 *Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over $[n]$, if $|A| \leq 3$ for all $A \in \mathcal{A}$, then $|\mathcal{A}| \leq (n^2 + 3n + 6)/6$.*

Proof. Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over $[n]$ such that $|A| \leq 3$ for all $A \in \mathcal{A}$. For each $A \in \mathcal{A}$, consider the set $(A \setminus \mathcal{B}) \setminus \{A\}$. These sets contain elements of size ≤ 2 , so the union of these sets can have at most $1+n+(n^2-n)/2 = (n^2+n+2)/2$ elements. Now we know by Corollary 3.4 that at most one set $(A \setminus \mathcal{B}) \setminus \{A\}$ can have size one, and furthermore we can see that at most n can have size 2 because this would imply one element of the set is a singleton. The remainder must consist of at least 3 sets each. By having the maximum number of sets of size 1 or 2, we can then obtain the maximum number of sets $(A \setminus \mathcal{B}) \setminus \{A\}$ we can have.

This gives us a maximum of $1 + n + (((n^2 + n + 2)/2) - 1 - 2n)/3 = 1+n+(n^2-3n)/6 = (n^2+3n+6)/6$ sets $(A \setminus \mathcal{B}) \setminus \{A\}$ and hence $|\mathcal{A}| \leq (n^2+3n+6)/6$.

■

Lemma 4.3.9 *Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over $\{1, \dots, 7\}$ such that $\max(|A| : A \in \mathcal{A}) = \max(|B| : B \in \mathcal{B}) = 3$, then Simonyi's Conjecture holds.*

Proof. Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over $\{1, \dots, 7\}$ such that $\max(|A| : A \in \mathcal{A}) = \max(|B| : B \in \mathcal{B}) = 3$. Then we know from Lemma 3.8 that both \mathcal{A} and \mathcal{B} have size $\leq (7^2 + 3 * 7 + 6)/6 = 76/6 < 13$, so both \mathcal{A} and \mathcal{B} have size ≤ 12 . However we know that if $|\mathcal{A}| \leq 11$ and $|\mathcal{B}| \leq 11$, then $|\mathcal{A}||\mathcal{B}| \leq 11^2 < 2^7$. So now

assume WLOG that $|\mathcal{A}| = 12$. By Lemma 3.6 we know that \mathcal{A} can have at most 4 elements of size ≤ 2 , so we know that \mathcal{A} has ≥ 8 elements of size 3. But because each set of size 3 contains 3 pairs and $\{1, \dots, 7\}$ contains 21 pairs, we know that two of our 3-element sets must have a pair in common. Therefore, by Lemma 3.1 we know that $|\mathcal{B}| \leq 2^3$, but we already know that $|\mathcal{A}| = 12 \leq 2^4$, so $|\mathcal{A}||\mathcal{B}| \leq 2^7$. ■

Theorem 4.3.10 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over a 7-element set. Then Simonyi's Conjecture holds.*

Proof. If $(\mathcal{A}, \mathcal{B})$ is not total, then we know $|\mathcal{A}||\mathcal{B}| \leq 2^n$ by Theorem 2.7 and the discussion of Theorem 3.2. If $(\mathcal{A}, \mathcal{B})$ is total, then we have the following: If either \mathcal{A} or \mathcal{B} have an element of size ≥ 4 , then by Theorem 1.9 we are done. If both \mathcal{A} and \mathcal{B} are composed of sets of size ≤ 3 , then we are done by Lemma 3.9. If either \mathcal{A} or \mathcal{B} consist of sets of size at most 2, then we are done by Corollary 3.7. Lastly, if \mathcal{A} or \mathcal{B} consist of sets of size at most one, then by Lemma 1.4 we are done. ■

4.4 Proof for $n = 8$

Lemma 4.4.1 *Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over $[n]$ where $n \geq 8$ such that neither \mathcal{A} nor \mathcal{B} contains any elements of size > 3 , then Simonyi's Conjecture holds.*

Proof. By Lemma 3.8 we know that $|\mathcal{A}|$ and $|\mathcal{B}|$ have size $\leq (n^2 + 3n + 6)/6$. It then follows from calculus that $((n^2 + 3n + 6)/6)^2 < 2^n$ whenever $n \geq 8$, thus completing the proof. ■

Lemma 4.4.2 *Let $(\mathcal{A}, \mathcal{B})$ be a total recovering pair over an 8-element set such that the largest element of either \mathcal{A} or \mathcal{B} has size 4, then Simonyi's Conjecture holds.*

Proof. WLOG assume that $\max(|\mathcal{A}| : A \in \mathcal{A}) = 4$.

Case 1. ($\max\{|B| : B \in \mathcal{B}\} = 1$) We know the lemma holds in this case by Lemma 1.4.

Case 2. ($\max\{|B| : B \in \mathcal{B}\} = 2$) From Lemma 3.6 we know that $|\mathcal{B}| \leq 5$ and because $|\mathcal{A}| \leq 2^{n-2}$ we can see that the theorem is true whenever $|\mathcal{B}| \leq 4$, so we are left to prove the case where $|\mathcal{B}| = 5$, so assume that this holds. Of these 5 sets in \mathcal{B} , only one may have the property that $\emptyset \in B \setminus \mathcal{A}$. Now consider the other 4 sets in \mathcal{B} , for each of which $\emptyset \notin A \setminus \mathcal{B}$. Because the recovering pair $(\mathcal{A}, \mathcal{B})$ is total, none of these sets can be singletons, because otherwise there would exist a set $A \in \mathcal{A}$ such that $B \setminus A = \emptyset$. So these sets must all be two-element sets, and furthermore, they must be disjoint. To demonstrate this, assume for a contradiction that there exists two sets $\{a, b\}, \{b, c\} \in \mathcal{B}$. Then there exists some $A \in \mathcal{A}$ such that $\{a, b\} \setminus A$ does not contain a , but then $\{a, b\} \setminus A = \{b\}$ because $\{a, b\} \setminus A$ cannot be empty. Now consider our set $\{b, c\} \in \mathcal{B}$ and note that there exists some $A \in \mathcal{A}$ such that $c \in A$. But then $\{b, c\} \setminus A$ is either $\{b\}$ or \emptyset , resulting in a contradiction. Now consider one of

these pairs $B \in \mathcal{B}$, and consider our sets A^* . Then $|\mathcal{A}| \leq 2^6 * (3/4)^3 = 3^3 = 27$, where the value $3/4$ comes from the fact that no element of \mathcal{A} can contain a two-element set from \mathcal{B} as a subset of itself. Therefore $|\mathcal{A}||\mathcal{B}| \leq 27 * 5 < 2^8$.

Case 3. ($\max\{|B| : B \in \mathcal{B}\} = 3$) Suppose that there exists $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $|A| = 4$, $|B| = 3$, and $A \cap B = \emptyset$. WLOG let $A = \{5, 6, 7, 8\}$ and let $B = \{1, 2, 3\}$. If there is any $A' \in \mathcal{A}$ such that $|A \cap A'| = 3$ or any $B' \in \mathcal{B}$ such that $|B \cap B'| = 2$, then by Lemma 3.1 we are done. Now assume that there is no such A' or B' , and consider the sets A^* and B^* with respect to B and A , respectively. Then $|A^* \cap \{5, 6, 7, 8\}| \leq 2$ and $\{4\}$ may or may not be a subset of A^* , resulting in 22 options for A^* other than A , therefore $|\mathcal{A}| \leq 23$. Similarly, $|B^* \cap \{1, 2, 3\}| \leq 1$ and $\{4\}$ may or may not be a subset of B^* , resulting in 8 options for B^* other than B , therefore $|\mathcal{B}| \leq 9$.

Now assume that for any $A \in \mathcal{A}$ such that $|A| = 4$ and $B \in \mathcal{B}$ such that $|B| = 3$, $A \cap B$ is non empty. Consider some $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $|A| = 4$ and $|B| = 3$ and consider the sets A^* and B^* with respect to B and A respectively. Then $|A^*| \leq 3$ for all A^* , and $|B^*| \leq 2$ for all B^* , therefore $|\mathcal{A}| \leq 26$ and $|\mathcal{B}| \leq 11$. Now we can note that if $|\mathcal{A}| \leq 23$ or if $|\mathcal{B}| \leq 9$, then $|\mathcal{A}||\mathcal{B}| \leq 2^8$. Now assume that $|\mathcal{B}| > 9$. Because each set $B \setminus \mathcal{A}$ must be disjoint from any other, we know that there exists a set $B' \in \mathcal{B}$ such that $B' \setminus \mathcal{A}$ does not contain the empty set, nor does it contain any singletons. Because $(\mathcal{A}, \mathcal{B})$ is total, we know that B' must contain three elements, and the size of its intersection with any $A \in \mathcal{A}$ is one. Now assume that \mathcal{A} contains at least 22 elements. Then by Lemma 3.8, we know that \mathcal{A} contains at least 7 elements of size 4, and hence there exists $A, A', A'' \in \mathcal{A}$ such that

$A \cap B' = A' \cap B' = A'' \cap B'$ and we have one of the following: $|A \cap A'| = 3$, $|A \cap A''| = 3$, or $|A' \cap A''| = 3$, and by Lemma 3.1 we have that $|\mathcal{A}||\mathcal{B}| \leq 2^n$. Otherwise we have that either $|\mathcal{A}| \leq 22$ or $|\mathcal{B}| \leq 9$, and hence $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Case 4. ($\max\{|B| : B \in \mathcal{B}\} = 4$) The proof of this case follows directly from Corollary 1.3. ■

Theorem 4.4.3 *Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over an 8-element set, then Simonyi's Conjecture holds.*

Proof. Assume $(\mathcal{A}, \mathcal{B})$ is a recovering pair over an 8-element set. If $(\mathcal{A}, \mathcal{B})$ is not total, then by Theorem 3.10 and the discussion of Theorem 3.2, $|\mathcal{A}||\mathcal{B}| \leq 2^n$. Otherwise, let k denote the size of the largest element of \mathcal{A} or \mathcal{B} . Lemma 4.1 proves the case when $k \leq 3$, Lemma 4.2 proves the case when $k = 4$, and Theorem 1.9 proves the cases when $k \geq 5$. ■

4.5 An Additional Theorem

Theorem 4.5.1 *The set of values of n for which Simonyi's Conjecture holds is either a set of the form $\{0, 1, \dots, k\}$ or consists of all natural numbers.*

Proof. Assume that there exists a value n_0 such that we can form a recovering pair $(\mathcal{A}, \mathcal{B})$ over $[n_0]$ such that $|\mathcal{A}||\mathcal{B}| > 2^{n_0}$. Furthermore, assume that n_0 is the smallest

possible value for which such a counterexample to Simonyi's Conjecture exists. Then we know that Simonyi's Conjecture holds over all numbers in the set $\{0, 1, \dots, n_0 - 1\}$. However, because we have a counterexample $(\mathcal{A}, \mathcal{B})$ over $[n_0]$, we can also find a counterexample over $[n]$ for any $n \geq n_0$ by repeating the following process $n - n_0$ times:

Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n_0]$ such that $|\mathcal{A}||\mathcal{B}| > 2^{n_0}$ and consider the recovering pair $(\mathcal{A}', \mathcal{B}')$ over $[n_0 + 1]$ where $\mathcal{A}' = \mathcal{A} \cup \{A \cup \{n_0 + 1\} : A \in \mathcal{A}\}$ and $\mathcal{B}' = \mathcal{B}$. Then $|\mathcal{A}'||\mathcal{B}'| = 2 * |\mathcal{A}||\mathcal{B}| > 2^{n_0+1}$. Therefore Simonyi's Conjecture only holds over numbers in the set $\{0, 1, \dots, n_0 - 1\}$.

If no such n_0 exists, then Simonyi's Conjecture holds over all natural numbers.

■

Another way of wording this theorem would be to say that if Simonyi's Conjecture holds for all recovering pairs over $[n]$, then it also holds for all recovering pairs over $[k]$ where $k < n$.

If in fact Simonyi's Conjecture is true in general, this theorem shows that we do not need to directly prove that it holds for every value of n , but rather it would be sufficient to prove it holds for arbitrarily large values of n . For example, it would be sufficient to show that the conjecture holds for all even values of n , or all values of n of the form 2^k . This creates more possibilities for an inductive proof.

4.6 A New Problem in Graph Theory

To begin this section, we first recall Lemma 2.5 as follows: Any graph on 6 vertices with at least 10 edges will have an induced subgraph with 5 vertices that is edge-wise triangle complete. This lemma was used in proving that Simonyi's Conjecture holds when $n = 6$. We will now generalize this and present the following problem:

Let G be a graph over n vertices. Then we define the function $\mathcal{F}(n, k, a, b)$ to be the minimum number e_0 such that if G contains $e \geq e_0$ edges, then G must have an induced subgraph G^* with k vertices such that every a -clique in G^* is contained in a b -clique. Given a set of values for n , k , a , and b such that $a \leq b \leq k \leq n$, what is the value of \mathcal{F} ?

Although is it easy to write down the function \mathcal{F} , it appears to be quite difficult to solve it when given specific input values. We will now discuss some results on the function \mathcal{F} .

To begin, let us consider Lemma 2.5, which proves that $\mathcal{F}(6, 5, 2, 3) \leq 10$ by showing that any graph G on 6 vertices with 10 edges will have an induced subgraph G^* with 5 vertices such that every edge (2-clique) is contained in a triangle (3-clique). To show that $\mathcal{F}(6, 5, 2, 3) = 10$, we must show that there exists a graph G over 6 vertices with 9 edges such that every induced subgraph G^* over 5 vertices contains an edge that is not part of a triangle. It is easy to verify that the complete bipartite graph $K_{3,3}$ satisfies this property.

We now consider the case when $n = k$.

Theorem 4.6.1 $\mathcal{F}(n, n, a, b) = \binom{n}{2} - n + b$.

Proof. Let G be a graph over n vertices, label its vertex set N , and let G contain an a -clique, where the set of vertices of this a -clique is labeled A . Because we know we have our a -clique, any other edge in G must either be between a member of A and a member of $N \setminus A$, or between two members of $N \setminus A$. We will construct graphs by considering the complete graph K_n over N and removing edges from this graph.

First consider edges of the form $\{a, x\}$ where $a \in A$ and $x \in N \setminus A$. For any such edge that is removed from our complete graph, we know that x cannot be a part of the b -clique we are aiming to obtain. If we have two edges $\{a_1, x\}$ and $\{a_2, x\}$ removed from K_n , then x is the only vertex that we now cannot have as part of the b -clique. By removing t edges of the form $\{a, x\}$, we can remove at most t vertices from being capable of being part of our b -clique.

Now consider edges of the form $\{x, y\}$ where $x, y \in N \setminus A$. If the edge $\{x, y\}$ is removed from K_n , then either x or y cannot be contained in our b -clique. By removing t edges of the form $\{x, y\}$, we can remove at most t vertices from being capable of being part of our b -clique.

Based on the above argument, let $t = n - b$. Then there will be at most t vertices removed from potentially forming a b -clique that contains A . But $n - t = b$, so such a b -clique must exist. Therefore, $\mathcal{F}(n, n, a, b) \leq \binom{n}{2} - n + b$.

Now consider the graph G that is formed in the following way. Begin with the complete graph K_n and consider any a -clique in K_n . Now pick a vertex a_1

from this a -clique, and remove the edges $\{a_1, x_1\}, \dots, \{a_1, x_{n-b+1}\}$ from K_n where every $x_i \in N \setminus A$. Then our a -clique is not contained in any b -clique. Therefore $\mathcal{F}(n, n, a, b) > \binom{n}{2} - n + b - 1$.

$$\text{Hence } \mathcal{F}(n, n, a, b) = \binom{n}{2} - n + b.$$

■

If we consider the case where $k = b$, then the value of \mathcal{F} is simply the number of edges our graph over n vertices must have to ensure it contains a k -clique. This problem is equivalent to Turán's Theorem, which is the following:

If a graph $G = (V, E)$ on n vertices has no $(k + 1)$ -clique, $k \geq 2$, then $|E| \leq \lfloor (k - 1)/k \rfloor (n^2/2)$. [9]

Furthermore, this theorem is best possible, and we can construct graphs over n vertices with $|E| = \lfloor \lfloor (k - 1)/k \rfloor (n^2/2) \rfloor$. For more information, the original work of Turán can be found in [14], however [2] is more accessible and provides a detailed overview.

$$\text{Hence } \mathcal{F}(n, k, a, k) = \lfloor \lfloor (k - 2)/[k - 1] \rfloor (n^2/2) \rfloor + 1.$$

Lastly, if $a = b$, then $\mathcal{F}(n, k, a, a) = 0$ because every a -clique is automatically contained in itself.

Chapter 5

Discussion

5.1 Summary

Throughout the time I spent researching Simonyi's Conjecture, my goal was to find a result or collection of results that could be generalized to larger values of n via induction or another form of proof. This proved to be very difficult, and I was not able to attain such a result. However, in my attempts I managed to find several special cases under which the conjecture held.

The main contribution that this thesis has provided is a proof that Simonyi's Conjecture holds for all values of $n \leq 8$. In addition there are several results that show the conjecture holds for any n given certain assumptions about the size of one or more elements contained in \mathcal{A} and/or \mathcal{B} . It was shown that if either \mathcal{A} or \mathcal{B} contain an element of size at least $n - 3$, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$, as well as when either \mathcal{A} or \mathcal{B} is restricted to containing sets of size at most one. If the recovering pair $(\mathcal{A}, \mathcal{B})$ is total and the largest element of either \mathcal{A} or \mathcal{B} has size 3, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

If we let $a = \max\{|A| : A \in \mathcal{A}\}$ and $b = \max\{|B| : B \in \mathcal{B}\}$, then we have that $|\mathcal{B}| \leq 2^{n-a}$ and similarly $|\mathcal{A}| \leq 2^{n-b}$. This then gives that if $a + b \geq n$, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$. Also, if $|A| = |A'| = a$ and $|A \cap A'| = a - 1$, then $|\mathcal{B}| \leq 2^{n-a-1}$.

It was found that there can be at most one element $A \in \mathcal{A}$ such that $A \subseteq B$ for some $B \in \mathcal{B}$ and similarly there can be at most one $B \in \mathcal{B}$ such that $B \subseteq A$ for some $A \in \mathcal{A}$.

If $(\mathcal{A}, \mathcal{B})$ is a recovering pair over $[n]$ such that only k elements of $[n]$ appear in both some element of \mathcal{A} and some element of \mathcal{B} , and Simonyi's Conjecture is known to hold over any k -element set, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$. Under the assumption that a recovering pair $(\mathcal{A}, \mathcal{B})$ over an n -element set is total, we now know that each of \mathcal{A} and \mathcal{B} each contain at most 1 set of size ≤ 1 , at most $\lfloor n/2 \rfloor + 1$ sets of size ≤ 2 , and at most $(n_2 + 3n + 6)/6$ size of size ≤ 3 .

In addition to these special cases, it was found that either Simonyi's Conjecture holds for all values of n , or it holds over all numbers in a set of the form $\{0, 1, \dots, k\}$ for some integer k .

Lastly, this thesis presented a new problem in graph theory that was inspired by the concept of an edge-wise triangle complete graph, which was used to verify Simonyi's Conjecture for $n = 6$. This problem involved the function $\mathcal{F}(n, k, a, b)$, and results were presented and/or discussed based on the cases where 2 of n , k , a , and b were equal.

5.2 Ideas for Further Research

Given the nature of recovering pairs and the intractable growth that the number of possible pairs of set systems over $[n]$ undergoes as n gets large, it would be completely infeasible to check all pairs of set systems, whether via human or computer

methods. For example, consider the fact that even for the case of $n = 7$ there are 2^{256} possible pairs of set systems that would then have to be sorted through to see which ones are recovering, then to calculate the product $|\mathcal{A}||\mathcal{B}|$.

In order to make further progress in attempting to verify the conjecture, there are a few methods that can be used, each of which will come with its advantages and disadvantages.

One method is to attempt to improve the currently known upper bound of $\approx 2.3264^n$ by means of various functions such as how the entropy function was used in [8]. One advantage to this type of approach is that they could potentially bring the bounds very close to 2^n with only a few results, or even bring it completely to 2^n , depending on the functions and equations used. A major challenge to using this approach is the fact that it is not clear which functions could be used, and therefore there would be very little direction in how to find new bounds.

Another approach to working on this problem is to prove that the conjecture holds under various restrictions on the members of the set systems \mathcal{A} and \mathcal{B} , as was focused on in this thesis. Attempting to solve the problem this way alone may be time consuming as many cases may be needed, however it would be very effective if a complete set of cases was found. Another advantage to this approach is that by splitting the problem into smaller cases can often result in the proofs being easier to find.

Alternately, research could be conducted that assumes a counterexample exists and finding results that would stem from this assumption. Using this method, it may be possible to construct a counterexample to the conjecture, or perhaps lead

to a contradiction which would prove that the conjecture is true.

Appendix A

Appendix

A.1 Outline of Results

The following is a listing of all lemmata, theorems, and corollaries presented in the results section of this thesis. After a result is listed, the results it calls upon in its proof are listed in parentheses.

Conjecture 1.1 (Repeat of Simonyi's Conjecture)

Lemma 1.2

Corollary 1.3 (Lemma 1.2)

Lemma 1.4

Lemma 1.5

Lemma 1.6 (Lemma 1.2, Corollary 1.3)

Lemma 1.7 (Lemma 1.2, 1.4)

Lemma 1.8 (Lemma 1.2, 1.4)

Theorem 1.9 (Lemma 1.5, 1.6, 1.7, 1.8)

Theorem 1.10 (Lemma 1.4, Theorem 1.9)

Lemma 2.1

Corollary 2.2 (Lemma 2.1)

Corollary 2.3 (Lemma 2.1)

Lemma 2.4

Lemma 2.5 (Lemma 2.4)

Lemma 2.6 (Corollary 2.2, 2.3, Lemma 1.2, 2.5)

Theorem 2.7 (Lemma 1.4, 2.6, Theorem 1.9)

Lemma 3.1

Theorem 3.2

Lemma 3.3

Corollary 3.4 (Lemma 3.3)

Lemma 3.5

Lemma 3.6 (Corollary 3.4)

Corollary 3.7 (Lemma 3.6)

Lemma 3.8 (Corollary 3.4)

Lemma 3.9 (Lemma 3.1, 3.6, 3.8)

Theorem 3.10 (Lemma 1.4, 3.9, Theorem 1.9, 2.7, 3.2, Corollary 3.7)

Lemma 4.1 (Lemma 3.8)

Lemma 4.2 (Lemma 1.4, 3.1, 3.6, Corollary 1.3)

Theorem 4.3 (Theorem 1.9, Lemma 4.1, 4.2)

Theorem 5.1

Theorem 6.1

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