Development of a Self-Consistent Gas Accretion Model for Simulating
Gas Giant Formation in Protoplanetary Disks

by

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ABSTRACT

DEVELOPMENT OF A SELF-CONSISTENT GAS ACCRETION MODEL FOR SIMULATING GAS GIANT FORMATION IN PROTOPLANETARY DISKS

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The number of extrasolar planet discoveries has increased dramatically over the last 15 years. Nearly 700 exoplanets have currently been observed through a variety of observation techniques. Most of the currently documented exoplanets differ greatly from the planets in our own Solar System, with various combinations of eccentric orbits, short orbital periods, and masses many times that of Jupiter. More recently, planets belonging to a new class of ‘distant gas giants’ have also been discovered with orbits of 30 to 100 times that of Jupiter. The wide variety of different planet formation outcomes stem from a complex interplay between gravitational interactions, hydrodynamic interactions and competitive accretion among the planets that is not yet fully understood.

Simulations performed using a series of modifications to an existing, widely used hydrodynamic code (FARGO) are presented. The main goal is to develop a more rigorous and robust gas accretion scheme that is valid and consistent for the ranges of exoplanetary gas giant masses, eccentricities and semimajor axes that have been observed to better understand the mechanisms involved in their formation. The resulting scheme is a more robust and accurate prescription for gas accretion onto planetary cores in a manner that is mostly resolution independent and valid over a
large range of masses (less than an Earth mass to multiple Jupiter masses). The modified scheme accounts for multiple, competing, dynamic accretion mechanisms (including atmospheric effects) and their associated time scales between an arbitrary number of protoplanets. This updated accretion scheme provides a means for exploring the entire formation process of gas giants out of a variety of initial conditions in a self-consistent manner. The modifications made to the code as well as simulation results will be discussed and explored.
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Chapter 1

Introduction and Background

There have been close to 700 extrasolar planets detected thus far\(^1\) by various advanced observational techniques. The majority of these extrasolar planets (exoplanets) are of sufficient mass to indicate they are gas giants. Some are in multiple planet systems and are often in resonant orbits (Marcy et al. 2005).

Protoplanetary disks have been observed around a very high fraction of stars in young star clusters (Haisch et al. 2001; Megeath et al. 2005). The fraction of Sun-like stars possessing at least one giant planet has been estimated at \((\sim 20\%)\) (Cumming et al. 2008) which suggests that the formation of giant gas planets must be a fairly robust process.

The primary goal and motivation of this thesis is to modify the FARGO code developed by Masset (2000) to more accurately and self-consistently model accretion across a wide range of planetary masses to better understand the formation of extrasolar gas giants. The first chapter will provide some background and topical introduction to accretion disk modelling as well as planetary formation mechanisms and underlying theory. Chapter 2 is an overview of the FARGO algorithm and code,

\(^1\)The Extrasolar Planets Encyclopaedia: \url{http://exoplanet.eu/}
with information on how the simulation space is modelled and implemented as well as important parameters and quantities used throughout. Chapter 3 is a detailed discussion of the sequential modifications made to the FARGO accretion algorithm and the resulting effects on planetary formation. Chapter 4 explores in detail two proof-of-concept simulations demonstrating relevant planetary formation scenarios. Chapter 6 describes possible future work and considerations regarding the modified code. Finally, the Appendix contains an brief description and overview of software used or written for analysis and visualization, as well as information on obtaining a copy of the modified code from the online repository. Note that Table 2.1 and the accompanying description in §2.5 details a number of units and important quantities used throughout the thesis.

### 1.1 Accretion Disks

The Solar Nebula Model and Planetesimal Hypothesis (see §1.3) are currently the most widely accepted models for explaining the formation and evolution of planetary systems in cosmology. Massive, dense, gravitationally unstable clouds of primarily molecular hydrogen coalesce and collapse to form stars and will naturally give rise to a flattened, gaseous protoplanetary disk from the collapse of rotating molecular cloud nebula (Hoyle 1960; Cameron 1962; Cassen and Moosman 1981; Terebey et al. 1984). Star formation is a rich and complex process, but is beyond the scope of this thesis. A Sun-like star usually takes on the order of $10^4$ years to form (Montmerle et al. 2006).

Accretion disks are structures formed by gaseous, diffuse material in orbital motion around a central body - typically a star. Gravitational forces cause the material in the disk to spiral inward toward the star. The material becomes heated, compressed and
emits electromagnetic radiation. Protoplanetary disks act as accretion disks because there is a net inward flow of gaseous material through the disk which is deposited onto the surface of the central star. However, it is worth noting that this process differs from the planetary accretion process (see §1.3).

If the rotation rate of the disk decreases with increasing radius, viscosity transports angular momentum outward (Balbus and Papaloizou 1999). A net inward flow of material implies the presence of disk viscosity which extracts orbital energy and angular momentum from the disk material. An order-of-magnitude estimate for the gravitational potential energy released by the accretion of a mass $m$ onto the surface of a star of mass $M$ and radius $R_*$ is (see Frank et al. 2002, §1.1):

$$\Delta E_{acc} = \frac{GMm}{R_*}$$  \hspace{1cm} (1.1)

where $G$ is the gravitational constant.

It can be shown that in a differentially rotating disk rotating with angular velocity $\Omega(r)$, viscosity $\nu(r)$ and surface density $\Sigma(r)$, the rate of angular momentum transport outward across a given radius $r$ of the disk is (Frank et al. 2002, §6.2):

$$\dot{L} = -2\pi r^3 \nu(r) \Sigma(r) \left(\frac{d\Omega}{dr}\right)$$  \hspace{1cm} (1.2)

If the rotation profile is Keplerian, $\Omega(r) = \sqrt{GM/r^3}$, then:

$$\frac{d\Omega}{dr} = -\frac{3}{2r} \Omega$$  \hspace{1cm} (1.3)

and so,

$$\dot{L} = 3\pi r^2 \nu \Sigma \Omega$$  \hspace{1cm} (1.4)

Thus, the direction of the flow of angular momentum is outward. This is the mechanism by which gaseous material in the protoplanetary disk loses angular momentum.
Figure 1.1: A perspective view of an idealized accretion disk showing exaggerated path of material spiralling in to the central star. Figure inspired by (Frank et al. 2002, §4.5).

and accretes onto the central star. The disk necessarily expands to account for the angular momentum lost by the accreting material removed from the disk. The rate at which accretion onto the central star and angular momentum transfer through the disk occurs is proportional to the viscosity. Hence, the evolution of the disk can be predicted if the viscosity is known (Hartmann et al. 1998). Figure 1.1 shows a perspective view of an idealized accretion disk and an exaggerated path of material spiralling in to the central star.

The typical lifespan of accretion disks is on the order of 10 million years (Haisch et al. 2001). By the time the star reaches the classical T-Tauri stage, the disk becomes thinner and cools (Hartmann et al. 1998). T-Tauri stars are a class of pre-main sequence stars found near molecular clouds which are identified by their optical variability and strong chromospheric lines. T-Tauri is a variable star in the constellation Taurus and is the prototype for T-Tauri stars.

Accretion disks may form through a variety of mechanisms ranging from massive, dense, gravitationally unstable gas clouds collapsing to form stars (see §1.3) to matter transfer between companion stars. In a binary star system, one star may increase in radius or the separation between them may shrink to the point where the gravitational influence of one star can liberate outer layers of the other in a process called Roche lobe overflow (Frank et al. 2002, §4). The Roche lobe is the region of space around
Figure 1.2: A profile view of the disk with (a) no flaring and (b) flaring. Both the aspect ratio and flaring have been exaggerated for visual clarity. Figure inspired by (Frank et al. 2002, §5.6).

A star where orbiting material is gravitationally bound to the star. Ejection of mass in the form of stellar winds by one companion star can be gravitationally captured by a second star. Transferred material usually must lose its angular momentum before being able to accrete onto the star - leading to the formation of accretion disks (Frank et al. 2002, §4).

The accreting material in many systems which undergoes mass transfer will have sufficient angular momentum to form an accretion disk. Often, the accretion disk flow is so closely confined to the orbital plane that the disk may be regarded as two-dimensional to a first degree - known as the “Thin Disk Approximation” (Frank et al. 2002, §5).

An accretion disk will, in fact, have a finite thickness. The ratio of the disk half-height $H$ (as measured from the disk mid-plane) to the radial distance $r$ is known as the disk’s aspect ratio (see also §2.6.1). The aspect ratio defines the height of the protoplanetary disk as well as describes how quickly the height changes with radial distance $r$. Protoplanetary disks typically have an aspect ratio $H/r \sim 0.1$ and thus fall into the thin disk regime. The height of the accretion disk can also be flared (see §2.6.1) at larger radii, where $H$ increases faster than linearly with increasing
radius $r$ (Bell et al. 1997). Figure 1.2 illustrates the effect of flaring on the height of a protoplanetary disk.

The thin disk approximation has been proven very successful in developing an elaborate theory for accurately modelling accretion disks (Frank et al. 2002, §5) and will be used as the basis for modelling and simulating protoplanetary disks in this thesis.

1.2 Alpha Disk Model

Viscosity must exist in the circumstellar disk in order to drive accretion and angular momentum transfer, however the actual source of the viscosity is not known. Pure molecular viscosity is many orders of magnitude too low to produce the observed rates of accretion, so disk viscosity is conjectured to arise from turbulence - chaotic variations exhibited by the fluid on arbitrarily short time and length scales (Frank et al. 2002, §4.7).

The leading candidate for the source of viscous turbulence in protoplanetary disks is a mechanism known as magnetorotational instability (MRI) (Balbus and Hawley 2002). In addition to hydrodynamical forces such as pressure and gravity, an element of magnetized fluid will be influenced by the Lorentz force $\vec{J} \times \vec{B}$, where $\vec{J}$ is the current density and $\vec{B}$ is the magnetic field vector. The stellar magnetic field interacts with the conductive gas disk and couples with the Keplerian rotation of disk material, thereby creating a destabilizing force proportional to the perpendicular displacement (Balbus 2009).

The onset of turbulence and the physical mechanisms involved are still not well
understood (Frank et al. 2002, §4.7). Nonetheless, the turbulent flow can be characterized by a length scale $\lambda_{\text{turb}}$ and a turnover velocity $v_{\text{turb}}$ of the largest eddies (see e.g. Landau and Lifshitz 1959). This turbulent viscosity can then be written as:

$$\nu_{\text{turb}} \sim \lambda_{\text{turb}} \cdot v_{\text{turb}} \quad (1.5)$$

General properties of the accretion disk allow for plausible limits on $\lambda_{\text{turb}}$ and $v_{\text{turb}}$. Clearly, the largest turbulent eddies cannot exceed the height of the disk, so $\lambda_{\text{turb}} \lesssim H$. Also, the turnover velocity is unlikely to be supersonic or else shocks would form, thus $v_{\text{turb}} \lesssim c_s$ (Frank et al. 2002, §4.7). Therefore, the viscosity can now be written as:

$$\nu = \alpha c_s H \quad (1.6)$$

where $\alpha \lesssim 1$. This is the well known $\alpha$-prescription of Shakura and Sunyaev (1973) where all ignorance of the viscosity mechanism has been parametrized in $\alpha$. Alpha disk models which have been fit to observations of T-Tauri disk accretion rates suggest an $\alpha$ value of $10^{-3} \sim 10^{-2}$ (Hartmann et al. 1998).

A cornerstone of accretion disk phenomenology, the $\alpha$-prescription still remains the central link between theory and observations and has encouraged a semi-empirical approach to the viscosity problem (Balbus and Papaloizou 1999; Frank et al. 2002).

1.3 Planetary Formation

Protoplanetary disks contain a mixture of gas and condensed matter consisting of surviving interstellar grains and solar nebula condensates (Lissauer 1993). Meteorite observations reveal relative elemental abundances in our Solar system to be very similar to the Sun (see e.g. Sears and Dodd 1988; Kerridge and Matthews 1988). Observations made by Walter et al. (1988); Haisch et al. (2001) reveal that the gaseous
part of these disks lasts only on the order of 1-10 Myr. Planets form in the circumstellar disks that accompany the star formation process and thus gas giants must form within this relatively short time window.

A Minimum Mass Solar Nebula (MMSN) refers to a protoplanetary disk containing the minimum amount of solids and condensed matter necessary for planetary formation (Hayashi 1981). For our Solar System, such a MMSN would contain a mass of approximately 0.01-0.02 M⊙ and have a gas surface density which decreases with distance from the Sun as \( \sim r^{-3/2} \) (Hayashi 1981; Lissauer 1993). The process of planetary growth is generally divided into several distinct stages: agglomeration of planetesimals, which then undergo runaway growth followed by a slower oligarchic growth stage (Thommes et al. 2003). According to the Planetesimal Hypothesis, planets grow within circumstellar disks via pairwise collisions and accretion of small bodies which are known as planetesimals (see Lissauer (1993) and references therein). Once bodies on the order of 1-100 kilometers in diameter form, gravitational interactions between pairs of these planetesimals dominate and continued agglomeration occurs via pairwise mergers, leading to the formation of exceptionally large planetesimals - or protoplanets.

As a protoplanet continues to grow, the depth and influence of its gravitational well increases and it becomes more capable of gathering and retaining substantial amounts of gases from the surrounding disk (Lissauer 1993; Montmerle et al. 2006). Sufficiently massive bodies can gravitationally capture large amounts of gas via the Nucleated Instability Model (see §1.3.1), thereby producing Jovian-type planets (Mizuno 1980; Hayashi et al. 1985; Bodenheimer and Pollack 1986; Pollack et al. 1996).

Throughout the planetary formation and growth processes, interactions between planetesimals and protoplanets can significantly affect accretion behaviour as well as
orbital properties through gravitational scattering events (see §1.3.2). Planetesimals which are never incorporated into protoplanets can remain in the circumstellar disk as asteroids, comets and debris (Lissauer 1993).

A competing planetary formation model known as “gravitational instability” has been invoked as a possible giant planet formation mechanism where density instabilities in the protostellar gas disk fragment into gravitationally-bound regions that eventually contract in a manner similar to star formation (Cameron 1962, 1978; Boss 1998). However, this model appears to suffer from a number of problems (e.g. Rafikov 2005) and there is considerable debate in the literature whether density instabilities survive to become protoplanets (Vorobyov and Basu 2006; Durisen et al. 2007).

1.3.1 Nucleated Instability Model

Core Accretion, also known as the Nucleated Instability Model, is thought to be the most promising mechanism for forming gas giants wherein rapid accretion of nebular gas is triggered by a solid protoplanet with some critical mass to form a massive gaseous envelope (Ikoma et al. 2000). It is currently the most widely accepted theory for how gas giants form in protoplanetary disks. It can explain the formation of gas giants in relatively low-mass ($< 0.1 \, M_\odot$) circumstellar disks and is supported by multiple observations (Marcy et al. 2005; Sato et al. 2005) including the fraction of planet-bearing stars increasing with stellar metallicity, implying that higher fractions of solid, core-building material contributes to forming gas giants (Gonzalez 1997; Fischer and Valenti 2003).

Core formation proceeds in a similar fashion as terrestrial planet formation - with agglomeration of planetesimals - only extended to larger masses. In this model, giant planet formation is divided into two stages where a rocky core several times
the mass of Earth forms and accretes a large envelope of gas from the surrounding protoplanetary disk (Montmerle et al. 2006).

According to the classical theory of Safronov (1972), accretion rate of planetesimals whose orbits cross the orbit of a solid protoplanet is roughly proportional to its orbital frequency $\Omega$. Kepler’s laws of motion state:

$$\Omega \propto a^{-3/2}$$

where $a$ is the semimajor axis of the protoplanet. Thus, planetary growth time scales increase steeply with distance from the central star and can exceed $10^9$ years in the Uranus / Neptune region (20~30 AU) (Safronov 1972; Weidenschilling 1977). However, the time scales necessary for giant planet formation can be achieved with a higher planetesimal surface mass density and if core growth primarily occurs during the runaway growth phase Lissauer (1993, 1987); Pollack et al. (1996)

As a protoplanet continues to grow, the depth and influence of its gravitational well increases. The ratio of the protoplanet’s gravitational cross section to the geometric cross section increases and it becomes more capable of gathering and retaining substantial amounts of gases from the surrounding disk (Lissauer 1993; Montmerle et al. 2006). Reaching or exceeding a critical mass leads to runaway growth where the accretion rate of the largest planetesimal(s) in an accretion zone increases dramatically (Kokubo and Ida 2002). The mass accretion rate of a protoplanet undergoing runaway growth is proportional to $\sim R^4$ and $\sim M^{4/3}$, where $R$ and $M$ are the radius and mass of the growing body, respectively (Pollack et al. 1996; Thommes et al. 2003). However, runaway accretion soon transitions to a slower, more orderly and self-limiting stage called “oligarchic” growth, in which protoplanets form at somewhat regular intervals in semimajor axis (see §1.3.2) (Lissauer 1993). A balance between planet-planet perturbations and damping by the disk keep planetary orbits relatively
circular. Hydrodynamic collapse of the protoplanet’s gas envelope can be avoided as long as the surrounding disk can supply gas rapidly enough to compensate for contraction of the outer envelope and expansion of the protoplanet’s gravitational sphere of influence Pollack et al. (1996).

The critical protoplanetary mass which triggers runaway gas accretion was evaluated by Ikoma et al. (2000) to be 5-20 \( M_\oplus \) based on hydrostatic calculations of the gaseous envelope (Mizuno 1980; Ikoma et al. 1998) as well as quasi-static evolutionary calculations (Bodenheimer and Pollack 1986; Pollack et al. 1996). However, this critical mass value is not always consistent with other theories and observations, as Rafikov (2006) found the critical mass in the region of giant planets can be as high as 20-60 \( M_\oplus \) as long as planetesimal accretion is initially fast enough for protoplanetary cores to form prior to dissipation of nebular gas.

Unimpeded dynamical accretion of gas onto the protoplanet becomes limited or terminates once residual gas is depleted either globally or locally in the form of a significant density gap in the protoplanetary gas disk (Ida and Lin 2004a; Papaloizou et al. 2007). Most mass growth takes place in the oligarchic growth regime (Kokubo and Ida 1998) which produces much larger mass planets in the giant planet region (>5 AU) than in the terrestrial region. This is partly due to being beyond the disk “snow line” where water ice dramatically increases the surface density of solids (Hayashi 1981) as well as a larger Hill radius (see §1.5) further from the central star providing more available material for accretion (Kokubo and Ida 2002; Thommes et al. 2003).

Though core accretion is widely accepted as the most promising mechanism for giant planet formation, it is not without its shortcomings. Core accretion appears to be too slow (5-10 Myr) for observed disk lifetimes (0.1~3 Myr) (Haisch et al. 2000; Marcy et al. 2005) and cores beyond 35 AU do not seem to be able to reach
critical mass to trigger runaway accretion (Dodson-Robinson et al. 2009). A likely factor involved in reducing the time scales required for core accretion are lower disk opacities, allowing for more rapid escape of radiation which will speed up the gas envelope collapse, resulting in shortened planet growth time scales (∼1 Myr) which are within protoplanetary disk lifetimes (Marcy et al. 2005).

The Core Accretion / Nucleated Instability Model and associated accretion mechanisms will be the primary focus of this thesis.

1.3.2 Planetary Interactions and Eccentricities

A remarkable overall feature of the observed exoplanet population is the high fraction of planets with significantly eccentric orbits. The origins of which remain poorly understood since the gas disk is thought to damp eccentric orbits (Artymowicz 1993; Ward 1993; Tanaka and Ward 2004). Gravitational interactions involving planets as well as the gas disk are thought to possibly give rise to observed orbital eccentricities after major stages of gas accretion (Marcy et al. 2005). A recent study of 152 exoplanets by Marcy et al. (2005) revealed that eccentric orbits are common in the population under examination and have a median eccentricity of ⟨e⟩ = 0.25. Even at larger distances (beyond 3 AU) many of the planets examined in the study have eccentric orbits which suggests that the near-circular orbits observed in our Solar System may be actually unusual.

Planet-planet scattering through gravitational interactions / encounters has been suggested as the origin of the observed orbital eccentricities in exoplanets (Rasio and Ford 1996; Chatterjee et al. 2008; Jurić and Tremaine 2008) as well as planet-planet resonant interactions due to planet migration driven by the gas disk (Lee and Peale 2002; Lee et al. 2009) (see §1.4). However, the source of exoplanetary eccentricity is
likely a combination of both effects (Thommes et al. 2008), occurring simultaneously throughout the evolutionary lifetime of the planet.

The planet-planet scattering scenario suggests that protoplanets form in the inner disk and are ejected to wide, eccentric orbits by a neighbouring inner protoplanet. Safronov (1972) recognized that the largest body in an accretion zone is special as it will act as the most effective scatterer of nearby bodies. Simulations performed by Thommes et al. (2008) showed that formation of a gas giant resulted in the scattering of neighbouring protoplanets to high eccentricities. The now eccentric orbit of the recently scattered protoplanet becomes damped by the gas disk through dynamical friction and re-circularizes (e.g. Muto et al. 2011; Bitsch and Kley 2010; Cresswell et al. 2007). Simplified 1-dimensional disk model simulations performed by Thommes et al. (2008) show that the formation of a gas giant is accompanied by violent scattering of neighbouring, smaller protoplanetary bodies. It has been found that planet-planet scattering can create unstable systems or extremely eccentric orbits (e.g. Dodson-Robinson et al. 2009).

In order to better understand planet-planet scattering and resonant interactions as possible sources of orbital eccentricity, it is important to be able to self-consistently simulate a population of interacting planets which are simultaneously growing and interacting within a 2-dimensional protoplanetary disk model (see §4).

1.4 Planetary Migration

Planets will exchange orbital energy and angular momentum through tidal interactions with the disk in which they are embedded. Typically, the planet’s semimajor axis decreases along with damping of any orbital eccentricity it may possess (e.g.
Gravitational torques arising from excitation of spiral density waves in the gaseous protoplanetary disk can potentially induce rapid radial migration of protoplanets (Goldreich and Tremaine 1980; Ward 1986).

Planetary migration is thought to occur by two primary processes: losing energy and angular momentum to the disk (Type I, see §1.4.1) causing inward migration, or viscous drag by the gas disk as it accretes onto the central star (Type II, see §1.4.2), dragging the planets inward (e.g. Bryden et al. 2000; Ida and Lin 2004b). Observations of orbital resonances among multi-planet systems provides strong evidence of planetary migration (e.g. Laughlin and Chambers 2001; Kley et al. 2005). A new, very fast migration scheme (Type III) which can result in inward and outward migration of massive planets has been investigated by others (e.g. Papaloizou et al. 2007), but is beyond the scope of this thesis.

Type I migration rates can be between one and two orders of magnitudes faster than Type II, but orbital decay is the prevalent outcome for both migration cases (Ward 1997). Inward planetary migration is thought to explain the significant population of extrasolar “hot Jupiters” - giant planets on very short-period orbits (see Marcy et al. 2005). Simultaneously, planet-planet gravitational interactions triggered by migration also likely plays an important role in producing large eccentricities observed in many systems (Lee and Peale 2002; Thommes et al. 2008).

1.4.1 Type I Migration

Type I migration occurs when a planet’s semimajor axis decreases over time due to an imbalance of torques acting on it from the disk (Ward 1986; Ida and Lin 2004b). The planet’s mass in the Type I regime is low enough to exert only small perturbations on the disk. The planet extracts angular momentum from the interior disk and loses
angular momentum to the outer disk through induced density waves which propagate away from the protoplanet inward and outward (e.g. Ward 1997; Thommes et al. 2008). The compensating net tidal torque exerted on the planet is not, in general, zero due to asymmetries in the planet-disk interactions that are inherent to a Keplerian disk (Ward 1997). The positive inner torques tend to be systematically smaller than the negative outer torques, resulting in a total negative torque (Papaloizou et al. 2007). Consequently, planetary migration is directed inwards leading to orbital decay onto the central object (Ward 1997; Papaloizou et al. 2007). The approximate time scale on which the protoplanet drifts relative to the disk material is inversely proportional to its mass (Ward 1997).

1.4.2 Type II Migration

As the planet becomes more massive, the flow perturbation on the disk becomes non-linear creating a shock in the vicinity of the planet (Papaloizou et al. 2007). Shock dissipation and disk viscosity leads to the deposition of angular momentum causing gaseous material to be pushed away from the planet, thus opening a gap in the disk and modifying its structure (Bryden et al. 1999; Papaloizou et al. 2007). For typical protoplanetary disk parameters (see §1.2), a planet with a mass roughly equal to or exceeding Saturn (∼95 M⊕) should open a significant gap (Papaloizou et al. 2007). The width of the gap is equilibrated by the balance of viscous and pressure forces which tend to close the gap and gravitational tidal torques which tend to open the gap (Papaloizou et al. 2007).

Accretion of gas onto the planet decreases sharply as the annular density gap surrounding the planet deepens, restricting the flow of gas across the planet’s orbit (e.g. Lin and Papaloizou 1993; Bryden et al. 1999). The density discontinuity which
forms across the orbit acts as a flow barrier to gaseous disk material (Ward 1997) and
the planet becomes locked into the overall accretion flow of the disk - referred to as
“Type II migration” (e.g. Lin and Papaloizou 1993; Bryden et al. 1999; Ida and Lin
2004a). The orbital migration of the planet becomes coupled to the viscous evolution
of the protoplanetary disk (Lin and Papaloizou 1985; Ida and Lin 2004a). The net
radial inflow of gas accreting through the disk onto the central star causes the planet
to migrate inward on a time scale determined by the disk’s viscosity Ward (1997).

1.5 Hill Radius

The Hill radius (or Roche lobe) is the region within which an astronomical body
dominates the attraction of satellites. It is an approximation of the gravitational
sphere of influence a smaller body has in the presence of perturbations from the
gravitational field generated by a more massive body. The approximate Hill radius
for a small body $M$ orbiting a large body $M_*$ is (e.g. Hamilton and Burns 1992):

$$R_H \approx a(1 - e)^{3/2} \frac{M}{3M_*}$$

where $a$ is the planet’s semi-major axis and $e$ is the planet’s orbital eccentricity.

The Hill radius of a planet on an eccentric orbit will vary between minimum and
maximum values as it orbits about the central mass. These values appear at periapsis
and apoapsis - the minimum and maximum distance, respectively, from the central
mass of an elliptical orbit. The term $a(1 - e)$ in equation (1.8) refers to the planet’s
periapsis. Here the planet’s Hill radius will be at a minimum due to the proximity of
the central mass. At apoapsis, $a(1 + e)$, the planet’s Hill radius will be at a maximum.
When eccentricity is negligible the planet’s mass and semi-major axis determine its
Hill radius and equation (1.8) becomes:

\[ R_H \approx a \sqrt[3]{\frac{M}{3M_\star}} \]  

(1.9)

which is the planet’s average Hill radius over the period of an orbit about the central mass.

A planet’s orbit may be affected by surrounding gas or gravitational interactions with another planet over relatively short timescales. In order to calculate a planet’s instantaneous Hill radius as a simulation progresses, the semi-major axis \( a \) in equation (1.9) can be replaced with the radial distance \( r \) to become:

\[ R_H \approx r \sqrt[3]{\frac{M}{3M_\star}} \]  

(1.10)

Note that although FARGO allows for an arbitrary number of embedded planets (see §2.1), only the gravitational influence of the central mass is taken into account when calculating a planet’s Hill radius.

### 1.6 Bondi Accretion

Bondi (1952) originally studied spherically symmetric accretion onto a point mass. The radius at which surrounding gas will accrete onto such a point mass is defined by the Bondi radius (see e.g. Edgar 2004):

\[ R_B = \frac{2GM}{c_s^2} \]  

(1.11)

where \( G \) is the gravitational constant, \( M \) is the point mass, \( c_s \) is the local sound (thermal) speed of the gas and the factor of 2 arises from numerical simulations performed by Shima et al. (1985).

If the body of mass \( M \) moves through a uniform gas cloud with velocity \( \Delta v \), the characteristic net velocity between planet and gas is \( v_{\text{net}} = \sqrt{\Delta v^2 + c_s^2} \). If \( v_{\text{net}} \) is
less than the gravitational escape speed of the body, $v_e = \sqrt{\frac{2GM}{r}}$, the gas can be considered bound to the body. Setting $v_{net} = v_e$ and solving for $r$ gives an impact parameter for gravitationally bound gas which can be thought of as an estimate of radius within which all gas ends up on the body. Thus, equation 1.11 can be rewritten as:

$$R_B = \frac{2GM}{v_{net}^2} = \frac{2GM}{\Delta v^2 + c_s^2} \quad (1.12)$$

The Bondi radius allows for an estimate of the accretion rate of a body of mass $M$ moving through a uniform gas cloud. Gravity will focus material behind the mass which can then accrete onto the planet (Edgar 2004). A planet moving through a gas cloud of density $\rho$ at a velocity $\Delta v$ has an accretion cross-section $\pi R_B^2$. The rate at which gas accretes onto the planet can be estimated as:

$$\dot{M}_B \sim \rho (\pi R_B^2) v_{net} = \frac{4\pi G^2 M^2 \rho}{(\Delta v^2 + c_s^2)^{\frac{3}{2}}} \quad (1.13)$$

which is the Bondi accretion rate (Edgar 2004). Note that large relative velocities imply small collision cross-sections and hence long accretion times. Lissauer (1993)
Chapter 2

FARGO Code

2.1 Description of FARGO Code

FARGO is a simple and fast 2-dimensional polar hybrid hydrodynamic and N-body code dedicated to the interplay between planet-disk tidal interactions and planet-planet gravitational interactions. Development of the FARGO code by Frederic Masset began in 1999 and has since become well tested and widely used among the astronomical community\(^1\). The FARGO acronym stands for “Fast Advection in Rotating Gaseous Objects” and refers specifically to the gas advection algorithm developed by Masset (2000). Details of the FARGO code, the FARGO algorithm as well as specific modifications made to the code will be discussed.

The code uses a 2-dimensional polar mesh to represent a non self-gravitating Keplerian disk modelled using the \(\alpha\)-disk formalism discussed in §1.2. This allows the general disk properties to be described by a small number of parameters (see §2.6). The mesh has fixed azimuthal spacing and arbitrary radial spacing between cell interfaces and allows for arbitrary resolutions for both the radial and azimuthal directions.

\(^1\)List of publications: http://fargo.in2p3.fr/spip.php?article3
to be defined, allowing for finer or coarser resolution depending on the specific problem requirements. By considering the disk to be 2-dimensional and harnessing the FARGO algorithm (see §2.2) the computational complexity involved in simulating the hydrodynamic and gravitational processes are simplified - greatly reducing the computation time required to numerically solve them. The justifications for using a 2D approximation are discussed in §2.3.

The reference frame is centered on the primary mass and therefore non-inertial. It can be non-rotating, rotating with a fixed angular velocity, co-rotating with a planet or its guiding-center. The additional forces that arise due to the rotating frame are conservatively implemented in a so-called potential indirect term. The Coriolis force is treated in a way that enforces angular momentum conservation (Kley 1998).

The disk is subject to the gravity of the central mass and that of an arbitrary number of embedded protoplanets. The N-body portions of the code handle the gravitational interactions between the embedded protoplanets, which are treated as point-like masses whose trajectories are determined using a fifth order Runge-Kutta integrator.

The hydrodynamic solver implemented in FARGO resembles the widely known ZEUS code solver (Stone and Norman 1992). It handles the flow and advection of gas by solving Navier-Stokes and continuity equations for a Keplerian disk subject to the gravity of the central object and that of embedded protoplanets, implementing the full viscous stress tensor in cylindrical coordinates.

It also features a locally isothermal equation of state with the option of an arbitrary temperature or, equivalently, sound speed radial profile. The code uses a van Leer upwind algorithm on a staggered mesh (see §2.4) and a harmonic, second-order slope limiter (van Leer 1977). Simulations may be run using either the FARGO
advection algorithm or standard advection algorithm. The inclusion of the standard advection technique is primarily for comparison and benchmarking purposes. The FARGO algorithm is exclusively used for the simulations throughout this thesis.

Parallelization of the code using MPI\(^2\) allows for efficient utilization of distributed systems with multiple processor nodes to numerically simulate the hydrodynamic and gravitational processes occurring in these protoplanetary disks. A detailed list of its properties can be found on the FARGO website\(^3\).

### 2.2 Description of FARGO Algorithm

The Fast Advection in Rotating Gaseous Objects (FARGO) algorithm developed by Masset (2000) involves a simple modification of the standard transport algorithm used in explicit Eulerian fixed polar grid codes, enabling a significantly larger time-step which saves computing time and reduces numerical diffusivity.

The Courant–Friedrichs–Lewy condition (CFL condition) is a necessary (but not necessarily sufficient) condition for convergence while numerically solving certain partial differential equations using the finite difference method (Courant, Friedrichs, and Lewy 1928). Specifically, the simulation time-step must be less than the time required for a fluid element to travel across adjacent grid points to prevent numerical instability. The time-step in numerical simulations involving differentially rotating gaseous disks is usually limited by the CFL condition at the inner boundary where the azimuthal fluid motion is fastest and the polar grid cells are narrow (Masset 2000). In a Keplerian disk, the orbital frequency decreases as \(r^{-3/2}\).

The standard transport method involves a source substep, radial transport substep


and residual azimuthal velocity transport substep (see e.g. Stone and Norman 1992). The time-step limitation arises from the constraints that a test particle in cell \([i, j]\) cannot sweep an azimuthal distance greater than \(\Delta \theta_i\) or radial distance \(R_{i+1} - R_i\) in one time-step (Richtmyer and Morton 1967).

FARGO implements an independently rotating frame on each annular ring of the disk which removes the average azimuthal velocity when applying the CFL condition. The modified algorithm limits the time-step by the perturbed velocity and shear arising from the differential rotation of the disk (Masset 2000). The azimuthal transport substep is decomposed into two parts: a classical azimuthal transport that is \(\leq |\frac{1}{2} \Delta \theta_i|\) and an integer cell shift. To borrow an example from Masset (2000), if some material at a radius \(r\) (in cell \([i, j]\)) must be azimuthally shifted 4.7 cells in one time-step, the transport is decomposed as \(4.7 = -0.3 + 5\), i.e. the material is classically transported by the remainder -0.3 then copied into cell \(j + 5\).

The modified algorithm reduces numerical error by minimizing the classical transport distance and shifting material an integer number of cells. Computationally, it also provides approximately an order of magnitude speed up compared to traditional transport schemes (Masset 2000).

### 2.3 2D Approximation of a Thin Disk

For a disk that is dominated by the gravity of a central object of mass \(M_\star\), the vertical density profile can be considered a Gaussian:

\[
\rho(r, z) = \rho(r, 0)e^{-\frac{z^2}{2H^2}}
\]

(2.1)

where \(H\) is the height of the disk as measured from the disk plane and \(r\) is the radial distance from the center of the disk (see figure 2.1). \(H\) is considered to be a pressure...
Figure 2.1: A profile view of the disk with an exaggerated aspect ratio and flaring for visual clarity. $H$ is the height from the disk plane, $r$ is the radial distance from the center of the disk.

Length scale (Masset, private communication) and can be estimated using:

$$H \approx \frac{c_s}{\Omega}$$

(2.2)

where $c_s$ is the local disk sound speed and $\Omega$ is the local orbital frequency:

$$\Omega = \sqrt{\frac{GM_*}{r^3}}$$

(2.3)

Protoplanetary disks typically have a ratio $H/r \sim 0.1$ and thus fall into the regime of what are known as ‘thin disks’ (see §1.1). Gas density is primarily concentrated in the vicinity of the disk mid-plane and as such can be reasonably well approximated using a 2D planar representation (Frank et al. 2002, §5). Though, such an approximation has important implications for how planet-disk gravitational interaction and gas accretion onto planets is handled and is discussed in §3.4.

Defining disk thickness is essentially equivalent to defining the sound speed profile throughout the disk since they are related by the approximation:

$$\frac{c_s}{v_k} \approx \frac{H}{r}$$

(2.4)

where $v_k$ is the local Keplerian orbital velocity of the unperturbed disk:

$$v_k = \sqrt{\frac{GM_*}{r^3}} = \Omega r$$

(2.5)
This estimate for $H$ is required for incorporating some effects of the disk’s finite thickness. Since the disk is treated as 2-dimensional, each point in the disk is actually a vertical integration that spans $\pm H$ above and below the disk mid-plane. Therefore, gas density $\Sigma$, radial and azimuthal velocities $v_r$ and $v_\theta$ are actually vertically averaged values for that point in the disk, \textit{i.e.}:

$$\Sigma(r) = \int_{-\infty}^{\infty} \rho(r, z) dz$$ \hspace{1cm} (2.6)

Since the radial sound speed profile $c_s$ only depends on $r$ (see above), it represents both a vertical and azimuthal average. It is calculated at the mid-points of each radial annulus.

The finite thickness of the disk must also be considered when evaluating the gravitational interaction between planet and disk. In FARGO this is done using a “smoothing length” which captures the fact that material in a cell is distributed throughout some vertical length scale. The smoothing length $\varepsilon$ is defined as a fraction of the local disk thickness $H$. The planet-disk interaction is then “softened” by adjusting the distance between a planet and the surrounding cells:

$$\tilde{r}_p = \sqrt{r_p^2 + (\varepsilon H)^2}$$ \hspace{1cm} (2.7)

where $r_p$ is the actual distance between a planet and cell, while $\tilde{r}_p$ is the softened distance. Thus, the softened planet potential can be written as follows:

$$\Phi_p = -\frac{GM}{\tilde{r}_p}$$ \hspace{1cm} (2.8)

where $M$ is the mass of the planet.

Additionally, a softened radial distance from the planet becomes useful and important for avoiding potential division by zero. In cases where $r_p \to 0$ for nearby or coincident cell-centers then $\tilde{r}_p \to \varepsilon H$ and there is no risk of division by zero when dividing by $\tilde{r}_p$. 

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2.4 Protoplanetary Disk Modelling in FARGO

The protoplanetary disk under consideration is Keplerian and non self-gravitating. It can optionally be modelled as a constant viscosity disk, but the alpha-disk formalism of Shakura and Sunyaev (1973) is used throughout (see §1.2). This allows the general disk properties to be described by a relatively small number of parameters.

The hydrodynamic portions of the code use discretized cellular coordinates to represent average gas densities on a polar grid. The N-body code makes use of a cartesian/polar coordinate system and is not limited to the cellular discretization of the hydrodynamic part. This allows for finer resolution of a planet’s location and planet-planet interactions in the disk limited only by the accuracy of double precision floating point numbers.

A notation similar to that used by Masset (2000) will be adopted for the purposes of describing the disk environment and how it is implemented in the code. The polar grid is composed of \( N_r \) rings and \( N_s \) azimuthal sectors. Each azimuthal sector is \( \Delta \theta = 2\pi/N_s \) wide. The rings, or radial separations, appear at \( R_k \) where \( 0 \leq k \leq N_r \),
so the inner boundary of the disk is located at $R_0$ and the outer boundary of the disk at $R_{N_r}$. The spacing of these radial zones can be defined as being linear, logarithmic or an arbitrary user-defined spacing can be provided. This polar grid is represented as a 2-dimensional array of size $N_r \times N_s$. The indices $i$ and $j$ are used to represent radial and azimuthal cell coordinates respectively, where $0 \leq i \leq N_r - 1$ and $0 \leq j \leq N_s - 1$.

Since the grid is considered to be polar, the azimuthal index $j$ wraps around so that the $j = N_s - 1$ azimuthal sector is beside the $j = 0$ azimuthal sector and that $j = N_s = 0$. Figure 2.2 illustrates both the rectangular and polar representations of the disk.

A 2D staggered mesh is used for the gas densities and velocities. The gas surface density is denoted $\Sigma_{ij}$ and is centered in the cells. The radial gas velocity $v_{r_{ij}}$ is located at the interface between cells $[i, j]$ and $[i - 1, j]$ and the azimuthal gas velocity $v_{\theta_{ij}}$ is located at the interface between cells $[i, j]$ and $[i, j - 1]$.

The use of a staggered mesh is common when using finite difference methods. This helps avoid a phenomenon known as “pressure decoupling”, where a solution to a partial differential equation discretized by centered differences is not uniquely determined. So instead of placing all variables on one centered grid, different variables are placed on different grids, which are each shifted half a grid point in a particular
coordinate direction. Figure 2.3 illustrates the layout of the staggered mesh and the locations of the quantities $\Sigma_{ij}$, $v_{r_{ij}}$ and $v_{\theta_{ij}}$.

2.5 Units and Important Quantities

There are a number of important quantities and symbols that will frequently be referred to throughout this thesis. Table 2.1 presented below lists these quantities and their corresponding symbols. Note that cgs (centimetre-gram-second) units are used throughout.

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>cgs Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solar Mass</td>
<td>$M_\odot$</td>
<td>$1.98892 \times 10^{33}$ g</td>
</tr>
<tr>
<td>Earth Mass</td>
<td>$M_\oplus$</td>
<td>$5.9742 \times 10^{27}$ g</td>
</tr>
<tr>
<td>Jupiter Mass</td>
<td>$M_J$</td>
<td>$1.8986 \times 10^{30}$ g</td>
</tr>
<tr>
<td>Astronomical Unit</td>
<td>AU</td>
<td>$1.49598 \times 10^{13}$ cm</td>
</tr>
<tr>
<td>Earth Year</td>
<td>yr</td>
<td>$31556926$ s</td>
</tr>
<tr>
<td>FARGO time-unit</td>
<td>$t_F$</td>
<td>$\approx 5024988$ s</td>
</tr>
</tbody>
</table>

Table 2.1: Important quantities used throughout this thesis.

It is typical to attempt to normalize some quantities to unity for many types of numerical simulations. This allows for maximal use of double precision floating point containers. In FARGO, the central mass $M_*$ is normalized to unity. The gravitational constant $G$ is also normalized to 1 instead of its regular value of $6.67300 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$. All force and accretion calculations are performed with $G$ and $M_*$ normalized.

Floating-point values representing physical quantities in FARGO are represented in a unitless form. One is free to choose desired units for physical quantities but certain choices are indeed more natural and/or convenient. Using arbitrary units, the orbital period, $T$, of a planet located at $r = 1$ about a central mass of 1 is defined as
This scenario is conveniently similar to Earth’s orbit about the Sun which takes 1 year at an average distance of 1 astronomical unit (AU) (see Table 2.1). It is therefore only natural (and convenient) to define radial distance in astronomical units and planetary masses in Solar mass units (M⊙). It follows that a planet orbiting a central mass of 1 M⊙ at a distance of 1 AU has an orbital period of 1 Earth year:

\[ 1 \text{ year} = 2\pi t_F \]  

A single time-unit in FARGO will therefore be defined as \(1/2\pi\) of an Earth year. If the central mass \(\neq M_\odot\), then quantities such as time, gas density (Σ) and planet mass(es) which are relative to (or expressed as a factor of) the central mass must be scaled accordingly. However, for the purposes of this thesis all references to the central mass \(M\) will imply 1 Solar mass (M⊙) unless otherwise stated.

### 2.6 Key Parameters in FARGO

There are 41 different parameters that define the FARGO simulations. However, a large number of these parameters are beyond or beside the scope of defining the accretion disk and the embedded planets. In this section, only the parameters required for adequately defining the accretion disk within the confines of the simulation space will be discussed. For full documentation of the parameters used in FARGO, as well as a parameter file template see the online documentation.

Though the FARGO simulation is 2-dimensional by construction (see §2.4), there are variables that attempt to approximate the finite thickness and shape of the accretion disk. Accretion disks are thought to have increasing thickness and flare vertically.
with respect to the plane of the disk as distance from the central mass increases (see §1.1). The reasoning behind defining a disk height for a 2-dimensional disk are discussed in §2.3.

The accretion disk and simulation properties as well as planets involved in the simulation are defined in two parameter files: \texttt{params.par}, which define the disk, mesh and numerical method parameters, and \texttt{planets.cfg} which describe the mass, locations and influences on each planet. The labels \texttt{params} and \texttt{planets} are merely place-holders and can be any desired filename.

A note regarding notation: a \texttt{monospace font} will be used below when referring directly to parameter names as required by FARGO or for labels identifying modified versions of the FARGO code (see §3).

### 2.6.1 Disk Parameters

There are two parameters required for defining the overall shape of the accretion disk. One parameter that partially defines the shape of the accretion disk is the \texttt{AspectRatio}, $h_0$, which defines the ratio of disk height $H$ to the radial distance $r$ as well as describes how quickly that height linearly changes with $r$. In order to define the shape of the disk everywhere, the flaring index $\lambda$ must also be defined. This defines the exponent for the power law behaviour of the aspect ratio:

$$\frac{H}{r} = h_0 \cdot r^{\lambda}$$

(2.10)

The default value of $\lambda$ is zero, corresponding to a uniform disk aspect ratio where the local disk thickness $H$ is proportional to the radius $r$. See figure 1.2 for an illustration of how these parameters define the shape of the accretion disk.
The height of the disk $H$ is related to the local soundspeed $c_s$ (see §2.3) through:

$$
\frac{c_s}{v_k} \approx \frac{H}{r}
$$

(2.11)

By defining both the aspect ratio and the flaring of the disk, the shape as well as the soundspeed profile of the disk are defined.

The gas surface density parameter $\Sigma_0$, denoted $\Sigma_0$, can be combined with the $\text{SigmaSlope}$ parameter, $\mu$, to define a power law gas surface density throughout the disk:

$$
\Sigma(r) = \Sigma_0 \cdot r^{-\mu}
$$

(2.12)

where $r$ is divided by distance units (AU in this case) to remain dimensionless. The default value for $\mu$ is 0, which describes a disk with uniform surface density.

The viscosity of the disk is defined using the $\text{AlphaViscosity}$ parameter, $\alpha$, from the alpha-disk formalism (see §1.2) of Shakura and Sunyaev (1973). The uniform coefficient $\alpha$ is used to define the kinematic viscosity $\nu$ used in the stress tensor which is calculated using equations (1.6) and (2.2):

$$
\nu = \alpha \cdot c_s^2 \cdot \Omega^{-1}
$$

(2.13)

where $c_s$ is the local sound speed and $\Omega$ is the local orbital frequency.

### 2.6.2 Planet Parameters

Each embedded protoplanet requires 5 parameters in order to define its gravitational behaviour and hydrodynamic influence on the disk. During initialization, each planet with mass $m$ is placed at a radial distance $r$ from the central mass along the $\theta = 0$ axis (equivalently $y = 0$ axis) with potentially eccentric orbits. Eccentricity, $e$, can only be defined in the simulation parameters file and is applied to all planets.
equally. It defaults to 0, making default initial orbits circular, but can be any value \( \geq 0 \). The planets are then initialized with \( x > 0, y = 0, v_x = 0, v_y > 0 \) so that each planet is at its apocenter at \( t = 0 \). Currently, FARGO does not allow for randomizing a planet’s initial \( \theta \) angle. Though, since the planets are on Keplerian orbits, their angular positions about the central star rapidly diverge.

An Accretion planet parameter \( k_a \) defines the accretion factor to apply to the accretion equations for each planet. It allows for linear modification of the accretion rates of each planet individually, defining the proportionality constant for the accretion rate. For instance, \( k_a = 1 \) refers to normal accretion as it is described in §2.8; whereas, for example, \( k_a = 2.3 \) would increase the accretion rate by 2.3 times. Note this factor only affects accretion rate when it is not being limited by Kelvin-Helmholtz accretion (see §3.6).

Whether or not a planet is influenced by the surrounding disk and other planets is determined by the Feels_Disk and Feels_Others boolean planet parameters. The Feels_Disk parameter determines whether the planet is gravitationally influenced by the disk tidal field or not. The Feels_Others parameter determines whether a planet is affected by the gravitational interactions it experiences from the central mass and other planets. Each planet can have any combination of true or false for these two parameters.

When calculating planet-disk forces, the FARGO code can optionally use a softening prescription to account for implicit disk height effects. When the ThicknessSmoothing parameter is defined to a non-zero value \( \varepsilon \), the potential arising from a planet is softened by a smoothing length \( \varepsilon H \) that scales with disk thickness at the location of the planet. This fraction \( \varepsilon \) applies to all planets simultaneously regardless of their location.
2.6.3 Mesh Parameters

There are 5 parameters which are required in order to completely define the properties of the 2-dimensional polar mesh which represents the protoplanetary disk. The number of radial and azimuthal zones must be defined in order to construct the hydrodynamic mesh. The parameters $N_{\text{rad}}$ and $N_{\text{sec}}$ are used to define the number of radial and azimuthal mesh zones, respectively (see §2.4). The inner and outer disk boundary radii are defined using parameters $R_{\text{min}}$ and $R_{\text{max}}$ respectively. Finally, in order to completely define the hydrodynamic mesh, the RadialSpacing parameter is used to define the radial zone spacing to either Arithmetic or Logarithmic. The first option sets the radial zone width as a constant over the whole disk, while the second option sets the radial zone width as proportional to the distance from the mesh center. The Logarithmic option has the advantage that all zones in the mesh have the same aspect ratio.

2.6.4 Numerical Method Parameters

The numerical method parameters define certain behaviours of the hydrodynamic and numerical schemes. The Transport parameter can be set to FARGO or Standard to define which azimuthal advection algorithm is to be used. The standard azimuthal advection is performed with the Courant–Friedrichs–Lewy (CFL) criterion includes the full azimuthal velocity, whereas the FARGO algorithm Masset (2000) relieves the CFL condition allowing for a much larger time-step resulting in a faster calculation (see §2.2).

The InnerBoundary parameter can be set either to Wall (or Rigid), Open, or Non-Reflecting and specifies the boundary condition at the inner disk boundary of the hydrodynamic mesh. The Open setting will be used for the purposes of this
thesis. Material can flow outside of the mesh, on its way to the central object. Thus, the total gaseous mass contained in the disk decreases (dissipates) with time. For an extended description of the other settings, see the online documentation\(^6\).

The Disk parameter simply allows one to disable the hydrodynamic disk and run a pure N-body simulation.

### 2.7 Standard Simulation Parameter Values

Simulations throughout this thesis are performed using the parameter values in Table 2.2 in order to provide a clear comparison across simulations performed with different code modifications. A standard value simulation is performed in §2.8.2 to provide a baseline comparison. The only exception is the “template” FARGO simulation performed in §2.8.1.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Variable Name</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas Surface Density</td>
<td>Sigma0</td>
<td>Σ₀</td>
<td>300 g cm(^{-2})</td>
</tr>
<tr>
<td>Sigma Slope</td>
<td>SigmaSlope</td>
<td>µ</td>
<td>1.0</td>
</tr>
<tr>
<td>Alpha Viscosity</td>
<td>AlphaViscosity</td>
<td>α</td>
<td>5.0 \times 10(^{-3})</td>
</tr>
<tr>
<td>Aspect Ratio</td>
<td>AspectRatio</td>
<td>h₀</td>
<td>0.05</td>
</tr>
<tr>
<td>Flaring Index</td>
<td>FlaringIndex</td>
<td>λ</td>
<td>0.25</td>
</tr>
<tr>
<td>Thickness Smoothing</td>
<td>ThicknessSmoothing</td>
<td>ε</td>
<td>0.6</td>
</tr>
<tr>
<td>Inner Disk Radius</td>
<td>Rmin</td>
<td>R(_{\text{min}})</td>
<td>1.0 AU</td>
</tr>
<tr>
<td>Outer Disk Radius</td>
<td>Rmax</td>
<td>R(_{\text{max}})</td>
<td>50.0 AU</td>
</tr>
</tbody>
</table>

Table 2.2: Standard simulation parameter values used throughout this thesis.

In addition to the table above, simulations involving a single planet have an initial semimajor axis 5 AU or 20 AU. The Jovian planets Jupiter, Saturn, Uranus and Neptune have semimajor axes ranging from approximately 5-30 AU; located at 5.2, 9.6, 19.2 and 30.1 AU respectively. These giant planets most likely form beyond \(\sim3\) AU.

\(^6\)InnerBoundary documentation: http://fargo.in2p3.fr/spip.php?article33
where their tidal influence allows accretion of large amounts of cool gas (Marcy et al. 2005).

A protoplanetary disk will have a surface density profile which goes as $\Sigma \propto r^{-1/2} \sim \Sigma \propto r^{-1}$ in the inner disk, steepening to $\Sigma \propto r^{-3/2}$ in the outer disk (Lissauer 1987; Wetherill 1996). Only a single power law density profile can be defined in FARGO, so $\mu = 1$ is chosen to give a density profile $\Sigma \propto r^{-1}$. The surface density $\Sigma_0 = 300 \text{ gcm}^{-2}$ together with the density profile give a total disk mass of approximately $0.01 \, M_\odot$ which is consistent with suggested protoplanetary disk masses of $10^{-2} \, M_\odot < M_{\text{disk}} < 10^{-1} M_\odot$ (Hayashi 1981; Lissauer 1993; Hartmann et al. 1998).

An alpha viscosity of $\alpha = 5 \times 10^{-3}$ fits with observations suggesting an $\alpha$ value of $10^{-3} \sim 10^{-2}$ (Hartmann et al. 1998).

The aspect ratio $h_0 = 5 \times 10^{-2}$ along with flaring index $\lambda = 0.25$ is consistent with typical aspect ratios and flaring of thin disks (Bell et al. 1997; Frank et al. 2002).

The outer boundary of the protoplanetary disk, $R_{\text{max}}$, was chosen to be 50 AU so as to include the Jovian planet region and because approximately 90% of exoplanets discovered thus far\footnote{Interactive Extra-solar Planets Catalog: \url{http://exoplanet.eu/catalog-all.php}} with a mass $\geq 1 \, M_J$ have a semimajor axis $\leq 50$ AU. Extending the grid inward of 1 AU is not necessary since it is not a region of great interest and quickly becomes needlessly expensive to compute.

### 2.8 FARGO Original Accretion Scheme

The original FARGO accretion scheme is straightforward: gas is removed from those cells falling within a specified accretion zone about the planet on a fixed timescale. There can be an arbitrary number of planets in the disk and each is essentially treated as a gravitational sink for the surrounding gas. The gas advection is handled...
by the hydrodynamic portion of the FARGO code. As the gas accretes, mass and momentum are transferred to the planet. The accretion is limited by the fixed discrete time-step length of the simulations but can be adjusted using the Accretion parameter mentioned in §2.6.

The specified eligible accretion zone for a planet is defined in terms of its instantaneous Hill radius $R_H$ - the region within which an astronomical body dominates the attraction of satellites (see §1.5), given by equation (1.10). Using parameters prescribed by Kley (1999), the eligible accretion zone is broken up into 3 sub-zones pertaining to different accretion rates. Parameters $f_1 = 0.75$ and $f_2 = 0.45$ describe the fractions of a planet’s Hill radius that are to be used for accretion zones (see figure 2.4). The area enclosed by $f_1R_H$ defines the largest zone from which gas may be accreted. Here, the accretable gas is limited to $1/3$ of the total gas contained within the area per unit time, where the unit of time is defined such that an orbit of radius $r = 1$ has an orbital period of $2\pi$ FARGO time-units. The smaller zone enclosed by $f_2R_H$ is limited to $2/3$ of the remaining gas within this area. This means that during one unit of time, cells within $r < f_2R_H$ of the planet would be accreted from twice. Note that no gas is accreted from the area between $f_1R_H$ and $R_H$ (see figure 2.4).

In order to determine which grid cells actually fall within the accretable zones, the zones must be discretized and approximated using the underlying polar grid. First, a rough rectangular grid section, guaranteed to contain the planet and its entire Hill radius, is isolated. This accretable grid narrows the region of interest to contain mostly cells that a planet will accrete from. The center of each cell, which is taken to be the location of the gas density $\Sigma_{ij}$ for that cell, is used to determine the cell’s distance from the planet. This planet-cell distance, $r_{pc} = |\vec{r}_{pc}|$, is used to determine which cells approximately fall within the accretion zones (see Figure 2.4).
Gas accretion for a planet occurs from all eligible cells contained within the planet’s accretion grid in a single simulation time-step. Each accretion step is performed in a number of sub-steps equalling the total number of cells contained in the accretable grid. For each cell \([i, j]\) in the accretable grid, if the planet-cell distance \(r_{pc}\) is within one or both accretion zones, then it is accreted from. As mentioned before, if the cell falls within the inner accretion zone defined by \(f_2R_H\) then it is accreted from twice: first removing \(\frac{1}{3}\) of the gas from that cell, then removing \(\frac{2}{3}\) of the remaining gas. The mass of the gas removed from a cell, \(\delta m_{ij}\), is calculated by:

\[
\delta m_{ij} = f_m \Sigma_{ij} A_{ij}
\]  

where \(f_m\) is one of the two Kley (1999) parameters \((m = 1, 2)\), \(\Sigma_{ij}\) is the cell surface density and \(A_{ij}\) is the cell surface area. The total mass removed from all accretable
cells per time-step, $\Delta m$, is summed and added to the planet after a complete time-step:

$$\Delta m = \sum_{ij} \delta m_{ij} \quad (2.15)$$

$$M_{n+1} = M_n + \Delta m_n \quad (2.16)$$

where $M_n$ and $M_{n+1}$ are the masses of the planet for the $n^{th}$ and $(n+1)^{th}$ time-steps respectively, and $\Delta m_n$ is the mass removed from all accretable cells for the $n^{th}$ time-step.

In order to determine momentum transfer from all accretable cells onto the planet, the cell-average gas velocities $\bar{v}_r$, $\bar{v}_\theta$, and $xy$ velocities must be calculated:

$$\bar{v}_{\theta ij} = \frac{1}{2} \left( v_{\theta ij} + v_{\theta i,j+1} \right) + \bar{R}_i \Omega_f \quad (2.17)$$

$$\bar{v}_{r ij} = \frac{1}{2} \left( v_{r ij} + v_{r i+1,j} \right) \quad (2.18)$$

$$v_{x ij} = \frac{\bar{v}_{r ij} x_{ij} - \bar{v}_{\theta ij} y_{ij}}{\bar{R}_i} \quad (2.19)$$

$$v_{y ij} = \frac{\bar{v}_{r ij} y_{ij} + \bar{v}_{\theta ij} x_{ij}}{\bar{R}_i} \quad (2.20)$$

where $\bar{R}_i$ is the radial mid-point of the $i^{th}$ annulus, $\Omega_f$ is the frame angular velocity, $x_{ij}$ and $y_{ij}$ are the cartesian coordinates of cell $[i, j]$’s center, and subscripts $r, \theta$ refer to radial and azimuthal respectively. These cell-average gas velocities along with the mass of the removed gas from each cell determine the total momentum transferred to the planet:

$$\Delta p_x = \sum_{ij} \delta m_{ij} v_{x ij} \quad (2.21)$$

$$\Delta p_y = \sum_{ij} \delta m_{ij} v_{y ij} \quad (2.22)$$

$$p_{x,n+1} = p_{x,n} + \Delta p_{x,n} \quad (2.23)$$

$$p_{y,n+1} = p_{y,n} + \Delta p_{y,n} \quad (2.24)$$
where $p_n$ and $p_{n+1}$ are the planet momenta for the $n^{th}$ and $(n+1)^{th}$ time-steps respectively.

One complete time-step signifies that the available mass and momentum of the surrounding accretable gas has been transferred to the planet. This work can be parallelized but requires checkpoints to sum together values calculated across different CPUs and for synchronized output of quantities.

### 2.8.1 FARGO Template Simulation

The standard “template” FARGO simulation involves a single Jupiter mass planet embedded in a thin, viscous protoplanetary disk with a circular orbit at a fixed radius $r = 1$. The planet’s accretion is disabled ($\text{Accretion} = 0.0$) and it is not influenced by the gravity of the disk ($\text{Feels.Disk} = \text{false}$). The disk spans from an inner radius of 0.4 to an outer radius of 2.5, has an aspect ratio $h_0 = 0.05$, uniform gas surface density $\Sigma_0 = 6.4 \times 10^{-4}$ and a uniform kinematic viscosity $\nu = 1.0 \times 10^{-5}$. The disk grid is comprised of 128 radial zones and 384 azimuthal zones. Note that there has been no mention or assumption of units, other than the planet mass. Since the central mass is normalized to 1, it can be considered to be in units of Solar mass and therefore $\frac{1}{1000} \ M_\star = 10^{-3} \ M_\odot \simeq 1 \ M_J$. The surface density $\Sigma_0$ and kinematic viscosity $\nu$ have units of mass/length$^2$ and length$^2$/time, respectively.

The output generated by FARGO will be used extensively throughout this thesis. An example output of the standard template simulation for the original FARGO code is shown in figure 2.5. The gas surface density for the entire disk is displayed in figure 2.5a and the logarithm of gas surface density in figure 2.5b. The radial and azimuthal gas velocities are displayed in figures 2.5c and 2.5d respectively. Figure 2.6 shows a zoomed-in view of the planet in figure 2.5b.
Figure 2.5: An embedded $1M_J$ mass planet at $r = 1$ in a disk after $\sim 2000$ orbits/years. The disk has an inner boundary $R_{min} = 0.4$ and outer boundary $R_{max} = 2.5$, an aspect ratio $h_0 = 0.05$, no flaring, a gas surface density $\Sigma_0 = 6.3661977237 \times 10^{-4}$, and a kinematic viscosity $\nu = 1.0 \times 10^{-5}$.
2.8.2 Standard Value Simulation

In order to examine the effects of the incremental code modifications which will be discussed in this section, a baseline simulation is required for comparison. Simulations are run without any code modifications using the standard parameter values discussed in §2.7 and listed in table 2.2. Standardising the disk and planet parameters across simulations allows for straightforward comparison of code modification effects on mass accretion and planetary migration.

Two sets of simulations are performed using single Jupiter-mass planets with initially circular orbits at \( r = 5 \) AU and \( r = 20 \) AU. The simulations are run for a period of approximately 6000 years in order to provide a reasonable amount of time for accretion, migration and a density gap to open in the protoplanetary disk. For the purposes of these simulations, a density gap will be defined as when the azimuthally
Figure 2.7: Mass accretion (left) and mass accretion rate (right) over time for an embedded planet of mass $1\,M_J$ with an initially circular orbit at 5 AU and 20 AU using the original accretion scheme with standard simulation parameters in table 2.2. The averaged density in the vicinity of the planet has decreased by roughly a factor of 10. Accretion for the planets is enabled and the Accretion parameter $k_a$ (see §2.6.2) is set to 1 for default accretion. The planets are also influenced by the disk and can therefore undergo migration (see §1.4).

The mass accretion for each simulation is shown in figure 2.7a. The surface density of the disk goes as $r^{-1}$ (see Eq. (2.12)) so there is higher initial density of gas in the vicinity of the 5AU planet, but overall there is more material available for accretion in the vicinity of the 20 AU planet. This results in approximately equal accretion for the two planets initially. However, since the disk is flared, the aspect ratio is smaller at 5 AU and thus the mass required to open a gap is lower (Bryden et al. 1999). The planet at 5 AU quickly opens up a gap in the disk accompanied by a reduction in accretion (see figure 2.7b). The planet at 20 AU is slower to open a gap in the disk as the aspect ratio at 20 AU is higher and thus continues accreting mass for longer.

Figure 2.8 illustrates the migration behaviour of each planet. The initially rapid migration exhibited by both planets is evidence of Type I migration (see §1.4) where
the planet loses angular momentum to the surrounding gas and the planet migrates inwards on relatively short time-scales. The 5 AU planet quickly opens up a gap in the disk at approximately 500 years (see figure 2.9a), transitioning to Type II migration, where migration slows down to time-scales of the viscous disk evolution. The 20 AU planet takes longer to open a gap in the disk and thus migrates farther inward under Type I migration. This also results in a more gradual transition to Type II migration for the 20 AU planet.

Figure 2.7b shows the mass accretion rate $\dot{M}$ of each planet. The mass accretion rates appear to be consistent with the Type I and Type II migration behaviour inferred from figure 2.8. Note the rapid reduction in accretion of the 5 AU planet due to early gap formation. There is a slight change in accretion rate at approximately 4200 years for the 20 AU planet which is most likely the approximate transition point from Type I to Type II migration.

Figure 2.9 shows the orbital eccentricity and semi-major axis with disk average gas density overlay for the 5 AU and 20 AU planets after approximately 500 and 4200
years respectively. This translates to approximately 45 orbits for both the 5 AU and
20 AU planets. Notice in figure 2.9a that the 5 AU planet has opened up a significant
gap in the disk as is evident from the comparison of average gas density at 500 years
to the original unperturbed gas density and has roughly doubled in mass. The 20 AU
planet mass increases by approximately 6 times and opens a much wider gap in the
disk.
Figure 2.9: Plots showing planet eccentricity vs. semi-major axis and average azimuthal gas density. Each planet is represented as a blue dot denoting its eccentricity on the vertical and semi-major axis on the horizontal, where the size of the dot is proportional to the planet mass. Each planet also has a label denoting its mass in $M_J$. The solid red line shows the azimuthally averaged gas density and the dotted black line shows the original, unperturbed gas density profile. Note that after $\sim 500$ years, the 5 AU planet has opened up a significant gap in the disk, whereas the 20 AU takes much longer ($\sim 4200$ years) to open a much wider gap.
Chapter 3

Accretion Scheme Modifications

The original FARGO accretion scheme, while well-suited for larger planetary masses ($\sim M_J$) on low eccentricity orbits, makes no consideration of accretion mechanisms arising from small planetary masses, eccentric orbits, atmospheric effects, relative gas velocities and associated time scales. A number of modifications to the original FARGO accretion scheme were made to address these issues relating to accretion mechanisms and time scales. The goal and motivation of these modifications is to ultimately handle accretion across a range of planetary masses in a self-consistent manner. The criterion for such an accretion scheme includes the ability to properly handle a large range of planetary masses - from on the order of an Earth mass to many Jupiter masses - on possibly eccentric orbits and the accretion time scales and mechanisms associated therein. The specific limitations of the original accretion scheme as well as the modifications made to address them will be further discussed in this chapter. These modifications will need to be clearly identified, since there are a number that have been made throughout this chapter. The original, unmodified version will be labelled using \texttt{orig}, and each subsequent modification will be labelled using \texttt{vn} where \texttt{n} is a number from 1-6.
3.1 Accretion Time Scale First Approach

There are many different time scales by which physical processes occur in protoplanetary disks - gravitational interactions between planets and the central mass, disk hydrodynamic evolution, planet-disk gravitational interactions, as well as different stages of gas accretion. As a first approach towards a rough estimate of the characteristic time scale of accretion, a logical place to start would be to consider the time scales intrinsic to the planetary system. Perhaps the most obvious time scale is that of the planet’s own motion around the star. This is the approach taken by Kley (1999), and adopted in the original FARGO accretion scheme.

The FARGO implementation of the Kley (1999) accretion scheme, however, assumes a planet on orbit of radius 1 (arbitrary units). The first modification made to the code was to generalize this scheme to any orbital radius. In the original FARGO accretion scheme, the amount of gas accreted in one timestep is determined by two elements: the Accretion factor $k_a$, and the simulation timestep length $dt$. Accretion is first and foremost limited by the simulation timestep length $dt$. This is intrinsic to any numerical simulation; any rate or time-dependent quantity will be limited by the advance of time in the simulation. The Accretion factor $k_a$ optionally defines a unique accretion time scale for each planet:

$$k_a = \tau^{-1}$$  \hspace{1cm} (3.1)

where $\tau$ is the characteristic accretion time scale of a planet. Note that this parameter, $k_a$, is given at run-time and remains constant throughout the simulations. Together, these two values determine the time scale on which available gas in each cell is transferred to the planet:

$$g_a = k_a \cdot dt$$  \hspace{1cm} (3.2)
Figure 3.1: Mass accretion over time for an embedded planet of mass 1 $M_J$ with an initially circular orbit at 5 and 20 AU using the original scheme and the modified v1 scheme.

where $g_a$ is the unitless, time- and parameter-limited quantity which defines the fraction of gas that can accreted from any cell per unit time and $dt$ is the length of time (in FARGO time-units) taken in a single simulation timestep. This factor $g_a$ is then multiplied by the Kley (1999) parameters discussed in §2.8 to give an amount of gas removed from any cell within the accretion zone.

The Accretion factor $k_a$, as it is originally defined (see §2.6.2), limits the possibility of including additional physical processes which may have different characteristic accretion time scales. For this reason, as well as other reasons which will become clear later on, it became more useful to re-define the Accretion factor $k_a$ for the purpose of providing a dimensionless linear scale factor to optionally adjust accretion rates on a per-planet basis. This allows for a dynamic accretion time scale that depends on the properties of the planet and its nearby disk neighbourhood and is not limited by a pre-defined, unchanging value.
Figure 3.2: Migration of an embedded planet of mass $1 \, M_J$ with initially circular orbits at 5 AU (left) and 20 AU (right). The modified version ($v1$) is compared to the original FARGO accretion scheme ($orig$).

However it is calculated, the time scale for which accretion is to be scaled by will be denoted $\tau$. The framework for scaling the accretion rate for the purpose of later modifications is therefore implemented as follows:

$$g_a = \frac{k_a \cdot dt}{\tau}$$  \hspace{1cm} (3.3)

Since $k_a$ has been re-defined as a dimensionless accretion scale factor, $g_a$ remains a unitless quantity. This allows for straightforward definition of the accretion time scale being used, which will become important later on.

We now set $\tau$ to equal the Keplerian orbital period of the planet, $T_{\text{planet}}$, to obtain the desired scaling:

$$\tau = T_{\text{planet}}$$  \hspace{1cm} (3.4)

$$T_{\text{planet}} = \frac{2\pi r}{v_k}$$  \hspace{1cm} (3.5)

where $r$ is the radial distance from the central star, and $v_k$ is the Keplerian orbital speed (see Eq. (2.5)). So effectively, equation (3.3) becomes:

$$g_a = \frac{k_a \cdot dt}{T_{\text{planet}}}$$  \hspace{1cm} (3.6)
Figure 3.3: Plots showing planet eccentricity vs. semi-major axis and average azimuthal gas density for v1 accretion modification. Refer to figure 2.9 for a detailed explanation of the plot elements.

which successfully scales the accretion rate by the planet’s orbital period about the central mass.

The label v1 will be used to identify a simulation run using the first version of the modified accretion scheme - scaling the gas accretion fraction $g_a$ by the planet’s orbital period discussed in this section. The label orig will be used to identify simulations using the original FARGO accretion scheme.

A comparison of the mass accretion over time for a 1 $M_J$ planet using the original FARGO accretion scheme and the v1 accretion scheme modification is performed. The disk properties are listed in table 2.2. The planets are initially located at 5 and 20 AU with circular orbits. Figure 3.1 shows the expected reduction in accretion due to being scaled by the planet’s orbital period about the central mass.

The significant reduction in accretion also manifests in an increased migration for both the 5 and 20 AU planets. Slower accretion will result in an extended time required to open up a significant gap in the gas disk, thereby exhibiting Type I migration for a longer period of time. Figure 3.2 shows the expected increased migration
and figure 3.3 shows the planet mass, orbital eccentricity and average gas density of the disk. In this case, the 5 AU planet takes approximately 4000 years to accrete the same amount of mass and open a significant gap in the disk as compared to the orig version simulations in figure 2.9. Note that, similar to the migration behaviour in figure 2.8, the planet initially at 20 AU continues to migrate rapidly since it is unable to open a gap as quickly as the planet at 5 AU. The transition from Type I to Type II migration is much smoother for the 5 AU planet and apparently absent for the 20 AU planet in figure 3.2.

3.2 Dynamic Selection of Accretion Type and Time-Scale

The original accretion scheme uses solely the Hill radius of a planet to determine which cells to accrete gas from. While it is crucial for surrounding gas to be gravitationally bound - or at least gravitationally dominated - by a planet, its mass is not the sole deciding factor regarding accretion. The planet’s orbit as well as the properties of the surrounding gas play significant roles in determining the accretion mechanism(s) at work. Here, Bondi accretion (see §1.6) is added as an additional consideration regarding the accretion mechanisms involved. The label v2 is given to simulations performed with the modifications discussed in this section.

The surrounding gas near the planet has a temperature and thus a thermal velocity, as well as a velocity relative to the planet’s motion. Bondi accretion essentially concerns the accretion of gas onto a body moving through such a gas cloud. The Bondi radius is effectively the point at which the net speed (thermal + relative) of the surrounding gas is equal to the escape speed of the planet. However, for the
purposes of this modification, the relative velocity will be ignored for the time being for reasons which are discussed below. So in order for accretion to occur, surrounding gas must be within a region that is gravitationally dominated by the planet as well as have a thermal velocity which is below the escape velocity of the planet.

The general Bondi radius is given by equation (1.12), however we initially neglect the relative gas velocity $\Delta v$. This amounts to assuming the planet to be on a circular orbit, moving at the same velocity as the mean velocity of the surrounding gas. So since $\Delta v \to 0$ then $v_{\text{net}} \to c_s$ and $R_B$, now determined solely by the planet mass and local sound speed $c_s$, reduces to equation (1.11):

$$R_B = \frac{2GM}{c_s^2} \quad \text{(3.7)}$$
where $M$ is the mass of the planet. Note that the sound speed in an accretion disk is
normally a function of $r$, the planet distance from the central star.

The Hill radius is affected by a planet’s semi-major axis and mass. Close proximity
to the central star will reduce the planet’s gravitational sphere of influence. Increasing
the distance away from the central star or the planet mass will increase the Hill radius
of the planet. However, as a planet’s mass increases, the Bondi radius increases faster
since $R_B \propto M$ whereas $R_H \propto M^{1/3}$, causing $R_H$ to become more restrictive. With
high thermal or relative gas velocities the Bondi radius $R_B$ can become much more
restrictive. Being inside $R_B$ means the gas is below thermal escape speed (sound
speed), and being inside $R_H$ means the planet has a stronger gravitational influence
on the gas than the central star.

The gravitational and thermal velocity effects are both necessary conditions for
accretion, so the more restrictive of $R_H$ and $R_B$ is selected as the accretion radius
(Pollack et al. 1996):

$$R_a = \min(R_H, R_B)$$  \hspace{1cm} (3.8)

This accretion radius, $R_a$, is used to determine the accretion zone from which gas
is removed from surrounding cells similar to the description in §2.8. However, the
Kley (1999) parameters $f_1$ and $f_2$ are no longer used to determine different accretion
fractions. It is assumed that gas within $R_a$ will definitely be accreted, albeit on a
different time scale which is discussed below. Relative velocities between the planet
and surrounding gas also affect Bondi accretion and will be addressed in §3.3.

Section 3.1 describes the attempt to estimate the planetary accretion time scales
using a time scale intrinsic to the planetary system - the planet’s orbital period about
the central star. This the time scale on which the planet’s accretion zone sweeps out
an annulus of the gas disk. However, the FARGO code handles this already since the
Figure 3.5: Mass accretion (left) and mass accretion rate (right) over time for an embedded planet of mass 1 $M_J$ with an initially circular orbit at 5 and 20 AU using the original accretion scheme and the modified v2 scheme with standard simulation parameters in table 2.2.

The planet is presumed to be on a Keplerian orbit which indeed sweeps through the disk in a self-consistent way. Thus, a different approach is preferred to estimate the time scale on which gas accretes onto the planet.

Other high-resolution 2D and 3D simulations show that the gas flow near the planet actually takes the form of a circumplanetary disk (e.g. Ward and Canup 2010). Figure 3.4 illustrates the formation of such a circumplanetary disk around a 1 $M_J$ planet within the planet’s accretion radius. A more relevant time scale for accretion is therefore one which relates to the gas once it falls within the planet’s accretion radius. Through viscous effects and angular momentum transfer to the surrounding gas (see §1.1), the gas comprising the circumplanetary disk will eventually accrete onto the planet. It follows that a more natural time scale to use for accretion is one on the order of the orbital period about the planet.

For simplicity, the accretion time scale $\tau$ is chosen as the orbital period about the
Figure 3.6: Migration of an embedded planet of mass $1 \, M_J$ with initially circular orbits at 5 AU (left) and 20 AU (right). The modified version (v2) is compared to the original FARGO accretion scheme (orig).

planet at a distance equal to the selected accretion radius:

$$\tau = T_{\text{orb}} = 2\pi \sqrt{\frac{R_a^3}{GM}}$$  \hspace{1cm} (3.9)

where $R_a$ is the minimum of the Bondi radius $R_B$ and Hill radius $R_H$ and $T_{\text{orb}}$ is the orbital period about the planet at $R_a$. Thus equation 3.3 now becomes:

$$g_a = \frac{a \cdot dt}{T_{\text{orb}}}$$  \hspace{1cm} (3.10)

Any gas contained within $R_a$ of the planet will therefore be accreted on a time scale roughly equal to the orbital period at a radius of $R_a$ about the planet. Figure 3.5a shows the expected reduction in gas accretion for the 5 and 20 AU planets, though the effect is more pronounced in the 20 AU case likely due to the higher mass required to open up a gap in the disk at this distance (see §2.8.2). The planet migration behaviour is shown in figure 3.6. The 5 AU planet begins with rapid Type I migration which promptly opens a gap, transitioning very quickly to Type II migration and appears to become locked in the density gap formed. The 20 AU planet, because of the scaled accretion rate and larger mass required to open a gap at 20 AU, appears to exhibit
Type I migration for the duration of the simulation. Mass accretion rate in figure 3.5b also shows the rapid reduction in accretion rate for the 5 AU planet as it opens a gap in the gas disk, and the roughly constant accretion of the 20 AU planet as it continues to accrete the surplus of available surrounding gas. Figure 3.7 shows that the 5 AU planet opens up a significant gap and roughly doubles its mass in \(~1200\) years whereas the 20 AU planet has also roughly doubled its mass but does not open a significant gap in the disk.

### 3.3 Cell-Specific Accretion Type and Time Scale

When a planet has a significantly eccentric orbital path, its velocity can deviate appreciably from the local Keplerian velocity over the course of an orbit. The planet-gas relative velocity \(\Delta v\) can possibly become much higher than the local sound speed, \(c_s\), which can result in a much more restrictive Bondi radius (see §1.6). However, the relative velocity can differ at various distances from the planet and hence the Bondi
radius potentially differs for each neighbouring cell that is part of the overall accretion zone. The original FARGO accretion scheme is only limited by the Hill radius and accounts for accretion rates at different radii by having constant accretion factors at different radii (see §2.8). By extending the method described in §3.2, the accretion time scales and restricting radius are calculated on a dynamic cell-by-cell basis to handle different accretion radii in a self-consistent manner. The label v3 is given to simulations performed with the modifications discussed in this section.

Since the relative gas velocity generally changes with distance away from the planet, a first-approximation of the accretion zone is still required to determine which cells will be potentially accreted from. This is performed in the same way as described in §3.2. The ‘maximum’ Bondi radius represents the maximum possible Bondi radius for a planet for a local sound speed and neglecting relative gas velocities is given by equation (1.11):

\[ R_{B_{\text{max}}} = \frac{2GM}{c_s^2} \]  

(3.11)

where \( c_s \) is the sound speed at the mid-point of the \( i^{th} \) radial annulus containing the planet. This is compared to the planet’s Hill radius and the more restrictive of these two radii is chosen as the first-approximation accretion radius:

\[ R_a = \min \left( R_{B_{\text{max}}}, R_H \right) \]  

(3.12)

where the instantaneous Hill radius given by equation (1.10) is used for \( R_H \).

The cells that fall within this first-approximation accretion zone are considered to be eligible accretion cells (e.g. see Fig. 2.4). In order to determine the cell-specific restricting accretion radius for each eligible accretion cell, the planet’s Bondi radius must be calculated using the gas velocities of that cell. Revisiting the general form
Figure 3.8: Mass accretion (left) and mass accretion rate (right) over time for an embedded planet of mass 1 $M_J$ with an initially circular orbit at 5 and 20 AU using the original accretion scheme and the modified v3 scheme with standard simulation parameters in table 2.2.

Figure 3.9: Migration of an embedded planet of mass 1 $M_J$ with initially circular orbits at 5 AU (left) and 20 AU (right). The modified version (v3) is compared to the original FARGO accretion scheme (orig).
for the Bondi radius in equation (1.12), a cell-specific substitution is made:

\[ R_{Bij} = \frac{2GM}{c_{si}^2 + \Delta v_{ij}^2} \]  

(3.13)

where \( c_{si} \) is the sound speed of the \( i^{th} \) radial annulus and \( \Delta v_{ij} \) is the relative velocity between the planet and gas residing in cell \([i,j]\):

\[ \Delta v_{ij} = \sqrt{(v_{px} - v_{xij})^2 + (v_{py} - v_{yij})^2} \]  

(3.14)

where \( v_{px,y} \) are the planet’s velocity components, and \( v_{x,y} \) are the gas velocity components of cell \([i,j]\). Calculations for \( v_x \) and \( v_y \) for each cell \([i,j]\) are shown in equations (2.19) and (2.20).

The cell-specific Bondi radius \( R_{Bij} \) is compared to the planet’s Hill radius and the more restrictive of the two is chosen as the accretion radius for the cell:

\[ R_{a_{ij}} = \min \left( R_{Bij}, R_H \right) \]  

(3.15)

The cell-specific accretion radius \( R_{a_{ij}} \) is used as the criterion for whether an eligible cell is accreted from. For each cell in the accretion zone, if the planet-cell distance is less than or equal to \( R_{a_{ij}} \), then gas is removed from the cell. For the purposes of clarity and brevity, the planet-cell distance will be written simply as \( r_{ij} \), referring to the distance from the planet to the centre of cell \([i,j]\).

The method for determining the accretion time scale in §3.2 is further extended to account for accretion taking place at different radii. Instead of accreting gas from all surrounding cells within the accretion zone over a single orbital time scale, an orbital time scale corresponding to each cell is calculated. For each cell \([i,j]\), an orbital period \( T_{r_{ij}} \) about the planet is calculated at a distance of \( r_{ij} \) from the planet and used as the accretion time scale for which accretion takes place:

\[ \tau_{ij} = T_{r_{ij}} = 2\pi \sqrt{\frac{r_{ij}^3}{GM}} \]  

(3.16)
This results in a dynamic range of accretion time scales for each cell where gas is accreted, updated for every unit of time throughout the simulation.

The resulting accretion rate can be approximated using the mass contained within the accretion volume and accretion time-scale $\tau$:

$$\dot{M} \approx \frac{\rho V}{\tau} \quad (3.17)$$

where $M$ is the mass of the planet, $\rho$ is the volume density of the surrounding gas and $V$ is the spherical volume of the accretion zone defined by the Bondi radius. If the accretion time-scale is approximately equal to the orbital period at a distance of $R_B$, then $\tau = 2\pi/\Omega$ where $\Omega = \sqrt{GM/R_B^3}$. Keeping in mind equation (1.12), the mass accretion rate becomes:

$$\dot{M} \approx \frac{\left(\frac{4}{3}\pi R_B^3\right) \rho}{2\pi \sqrt{R_B^3/GM}} \approx \frac{\sqrt{2}}{3\pi} \frac{4\pi G^2 M^2 \rho}{(\Delta v^2 + c_s^2)^{3/2}} \quad (3.18)$$

which resembles the Bondi accretion rate from equation (1.13) apart from a factor of $\sqrt{2}/3\pi$. Thus,

$$\dot{M} \approx \frac{\sqrt{2}}{3\pi} \dot{M}_B \quad (3.20)$$

The mass accretion rate using the Bondi radius and orbital time-scale approximation is actually significantly slower than the Bondi accretion rate in equation (1.13). The actual rate will equal the sum of masses accreted from each individual cell in the accretion zone divided by the time-step length:

$$\dot{M} = \sum_{ij} \frac{\delta m_{ij}}{dt} \quad (3.21)$$

where the mass accreted from a specific cell, $\delta m_{ij}$, depends on the cell’s relative gas velocity $\Delta v_{ij}$, the local annular sound speed $c_{s_i}$, as well as the actual cell gas surface
Figure 3.10: Plots showing planet eccentricity vs. semi-major axis and average azimuthal gas density for v3 accretion modification. Refer to figure 2.9 for a detailed explanation of the plot elements.

density $\Sigma_{ij}$ and height $H_{ij}$. Thus, the actual rate will tend to be higher than the estimate in equation (3.20) since cells toward the interior of the accretion zone, where $r_{ij} < R_B$, will be accreted from on a time-scale shorter than $\tau$.

Figure 3.8 shows the mass accretion and mass accretion rate of embedded Jupiter-mass planets at 5 and 20 AU initial orbits. Both planets exhibit similar mass accretion trends to the original accretion scheme (see Fig. 2.7), especially the 5 AU planet which exhibits similar early gap formation. The planet initially at 20 AU has lower overall accretion than the original scheme - a result of the lower accretion rate visible in figure 3.8b. Both planets have a lower accretion rate than the original scheme prior to gap formation (see Fig. 2.9), and very similar accretion rates afterwards which indicates that after the gap-opening, the accretion is strongly limited by the rate at which the disk resupplies gas to the accretion zone.

The migration behaviour shown in figure 3.9 suggests the 5 AU planet opens a gap and begins to transition from Type I to Type II migration at approximately 750 years, whereas the 20 AU planet appears to begin transitioning at approximately
3000 years. The 5 AU planet initially migrates faster than in the original scheme due to lower accretion rate resulting in a longer gap-opening time scale. The lower accretion rate of the 20 AU planet results in less mass accreted onto the planet than in the original scheme and therefore higher overall migration. The state of the protoplanetary disks and planets at are shown in figure 3.10. The 5 AU planet has opened up a significant gap and nearly doubled in mass while the 20 AU planet mass has increased by approximately 3.6 times.

3.4 Implicit Disk Height Effects

The protoplanetary disk is a thin disk and can be accurately modelled as a 2-dimensional planar disk (see §2.3). However, there is an implicit disk height $H$ defined by the disk aspect ratio and flaring (see Eq. (2.10)). The cell properties like gas surface density $\Sigma$ and gas velocities $v_r$ and $v_\theta$ therefore must be considered as vertical averages over $\pm H$. The label v4 is given to simulations performed with the modifications discussed in this section.

The FARGO code uses a softening prescription to account for implicit disk height effects when calculating planet-disk forces. When the ThicknessSmoothing parameter (see §2.6.2) is defined to a non-zero value $\varepsilon$, the potential arising from a planet is softened by a smoothing length $\varepsilon H$ that scales with disk thickness. This same approach is adopted to further extend the cell-specific accretion and time scale modifications made in §3.3.

This softening prescription is used to calculate a vertically-averaged value for the orbital period of all the gas in the cell about the planet using a “softened” planet-cell
distance:
\[ \tilde{r}_{ij} = \sqrt{r_{ij}^2 + (\varepsilon H)^2} \] (3.22)

The planet-cell distance \( r_{ij} \) is still used to determine which cells fall within the accretion zone as described in §3.3. However, the accretion time scale for which gas from cell \( [i, j] \) is accreted onto the planet is now calculated as the orbital period about the planet at the softened planet-cell distance \( \tilde{r}_{ij} \):
\[ \tau_{ij} = \tilde{T}_{ij} = 2\pi \sqrt{\frac{\tilde{r}_{ij}^3}{GM}} \] (3.23)

where \( \tau_{ij} \) is the cell-specific accretion time scale and \( \tilde{T}_{ij} \) is the softened orbital period about the planet.

Since the properties of each cell actually span a vertical column of \( \pm H_{ij} \), the mass contained within a cell can be considered as the total mass enclosed by the volume of the cell column. This leads to an implicit volume density of the gas that can be considered as a surface density \( \rho \) divided by the total column height:
\[ \rho = \frac{\Sigma}{2H} \] (3.24)

Now the total mass in a particular cell \( [i, j] \) may be written as:
\[ m_{ij} = \rho_{ij} A_{ij} (2H_{ij}) \] (3.25)
Figure 3.12: An exaggerated side-view geometric representation of how implicit accretion volume is calculated. The disk is flared as in figure 1.2 with a solid line representing the disk mid-plane. Each outlined column represents the disk height at a radial midpoint $R_i$ whereas the shaded columns represent the approximate volume of the accretion zone defined by the accretion radius $R_a$. Note that the accretion zone need not be spherical.

where $A_{ij}$ is the surface area of the cell. If the height of the disk is greater than the accretion radius of the planet, then there is artificially more mass available for accretion onto the planet since the entire amount of gas contained within a cell may not be within the implicit accretion volume (see Figure 3.12).

At large distances away from the central star, the disk height $H$ can become much larger than the accretion radius $R_a$ (see Figure 3.12). This leaves the possibility that a given cell may lie within the accretion zone, but the entirety of its vertical extent may not necessarily be contained within the accretion zone. In order to approximate how much mass is actually contained within the accretion zone of the planet, an ‘accretion height’ is calculated for each cell within the accretion zone (see Fig. 3.11):

$$H_{a_{ij}} = \sqrt{R_{a_{ij}}^2 - r_{ij}^2} \quad (3.26)$$

where $R_{a_{ij}}$ is cell-specific accretion radius (see Eq. (3.15)) and $H_{a_{ij}}$ is the cell-specific accretion height which spans $\pm H_{a_{ij}}$ vertically in cell $[i, j]$ since the planet is considered to be co-planar with the disk midplane. The ratio of the accretion height $H_{a_{ij}}$ to the
Figure 3.13: Mass accretion over time for an embedded planet of mass 1 M_J with an initially circular orbit at 5 AU (left) and 20 AU (right) using the original accretion scheme, the modified v3 scheme, and modified v4 scheme with standard simulation parameters in table 2.2.

disk height $H_{ij}$ can be used to approximate the actual amount of material that is available for accretion onto the planet in a particular cell column. Referring back to equation (3.25), the disk height $H$ can be replaced with the accretion height $H_a$ to get a fractional mass:

$$\tilde{m}_{ij} = \left( \frac{\Sigma_{ij}}{2H_{ij}} \right) A_{ij} \left( 2H_{a_{ij}} \right)$$  \hspace{1cm} (3.27)

$$= \left( \frac{H_{a_{ij}}}{H_{ij}} \right) \Sigma_{ij} A_{ij}$$  \hspace{1cm} (3.28)

$$= h_{ij} \Sigma_{ij} A_{ij}$$  \hspace{1cm} (3.29)

where $h_{ij}$ is a unitless fraction, $\tilde{m}_{ij}$ is the fractional mass, $H_{a_{ij}} \leq H_{ij}$ and $h_{ij} = 1$ if $H_{a_{ij}} > H_{ij}$. This way if the accretion radius happens to be larger than the disk height, the fraction of accretable mass cannot exceed the available mass in a cell. In effect, this process approximates the total volume of the dynamic accretion zone in relation to implicit volume of accretable mass contained within the polar grid. Figure 3.12 illustrates this concept.

The surrounding cells from which gas is accreted and the time scale by which
Figure 3.14: Mass accretion rate over time for an embedded planet of mass 1 $M_J$ with an initially circular orbit at 5 AU (left) and 20 AU (right) using the original accretion scheme, the modified v3 scheme, and modified v4 scheme with standard simulation parameters in table 2.2.

Accretion takes place has not changed relative to the modifications (v3) discussed in §3.3. The only change that has been made is to modify how much gas in each surrounding cell is available for accretion onto the planet. Also note that the simulations performed in this section have a higher resolution than previous sections for comparison purposed as well as reasons discussed in §3.5.

The mass available for accretion onto the planet now becomes largely dependent on the height of the disk relative to the planet’s accretion radius. Figure 3.13 compares the mass accretion of a Jupiter-mass planet using the original scheme, v3 code modification and v4 modifications discussed in this section. Mass accretion is slightly reduced at 5 AU but it is clear that at 20 AU, where the height of the disk is much larger due to the aspect ratio and disk flaring, the overall accretion decreased by almost a factor of 2 compared to the v3 modification. It is also clear from figure 3.14 that, as mentioned, the mass accretion rate at both distances has decreased. However, this does not seem to drastically affect the planet migration compared to the v3 modification seen in figure 3.15.
Figure 3.15: Migration of an embedded planet of mass 1 \( M_J \) with initially circular orbits at 5 AU (left) and 20 AU (right). The modified version (v4) is compared to the original FARGO accretion scheme (orig).

3.5 Resolution Considerations

The choice of resolution for any numerical simulation can be an important one. There is always a trade-off between resolution and computation time coupled with how much useful information is actually gained from increasing the resolution. Resolution dependence of the code is ideally non-existent, but is often unavoidable. Usually a reasonable compromise between resolution, computation time, and minimal resolution dependence must be made.

FARGO simulations allow for specifying arbitrary radial and azimuthal resolutions with the option of linear or logarithmic radial spacing. Logarithmic spacing of the radial zones allows for a sort of resolution gradient that scales with distance from the central star. The radial width of the zone is proportional to its distance from the central mass. If there are \( N \) radial sectors per logarithmic decade, meaning there are \( N \) rings between 1 and 10 AU, \( N \) rings between 10 and 100 AU, and so on, then for a disk with an inner edge of \( R_{\text{min}} \) and outer edge \( R_{\text{max}} \), the number of radial sectors...
Figure 3.16: Comparison of mass accretion for 1 M\textsubscript{J} initial mass using code modification v4 (see §3.4) for different resolutions using standard values in table 2.2, except without flaring. Note the convergence of mass accretion for both 5 and 20 AU at 3\times, 4\times, and 5\times resolutions.

\(N_r\) is calculated:

\[
N_r \approx \log(R_{\text{max}} - R_{\text{min}}) \cdot N
\] (3.30)

The number of azimuthal sectors \(N_s\) is then chosen to keep the radial span of each cell approximately equal to the azimuthal span:

\[
\Delta r \approx r \Delta \theta
\] (3.31)

Keeping the aspect ratio the same results in approximately square cells for all disk radii.

An initial choice of \(N=100\) resulted in an output which offered acceptable resolution for testing and analysing code modifications with reasonable computation time. This produces \(N_r=170\) radial zones logarithmically spaced and \(N_s=314\) azimuthal zones which, for the purposes of this discussion, will be referred to as ‘1\times’ resolution.

Simulations in sections 3.1 to 3.3 were performed using 1\times resolution. However, the final modifications to the FARGO code required examination of the resolution dependence of the code. The metric used for determining resolution dependence was
Figure 3.17: A close up of a planet at 5 AU in a protoplanetary disk with standard simulation parameters in Table 2.2 at 1× and 3× resolution.
Figure 3.18: A close up of figure 3.17 showing circumplanetary disk formation around the planet at $1 \times$ and $3 \times$ resolution with relative gas velocity vector fields superimposed.
that of mass accretion and subsequently, mass accretion rate. Identical simulations were performed using three additional resolutions - denoted $3\times$, $4\times$ and $5\times$ pertaining to $N=300$, 400 and 500 radial zones per logarithmic decade, respectively. Since both radial and azimuthal resolution is scaled by the same factor, the total number of grid cells scales as $N^2$.

The overall mass accretion of a $1 \, M_J$ planet at 5 and 20 AU for each resolution is compared in figure 3.16. It is clear that the mass accretion for both planets begins to converge at $3\times$, $4\times$ and $5\times$ resolutions. The mass difference between resolution increments becomes smaller as resolution increases, with the smallest difference after 2000 years being between $4\times$ and $5\times$ resolution. The difference in mass accretion between $1\times$ and $3\times$ resolutions is approximately $0.2 \, M_J$ and $0.5 \, M_J$ for the 5 and 20 AU planets respectively, whereas the mass accretion difference between $3\times$ and $4\times$ as well as between $4\times$ and $5\times$ is on the order of only a few hundredths of a Jupiter mass.

Though the code is parallel, the increased computational cost of higher-resolution simulations can only be partially offset by increasing the number of CPUs. In order to compromise between computation time and resolution which offers reasonable convergence for mass accretion, the $3\times$ resolution will now be used for any further simulations. This gives $N_s=942$ azimuthal sectors and $N_r=510$ radial zones resulting in a total of 480,420 cells. Comparing figures 3.17a and 3.17b shows the improvement this brings when resolving the protoplanetary disk. Furthermore, figures 3.18a and 3.18b illustrates the dramatic improvement in resolving the circumplanetary disk gas flow that forms around the planet by increasing the resolution. The resemblance of a circumplanetary disk and the associated velocity field in figure 3.18a is resolved in
much greater detail in figure 3.18b revealing numerous ring-like configurations surrounding the planet.

3.6 Atmospheric Effects on Mass Accretion

In the nucleated instability model (or core instability model, see §1.3.1), rapid gas accretion is triggered when the planet core mass, $M_c$, exceeds some critical mass $M_{c,crit}$ (Pollack et al. 1996; Ikoma et al. 2000). Based on hydrostatic calculations of the gaseous envelope (Mizuno 1980; Ikoma, Emori, and Nakazawa 1998) and quasi-static evolutionary calculations (Bodenheimer and Pollack 1986; Pollack et al. 1996), the critical mass for rapid gas accretion is conventionally taken as approximately 5-20 $M_\oplus$. However, this value does not always seem to be consistent with other more recent theories and observations (Ikoma et al. 2000). The critical core mass is dependent on the rate at which planetesimals are accreted ($\dot{M}_c$) and the opacity ($\kappa$) of the disk gas (Ida and Lin 2004a). Rafikov (2006) shows that the critical mass in the gas giant planet region ($\gtrsim 5$ AU) can be as high as 20-60 $M_\oplus$ as long as protoplanetary cores form prior to the dissipation of nebular gas.

Initially, gas accretion onto the planet is limited by the gravitational contraction time scale of the planet’s gas envelope (Ida and Lin 2004a). This is usually referred to as the Kelvin-Helmholtz time scale and regulates the planet’s ability of its atmosphere to accept new gas. This effect results in a growth time of the gas envelope mass on the order of the characteristic time of the contraction while the planet mass is below critical mass (Ikoma et al. 2000). When a planet core reaches or exceeds a critical mass of several $M_\oplus$, the atmospheric pressure gradient can no longer withstand the planet’s gravity. This causes the gas envelope to collapse onto its core and rapid

Until this point in the accretion scheme modifications, considerations involving atmospheric effects have not been made. Simulations performed thus far have initial planetary masses that are beyond any critical value mentioned above and are thus already in the regime of rapid gas accretion. It has been necessary, though useful, to perform simulations with larger initial planetary masses (∼M_J) since the original accretion scheme lacks any mechanisms to accurately model gas accretion onto a low-mass planetary core.

The FARGO simulations begin by initializing the gas disk with a power-law gas surface density that is only dependent on radius (see §2.6.1). Adding a planetary mass (∼M⊕ or higher) at the beginning of a simulation can be a significant perturbation to the gas disk. It is unrealistic to expect a multiple Earth-mass planetary core to suddenly be in the vicinity of relatively high-density, uniformly-distributed gas and to accrete at a rate that is beyond the Kelvin-Helmholtz contraction time scale. In order to extend the applicability of the current accretion scheme any further, it is necessary to account for low initial mass planetary cores (∼M⊕) to allow for self-consistent growth to occur in order to better understand planetary growth from a small core to gas giant. The label v5 is given to simulations performed with the modifications discussed in this section.

The Kelvin-Helmholtz gravitational contraction time scale τ_KH represents the characteristic growth time of a planet’s gas envelope. Ikoma et al. (2000) found that τ_KH is strongly dependent on the core mass and is expressed approximately as:

$$\tau_{\text{KH}} \simeq 10^b \left( \frac{M}{M_\oplus} \right)^{-c} \left( \frac{\kappa}{1 \text{ cm}^2 \text{ g}^{-1}} \right) \text{ yr}$$

(3.32)

where M is the mass of both the solid core and gaseous envelope, and κ is the grain
opacity with power-law index $b \approx 8$ and $c \approx 2.5$. Here, however, a similar approach to that taken by Ida and Lin (2004a) is adopted where $b = 8$, $c = 3.0$ and $\kappa$ dependence is neglected, resulting in:

$$\tau_{\text{KH}} \approx 10^8 \left( \frac{M}{M_\oplus} \right)^{-3.0} \text{yr}$$

This gives an approximate characteristic growth time of the gas envelope for a planet with low mass. Note that the coefficient $10^b$ and exponent $c$ can be set at run-time using FARGO parameters `KH_Coefficient` and `KH_Exponent` respectively.

The primary modification made to the accretion scheme in this section is to compare the mass accretion rate determined from the accretion scheme prescribed in §3.4 using the v4 modification and the so-called Kelvin-Helmholtz mass accretion rate determined using the time scale in equation (3.33). The lesser of the two rates is used to determine how quickly gas is accreted onto the planet. For small planetary masses, the limiting accretion time scale will be the Kelvin-Helmholtz time scale $\tau_{\text{KH}}$. The code modification v5 incorporates the fact that no matter what is happening
Figure 3.20: Logarithm of mass accretion rate over time for an embedded planet of mass $50 \, M_\oplus$ with an initially circular orbit at 5 AU (left) and 20 AU (right) using the original accretion scheme and modified v5 scheme with standard simulation parameters in table 2.2. The three v5 mass accretion rates shown correspond to the ‘would-be’ rate, the Kelvin-Helmholtz limited rate, and the actual rate calculated using Eq. (3.36). Note the strong mass dependence for $\dot{M}_{KH}$.

Figure 3.21: Plots showing planet eccentricity vs. semi-major axis and average azimuthal gas density for v5 accretion modification for 5 AU planet after 3000 years (left) and 20 AU planet after 6000 years (right). Refer to figure 2.9 for detailed explanation of the plot elements.
in the planet’s accretion zone, its accretion is ultimately limited by the ability of its atmosphere to accept new gas.

The FARGO code was initially written to be a one-pass algorithm - removing gas (if necessary) from each cell in the accretion zone as it is examined. A two-pass modification to the algorithm is required to compare the two differing accretion rates. This was the optimal solution which would minimize bookkeeping and code re-factoring while allowing for the parallel implementation of the code to continue to function properly. The extra overhead required in order to loop over the cells in a planet’s accretion zone twice is minor even when compared to the slight temporal overhead caused by an extra parallel synchronization point in the code.

The first pass performs the calculations necessary for accreting gas from surrounding cells without actually removing any mass. This provides an unbound, would-be mass accretion rate onto the planet for that time step:

\[ \dot{M}_u = \frac{\Delta m}{dt} \]  

(3.34)

where \( \Delta m \) is the total accreted mass that would have been removed from all cells in the planet’s accretion zone (see Eq. (2.15)) and \( dt \) is a single time-step for the simulation.

The mass accretion rate \( \dot{M}_u \) is the unbound rate using the \( v4 \) modification and is only limited by how quickly the gas disk can hydrodynamically resupply gas into the accretion zone of the planet under the accretion scheme modifications put forth in §3.4.

The unbound mass accretion rate in equation (3.34) is then compared to the Kelvin-Helmholtz accretion rate calculated for that time step, which is calculated as follows:

\[ \dot{M}_{KH} = \frac{1}{2\pi} \frac{M}{\tau_{KH}} \]  

(3.35)
where \( M \) is the planet mass (before any accretion occurs) for that particular simulation timestep, \( \tau_{KH} \) is the Kelvin-Helmholtz time scale in equation 3.33, and division by \( 2\pi \) is required for proper unit consistency since \( \tau_{KH} \) is in units of years and \( dt \) is in units of \( t_F \) (FARGO time units).

The rate which limits gas accretion onto the planet is determined as the minimum between the would-be mass accretion rate and the Kelvin-Helmholtz mass accretion rate for a particular time step:

\[
\dot{M} = \min \left( \dot{M}_U, \dot{M}_{KH} \right)
\]

(3.36)

The limiting accretion rate determines what ratio of gas to remove from cells within a planet’s accretion zone on the second pass. The mass accretion ratio \( \mu \) is calculated as follows:

\[
\mu = \frac{\dot{M}_{KH}}{\dot{M}_U}
\]

(3.37)

where \( \mu \leq 1 \). The mass removed from each cell (\( \delta m_{ij} \)) in a planet’s accretion zone is then potentially reduced by the factor \( \mu \):

\[
\delta m_{ij} = \mu \Sigma_{ij} A_{ij}
\]

(3.38)

where \( \Sigma_{ij} \) is the gas surface density, \( A_{ij} \) is the cell surface area. This means that if \( \dot{M}_{KH} < \dot{M}_U \) then \( \mu < 1 \) and accretion is limited by the Kelvin-Helmholtz rate. Otherwise, \( \mu = 1 \) and the gas accretion proceeds as prescribed in \( \S 3.4 \).

Figure 3.19 shows the mass accretion for a 50 M\(_{\oplus} \) planetary core on an initially circular orbit at 5 and 20 AU and figure 3.20 shows the corresponding unbound, Kelvin-Helmholtz, and actual \( (v5) \) mass accretion rates. Note the strong mass dependence for \( \dot{M}_{KH} \). For both the 5 and 20 AU cases, the actual rate is initially coincident with the limiting Kelvin-Helmholtz rate, then becomes coincident with the unbound rate.
once the Kelvin-Helmholtz rate exceeds the (unbound) rate at which available gas can flow onto the planet. An increase in mass accretion for the 5 AU planet can be seen to occur at \( \sim 2000 \) years due to strong mass dependence of the Kelvin-Helmholtz accretion. The unbound mass accretion rate becomes the limiting rate much sooner for the 50 M\( \oplus \) planet at 20 AU as seen in figure 3.20b. Once the Kelvin-Helmholtz rate is no longer limiting mass accretion, the accretion rate becomes approximately constant for the 5 AU planet but continues to increase roughly exponentially for the 20 AU planet as seen in figures 3.19 and 3.20. Figure 3.21a shows that the 5 AU planet begins to open up a gap in the disk at \( \sim 3000 \) years, limiting accretion further, whereas figure 3.21b shows that no such gap has formed in the vicinity of the 20 AU planet. These figures show that it is important to impose the Kelvin-Helmholtz contraction time scale as a limit on the accretion rate since the difference can be quite significant at low planetary masses.

3.7 Very Low Initial Mass: Single-Cell Accretion

The nature of the underlying protoplanetary disk representation used by FARGO requires a planet’s accretion zone to be approximated using the 2-dimensional polar grid cells. Effects arising from the implicit height of the disk are also taken into account (see §3.4) and must also be approximated by projecting their effects onto the 2-dimensional grid representation. However, approximating the accretion zone with the polar grid has limitations when confronted with small planetary masses which give rise to accretion zones occupying on the order of a single cell or less.

One approach to this problem would be to increase the resolution of the simulation. While valid, this approach only manages to regress the issue of resolution dependence
Figure 3.22: Side view of a small accretion volume contained within a larger cell volume. Note the overestimation of the accretion volume using the accretion-height approach (see §3.4).

on initial mass. For very small masses ($< \sim M_\oplus$), one may attempt to equalize single-cell surface area to a planet’s accretion zone for a better approximation (e.g., see Fig. 2.4), but then the simulation resolution becomes dependent on the initial planetary mass - a smaller accretion zone necessarily leads to an increased resolution.

Attempting to combat the resolution dependence of the accretion zone on small planetary mass by increasing the resolution is further compounded by the unnecessary computational expense added by a higher resolutions. FARGO demonstrates reasonable mass accretion convergence at $3\times \sim 4\times$ resolution (see §3.5). A resolution increase for the purpose of better accretion zone approximation results in needless extra computational expense later on when sub-cellular accretion zone approximation is no longer an issue.

The implicit height modifications discussed in §3.4 break down at low planetary masses ($< \sim M_\oplus$) when there is a possibility that the accretion zone may be much smaller than the surface area of the cell containing the planet. This is especially important for small planetary masses at larger radial distances since grid cells that are further away from the inner disk edge have a larger surface area due to equal azimuthal spacing and logarithmic radial spacing of cell interfaces. Using the accretion-height ($H_a$) approach has a potential for overestimating the volume of gas available
for accretion onto the planet. Figure 3.22 illustrates this by showing how the volume approximated using $H_a$ is much greater than the actual accretion zone volume. In order to further extend the accretion scheme to self-consistently handle lower planetary masses ($\sim < M_\oplus$), an approach similar to the accretion-height modifications made in §3.4 is taken where a ratio of volumes is used to determine the amount of available gas for accretion onto the planet. The label v6 is given to simulations performed with the modifications discussed in this section.

If the planet and its accretion zone are determined to be within a single cell, a ratio of the volume of the planet’s accretion zone to the volume of the cell column containing the planet is calculated:

$$h = \frac{V_a}{V_{cell}}$$

$$= \frac{4\pi R_a^3}{2HA}$$

where $h \leq 1$; $V_a$, $V_{cell}$ are the respective volumes of the accretion zone and cell containing the planet, $R_a$ is the accretion radius determined for the cell which contains the planet (see Eq. (3.15)), $H$ and $A$ are the height and surface area, respectively, of the cell containing the planet.

The criteria for a planet’s accretion zone to be within a single cell are that the planet’s accretion radius does not overlap the center point of any surrounding cells. More specifically, the accretion radius does not exceed half of the diagonal distance of the cell containing the planet:

$$\Delta R = R_{sup} - R_{inf}$$

$$s = R_{mid} \left( \frac{2\pi}{N_s} \right)$$

$$l = \sqrt{\Delta R^2 + s^2}$$

where $R_{sup}$, $R_{mid}$ and $R_{inf}$ are the superior, mid, and inferior radii pertaining to the
cell which contains the planet and $N_s$ is the number of azimuthal grid zones. Thus $\Delta R$ describes a radial cell side-length, $s$ describes an average azimuthal cell arc-length. Figure 3.23a illustrates the geometric representation of the single-cell accretion length condition $l$. The radial mid-distance $R_{\text{mid}}$ is chosen for calculating the ‘average’ cell arc-length since the cells have a slight keystone shape due to polar coordinate mapping and $R_{\text{sup}} > R_{\text{mid}} > R_{\text{inf}}$. The resulting condition for single-cell accretion is:

$$R_a \leq \frac{l}{2} \quad (3.44)$$

Figure 3.23b shows an example of a scenario that would be considered as single-cell accretion.

The fractional mass available for accretion onto the planet within the cell containing the planet, $\tilde{m}$, is then calculated using equation (3.40):

$$\tilde{m} = \rho V_a \quad (3.45)$$

$$= \left( \frac{\Sigma}{2H} \right) \frac{4}{3}\pi R_a^3 \quad (3.46)$$

$$= h\Sigma A \quad (3.47)$$
where $\rho$ is the implicit cell volume density, $\Sigma$ is the gas surface density of the cell containing the planet. The planet will accrete gas from the entirety of the cell, but the fraction of available mass is reduced by $h$.

In previous sections discussing multi-cell accretion, time scales for accretion were calculated using the planet-cell distance $r_{ij}$. However, since mass is only removed from the cell containing the planet for single-cell accretion, the planet is effectively located in the middle of the cell. The time scale for accretion, $\tau$, is then equal to the orbital time about the planet at a distance equal to the softened accretion radius calculated for the cell containing the planet:

$$\tilde{R}_a = \sqrt{R_a^2 + \varepsilon^2} \tag{3.48}$$

where $\varepsilon$ is the smoothing length (see §2.6.2) and $R_a$ is calculated using the cell-specific accretion radius (see Eq. (3.15)). This modification will allow for very low planetary masses and approximate sub-cellular accretion zones and gracefully transition to multi-cell accretion (see §3.4) with sufficient planetary mass, though the limits of the single-cell approximation will require further examination.
Chapter 4

Simulation Demonstrations

Additional simulations intended to further demonstrate the capabilities and robustness of the modified accretion algorithm are presented. Simulations performed thus far have been single planet simulations dealing with accretion mechanisms at different radii on circular orbits. The modified FARGO code discussed in chapter 3 will be used to test accretion onto planets with eccentric orbits as well as multi-planet scenarios where planet-planet interactions are possible.

4.1 Eccentric Orbits

The relative velocity between a planet and the surrounding gas increases with eccentricity as a result of the planet’s elliptical transit through the gas disk. The modifications discussed in §3.3 dealt with this possibility. The goal of this section is to conduct tests of gas accretion onto an eccentric planet. Comparable simulations were performed by Thommes et al. (2008) using a simplified 1-dimensional disk model; here, simulations of a planet with a significantly eccentric orbit are performed with a 2-dimensional disk model.
Figure 4.1: Planet orbital eccentricity over time for a 1$M_J$ planet on initially eccentric orbits ($e = 0.3$) at semi-major axes of 5 and 20 AU using standard simulation parameters in table 2.2.

A number of recent numerical simulations have shown that eccentricity of planetary orbits is damped (Muto et al. 2011; Bitsch and Kley 2010; Cresswell et al. 2007, e.g) for a variety of masses. However, very high eccentricities can require long periods of time to damp, and can also be countered by ongoing planet-planet interactions. Thus a planet may potentially spend a significant part of its growth time in an eccentric orbit. In order to examine the effects of eccentric orbits on planetary gas accretion, simulations were performed with high-mass 1$M_J$ planets on initially high-eccentricity orbits ($e = 0.3$) at 5 and 20 AU. Jupiter-mass planets ensure gas accretion would not be limited by the Kelvin-Helmholtz time scale to assist in accelerated completion times for the simulations, though slow initial growth is expected due to the high relative gas velocities arising from an eccentric orbit. Note also that the Kelvin-Helmholtz time scale is independent of eccentricity and only depends on mass (see §3.6).
A rough estimation for when a planet will experience faster growth can be made by examining when the eccentricity is comparable to the disk aspect ratio. The sound speed $c_s$ and disk height $H$ are related by equation (2.4):

$$\frac{c_s}{v_k} \approx \frac{H}{r}$$ \hspace{1cm} (4.1)

and the relative velocity between the planet and surrounding gas can be thought to be on the order of the Kepler velocity multiplied by the eccentricity:

$$\Delta v \sim ev_k$$ \hspace{1cm} (4.2)

At higher eccentricities, high relative gas velocities ($\Delta v$) become the dominant factor in determining how restrictive the Bondi radius is, but when the planet’s eccentricity is on the order of the aspect ratio of the protoplanetary disk:

$$e \sim \frac{H}{r}$$ \hspace{1cm} (4.3)
Figure 4.3: Logarithm of mass accretion rate over time for an embedded planet of mass $1M_J$ with an initially eccentric ($e = 0.3$) orbit at 5AU (left) and 20AU (right) using standard simulation parameters in table 2.2.

then the contributions from $\Delta v$ and $c_s$ are comparable (see Eq. (1.12)) and the planet will experience greater gas accretion than while on a highly eccentric orbit. Equation (4.3) gives a reasonable estimate for the eccentricity below which planetary gas accretion and growth will speed up.

Figure 4.1 shows the eccentricity over time of planets at 5 and 20 AU with initially eccentric orbits. The eccentricity of both planets is damped by the gas disk, with the eccentricity of the 20 AU planet taking longer to damp, as expected (Bitsch and Kley 2010, e.g.). The time scale for the eccentricity of the 20 AU planet is also expectedly longer due to the larger disk aspect ratio at 20 AU. The disk aspect ratio (with flaring, see Eq. (2.10)) is approximately 0.08 at 5 AU (see Eq. (2.10) and Table 2.2), and approximately 0.1 at 20 AU. This gives a rough estimation of $e \sim 0.1$ where planetary accretion should increase.

Figure 4.2 shows a sudden increase in gas accretion for the 5 AU planet at $\sim 200$ years and at $\sim 800$ years for the 20 AU planet which corresponds with increases
Figure 4.4: Planetary migration over time for an embedded planet of mass $1M_J$ with an initially eccentric ($e = 0.3$) orbit at 5AU (left) and 20AU (right) using standard simulation parameters in table 2.2.

in mass accretion rate at the same times in figure 4.3 for both planets. These correspond with an eccentricity of approximately 0.2 for both planets, which is roughly within a factor of 2 of the estimate. The planetary migration for the 5 and 20 AU planets is shown in figure 4.4. There is inward migration of these high-mass planets which have larger values of $e > 0.1$, as expected (e.g. Bitsch and Kley 2010; Cresswell et al. 2007). The noticeably rapid initial migration is on a similar time scale as eccentricity damping for the 5 AU planet before it opens up a significant gap in the disk at $\sim 500$ years as shown in figure 4.5. The eccentricity damping and migration time scales are also expectedly longer for the 20 AU planet, due to larger disk height. The 5 AU planet opens up a gap in the gas disk fairly early while the 20 AU planet begins to form a significant gap closer to $\sim 3000$ years, shown in figure 4.6.

A planet on an eccentric orbit will experience higher relative gas velocities which leads to more restrictive Bondi radius and overall decreased gas accretion. However, as the orbital eccentricity is damped by the gas disk, accretion onto the planet will
Figure 4.5: Planet eccentricity, semi-major axis and average gas disk density (above) and actual gas disk density (below) for 5 AU planet on eccentric orbit after 600 years.
Figure 4.6: Planet eccentricity, semi-major axis and average gas disk density (above) and actual gas disk density (below) for 20 AU planet on eccentric orbit after 2700 years.
increase resulting in potentially multiple competing accretion mechanisms. A low-mass planet on an eccentric orbit may be initially limited by its Bondi radius due to high relative gas velocities, then become limited by the Kelvin-Helmholtz time scale once sufficient eccentricity damping has occurred and finally become limited by the rate at which the disk can supply gas into the accretion zone.

The main purpose of these simulations is to highlight the importance of accounting for the potential effects that arise from an eccentric orbital path. It has been shown that the planetary accretion dynamics that arise from competing accretion mechanisms can be largely dependent on a planet’s orbital properties. The modified FARGO accretion scheme provides a tool for accurately modelling gas accretion onto an eccentric planet orbiting within a protoplanetary gas disk in a self-consistent manner.

4.2 Randomly Seeded Planetary Cores

Simulations involving only a single planetary core have been useful in developing, testing and demonstrating the concepts and modifications integrated into the FARGO accretion algorithm. However, simulating the growth of a system of planets starting out as cores in a disk is not only a more realistic scenario, but employs all of the code modifications undertaken thus far.

The protoplanetary disk is seeded with planetary cores which have randomly generated properties. Each core accretes gas from the disk in a self-consistent way that is determined by the planet mass, surrounding relative gas velocity and orbital eccentricity. The cores are distributed in roughly such a manner that is expected after some period of oligarchic planetary growth. Significant eccentricities tend to
<table>
<thead>
<tr>
<th>Planet Name</th>
<th>Radius (AU)</th>
<th>Mass ($\times 10^{-5} M_\star$)</th>
<th>Mass ($M_\oplus$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core0</td>
<td>5.02</td>
<td>8.23</td>
<td>27.43</td>
</tr>
<tr>
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<td>6.35</td>
<td>8.77</td>
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<td>Core3</td>
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<td>25.62</td>
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<td>8.22</td>
<td>27.39</td>
</tr>
<tr>
<td>Core5</td>
<td>13.42</td>
<td>8.51</td>
<td>28.37</td>
</tr>
</tbody>
</table>

Table 4.1: Planet parameters for randomly seeded planetary core simulations for demonstration #1. Values have been rounded to 2 decimal places for readability.

arise when multiple giant planets grow in close proximity, and continued planet-planet scattering can maintain these for a long time despite damping by the disk (e.g. Thommes, Matsumura, and Rasio 2008; Bitsch and Kley 2010). Accretion occurs from an accretion zone radius which is less than disk half-thickness ($H$) and depending on the chosen resolution and initial planet mass, it is possible that planet accretion may occur in a single cell.

Since these simulations are meant as a demonstration, the initial conditions are somewhat biased to shorten the initial slow core growth and instead produce rapid, early planet growth. Initial planetary masses are fairly large (slightly above Uranus/Neptune mass) and the Kelvin-Helmholtz time scale is reduced by a factor of $2\pi$, which remains a possible value in low-opacity disks, for example. The planetary masses correspond to cores which have already undergone significant gas accretion. However, the planetary masses are still low enough to initially limit gas accretion by the Kelvin-Helmholtz gravitational collapse time scale and thus are well below the runaway gas accretion threshold.

The planetary parameters for the simulations were randomly generated using an algorithm. Each planetary core is given an initial mass between 25-30 $M_\oplus$. The first core is placed at a distance between 4.5-5.5 AU away from the central star on
<table>
<thead>
<tr>
<th>Planet Name</th>
<th>Radius (AU)</th>
<th>Mass ($\times 10^{-5} M_\star$)</th>
<th>Mass (M$_\oplus$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core0</td>
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<td>8.82</td>
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<tr>
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<td>8.52</td>
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<td>7.70</td>
<td>25.67</td>
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<tr>
<td>Core4</td>
<td>11.37</td>
<td>8.55</td>
<td>28.50</td>
</tr>
</tbody>
</table>

Table 4.2: Randomly generated planet parameters for seeded planetary core simulations for demonstration #2. Values have been rounded to 2 decimal places for readability.

an initially circular orbit. Each subsequent planet is placed at a distance of 5-10 Hill radii from the previous planet for a total of 5-6 cores. Placing the planets within $< 10R_H$ of each other is consistent with the expected spacing of planets in the orderly oligarchic growth regime which precedes runaway gas accretion (see §1.3.1). A small initial mass ($<30 M_\oplus$) and places the planetary cores in the Kelvin-Helmholtz limited growth regime.

The simulations are inherently chaotic and different realizations with only slight variations in initial conditions can produce very different results. In fact, when running FARGO simulations in parallel, even identical initial conditions can produce stochastic results (see §A.3). A total of ten simulations were performed, each involving 5-6 planetary cores with randomly generated properties using the standard simulation values in Table 2.2. Two simulations were chosen as demonstrations (hereby denoted demonstration #1 and #2) which exhibited evidence of competitive accretion, significant growth, and planetary orbits remain within the bounds of the protoplanetary disk throughout the simulations. Table 4.1 and 4.2 show the randomly generated parameters of the embedded protoplanets for the two multi-core simulation demonstrations. For each simulation, the planetary cores are numbered starting at 0 and increase with initial radial distance away from the central star.
Figure 4.7: Planetary core mass accretion (top-left), mass accretion rate (top-right), planetary migration (bottom-left) and orbital eccentricity (bottom-right) for demonstration #1.
Figure 4.8: Planetary core mass accretion (top-left), mass accretion rate (top-right), planetary migration (bottom-left) and orbital eccentricity (bottom-right) for demonstration #2.
Random variation in initial core masses combined with the steep dependence of accretion rate on mass during the Kelvin-Helmholtz limited phase causes the different planets for grow at different rates. This tends to result in one or two planets reaching runaway accretion before other cores. When this happens, stronger gravitational perturbations from the larger planet(s) tend to give rise to increased eccentricities of neighbouring planets. The planet(s) which achieve runaway accretion early on can cause competitive accretion in which the growth of their neighbours is inhibited since eccentric planets accrete gas more slowly, thus further widening the gap in planet mass. Another competitive accretion mechanism occurs once a planet becomes massive enough to open a significant gap, reducing the supply of gas to any planets orbiting interior to it since material in the disk is gradually spiralling inward as it loses angular momentum.

Figures 4.7 and 4.8 show the planetary core mass accretion, mass accretion rates, planetary migration and orbital eccentricity for demonstrations #1 and #2. Both simulations exhibit anti-correlated shifts in semi-major axis as well as correlated peaks of orbital eccentricity between neighbouring cores, indicating ongoing planetary interactions throughout the simulation. The anti-correlation in semi-major axis occurs due to interacting planets transferring orbital energy and angular momentum. Shifts in the semi-major axis do not correspond to instantaneous jumps in planetary position, but rather a change in planet velocity which indicates a change in orbital path.

There is clear evidence of distinct interactions in demonstration #1 between core 0 and core 1 at approximately 3400 years where figure 4.7c shows a slight change in semi-major axis of both planets and a corresponding increase in orbital eccentricity in figure 4.7d. The spike in orbital eccentricity of core 0 also appears to correlate to a slightly delayed decrease in mass accretion rate of core 0 shown in figure 4.7b. This
Figure 4.9: Random core disk profile and orbital eccentricity for demonstration #1. Refer to figure 2.9 for a detailed explanation of the plot elements.

demonstrates the adamant nature of competitive accretion - even the planet(s) that achieve runaway accretion early on continue to compete with each other. Additionally, there are a number of interactions between cores 2 and 3 at approximately 5000 years in demonstration #1. Figure 4.7c shows distinct shifts in semi-major axis at this time which corresponds to significant increases of orbital eccentricity of both planets, with peak eccentricities of $\approx 0.07$ and $\approx 0.15$ for core 2 and 3 respectively.

There are also a number of planetary interactions between neighbouring cores in demonstration #2. Figure 4.8c shows evidence of a distinct interaction between core 1 and 2 at approximately 4000 years with correlated peaks in orbital eccentricity approaching $e \sim 0.1$ in figure 4.8d. There is evidence of interactions between all five planetary cores at approximately 5300 years and later. However, there are slight temporal offsets of the shifts in semi-major axis for the planets involved. The strongest and earliest interaction appears to be between cores 1 and 2 at approximately 5300
years with correlated peaks in eccentricity exceeding $e \sim 0.1$ at this time. Figure 4.8c shows that the shift in semi-major axes of cores 1 and 2 due to this interaction triggers additional, ongoing planetary interactions to propagate to neighbouring cores. This leads to a very strong gravitational encounter between cores 2 and 3 followed by another encounter between cores 1 and 2.

The mass accretion and mass accretion rates of the planets are shown in figures 4.7a, 4.7b and 4.8a, 4.8b for demonstration #1 and #2 respectively. They reveal that the innermost cores tend to have higher mass accretion rates than neighbouring cores even though mass accretion is initially limited by the Kelvin-Helmholtz collapse time scale. The outermost planets tend to be limited by the Kelvin-Helmholtz time scale for much shorter or not at all due to overall lower gas densities. The innermost planets tend to grow the fastest initially and reach runaway accretion first, while the outermost planets tend to grow slower and more gradually. However, all of these
trends can be affected by competitive accretion between neighbouring planets.

The mass and accretion rates of the innermost cores (0, 1) in demonstration #1, shown in figures 4.7a and 4.7b, rapidly increase with mass during the initial phase of accretion. Core 1 has an initially higher mass accretion rate than core 0 and hence is able to accrete more mass and achieve runaway accretion sooner. This is visible as a change in mass accretion at $\sim$1200 years for core 1 and at $\sim$1700 years for core 0 in figure 4.7a. Accretion rates for both inner cores stay approximately constant for a time and then begin to steadily decline for the remainder of the simulation. The intermediate cores (2, 3) are also initially limited by the Kelvin-Helmholtz collapse time scale, experience a period of approximately constant mass accretion, then a decreasing mass accretion rate. The outermost cores (4, 5) have gradually increasing mass accretion rates for the entire duration of the simulation and actually surpass the accretion rate of core 1 at $\sim$5000 years. The reduction in mass accretion rate experienced by planets interior to cores 4 and 5 is most likely due to gap opening by those cores.

Similar trends in mass and accretion rates can be seen in demonstration #2, shown in figures 4.8a and 4.8b. Core 0 has the highest overall accretion rate, while core 2 and 1 have the second and third highest respectively, each showing periods of rapid initial growth, approximately constant accretion, and declining accretion over the simulation time window. The outermost core (4) has nearly identical initial mass accretion rate as core 2, which is expected given the close initial masses of the two cores, and then surpasses the rate for core 2 at $\sim$3500 years and continues to rise, surpassing the accretion rate of core 0 at $\sim$5500 years. The simulation begins with core 4 actually more massive than core 1 and 2 (but less massive than core 0), then becomes less massive than both and then exceeds both core masses by the end of the
Competitive accretion and planet-planet interactions all affect orbital properties, planetary gas accretion and the evolution of the disk. Figures 4.9 and 4.10 show the gas density profile and orbital properties of the planetary system at the end of each simulation. The figures show that significant density gaps form in the vicinity of the planets and that appreciable eccentricities can form through planetary interactions. The effects of competitive accretion between planetary cores can be seen in these simulation demonstrations, but these effects can also occur on longer time scales (e.g. Thommes et al. 2008). Gas accretion will depend on the surrounding gas environment and relative velocity, planetary mass and orbital properties of the planet. Though these simulations will require longer run times for further investigation, they demonstrate the self-consistent modelling of multiple, competing planetary gas accretion mechanisms in the face of perturbations from potentially large neighbouring masses and the evolutionary effects on the planetary system as a whole.
Chapter 5

Summary

The final numeric gas accretion scheme is an aggregation of the modifications discussed in §3.2 - §3.7 and is a more robust prescription for gas accretion onto protoplanetary cores embedded in a gaseous accretion disk. It accounts for multiple, competing, dynamic accretion mechanisms dependent on a planet’s mass, semimajor axis, orbital eccentricity and surrounding gas properties. The accretion length scales associated with a planet’s Bondi radius and Hill radius are calculated on a cell-by-cell basis and used to approximate the dynamic accretion zone of a planet within the hydrodynamical grid. The time-scales involved in accreting gas onto a planet are also calculated on a cell-by-cell basis to be on the order of the orbital period about the planet, and an accretion scaling parameter allows this time-scale to be adjusted if required.

Implicit effects arising from the approximation of the thin protoplanetary disk as a 2-dimensional planar disk are accounted for when calculating planet-disk forces as well as dynamic planetary accretion zones and time-scales. Single-cell accretion approximations allows for accretion onto a large range of planetary masses to be
examined, from a few Earth masses to several Jupiter masses. Addition of the Kelvin-Helmholtz collapse time-scale allows for approximating a planet’s gas accretion rate at small masses as well as the point where a planet’s atmospheric pressure gradient can no longer withstand the planet’s gravity - the point where the planet’s gas envelope begins to collapse, resulting in rapid gas accretion from the surrounding disk. This updated accretion scheme provides a means for exploring the entire formation process of gas giants out of a variety of initial conditions in a self-consistent manner that is mostly resolution independent.
Chapter 6

Future Work and Considerations

In this section, some additional code modifications as well as future considerations are proposed and discussed.

Initializing multiple planets on uniquely eccentric orbits is currently not possible due to a code-based limitation in FARGO. A single Eccentricity parameter (see §2.6.2) is used to define initial orbital eccentricity for all planets in a simulation. An ideal solution would allow for planet-specific orbital eccentricity to be defined within the planet configuration file. This would allow for specific case studies of multi-planet systems with unique eccentricities provided as initial conditions.

Planet-planet interactions are calculated using whatever time scale is dictated by the hydrodynamic part of the code (see §2.2). Typically this time-step is short enough, but very close encounters between planets might require smaller time-steps in order to properly resolve the interaction. An adaptive time-step scheme for planet-planet interactions would be an important improvement to ensure they are properly resolved. So, between hydrodynamic time-steps \( t_n \) and \( t_{n+1} \), there could be \( N \) gravitational time-steps of length \( dt_g \) such that \( t_n + N \cdot dt_g = t_{n+1} \), where \( N \) may be calculated adaptively depending on the conditions of the encounter. This could be
accomplished using the popular GNU Scientific Library\(^1\). Additionally, including a
softening length when calculating planet-planet interactions would ensure a minimum
approach distance and eliminate unrealistically close encounters.

It would be ideal to run these simulations over the entire lifetime of the proto-
planetary disk. Longer simulations would capture more of the relatively slow and
orderly initial growth regime with only a population of cores embedded in the gas
disk until the end of its lifetime when it has been depleted through a combination
of gas accretion onto planets and onto the central star. Longer simulations would
capture more of the relatively slow and orderly initial growth regime. The desire is
to evolve the entire protoplanetary system from the stage with only a population of
cores embedded in the gas disk, to the appearance of the first gas giant(s), to the end
of its lifetime when it has been depleted through a combination of gas accretion onto
planets and onto the central star on as realistic a time scale as possible. In practice,
though, such simulations would require simulations on the order of millions of years,
thus are likely to be restricted to smaller snapshots of the disk’s evolutionary lifetime.

Lastly, in the near future many fields, namely astronomy, will benefit greatly from
inexpensive, high-performance parallel graphics processing (GPU) based computing.
Astronomical simulations involving complex hydrodynamic interactions can achieve
speed increases of multiple orders of magnitude after proper GPU-oriented algorithm
parallelization. Recently, Masset has released a proof-of-concept GPU version of
FARGO aptly named GFARGO that achieves up to \(90\times\) performance speed-up\(^2\).
Unfortunately, gas accretion onto planets is currently missing in GFARGO. Further
development on GFARGO as well as other GPU-based hydrodynamic solvers will
undoubtedly yield new and exciting simulation possibilities in the future.

Appendix A

Software

A.1 Code Modifications

All code modifications discussed throughout this thesis are available for viewing and download online at: https://bitbucket.org/jrussell/fargo. The online repository tracks changes as well as allows users to download any desired revision for testing.

A.2 Farpyplot

Figures which portray 2-D gas disk density, 1-D averaged gas density with planet positions, or 2-D gas density with gas velocity field were generated using Farpyplot software written by John Russell. Farpyplot was inspired by the FARGO plotting tool Fargnuplot\textsuperscript{1} written by Hanno Rein.

Farpyplot extends on many of the features of Fargnuplot. It allows for real-time visualization of FARGO output files, while allowing the user to advance forward

\textsuperscript{1}Fargnuplot website: http://hanno-rein.de/science/scientific-software/fargnuplot
and backward between output files in a sequence using the keyboard. Rectilinear or polar colour maps of gas density, radial gas velocity or azimuthal gas velocity can be generated with the option for contour lines highlighting constant values. In gas density mode, it allows for plotting local gas velocity fields in the vicinity of any embedded protoplanet(s). Planet positions as well as approximate accretion zones can be shown as well. In 1-D mode, a disk profile view is generated showing the planet(s) semimajor axis, eccentricity and mass superimposed on an azimuthally averaged gas density profile.

The code, which includes documentation, is available for viewing and download online at: https://bitbucket.org/jrussell/farpyplot.

A.3 Parallelism and Determinism

Floating-point numbers are inherently an approximation of any number. Some numbers are able to be exactly represented in binary (such as any power of 2), however most numbers are approximated using whatever bit precision is available on the hardware and/or representational container (i.e. float32, float64). For example, the number 0.01 (decimal) cannot be exactly represented in binary - the closest (single precision, 24-bit) representable number is 0.009999999776482582092285156250 exactly. Also, floating-point addition and multiplication are commutative, but not associative. That is, \((a + b = b + a)\) and \((a \times b = b \times a)\), however \((a + b) + c\) is not necessarily equal to \(a + (b + c)\).

Algorithms and code can be parallelized using MPI, OpenMP or POSIX threads, just to name a few. FARGO uses MPI to distribute computations across multiple CPUs connected by a network. In one parallel programming technique, one CPU
deemed the ‘master’ CPU delegates computational tasks to ‘slave’ or ‘worker’ CPUs. Once a CPU is finished performing a calculation, its result is sent back to the master CPU over the network. The master CPU then performs some operation (addition, for example) on the results as they are returned - a process called reduction. However, due to a myriad of reasons (network latency, I/O interrupt, etc.) the order in which the master CPU receives results from all slave CPUs is non-deterministic.

So, depending on the order of reduction from different processors, floating point representation may differ due to non-associativity - thus producing slightly different results which can diverge over time. In other words, identical initial conditions on the same set of parallel processors can produce different results! (see e.g. Lee 2006)
Bibliography


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