TECHNOLOGY IN MATHEMATICS EDUCATION: 
IMPLEMENTATION AND ASSESSMENT 

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ABSTRACT

TECHNOLOGY IN MATHEMATICS EDUCATION:IMPLEMENTATION AND ASSESSMENT

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The use of technology has become increasingly popular in mathematics education. Instructors have implemented technology into classroom lessons, as well as various applications outside of the classroom. This thesis outlines technology developed for use in a first-year calculus classroom and investigates the relationship between the use of weekly formative online Maple T.A. quizzes and student performance on the final exam. The data analysis of the online quizzes focuses on two years of a five-year study. Linear regression techniques are employed to investigate the relationship between final exam grades and both how a student interacts with and performs on the online quizzes. A set of interactive class notes and a library of computer demonstrations designed to be used in and out of a calculus classroom are presented. The demonstrations are coded in Maple and designed to give geometric understanding to challenging calculus concepts.
To the great teachers who have inspired me
to follow this path and to my family and friends
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Chapter 1

Introduction

There are various ways of integrating technology into a mathematics classroom. Some instructors choose to utilize a computer algebra system for in-class demonstrations, or have students use the system to complete assignments. One may also post learning modules on the internet for students to use as a self-study guide. The use of online testing is also wide-spread. Online testing systems can be used as a diagnostic tool for placing students into the correct course, for summative testing, and for formative assessment. With all of these options available, the assessment of technology in education becomes very important. The assessment of technology use in education is challenging, as measuring learning is difficult and with many confounding factors it is hard and probably wrong to attribute that learning to any one cause.

The development of new uses of technology to engage students is an important task. Interactive class notes give the instructor an opportunity to present the mathematical concepts in the classroom in an innovative way. Notes can be developed
using mathematical software to display graphics or answers that the software gives to problems completed in class. Computer demonstrations can be utilized to help students visualize concepts in the classroom and explore topics on their own time. One effective method is to design demonstrations that bring a geometric understanding to challenging mathematical concepts.

The structure of this thesis is as follows. Chapter 2 presents a review of relevant literature pertaining to technology use in mathematics education. It also discusses the difference between formative and summative assessment and the challenges in designing an educational experiment. In Chapter 3, results of a five-year study of the use of the online testing system Maple T.A. in a first-year University of Guelph calculus course are presented. The testing system was utilized to provide students with regular formative quizzes. Based on techniques learned through the literature search, the data analysis focuses on how students interact with the testing system and not necessarily just their performance on the online quizzes. The availability of information such as the number of times a student attempted each quiz and the date on which the student first attempted each quiz allows our data analysis to test the effect of how the student interacts with the system in ways not found in the literature. Student attitude towards the online testing is assessed through survey results. In Chapter 4, a set of interactive class notes and a small library of computer demonstrations designed for a first-year calculus class are presented. This gives the reader examples and inspiration for how computer-aided instruction may be beneficially uti-
lized in a mathematical setting. Finally, Chapter 5 gives a summary of the results and some discussion of future work.
Chapter 2

Review of the Literature

The literature regarding the use of technology in mathematics classrooms is diverse. One can find many papers discussing implementations of new technologies for various courses (for instance, see Jacobs [16]), but it is difficult to find papers that evaluate the success or failure of the technologies in a statistically meaningful way. This could be due to the difficulty in designing a proper experiment in an educational setting. Many authors have attempted to measure the effect of technology in the classroom using a control versus treatment design, (i.e., Bonham et. al [3], Brown and Liedholm [4], Ashton et. al [2], Dinov et. al [9], McSweeney and Weiss [18] and Spradlin [23]). Generally these experiments involve having one section of a course presented in a traditional way and another presented with technology. A common issue in designing a study this way is students cannot be randomly placed into the sections, since students must be free to arrange their own schedules, resulting in a quasi-experimental design. There are also numerous sources of variation in an
educational setting that can be very difficult to account for; for instance, different sections having different instructors, various levels in student preparation coming into the course, differences in student effort in the given subject and students not using the learning tools in the way they are intended. There are also ethical concerns in performing experiments in an educational setting. For example, restricting access to the learning tool to all students in the control group eliminates a possible positive learning experience for those students. Some studies have even limited some students’ access to online material to reduce variability in students’ willingness to work as a factor in the experiment. Many educational institutions require student consent before s/he is involved in the study, which can lead to the issue that Stillson and Alsup [25] had where only 55% of their class agreed to be involved in the study (nearly half of their data needed to be discarded). With the issues involved in conducting an experiment, some studies, such as the study presented in this thesis and the one of Angus and Watson [1] have chosen instead to perform an observational study in which the instructor presents the course with the technology as s/he normally would. Linear regression techniques are then used to attempt to account for sources of variation. This does not give the benefit of comparing to a control group, but it also avoids the aforementioned issues.

Since many of the studies that we focus on will discuss formative versus summative assessment, let us define these terms. In the broadest sense, formative assessment is assessment for learning, whereas summative assessment is assessment of learning.
Formative assessment typically occurs during the course by allowing students to complete exercises in an environment that is less stressful than a formal test and giving students feedback to allow them to identify strengths and weaknesses. Ideally, formative assessment would not count towards the students final grade; however, many instructors find that a small percentage of the course grade must be allocated to the formative assessment in order for the student to take it seriously and complete it. Summative assessment is any assessment designed to evaluate the student’s knowledge of the subject material and produce a grade for the student. The most common form of summative assessment is a final exam.

2.1 Assessment of Computer-Assisted Instruction

Computer-assisted instruction (CAI) refers to any instruction done in or out of the classroom that utilizes computers or technology. Many authors have investigated the effects of this type of instruction. A report on their findings follows.

Bruce and Ross [5] investigated the use of the interactive program CLIPS (Critical Learning Instructional Paths Support, see http://www.oame.on.ca/clips/index.html) in several grade 11 and 12 classrooms to further understanding in the trigonometry unit. This program was designed to give students independent lessons, including video demonstrations and assigned readings, followed by quizzes and other activities. They found that students benefitted the most when they were exposed to the learning tool after having the lesson presented in class, not before the in-class lesson. They
also found CLIPS had little effect on student attitude. Students were less positive about using the technology when exposed to it before the in-class lesson.

Brown and Liedholm [4] ran three versions of a microeconomics course: a control group with three hours of lecture each week; a hybrid class with two hours of lecture each week, supplemented with various online components including a large interactive problem set; and a fully online course. The online course had access to the same online components as the hybrid course, as well as live streaming of the in-class lectures. A common exam was written by all three class types at the completion of the course. They found that the control class outperformed the online class, and all other pairwise comparisons were not significant. There was no difference between the class types on the basic definition questions, but there was a marked difference between the online course and the in-class courses on the questions that called for more in-depth knowledge of the subject material.

Dinov et al [9] performed various experiments using an online computational resource in an introductory statistics course. Comparisons were made between a traditional class and a class that viewed many in-class demonstrations and completed assignments with the computational tool. They found a statistically significant difference in performance between students who were exposed to the resource and those who were in the control group. While students who were exposed to the resource performed better, the size of this difference was small.

A meta-analysis performed by Timmerman and Kruepke [26] found an overall
positive effect of students being exposed to a computer-assisted instruction environment. The analysis showed that CAI packages that were designed specifically for one course had a larger effect on student performance than that of generally published CAI packages. They also found gains were higher when the technology was used consistently throughout the course, not just as a one-time instance. The study compared CAI packages that provided feedback to students with ones that did not. There was no evidence to support the hypothesis that CAIs that provided feedback to students generated a greater improvement in student performance. A possible explanation for this result may come from the research of Cazes et al. [8], which measured the amount of time students spent reading the CAI feedback page. This study noted that students only briefly viewed the feedback screen before clicking to move on. We see one of the challenges of using a CAI package is motivating students to use the system as the instructor intends.

2.2 Assessment of Regular Online Testing

Computer-assisted assessment is any assessment method the students complete via a computer, typically automatically graded by the computer program. There are three main research questions in the literature regarding computer-assisted assessment:

1. Can online testing adequately replace traditional paper-based testing methods?

2. Do students embrace the online testing tool as a learning opportunity?
3. Do regular online formative assessments enhance student learning?

The first question has been discussed by many authors, but will not be the focus of this study. With increasing class sizes and the lack of funds to hire markers, many instructors are not faced with the question of “Should we have regular online quizzes or weekly paper-based quizzes?” but rather “Should we have regular online quizzes or no quizzes at all?”. The second and third questions must both be answered “yes” in order to have a successful implementation of online formative testing. The second question is of particular importance because students and instructors may reject the online assessment tool if they are faced with technical difficulties. An example of this is noted by Bonham et al. [3], where a sluggish server and errors in the programming of quiz questions led to a rejection of the technology by both instructor and students.

2.2.1 Comparing online testing with paper-based assessment

The main disadvantage of current online assessment techniques is that the computer grades only the final answer and does not follow through the student’s solution to award partial credit as a human grader would. Ashton et al. [2] attempted to incorporate partial credit in online assessment in two different ways. The first was based on having students enter information at each key step of the problem; for instance, if the question asked for the equation of the tangent line to a function at a particular point, the user would be asked to enter the derivative of the function, the value of the function at the point and finally the equation of the tangent line. The issue they
noted with this approach was that students were given the strategy to solve each question. To attempt to fix this issue, the questions were designed to ask only for the final answer, but give students the option to press a button labeled “Steps”, which would then ask for the additional information. If the student chose to use “Steps”, they were deducted any marks reserved for the strategy of the question; for instance, in the tangent line example they would lose one of four marks. The second method of accounting for the lack of partial credit was to ask only for the final answer, but allow students to see if their answer is correct and change it if it is not. This method was based on the belief that most errors would be simple arithmetic errors and if the student was aware there was a mistake, s/he would be able to look over the work and correct it. Groups of students at five different schools wrote a standardized mathematics test using one of the above methods or a traditional paper-based test. They found very few students in the “Steps” group actually used the “Steps” option (it was used 75 times out of a total of 520 opportunities). Overall they found no significant difference in student performance between the three testing methods; however, the experiment had many issues. For instance, many students were not able to finish the computer version of the test in the allotted time as they were unfamiliar with the system, which limited the analysis to only those questions that all students were able to complete. Unfamiliarity with the system also resulted in students making many input errors, so many tests had to be manually graded anyway. An approach to online testing that avoids these issues with partial credit is to ask only fundamental
questions that require only one or two steps and reserve the more in-depth, multi-part questions for written exams.

Some instructors have found ways around issues with grading in an online environment. For example, Cann [6] developed “extended multiple choice” questions with ten or more possible answers. This helped to avoid problems with grading rounded numerical answers and issues with students guessing the correct answer in typical multiple choice questions. However, new technologies and advancements in online testing systems that allow for mathematical grading have enabled instructors to ask the kind of questions that would have previously been reserved for hand-marked assignments or, in some cases, would not have been possible to pose before. For instance, Sangwin [20] utilized the online testing system AIM (Alice Interactive Mathematics) which gives the question designer access to the Maple mathematical engine to ask students very challenging, open ended questions, such as ‘Find a singular $5 \times 5$ matrix with no repeated entries’. However, some may consider this to be an abuse of the power that these testing systems give, as challenging multi-step questions should be graded by hand to award students points for the logic they use to solve such problems.

Bonham et al. [3] compared two groups of students, one submitting weekly written assignments and one completing weekly randomized online quizzes. The students were able to submit the online quizzes multiple times, leading to a higher average on the online quizzes than the written homework. There was no significant difference in exam score between the two groups. They also noted that even though the written
homework group had more experience submitting full written solutions, their exam papers did not have a higher level of detail compared to the online homework group (as measured by the number of words, equation signs, written variables and units given in the written portion of the exam).

Spradlin [23] performed a study at a small university that compared two sections of developmental mathematics students submitting daily written homework with two sections completing regular online quizzes using a tool that was provided with the course textbook. The online quizzes were mostly multiple choice and matching questions. The experiment performed was a non-randomized control group versus treatment pre-test post-test design. They found no significant difference in post-test scores after adjusting for pre-test scores in the two groups.

The above studies give some indication that online testing can adequately replace paper-based methods. Comparisons of performance between students completing homework by hand and online have not yielded significant differences. Current testing systems give instructors the ability to ask sophisticated mathematical problems and grade them accurately.

2.2.2 Analysis of student attitude towards online testing

Typically, student attitude towards online testing is measured via surveys or testimonials of students in the classroom. Reported results are very positive. For instance, Engelbrecht and Harding [11] combined weekly online quizzes and had online com-
ponents to the midterms and final exam. They found that 56% of students surveyed preferred the online assessment over paper assessment. Shafie and Janier [21] utilized an online testing package provided by the textbook and found that approximately 80% of students agreed that the online component of the course had a positive impact on their learning. The study by Angus and Watson [1], which is discussed in detail in the next section, resulted in over 90% of survey respondents agreeing that the online quizzes were a useful tool to help them study consistently throughout the semester.

Cassady et al. [7] suggested using formative testing to allow students to write in a stress-free environment in order to identify strengths and weaknesses. They found that, although changing the testing format from written to online showed no difference in test anxiety, the online tests had a lower perceived threat. They also found students with limited or no use of the formative online quizzes that were available before the written tests had a high level of test anxiety.

Instructors at Macquire University in Sydney, Australia, have developed the powerful randomized testing system MacQ\TeX, as reported by Griffin and Gudlaugsdottir [13][14]. Quizzes are generated as PDF documents via \LaTeX with javascript processing built in to handle randomization and grading of student responses. The main advantage of this system is that server communication is required only to initialize and submit quizzes and not continuously while the student is writing the quiz, as with most testing systems. A disadvantage of this system is that question authoring
requires a good deal of programming savvy, but a large bank of questions is available for instructors to choose from. The quizzes are designed to encourage students to review previous mathematical topics and test only technical ability, with questions requiring deeper understanding reserved for written tests. Griffin and Gudlaugsdottir chose to implement the quizzes using a pass/fail system, where passing was defined as committing no more than two errors. Students were allowed unlimited attempts at each quiz, and full solutions to all questions were displayed after the quiz was submitted. This was designed to encourage students to immediately try the quiz again if they did not pass. The classes had a very high pass rate on the quizzes. Although no attempt was made to measure whether the quizzes had a positive impact on student learning, survey data indicated that students had a positive attitude towards the testing system. More than 75% of students agreed that the quizzes helped them understand the concepts in the course, 70% believed the quizzes helped the students express their solutions in a clear and logical way, and 75% agreed that completing the quizzes helped them remember the processes for solving particular problems. Surprisingly, more than half of the class claimed they did quizzes again for more practice even after they had passed.

The above demonstrates that it is possible to have students recognize and embrace online testing as a means of better understanding course materials. Of course, this depends on the particular implementation of online testing, but it is a positive sign to see these successful outcomes.
2.2.3 Analysis of the use of formative online quizzes on student learning

Determining whether online testing has a positive outcome on student learning is very difficult. This is due to the many confounding factors involved in education, as well as the lack of ability to determine if the students are using the online quizzes as designed. Authors have compared performance on online quizzes to performance on written exams; for instance, Smith [22] found that final exam grades correlated higher to online assignment grades than to written assignments or laboratory report scores. This, however, is not conclusive in determining whether the online quizzes helped students learn the material, as perhaps the online quiz questions were more similar to the questions tested on the final exam.

Stillson and Alsup [25] used the online testing system ALEKS (Assessment and Learning in Knowledge Spaces), which is designed to identify student strengths and weaknesses and allow students to master all material in the course. They found that performance on the online material did not correlate highly to final exam grades. They also found time spent working in the system correlated highly with exam scores. Unfortunately, this study looked only at the correlation between time spent on the system and exams scores without considering other variables, such as those related to past performance of the students. This made it impossible to determine whether using ALEKS resulted in better learning or if stronger students were using ALEKS more.
McSweeney and Weiss [18] performed an experiment to determine the effect of the Math Online system in a first-year calculus course. The system was designed to allow students to review fundamentals and pre-calculus topics independently, as the instructors found these issues took up too much class time. The online assignments were randomly generated multiple choice questions. Multiple instructors each taught two sections of the course concurrently, one section using the system and one section without. The study included data from 12 different sections with 25 to 35 students in each section. They found the averages on the written tests were statistically significantly higher in the Math Online group. A multiple-choice pre-test was given to all students during the first week of classes and the same test was given to all students at the end of the semester. The difference in a student’s pre-test and post-test scores was interpreted as their algebraic improvement over the semester. The study showed evidence that the students using Math Online had a higher average pre-test to post-test improvement score. They also found that within the Math Online group, each online quiz a student completed resulted, on average, in an additional half point improvement score from pre-test to post-test. Additionally, they noted that to see these improvements, the instructor needed to be actively encouraging students to complete the quizzes, as there were no differences in the sections taught by a more passive instructor.

In the study most comparable to the present work, Angus and Watson [1] utilized retrospective regression methodology to test whether exposure to online formative
quizzes had a positive effect on student learning. They introduced four formative online quizzes (worth 2% each) to a first-year applied mathematics course. Data was collected from two sections of the course in two subsequent years, the first having approximately 400 students, the second approximately 1200. The quizzes were randomized and required the input of a single calculated number, with grading tolerances set to avoid grading issues with rounded numbers. Students were allowed two attempts at each quiz. Unlike many authors, they did not focus on student performance on the quizzes, just on the amount the student engaged with the quizzes. They attempted to account for various sources of variation, such as previous mathematical ability, in-course mastery of subjects and the level of effort put towards the course by including many explanatory variables. Previous mathematical ability was accounted for using variables indicating which high school mathematics courses the student had taken; in-course mastery was measured by midterm grades; effort towards the course was measured by attendance at voluntary peer-led study groups. Each student’s interaction with the online system was measured by whether or not the student attempted all four online quizzes. They found a significant positive impact on students completing all four quizzes, even after accounting for the explanatory variables mentioned. Completion of all quizzes was associated with 2.5% improvement on the final exam for given values of the explanatory variables. Limitations of this study are that they looked only at the number of quizzes completed, but not other indicators of how the student interacted with the online quizzes, such as when the student wrote the
quizzes or the number of attempts the student had on each quiz.

The current literature has shown much promise that formative online testing can have a positive impact on student learning. Care must be taken to control or account for the various sources of variation in studies, as it is very easy to confound the effects of the online instrument with other factors. The study discussed in the next chapter of this thesis uses lessons learned in the literature to attempt to investigate the relationship between student usage of online testing and final exam performance.
Chapter 3

Analysis of Data

3.1 Introduction of Technology Used in the Classroom

Beginning in the winter semester of 2006, there has been a wide range of technology used in two first-year calculus courses at the University of Guelph (Math*1200 and Math*1210). Through a joint research program with Maplesoft, a Waterloo-based mathematics software company, Professor Jack Weiner utilized Maplesoft products in the classroom and beyond. These first-year calculus classes included students from a wide variety of programs including Arts, Chemistry, Physics, Computer Science, Engineering and Mathematics. During this research period (2006-2011), typical class sizes ranged from 400 to 600 students. During the period of this study in both courses, students’ final grades were determined by three different types of assessments.

1. Three midterm tests, worth 15% each. These were closed-book and given during the teaching semester.
2. Numerous online Maple T.A. quizzes, worth 20% for the semester. These will be described in detail later.

3. A final examination, worth 35%, closed-book, given after the teaching semester.

Students were required to purchase a course manual, with the class notes 70-80% completed, the remaining as blanks to be filled in during lecture. These notes were also created as Maple documents and made available for download from the course website. The Maple class notes were fully executable, meaning that students could investigate each example during and after class to see what answer the software gave to a particular question. They could also change the parameters within questions and immediately see how the answer was affected. Maple was also used by the instructor in class to display plots and animations to the students.

The course had an associated website. The website gave students access to completed class notes, solutions to previous tests, announcements and grades achieved to date. The website also featured a discussion forum. Here students were able to post questions they had on the class material, which could be answered by the Professor, the Teaching Assistants or fellow classmates.

The final use of technology was the weekly online Maple T.A. quizzes. The quizzes were designed to be formative in nature, meaning they were designed to help students master the course material and discover which topics they needed to review, not to simply generate grades for the students. The small grade attached to the quizzes (2% per quiz) was intended to motivate students to complete the quizzes. The online
quizzes could be thought of as enforced homework, providing motivation to complete numerous practice problems on each topic. Each quiz typically had ten short questions, each focusing on one specific learning objective that was covered in lecture. A sample version of each quiz, with full solutions, was available in the course manual. Students were allowed an unlimited number of attempts at each quiz. A practice version of each quiz could be attempted anonymously by students at any time during the semester and did not count for grades. The practice version allowed students to check their answer to each question as they went along. They were thus able to focus on specific question types that were giving them difficulties. The homework version of each quiz was only available to students for one to two weeks, directly after the material needed for that quiz had been covered in lecture. Upon completing a quiz, a student received their grade as well as question-by-question feedback. The feedback included the student’s answer, the correct answer and, at the instructor’s discretion, comments on common errors or an explanation of the correct strategy. Sample feedback is shown in Figure 3.1.

Maple T.A. was chosen as the online quiz software for one very important reason: the access to Maplesoft’s mathematical engine. Many authors have noted that other online assessment tools are limited to multiple choice or fill in the blanks style of questions (for instance see Engelbrecht and Harding [12]). Maple T.A. questions can be designed to be algorithmically generated and mathematically graded. The algorithmic design of the questions and quizzes means that almost every quiz opened
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Figure 3.1: An example of feedback the user can see after a Maple T.A. quiz has been completed.

up is unique. It gives the instructor the power to randomly select quiz questions from the question bank and also randomize the numbers or functions within each question.

With high probability, each student receives a different quiz from their peers and with every attempt. This reduces problems of academic misconduct. It also gives students access to a very large set of sample problems. The software also allows the instructor to set up questions requiring students to enter information at each key step of the
solution. This not only more closely mimics how the question would be graded by hand, but it also allows students to see where they are making their mistakes. An example of this type of question can be found in Figure 3.2.

**Question 5: (1 points)**

Below is the graph of $y = x^2 + 3x$ and $y = x + 3$.

![Graph of $y = x^2 + 3x$ and $y = x + 3$.](image)

(a) Find the intersection points of $y = x^2 + 3x$ and $y = x + 3$, entering the point with the smaller $x$ value first.

(b) The integral which gives the area bounded by $y = x^2 + 3x$ and $y = x + 3$ is

(The format for entering an integral, in text, is $\int a f(x) \, dx \left[ \begin{array}{c} a \\ b \end{array} \right]$)

(c) Now find the area.

Figure 3.2: An example of a Maple T.A. quiz question, asking the user to enter information at some key steps of the problem.

As a guide on how to complete these online quizzes effectively, students receive a "T.A. Syntax Sheet" and a "T.A. Protocol Sheet" at the beginning of each semester. The syntax sheet shows students how to correctly enter common mathematical symbols and functions into the online testing system. The protocol sheet outlines a suggested strategy for completing these quizzes that should lead to mastery of the
material. Both the “T.A. Syntax Sheet” and the “T.A. Protocol Sheet” can be found in Appendix A, but I will outline the strategy suggested by the protocol sheet.

The suggested strategy is based on three steps. The first is to work out full solutions to a sample quiz that can be found in the course manual. The course manual also contains full solutions to this quiz. This step is designed to help the students understand the mathematics in these questions and allow them to recognize when they need to review the material in the notes before attempting the quiz. The second step is to do a practice version of the online quiz, checking their answer to each question as they complete the quiz. This step is designed to get the students comfortable with entering the answers into the online system, to avoid common syntax errors and to allow students to complete questions that they have not seen before. The students are asked to delay attempting a homework version of the quiz until they have achieved a perfect score on the practice version of the quiz. The final step is to then do a homework version of the quiz.

The aim of this study is to investigate the relationship between the use of technology and student test performance. Data has been collected every year since 2006; however, we will report on two specific semesters (Winter 2010 and Winter 2011). The results from other semesters were similar. We will look at correlations between certain variables, as well as several linear models. Student attitude toward the technology will be analyzed through the results of surveys administered each winter semester.
3.2 Analysis of Winter 2010 Data

In the Fall 2009 (F09) and Winter 2010 (W10) semesters, the first year calculus courses in question were not taught by the creator of the Maple content used in the class. The instructor of these courses did implement all the technology listed at the beginning of this chapter. The new instructor implemented one major change in W10: instead of having ten weekly online quizzes consisting of 8-12 questions each, there were 18 quizzes consisting of 2-4 questions each. The questions used on the quizzes were the same as in previous years.

We wish to investigate the effect of technology on student learning. This is difficult to assess, as there are many confounding factors involved. These factors include a student’s previous mathematical preparation, the student’s willingness to utilize the technology the way the instructor intends and overall student effort to the subject material. We will include several explanatory variables in our models to try to account for some of these factors.

We focus on the winter semesters for two reasons, the first being that students should be well acquainted with the technology used and more importantly, we can use the student’s grade from the fall semester F09 as a measure of student preparedness. One measure of student learning over the semester is their score on the final exam. We will use this score as our response variable.

The explanatory variables we will investigate are the student’s overall grades on the Maple T.A. quizzes, the combined grade of all three midterms, the student’s
grades in the F09 calculus course, the average number of attempts on each online quiz and the total number of the online quizzes the student attempted. All grade data has been scaled to be out of 100 for interpretability. A total of 484 students wrote the final exam in the winter 2010 semester; of these, 423 completed Calculus I in the F09 semester.

We begin with correlations between our variables of interest, which are found in Table 3.1. We see a very strong relationship between the final exam grade and midterm grades. This is expected as both are closed-book, hand-written and hand-marked tests, making them the most similar of our variables. We see that there is some relationship between the T.A. scores and the grades on the final exam and midterms. This seems to indicate that on average a better score on the online quizzes is associated with a better score on the written tests. There is also a relationship between the number of quizzes completed and the final exam score. This may be evidence of the success of the formative nature of the quizzes; students who complete all quizzes tend to do better on the final exam. However, at this point, we cannot discern whether the quizzes are helping students learn or if the stronger students are completing all quizzes. Finally, we see a weak negative correlation between final exam scores and the average number of attempts on each quiz. This indicates that students with fewer attempts at each homework quiz tend to do better on the final exam. We will now try gain an understanding of the nature of these effects through linear regression techniques.
Variable Pair                           Correlation  p-value
Final exam with midterms              0.82        < 0.001
Final exam with T.A. grade            0.51        < 0.001
Final exam with average attempts/T.A. quiz -0.23 < 0.001
Final exam with number of T.A. quizzes 0.48        < 0.001
Midterms with T.A. grade              0.56        < 0.001

Table 3.1: A table of various correlations between pairs of variables for the W10 semester.

3.2.1 Model 1: Analysis of number of quizzes completed

The first model we will use was adapted from Angus and Watson [1]. We look to explain student learning through only the usage of the formative online quizzes, not the student’s performance on these quizzes. The model accounts for the student’s previous mathematical preparedness using the grade from the previous semester, and the student’s in-course mastery of the topics in the course using the midterm grades. We measure each student’s usage of the online quizzes by the number of the 18 quizzes the student completed. The main advantage of this model over the model used by Angus and Watson is that we can measure previous student preparedness much more precisely with the grade from the fall semester F09 calculus course. The model used by Angus and Watson did this through the use of dummy variables that indicated which mathematics courses the students took in high school, but not individual performance in these courses. The model we will use to analyze the data is as follows,

\[ FE_i = \alpha_0 + \alpha_1 MT_i + \alpha_2 FALL_i + \alpha_3 LOW_i + \alpha_4 HIGH_i + \varepsilon_i \quad (3.1) \]
where the response variable is the final exam grade, denoted as FE. The midterm grade is denoted MT, the grade from the fall course is labeled FALL. The variables of interest are LOW and HIGH. The variable LOW is assigned a value of 1 if the student completed 9 or less of the 18 online quizzes during the semester, 0 otherwise. HIGH is assigned a value of 1 if the students completed 16 or more of the online quizzes, 0 otherwise.

These variables group students into three groups: low use, moderate use and high use of the online quizzes. There were 29 students with low usage, 52 with moderate usage and 349 with high usage. Figure 3.3 gives some indication that students who wrote a higher number of quizzes tended to perform better on the final exam. The goal of this model is to assess whether simply using the online quiz system is associated with a higher final exam grade.

Figure 3.3: Box plot of final exam scores vs. number of quizzes completed.
This model was run using ordinary least squares regression techniques. Issues with heteroscedasticity were observed (see Figure 3.4), as the variance appears to be smaller for larger fitted values. Common transformations, such as the logit, were considered and carried out, but these did not result in an improved fit. Interpretability of the parameters on the original scale of measurement was desired, so the decision was made to run models on the untransformed data. In the inference procedures, heteroscedasticity was accounted for by using robust standard errors based on the heteroscedastic consistent covariance matrix (HCCM) (White [27]). Based on the recommendation of Long [17], the HC3 variation of the HCCM was chosen. Computations were carried out in the statistical package R, using the package car. The results given by this model can be found in Table 3.2.

Figure 3.4: Studentized Residuals versus fitted values for Model 1 in the W10 semester, using ordinary least squares regression techniques.
| Variable | Estimated Coefficient | Std. Error | t-value | $Pr(>|t|)$ |
|----------|-----------------------|------------|---------|------------|
| HIGH     | -0.53                 | 1.42       | -0.37   | 0.71       |
| LOW      | -4.87                 | 3.86       | -1.26   | 0.21       |
| FALL     | 0.51                  | 0.09       | 6.07    | < 0.001    |
| MT       | 0.70                  | 0.06       | 11.22   | < 0.001    |
| (Intercept) | -23.56              | 4.36       | -5.41   | < 0.001    |

$R^2 = 0.7216$

Table 3.2: Results from Model 1 for the W10 semester (Equation (3.1)).

We note from Table 3.2 that most of our explanatory variables are behaving as expected. A higher grade on the midterms or in the previous course is associated with a higher exam score. There is no significant relationship between students with high and moderate usage or between low and moderate usage after accounting for the fall grade and midterm scores. The comparison between low and high usage also yielded no significance after accounting for fall and midterm grades (estimated coefficient for HIGH 0.50, $p$-value 0.79).

### 3.2.2 Model 2: Analysis of performance on Maple T.A. quizzes

Here the model of interest involves student performance on the online quizzes. We wish to see if there is a relationship between scores on the Maple T.A. quizzes and final exam grades, after accounting for the midterm scores and final grade from the F09 semester. We introduce the following linear model,

$$FE_i = \gamma_0 + \gamma_1 TA_i + \gamma_2 MT_i + \gamma_3 FALL_i + \varepsilon_i.$$  

(3.2)
We continue to use final exam score as our response variable. We account for student mathematical preparedness using the student’s grade from the F09 semester and midterm grades as we did in Model 1. The variable of interest is each student’s combined grade on all Maple T.A. quizzes from the semester (denoted as TA). The results provided by this model are available in Table 3.3.

| Variable | Estimated Coefficient | Std. Error | t-value | Pr(>|t|) |
|----------|-----------------------|------------|---------|---------|
| TA       | 0.01                  | 0.04       | 0.28    | 0.78    |
| MT       | 0.71                  | 0.06       | 11.20   | <0.001  |
| FALL     | 0.51                  | 0.09       | 5.95    | <0.001  |
| (Intercept) | -25.67            | 4.15       | -6.18   | <0.001  |

\[ R^2 = 0.7189 \]

Table 3.3: Results from Model 2 for the W10 semester (Equation (3.2)).

We see that the estimated coefficient for T.A. score is positive, but not significantly different from zero. The fact that these scores are not significant after accounting for the written midterm grade and the grade from the previous course is not surprising. The students have unlimited attempts and access to their class notes while completing the Maple T.A. quizzes, making these grades not an accurate representation of how the student will perform on a closed-book final exam. The midterm tests are very similar to the final exam in both the way they are administered and question style, which is one reason why the midterm grades are the best predictor of final exam grades.

If we run the linear model with just the T.A. and fall semester grades as explanatory variables, we do find a significant positive relationship between Maple T.A. and
final exam grades (estimated coefficient 0.17, \( p \)-value < 0.001). There is significant evidence of a relationship between T.A. grades and final exam grades, after adjusting for the fall semester grades. If midterm grades are also included in the model, the T.A. effect is no longer significant.

### 3.2.3 Model 3: Analysis of number of attempts per quiz

Students were allowed an unlimited number of attempts at each online homework quiz; the class averaged 4.17 attempts on each quiz (see Figure 3.5). Our main research question that this model will address is, “What is the relationship between taking extra attempts to achieve the same grade on the online tests and final exam performance?”. The plot of average attempts versus final exam score (see Figure 3.6) seems to indicate a negative relationship. To address this question, we introduce a linear model,

\[
FE_i = \beta_0 + \beta_1 TA_i + \beta_2 AvgAttempts_i + \epsilon_i. \tag{3.3}
\]

The response variable is the final exam score. Our explanatory variables are TA (each student’s grade on the Maple T.A. quizzes) and the average number of attempts per quiz completed (denoted as AvgAttempts). The point of interest for this model is estimating the relationship between the number of attempts per quiz and final exam grade, after accounting for the student’s score on the online quizzes.

The results provided by this model can be found in Table 3.4. We see the coefficient for our variable TA is positive and significantly different from zero, indicating that a
Figure 3.5: Histogram of average attempts per Maple T.A. quiz.

Figure 3.6: Scatter plot of average attempts per online quiz and final exam grade
Table 3.4: Results from Model 3 for the W10 semester (Equation (3.3)).

| Variable       | Estimated Coefficient | Std. Error | t-value | Pr(>|t|) |
|----------------|-----------------------|------------|---------|---------|
| TA             | 0.50                  | 0.04       | 13.15   | < 0.001 |
| AvgAttempts    | -4.06                 | 0.53       | -7.72   | < 0.001 |
| (Intercept)    | 39.75                 | 3.50       | 11.34   | < 0.001 |

$$R^2 = 0.3548$$

high grade on the online quizzes is related to a higher grade on the final. We also see our variable of interest, AvgAttempts, has a negative coefficient that is significantly different from zero. This indicates that each extra attempt per quiz students took to achieve the same score on the online quizzes was associated with a 4% decrease in final exam score. The 95% confidence interval for the size of this effect is -5.10% to -3.03%. We offer two possible explanations for the sign and magnitude of this coefficient. The first explanation is that the stronger students in the class will achieve a high grade on the T.A. quizzes in a lower number of attempts, and the second is that students following the “T.A. Protocol” outlined at the beginning of this chapter may both lower their number of attempts required and improve their understanding. Based on the success of students who are known to be following the protocol, it is the belief of the author that following this protocol would result in mastery of the course material. These speculations cannot be tested formally as this would require each student’s work habits to be monitored in order to determine which strategies they are implementing.
3.2.4 Results of the attitudinal survey

Students were asked to complete a survey near the end of the Winter 2010 semester to measure student attitude towards the technology in the classroom. There were 324 respondents out of 483 registered students. A full version of this survey is available in Appendix B of this thesis. The responses to the Maple T.A. quizzes were very positive. Perhaps the most telling statistic from the survey is that 90% of students agreed with the statement “The T.A. quizzes helped me learn” (50% strongly agree, 40% agree, 7% no opinion, 3% disagree, 0% strongly disagree). This shows that the students see the value in the learning tool, which is a big step in getting students to use it properly. As for the rest of the technology used in the classroom, 83% of students believed that “Overall, I benefited from the inclusion of technology in the course” (34% strongly agree, 49% agree, 11% no opinion, 5% disagree, 1% strongly disagree).

The survey also shows the importance of making these quizzes count towards the student’s final grade, since 56% of the students admitted that if the quizzes did not count for grades, they would not make time to write them. This shows that even though the quizzes are designed to be formative, it will be difficult to get students to complete them if they are not treated as a summative assessment.

The survey also highlighted the difficulty of getting students to use new technology on their own. Even though 62% of the respondents agreed that the in-class Maple demonstrations helped them learn, only 18% of the class said that they used the Maple
version of the course notes, modifying them as needed, for their own investigations after class. This is a frustrating aspect of using technology in the classroom; the students know the power of the software and are provided with the opportunity to use it on their own, but are resistant to adopting it for their own independent use.

### 3.3 Analysis of Winter 2011 Data

In the Fall 2010 (F10) and Winter 2011 (W11) semesters, Math*1200 and Math*1210 were taught by the creator of the Maple content used in the classroom, Professor Jack Weiner. We will again focus our analysis on the winter semester. In the W11 semester, 562 students wrote the final exam; of these students 490 took Math*1200 in the previous fall semester. There were a total of ten online quizzes during the semester. There was one major change in the use of the online quizzes during this school year; students were allowed a maximum of five attempts at each quiz. Students still had unlimited access to the practice versions of the quizzes. The limited number of attempts was to encourage students to treat the quizzes in a serious manner and to follow the “T.A. Protocol” outlined earlier in this chapter.

Our analysis will follow a similar path as with the W10 semester; however, the availability of new data, such as the number of days before the due date students start the quiz, allows us to explore new research questions. We begin by observing correlations between variables of interest in Table 3.5.

We see very similar correlations between final exam grade and midterm grade
Variable Pair | Correlation | p-value
--- | --- | ---
Final exam with midterms | 0.77 | < 0.001
Final exam with T.A. grade | 0.54 | < 0.001
Final exam with average attempts/T.A. quiz | 0.04 | 0.342
Final exam with number of T.A. quizzes | 0.34 | < 0.001
Final exam with average number of days before due date quiz is started | 0.43 | < 0.001
Final exam with proportion of quizzes started on due date | -0.40 | < 0.001
Midterms with T.A. grade | 0.57 | < 0.001

Table 3.5: A table of various correlations between pairs of variables for the W11 semester.

and final exam with online quiz grade as we did in the Winter 2010 semester. This indicates that better performance on the midterms or online quizzes is associated with better performance on the final exam. The relationship between midterm grades and final grades is again stronger than that of the relationship between final exam and T.A. grades. We also see a positive relationship between the number of the ten online quizzes the students completed and the final exam grade.

With the limit of five attempts per quiz, we see no relationship between final exam grade and average number of attempts per quiz (correlation 0.04, compared to -0.23 in W10 when attempts were unlimited). We see a positive relationship between final exam grade and the average number of days before the due date a student first attempts an online quiz. This indicates that starting quizzes early is correlated with higher final exam scores. There is a negative relationship between the number of quizzes not attempted prior to the due date and final exam. We will investigate these effects further using linear regression techniques.
3.3.1 Model 1: Analysis of number of quizzes completed

We wish to analyze the effect of the number of quizzes completed using Model 1 in (3.1) as before. Here we define the variable HIGH to have a value of 1 if the student completed nine or more of the ten online quizzes and 0 otherwise and LOW to have a value of 1 if the student completed five or less of the ten online quizzes, 0 otherwise. These values were chosen to group students into three groups, low usage, moderate usage and high usage of the online quizzes. There were 19 students with low usage, 53 with moderate usage and 490 with high usage of the online quizzes. The results provided by this model can be found in Table 3.6.

| Variable | Estimated Coefficient | Std. Error | t-value | Pr(>|t|) |
|----------|------------------------|------------|---------|---------|
| HIGH     | 4.05                   | 1.91       | 2.12    | 0.03    |
| LOW      | 3.13                   | 4.52       | 0.69    | 0.49    |
| FALL     | 0.36                   | 0.06       | 5.68    | < 0.001 |
| MT       | 0.61                   | 0.05       | 12.94   | < 0.001 |
| (Intercept) | -8.62               | 3.51       | -2.45   | 0.01    |

$R^2 = 0.6476$

Table 3.6: Results from Model 1 for the W11 semester (Equation (3.1)).

We find that the effects of the midterm grades and grades from the F10 calculus course are as expected. We have a positive and significant impact of the online quizzes. Students who completed a high number of the quizzes performed on average 4% higher (95% confidence interval 0.30% to 7.81%) on the final exam for given midterm and F10 grades. The coefficient for LOW is positive, but it is not significantly different than zero, giving no evidence of a difference between students who completed a low versus
a moderate number of quizzes after accounting for midterm and F10 performance.

3.3.2 Model 2: Analysis of performance on Maple T.A. quizzes

We repeat the analysis of performance on the T.A. quizzes from W10 on the W11 data using Model 2 in (3.2). The results can be found in Table 3.7.

| Variable | Estimated Coefficient | Std. Error | t-value | Pr(>|t|) |
|----------|----------------------|------------|---------|---------|
| TA       | 0.10                 | 0.04       | 2.38    | 0.002   |
| MT       | 0.58                 | 0.05       | 12.45   | < 0.001 |
| FALL     | 0.32                 | 0.06       | 5.01    | < 0.001 |
| (Intercept) | -8.49               | 3.49       | -2.43   | 0.02    |

\[ R^2 = 0.6504 \]

Table 3.7: Results from Model 2 for the W11 semester (Equation (3.2)).

We find a small but significant positive effect of the grade achieved on the Maple T.A. quizzes even after accounting for midterm and previous course grades. We did not see this effect in the W10 semester. The magnitude of the effect may be small, but it does suggest that even after accounting for midterm grades and previous mathematics preparedness, better performance on the online quizzes is associated with better performance on the final exam.

If we run the same model omitting the midterm grade as an explanatory variable, we find that the Maple T.A. grades have a significant and positive relationship with final exam grades (estimated coefficient 0.20, p-value < 0.001).
3.3.3 Model 3: Analysis of number of attempts per quiz

In the F10 and W11 semesters, students were limited to five attempts per online homework quiz. In the W11 semester, students averaged 2.16 attempts per quiz, compared to 4.17 in W10 when attempts were unlimited. We analyze the effect of these attempts using Model 3 in (3.3). These results can be found in Table 3.8.

| Variable       | Estimated Coefficient | Std. Error | t-value | Pr(>|t|) |
|----------------|-----------------------|------------|---------|---------|
| TA             | 0.56                  | 0.04       | 15.23   | < 0.001 |
| AvgAttempts    | -4.01                 | 1.06       | -3.77   | < 0.001 |
| (Intercept)    | 31.71                 | 3.67       | 8.64    | < 0.001 |

\[ R^2 = 0.3057 \]

Table 3.8: Results from Model 3 for the W11 semester (Equation (3.3)).

We observe very similar effects to the W10 semester, despite the restriction on the number of attempts per quiz. Each extra attempt taken per quiz to achieve the same grade on the T.A. quizzes is associated with a decrease in final exam grade of 4%. The 95% confidence interval for the size of this effect is -6.10% to -1.92%. This relationship is similar to that seen in the W10 semester, despite the fact that the number of attempts allowed per quiz was restricted.
3.3.4 Models 4 and 5: Analysis of date of first attempt of quiz

Typically the online quizzes were available for 14-16 days; despite this, many students left the homework quiz to the last day. For each quiz there was an average of 182 students that made their first attempt on the homework quiz the day it was due. We wish to find what effect starting the quizzes early has and whether there is a negative effect of waiting until the last day. We will investigate this through two models, the first being Model 4 given below:

\[ FE_i = \kappa_0 + \kappa_1 TA_i + \kappa_2 \text{AvgAttempts}_i + \kappa_3 \text{AvgDays}_i + \varepsilon_i, \]  

(3.4)

where \( TA \) denotes the student’s grade on the online quizzes and \( \text{AvgAttempts} \) denotes the average number of attempts per T.A. quiz as before. The variable \( \text{AvgDays} \) represents the average number of days before the due date the student first attempts the online quiz. The results provided by this model can be found in Table 3.9.

| Variable       | Estimated Coefficient | Std. Error | \( t \)-value | \( Pr(> |t|) \) |
|----------------|-----------------------|------------|---------------|----------------|
| AvgDays        | 1.59                  | 0.21       | 7.56          | < 0.001        |
| TA             | 0.47                  | 0.04       | 10.62         | < 0.001        |
| AvgAttempts    | -4.51                 | 1.01       | -4.45         | < 0.001        |
| (Intercept)    | 34.06                 | 3.58       | 9.51          | < 0.001        |

\( R^2 = 0.3703 \)

Table 3.9: Results from Model 4 for the W11 semester (Equation (3.4)).

We see that starting the online quizzes earlier is associated with higher final exam grades. Given a student’s T.A. grade and average attempts on each quiz, for each
Figure 3.7: Histogram of the average number of days before the due date that students first attempted the online quiz.

day before the due date the student first attempts the quiz on average is associated with a 1.6% increase on the final exam. The 95% confidence interval for the size of this effect is 1.18% to 2.01%.

We also wish to investigate the effects of leaving a quiz to the due date. We do this with Model 5 given below:

\[ FE_i = \eta_0 + \eta_1 TA_i + \eta_2 AvgAttempts_i + \eta_3 pLast_i + \varepsilon_i, \quad (3.5) \]

where the explanatory variables TA and AvgAttempts are as before, pLast represents the proportion of quizzes the student did not attempt until the due date. The results provided by this model are given in Table 3.10.
| Variable    | Estimated Coefficient | Std. Error | t-value | Pr(>|t|) |
|-------------|-----------------------|------------|---------|---------|
| pLast       | -15.42                | 2.57       | -6.02   | < 0.001 |
| TA          | 0.44                  | 0.05       | 9.40    | < 0.001 |
| AvgAttempts | -4.34                 | 1.02       | -4.23   | < 0.001 |
| Intercept   | 47.35                 | 4.69       | 10.10   | < 0.001 |

$R^2 = 0.3572$

Table 3.10: Results from Model 5 for the W11 semester (Equation (3.5)).

We observe that leaving quizzes until the last day has a negative impact on final exam performance. For a fixed T.A. grade and average number of attempts per quiz, each quiz not attempted until the due date is associated with a 1.5% decrease on the final exam (assuming the student completed all ten quizzes). The 95% confidence interval for the size of this effect is -2.05% to -1.04%.

### 3.3.5 Results of the attitudinal survey

The W11 survey had 368 respondents. The survey was slightly modified from the previous year. We again see a very positive reaction to the Maple T.A. quizzes. 90% of students agreed that the T.A. quizzes helped them learn (57% strongly agree, 33% agree). 46% of respondents claimed that they followed the “T.A. Protocol” outlined in Section 3.1 (completing the sample quiz in the course manual, followed by practice quizzes online, then homework quizzes); 37% stated that they completed practice quizzes online before attempting homework quizzes but did not try the sample quiz in the course manual, while 17% claimed they did only homework versions of the
online quizzes. A total of 223 students estimated their grade in the course to be 80% or higher (at the time of the survey, the students only had the 35% final exam to write, knowing all other grades); of these students, 120 followed the “T.A. Protocol” and only 28 did homework versions of the quizzes exclusively.

3.4 Conclusion and Discussion

All of the above linear models were run again using a logistic transformation, and this did not improve the fit or change the conclusions in any meaningful way. We also ran the models investigating the number of attempts and starting dates of quizzes including the grade from the fall course as a proxy for student mathematical preparedness, which did not change the effects in any significant way. Noting a possible introduction of bias using the average attempts and average starting date only on quizzes the student actually completed, we also ran these models for the W11 semester using only the 414 students who completed all 10 quizzes; this did not change the results in any notable fashion.

We also investigated but did not report on the relationship between the T.A. quizzes and the midterm grades. We ran simple models using each midterm grade as the response variable with the explanatory variables being the student’s grade in the fall course and the student’s grade on the T.A. quizzes taken before that midterm test. These models all showed a significant and positive relationship between the online quiz performance and midterm score (estimated coefficients on T.A. scores
between 0.1 and 0.2, p-value < 0.001).

A final point of discussion is the treatment of missing data. Students who dropped the course during the semester and did not write the final were omitted; to our knowledge, no student dropped the course because of the technology used in the course. Students were allowed to count only two of the three midterms and have a more heavily weighted final. The midterm grade for any student who chose to only write two midterms was taken to be the average grade on those two tests, ignoring the missed test. Two students in W10 and one student in W11 did not complete any online quizzes, these students were omitted from the models including the average attempts per quiz. Finally, only students who took Math*1200 in the fall semester previous to the semester in question were included in any models using the fall grade as an explanatory variable; as mentioned before in both semesters discussed, approximately 88% of the class completed Math*1200 in the previous fall semester.

We have seen evidence of a significant and positive relationship between the use of online Maple T.A. quizzes and student performance. We observed even after accounting for in course mastery of the material and previous mathematical preparedness that students who complete more of the online quizzes tend to perform better on the final exam. We have also seen a positive impact of taking a fewer number of attempts to achieve the same score on the online quizzes. We found that although performance on T.A. is related to performance on the final exam, it is not as strong of an indicator as midterm grades or grades from the previous course. Finally we found students who
attempted quizzes before the due date performed better on the exam.

We have also seen that student attitude towards the technology is positive, with most students believing that the quizzes help them learn. We see that although the quizzes are intended to be a formative assessment, it is important that there is a small grade attached to each quiz to motivate students to complete them.

The take-home message of this analysis is that students who use the online quizzes the way they are intended (i.e., start quizzes early, reduce the number of attempts required per quiz, complete each weekly quiz) tend to perform better on the final exam.
Chapter 4

Presentation of Material Developed for First-Year Calculus Topics

This chapter presents interactive class notes and various demonstrations designed for use in a first-year calculus course. For more in-depth information on the topics, see Stewart [24]. The class materials are all developed in Maple and available on the CD provided. A sample class from the interactive notes is available in Appendix C. Printed code for one demonstration is available in Appendix D.

The interactive notes were developed for a first semester calculus course at the University of Guelph. The notes were designed to be used during 34 one-hour classes. The notes comprised the body of the course manual students were required to purchase. The notes were 60-80% complete with the remaining to be filled in by students in class. The notes were fully executable, meaning that every problem could be answered using the software. This could be used by the instructor as an opportunity to discuss why the answer found on paper may differ from the answer given by the
software, and why both are correct. The instructor may also change the information within a question and display to the class how the answer is affected instantaneously.

The notes also included commands that would display plots or animations of the functions being used in class. The students were given the option to download these files and use them to explore the examples outside of class time. These notes were first implemented in the Fall 2010 semester, in a class of approximately 600 students. The remainder of this chapter is dedicated to the presentation of computer demonstrations designed for a first-year calculus class.

During the design phase of the demonstrations, two requirements were kept in mind. The first was that the tools should give a geometric representation of each topic to allow students a better understanding of the material and the second was that the programs must be easy to use. If the second goal is not met, students may be intimidated by the program and this would discourage them from exploring the examples on their own. Each topic covered by the demonstration will be introduced, followed by a description of the tool and finally a discussion on how this may be adapted to teach in and out of the classroom. It is important to note that the discussions in this chapter only give some suggestions on the use of these programs and that if instructors choose to utilize these tools, they will likely find other uses that better fit their own teaching style and course to maximize the potential benefit. This chapter will give insight into the uses and benefits of computer-aided instruction (CAI) in a mathematics classroom.
4.1 The Formal Definition of a Limit

4.1.1 Description of the Lesson

The formal mathematical definition of a limit is typically presented after students have experience calculating limits. This topic is notoriously difficult for first-year students to grasp. There are two common issues that lead to trouble understanding this topic. The first is the fact that there are various forms of the definition depending on if the point of interest is finite or whether the limiting value tends to infinity. The second is that the definitions are dense with mathematical notation making it difficult to find a geometric understanding. We now introduce some of these limit definitions.

Definition 4.1 Assume that $f(x)$ is a real-valued function defined in some open neighbourhood of a real number $a$. The limit of the function $f(x)$ as $x$ approaches $a$ is the finite number $L$, written $\lim_{x \to a} f(x) = L$, if $\forall \, \varepsilon > 0 \, \exists \, \delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Definition 4.2 Assume that $f(x)$ is a real-valued function defined on some open interval $(a, b)$, $b > a$. The one-sided limit of the function $f(x)$ as $x$ approaches $a$ from the right is the finite number $L$, written $\lim_{x \to a^+} f(x) = L$, if $\forall \, \varepsilon > 0 \, \exists \, \delta > 0$ such that if $0 < x - a < \delta$ then $|f(x) - L| < \varepsilon$. 
Definition 4.3 Assume that \( f(x) \) is a real-valued function defined on some open interval \((a, \infty)\). The limit of the function \( f(x) \) as \( x \) approaches infinity is the finite number \( L \), written \( \lim_{x \to \infty} f(x) = L \), if \( \forall \, \varepsilon > 0 \, \exists \, N > 0 \) such that if \( x > N \) then \( |f(x) - L| < \varepsilon \).

Definition 4.4 Assume that \( f(x) \) is a real-valued function defined in some open neighbourhood of a real number \( a \). The limit of the function \( f(x) \) as \( x \) approaches \( a \) is positive infinity, written \( \lim_{x \to a} f(x) = \infty \), if \( \forall \, N > 0 \, \exists \, \delta > 0 \) such that if \( 0 < |x - a| < \delta \) then \( f(x) > N \).

These definitions can be easily modified to include one-sided limits from the left and limits involving negative infinity. Typical problems students are asked on this topic involve finding the maximum \( \delta \) value (or minimum \( N \) value) for a given limit or to illustrate the \( \delta \) value on a graph for a given \( \varepsilon \). The following section introduces demonstrations that attempt to bring these definitions to life.

4.1.2 Introduction of the Learning Tool

Three separate Maple procedures were developed to illustrate the formal definition of a limit: one for finite limits, one for when the limit is taken at infinity and one for when the limit approaches infinity.

The procedure for finite limits, defined in Definition 4.1, is called FormalLimit. The user must enter a function, the point at which the limit is taken and a value for
ε. The program then calculates the limit of the function at the given point and the maximum δ value for the given ε. Using this information, an animation is displayed showing the function, horizontal bars indicating the range |y − L| < ε, and vertical bars outlining the range |x − a| < δ. The animation shows δ increasing from zero to the maximum δ value. A series of frames from a typical animation can be found in Figure 4.1. The program also shows small dashed horizontal lines indicating the bounds of where the function values lie, showing that smaller values of δ often result in the function being closer than ε to the limit value. This procedure can also be used to animate one-sided limits, defined in Definition 4.2.

![Figure 4.1: A series of frames from the FormalLimit animation for \( \lim_{x \to \frac{1}{3}} (x^3 - x) = -\frac{8}{27} \) for a given ε = 0.3.](image)

The procedure for limits taken at infinity, defined in Definition 4.3, is called **FormalLimitAtInfinity**. The user enters a function, specifies positive or negative infinity and a value for ε. The program then calculates the limit specified and the minimum N value for the given ε. An animation displaying the function, horizontal
lines showing the range \( |y - L| < \varepsilon \) and a vertical line showing the minimum \( N \) value. A second vertical line moves from the minimum \( N \) to infinity showing that for any values larger than the minimum \( N \), the function is still within \( \varepsilon \) of the limit value (in the case that the limit is taken as \( x \to +\infty \)). A series of frames from an example animation can be found in Figure 4.2.

Figure 4.2: A series of frames from the FormalLimitAtInfinity animation for \( \lim_{x \to \infty} \frac{6x + 2}{3x^2} = 0 \) for a given \( \varepsilon = 0.1 \).

The final procedure discussed in this section illustrates limits that tend to infinity, defined in Definition 4.4, and is called FormalLimitToInfinity. The user must enter a function, a point to take the limit and a value for \( N \). The procedure then calculates the limit (which must be positive or negative infinity) and maximum \( \delta \) value for the given \( N \). An animation showing the function, a horizontal line showing the given \( N \) and vertical bars showing the current \( \delta \) range is displayed. The animation shows \( \delta \) starting at zero and increasing until the maximum \( \delta \) value is reached. The students will see that for any \( \delta \) smaller than the maximum \( \delta \), the function is always above the
Figure 4.3: A series of frames from the FormalLimitToInfinity animation for $\lim_{x\to1} \frac{1}{(x-1)^2} = \infty$ for a given $N = 100$.

4.1.3 Discussion and Suggested Use

These tools have great potential to give students a geometric understanding of the formal definition of a limit. In order to stress that the definition of a limit changes when infinity is involved, the choice was made to create separate programs to illustrate the different limits. It would have been possible to have the user enter a function and a point and have the program decide which definition needed to be used, but we wanted the students to make this decision if they are using the programs on their own.

The animation to demonstrate limits where $x$ tends to infinity is qualitatively different from the other demonstrations. The limits taken at a finite point show the smaller $\delta$ values first, then stop at the optimal $\delta$ value. The animation for limits where
$x$ tends to infinity shows the optimal (minimum) $N$ value and then shows larger $N$ values. Going from the optimal to larger $N$ seemed to better illustrate a limit at infinity.

It is important to not let these demonstrations stand on their own, but rather as a tool to bring life to the in-class explanations. For instance, these animations can also lead to conversations with the class as to why a particular definition works to describe a certain limit. For instance consider a limit taken at infinity, as defined in Definition 4.3 and illustrated in Figure 4.2. The instructor may focus on just the first frame and point out that as long as $x$ is to the right of the vertical blue line, the function is always within $\varepsilon$ of the limit value. The instructor could then stress that this is exactly what “if $x > N$ then $|f(x) - L| < \varepsilon$” means. Similar observations could help reinforce why other limit definitions are used.

A typical problem students may be asked to complete, along with a common error students may make, is presented below in Figure 4.4. The student has outlined the largest possible interval around $x = a$ that satisfies the condition $|f(x) - L| < \varepsilon$. The issue with this solution is obvious: the region the student has outlined is not symmetric about the point $x = a$, but any region of the form $|x - a| < \delta$ must be. The FormalLimit procedure may be used to help students understand this particular error. The student will see that as the $\delta$ value increases, the interval $|x - a| < \delta$ grows, but remains centered at $x = a$. The student will also observe that once the function crosses one of the horizontal lines representing $L + \varepsilon$ or $L - \varepsilon$, both sets of
vertical bars representing $a + \delta$ and $a - \delta$ stop.

**Question:** On the graph of the function $f(x)$ below, illustrate the region for the largest possible $\delta$ such that if $|x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

![Graph](image)

Figure 4.4: A common sample problem in a first-year formal limit definition unit, along with a common error students may make.

### 4.2 Rolle’s Theorem, the Mean Value Theorem and the Intermediate Value Theorem

#### 4.2.1 Description of the Lesson

Three theorems commonly introduced in a first-year calculus course are Rolle’s theorem, the mean value theorem and the intermediate value theorem. These theorems are all very important and are not only necessary to prove theorems later in a first-year course (such as the fundamental theorem of calculus), they will be utilized in many upper-year mathematics courses. All three theorems have a geometric inter-
pretation which the programs presented in this chapter are designed to demonstrate.

We begin by stating the theorems in question.

**Theorem 4.1 : Rolle’s Theorem**

Let $f : [a, b] \to \mathbb{R}$. If

1. $f$ is continuous on the closed interval $[a, b]$
2. $f$ is differentiable on the open interval $(a, b)$
3. $f(a) = f(b)$

then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

**Theorem 4.2 : The Mean Value Theorem**

Let $f : [a, b] \to \mathbb{R}$. If

1. $f$ is continuous on the closed interval $[a, b]$
2. $f$ is differentiable on the open interval $(a, b)$

then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

**Theorem 4.3 : The Intermediate Value Theorem**

Let $f : [a, b] \to \mathbb{R}$. If $f$ is continuous on the closed interval $[a, b]$ and $u$ is some number between $f(a)$ and $f(b)$, then there exists some $c \in (a, b)$ such that $f(c) = u$.

It is easy to see that Rolle’s theorem is a special case of the mean value theorem.

We now introduce the tools that have been developed to help explain these theorems.
4.2.2 Introduction of the Learning Tool

The tool to demonstrate the mean value theorem (Theorem 4.2) is called MVT. The code for this demonstration is available in Appendix D. The user must enter a function $f$ and a range $[a,b]$. The program checks if the function satisfies the hypotheses of the theorem. If it does not, an error message is displayed to the user explaining which conditions were not satisfied. If the theorem does apply, all values of $c \in (a,b)$ for which $f'(c) = \frac{f(b) - f(a)}{b-a}$ are found. The conclusion of this theorem states that the slope of the secant line joining $(a,f(a))$ and $(b,f(b))$ equals the slope of the tangent to $y = f(x)$ at some $c \in (a,b)$. An animation showing the function and the secant line joining the points $(a,f(a))$ and $(b,f(b))$ is displayed. The animation shows the tangent line as $x$ moves from $a$ to $b$. When $x$ reaches a $c$ value for which the conclusion of the theorem holds, the tangent line changes colour to match the secant line and the animation holds for 10 frames (with default settings this would take 1 second, but the frame rate can be adjusted). After the pause, the tangent line corresponding to the $c$ value remains, and a second tangent line begins to move across the function. The function will show all values of $c$ which satisfy the theorem. Several frames from an example animation are shown in Figure 4.5.

The program to demonstrate Rolle’s theorem (Theorem 4.1) is called RollesTheorem. As this theorem is a special case of the mean value theorem, the demonstration is similar. The only difference is the program now confirms that $f(a) = f(b)$ and displays an error message to the user if this is not the case.
Figure 4.5: A series of frames from the MVT animation demonstrating the mean value theorem (Theorem 4.2) for the function $x^3 + 5$ on the interval $[-1, 1]$.

The procedure to demonstrate the intermediate value theorem (Theorem 4.3) is called IVT. The user must enter a function $f$, a range $[a, b]$ and the intermediate value $u$. The program finds all $c \in (a, b)$ such that $f(c) = u$ and displays an animation. The animation shows the horizontal lines $y = f(a)$, $y = f(b)$ and $y = u$. The animation shows the function being drawn from $a$ to $b$; every time the function crosses the line $y = u$, the point of intersection is labeled with the corresponding $c$ value. A selection of frames from an example animation are shown in Figure 4.6.
Figure 4.6: A series of frames from the IVT animation demonstrating the intermediate value theorem (Theorem 4.3) for the function $x^3 - \frac{3}{2}x^2 - \frac{3}{2}x + 15$ on the interval $[-2, \frac{5}{2}]$ with the intermediate value $u = 14$.

4.2.3 Discussion and Suggested Use

These programs are designed to be used shortly after students are introduced to each theorem. The demonstrations should hopefully give the students a geometric understanding of what each theorem states. The programs can also be used to reinforce the hypotheses of each theorem through the error messages displayed when the
theorem does not apply to the given function.

When showing a mean value theorem demonstration to the class, the instructor could stress that typically on one side of the \( c \) value the slope of the tangent lines are larger (or smaller) than the slope of the secant line and on the other side the slopes are smaller (or larger); in order for this to happen, the slope of the tangent must match the slope of the secant somewhere in between. A similar explanation could accompany a Rolle’s theorem animation, illustrating that on one side the slopes are positive and on one side the slopes are negative (in the case of a local maximum or minimum).

The programs can be used to make clear to students in two ways that Rolle’s theorem is a special case of the mean value theorem. The first would be to run both programs using the same function to show that the animations are the same. The second would be to show the actual code of both programs and point out that the only difference in the code is that with Rolle’s theorem you must check that \( f(a) = f(b) \).

When demonstrating the intermediate value theorem, the instructor could stress that there is no way of drawing a continuous function from \( f(a) \) to \( f(b) \) without crossing the line \( y = u \). This would help give students an intuitive understanding of why the theorem must work. This is important since the proof of the intermediate value theorem requires tools from an upper-year analysis class that students will not have at the first-year level.
4.3 Graphing Equations in Polar Coordinates

4.3.1 Description of the Lesson

Polar coordinates provide an alternative method to rectangular coordinates for labeling points in two-dimensions. Instead of giving a horizontal value $x$ and a vertical value $y$, a point is specified by giving a distance from the origin $r$ (referred to as the radius) and an angle $\theta$ from the positive $x$ axis. The coordinate system can be extended to allow negative values in the radial coordinate. A polar point with a negative radial coordinate is plotted by plotting the point as if the radius were positive and then reflecting the point through the origin. Points do not have a unique representation in polar coordinates; for instance, the point $(\sqrt{2}, \pi/4)$ (corresponding to $(1,1)$ in rectangular coordinates), can also be written $(\sqrt{2}, 9\pi/4)$ or with negative radial coordinate $(-\sqrt{2}, 5\pi/4)$.

Typically, after gaining some experience plotting points in polar coordinates, students are taught to graph polar equations of the form $r = f(\theta)$. The function $f$ is often chosen to be trigonometric because intervals where the function is positive and negative are well-known and trigonometric functions are periodic. Special care must be taken on intervals where $f$ is negative. A common practice is to plot the curve $r = |f(\theta)|$ with a dashed line, then reflect this dashed curve through the origin. The program described in the next section gives students the ability to do this programmatically.
4.3.2 Introduction of the Learning Tool

The Maple procedure created to demonstrate plotting polar functions is named \texttt{PolarAnimate}. The user must enter the function $f(\theta)$ and the interval of $\theta$ values they want to be plotted. An animation shows the function being plotted along with the radial arm to show the current angle $\theta$. When the given $f$ is negative, the program plots simultaneously the actual plot and a dotted line showing the function plotted with a positive radius. Several frames from an example animation are available in Figure 4.7.

4.3.3 Discussion and Suggested Use

These animations are designed to give students an understanding of plotting negative radii. This is achieved through plotting both the function and the function with a positive radius. This not only gives students a method for dealing with regions of negative function values, but also allows them to see the positive radius being reflected through the origin.

The addition of the radial arm and the dotted lines also helps students who may be confused by the order in which the separate pieces of the function are displayed. For instance, if the animation only showed the function when plotting $r = 3 \sin(2\theta)$, (shown in Figure 4.7) the student would see the function plotted in the first quadrant, then the fourth, then the third and finally the second quadrant. A student may wonder why the angle $\theta$ is jumping around in this manner. With the inclusion of the
Figure 4.7: A series of frames from the PolarAnimate animation for the function \( r = 3 \sin(2\theta) \) for \( \theta \in [0, 2\pi] \).

radial arm and the dotted lines, it should be clear that \( \theta \) is always increasing from 0 to \( 2\pi \) and the function is displayed in that order due to the negative radius.

Another advantage of this program is that it greatly simplifies the command to obtain a polar animation in Maple. Using standard Maple commands to obtain the animation of the function without the radial arm or dotted lines, the user would have
to enter:

\[
\text{plots[animate]}(\text{plot, } [3\times \sin(2x), x = 0..s, \text{coords = polar}], s = 0..2\times \Pi)
\]

versus the program call which is simply \text{PolarAnimate}(3\times \sin(2x), x=0..2\times \Pi).

This can help combat the intimidation factor of the code that may discourage students from trying the program on their own.

4.4 The Proof of the Fundamental Theorem of Calculus

4.4.1 Description of the Lesson

The fundamental theorem of calculus, as the name suggests, is one of the most important theorems a first-year calculus student will learn. What follows in the next section is the statement and proof of the first part of the fundamental theorem of calculus, as it would be presented in a first-year course in a computer-aided instruction classroom. The lesson includes an animation designed to explain a key step of the proof.

4.4.2 Introduction of the Learning Tool

Before we state the theorem, we assume that the following three properties have already been covered in class. Assume that \( f \) and \( g \) are continuous on \([a,b]\) and \( c \in [a,b] \).

\[
\int_{x=a}^{b} kdx = k(b - a) \text{ where } k \text{ is constant.} \quad (4.1)
\]
\[
\int_{x=a}^{b} f(x)dx = \int_{x=a}^{c} f(x)dx + \int_{x=c}^{b} f(x)dx
\] (4.2)

If \( f(x) \leq g(x) \) for all \( x \in [a,b] \), then
\[
\int_{x=a}^{b} f(x)dx \leq \int_{x=a}^{b} g(x)dx
\] (4.3)

We also require that the extreme value theorem, given below, has been introduced.

**Theorem 4.4 : The Extreme Value Theorem**

Let \( f : [a,b] \to \mathbb{R} \). If \( f \) is continuous on the closed interval \( [a,b] \), then there exists \( m, M \in [a,b] \) such that \( f(m) \leq f(x) \leq f(M) \) for all \( x \in [a,b] \).

We now state the first part of the fundamental theorem of calculus.

**Theorem 4.5 : The Fundamental Theorem of Calculus (Part One)**

Let \( y = f(x) \) be a function continuous on the interval \( [a,b] \). Define \( A(x) = \int_{t=a}^{x} f(t)dt \).

Then \( A'(x) = f(x) \), for \( x \in [a,b] \).

**Proof:** (as would be presented in a first-year CAI classroom)

Choose \( x \in [a,b] \). We wish to show that \( \lim_{h \to 0} \frac{A(x+h)-A(x)}{h} \) exists and it equals \( f(x) \).

**Case 1:** Let \( h > 0 \) such that \( x + h \leq b \).

\( A(x) = \int_{t=a}^{x} f(t)dt \) and \( A(x + h) = \int_{t=a}^{x+h} f(t)dt \).
\[
\therefore A(x+h) - A(x) = \int_{t=a}^{x+h} f(t) \, dt - \int_{t=a}^{x} f(t) \, dt \\
= \int_{t=a}^{x} f(t) \, dt + \int_{t=x}^{x+h} f(t) \, dt - \int_{t=a}^{x} f(t) \, dt \quad \text{by (4.2)} \\
= \int_{t=x}^{x+h} f(t) \, dt \quad (4.4)
\]

Figure 4.8: Illustrations of \( A(x), A(x+h) \) and \( A(x+h) - A(x) \).

Now since \( f \) is continuous on \([a, b]\), it must be continuous on \([x, x+h]\). By the extreme value theorem, there exists \( m \) and \( M \) in \([x, x+h]\) such that

\[
f(m) \leq f(t) \leq f(M) \quad \text{for all} \quad t \in [x, x+h].
\]
By (4.3) we have
\[
\int_{t=x}^{x+h} f(m) dt \leq \int_{t=x}^{x+h} f(t) dt \leq \int_{t=x}^{x+h} f(M) dt.
\]

By (4.1) this gives, \( f(m)h \leq \int_{t=x}^{x+h} f(t) dt \leq f(M)h \). Substituting (4.4), we obtain
\[
f(m)h \leq A(x + h) - A(x) \leq f(M)h
\]

Now let \( h \to 0^+ \). ∴ \( x + h \to x \), which forces \( m, M \to x \). (Note: At this point in the proof, an animation would be shown depicting \( h \) going to zero and \( m \) and \( M \) being squeezed to \( x \). Sample frames are shown in Figure 4.9).

\[
\lim_{h \to 0^+} f(m) \leq \lim_{h \to 0^+} \frac{A(x + h) - A(x)}{h} \leq \lim_{h \to 0^+} f(M)
\]

\[
f(x) \leq A'_+(x) \leq f(x)
\]

Figure 4.9: A series of frames from the animation used to motivate a key step in the proof of the fundamental theorem of calculus.

To complete the proof, a second case must be taken where \( x \in (a, b] \) and \( h < 0 \) such that \( x + h \geq a \).
4.4.3 Discussion and Suggested Use

This is an example of how a computer algebra system may be integrated into the presentation of the proof of the fundamental theorem of calculus. The images in Figure 4.8 could be displayed to the class to illustrate what $A(x)$ and $A(x + h)$ represent. These images assume that $f$ is positive, but the proof works without this assumption.

The animation depicted in Figure 4.9 is designed to illustrate a key step in the proof, one that many students struggle to understand. The animation shows $h \to 0^+$, and with each new value of $h$, labels the absolute minimum and maximum on the interval $[x, x + h]$. The students will see the interval shrink to the single point $x$ and, with that, the points $x + h$, $m$ and $M$ all converge to $x$.

Unlike the other animations in this chapter, this program produces the same animation every time; the user cannot change the function in the animation. This choice was made because the program was not designed to be used by students on their own to deepen their understanding of a concept; rather, it was designed to illustrate one key step during a single class. The function was chosen such that the values of $m$ and $M$ would change a few times each as the interval shrinks.
Chapter 5

Results and Future Work

The data analysis revealed evidence of a positive relationship between the use of online Maple T.A. quizzes and student performance. We observed that students who complete more of the online quizzes tend to perform better on the final exam even after accounting for midterm grades and previous mathematical preparedness. We have also seen a positive correlation between taking a fewer number of attempts to achieve the same score on the online quizzes with final exam grades. We found that although performance on T.A. gives some indication of performance on the final exam, it is not as strong of an indicator as midterm grades or grades from the previous course. Finally a positive relationship was found between attempting quizzes before the due date and exam scores. Student attitude towards the technology was overwhelmingly positive. This analysis leads us to the conclusion that students who use the online quizzes as intended have benefited from the positive learning experience.

Several examples of how computer technology may be utilized in a mathematics
classroom were presented. The interactive notes give an example of how one may integrate technology into the classroom. The computer demonstrations focused on providing geometric interpretations for difficult calculus topics through the use of animated plots. Suggested uses of each program were given that may lead students to achieving a better understanding of each topic.

5.1 Future Work

The use of Maple T.A. is expanding at the University of Guelph. Beginning in the Winter 2012 semester, four more courses will be utilizing Maple T.A. quizzes. These courses include a first-year calculus course focusing on modeling for the biological sciences, an electricity and magnetism course for physics and engineering students, a large introductory statistics course and a mathematical economics course. With such a large and diverse implementation of the online testing system, a much richer data analysis may be performed building on techniques utilized in this thesis. This may also give us the opportunity to design and carry out a control versus treatment experiment, which may result in stronger conclusions.

The extension of Maple T.A. into smaller courses at the University of Guelph may also allow a comparison of weekly online formative quizzes with paper-based assignments or short in-class quizzes. This could help determine if the medium that delivers the practice problems matters, or if any practice benefits student learning.

Currently the online system does not save information regarding practice quizzes.
In the future these quizzes should be created in a manner such that the system saves this data, allowing for an exploration of the effect of attempting practice quizzes.

To this point, only the formal definition of a limit and polar animation computer demonstrations have been field-tested in the classroom. The remaining computer demonstrations should be utilized in the classroom and a method of retrieving student feedback, such as a focus group or a survey should be performed. This would allow for the demonstrations to be improved and for ideas for new programs to be developed. The effectiveness of these demonstrations may be examined by looking at student grade data on exam questions focusing on each of the demonstration topics. Comparisons could be made between grades on common exam questions in years the demonstrations were not used and in years they were used.

The demonstrations presented in this thesis may be featured on the Maplesoft Application Center (see http://www.maplesoft.com/applications/) in the future. This would make the demonstrations accessible worldwide for instructors to implement them in their own courses. There have been preliminary discussions with Nelson Learning to develop the interactive notes, T.A. tests, and Maple demonstrations into a marketable package.
Bibliography


[5] Bruce, C. and Ross, J., “Trigonometry and linear growing patterns CLIPS field


[22] Smith, G., “How does student performance on formative assessments relate to


Appendix A

T.A. Protocol Sheet

PLEASE FOLLOW THIS TA PROTOCOL!

1) Work through the sample TA test in your course manual.

2) Do a couple of “Practice” tests. Use “How did I do?” to check your answers on the go. Use “Preview” to check your syntax.

**Note:** Preview does not recognize interval notation. So don’t use preview to check questions requiring an interval.

**Hint:** If the answer involves “complicated” math, enter it in Maple, then copy and paste this into TA. TA will translate your answer into correct syntax. Neat. Please don’t abuse this suggestion by getting Maple to DO the questions for you. By all means, use Maple to check your answers.

3) Now you are ready for prime time. You should be able to get perfect on a “Homework” quiz in one or two attempts. You will be allowed FIVE attempts. Only your BEST mark will count on TA.

4) **DO NOT LEAVE TA TILL THE LAST DAY THE QUIZ IS OPEN!**

5) **ALWAYS GRADE YOUR TEST WHEN YOU ARE FINISHED.** If you didn’t do all questions when you grade, TA will inform you. Then you MUST click grade again. If you click, “View Details”, you will see your entire test and be given the option of printing it.
6) Always “QUIT AND SAVE” after you finish a test whether it is homework or practice.

All your homework tests are saved in the system and you can retrieve and view them at any time.

Unless indicated otherwise in class, do NOT switch math entry mode to symbolic math. Continue to use text entry.

**Always include arithmetic operations.** For example, don’t enter xy when you mean x*y. (TA and Maple will treat xy as a single symbol.) Use brackets generously but only “(” and “)” unless otherwise specified in the instructions to a question.

Please pay attention to those extra instructions when they are included.
# T.A. Syntax Sheet

(Keep this sheet with you whenever you work on at TA test.)

<table>
<thead>
<tr>
<th>Math Expression</th>
<th>TA text entry syntax</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \cdot y; \frac{x}{y}$</td>
<td>x*y; x/y</td>
</tr>
<tr>
<td>$x^y$</td>
<td>x^y</td>
</tr>
<tr>
<td>$\frac{a}{b \cdot c}$</td>
<td>a/(b*c) or TA likes a/b/c (but I don’t)</td>
</tr>
<tr>
<td>$\sqrt{x}$</td>
<td>sqrt(x) or x^(1/2) Do not use x^.5!</td>
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<tr>
<td>$x^{\frac{2}{3}} = \frac{\sqrt[3]{x^2}}{2}$</td>
<td>x^(2/3)</td>
</tr>
<tr>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$\ln(x); \log_2(x)$</td>
<td>ln(x); log<a href="x">2</a></td>
</tr>
<tr>
<td>$e^x; e; \pi; \infty$</td>
<td>exp(x); e or exp(1); pi or Pi; infinity</td>
</tr>
<tr>
<td>$\sin^2(x) = (\sin(x))^2$</td>
<td>sin(x)^2 or (sin(x))^2</td>
</tr>
</tbody>
</table>

TA always uses $1+ \tan(x)^{\cdot2}$ for $\sec(x)^{\cdot2}$ but $\sec(x)^{\cdot2}$ is fine.

TA always uses $1+ \cot(x)^{\cdot2}$ for $\csc(x)^{\cdot2}$ but $\csc(x)^{\cdot2}$ is fine.

TA always uses $\sin(x)/\cos(x)^{\cdot2}$ for $\sec(x)*\tan(x)$ but $\sec(x)*\tan(x)$ is fine.

TA always uses $\cos(x)/\sin(x)^{\cdot2}$ for $\csc(x)*\cot(x)$ but $\csc(x)*\cot(x)$ is fine.
Appendix B

MATH 1200/1210 TECHNOLOGY SURVEY AND RESULTS

1. (a) I am female. (b) I am male.

**Results**
2010: (a) 128 (b) 195

2011: (a) 138 (b) 224

2. (a) I successfully installed Maple and the Maple MSK on my computer.
(b) I installed Maple but was unable to install the Maple MSK.
(c) I tried but was unable to install either program.
(d) I have not tried to install the programs.

**Results**
2010: (a) 227 (b) 56 (c) 14 (d) 26

2011: (a) 292 (b) 44 (c) 13 (d) 13

3 and 4. I like mathematics.

**As of September, 2010**

(a) Strongly agree
(b) Agree
(c) No opinion
(d) Disagree
(e) Strongly Disagree

**As of April, 2011**

(a) Strongly agree
(b) Agree
(c) No opinion
(d) Disagree
(e) Strongly Disagree
Results (Question 3)  
2010: (a) 113 (b) 151 (c) 39 (d) 19 (e) 2  
2011: (a) 131 (b) 162 (c) 51 (d) 16 (e) 3  

Results (Question 4)  
2010: (a) 135 (b) 157 (c) 24 (d) 6 (e) 2  
2011: (a) 138 (b) 156 (c) 44 (d) 21 (e) 5  

5 and 6. I am confident of my ability to do well in math.

As of September, 2010  
(a) Strongly agree  
(b) Agree  
(c) No opinion  
(d) Disagree  
(e) Strongly Disagree  

As of April, 2011  
(a) Strongly agree  
(b) Agree  
(c) No opinion  
(d) Disagree  
(e) Strongly Disagree  

Results (Question 5)  
2010: (a) 103 (b) 153 (c) 37 (d) 26 (e) 5  
2011: (a) 134 (b) 152 (c) 35 (d) 36 (e) 6  

Results (Question 6)  
2010: (a) 123 (b) 154 (c) 30 (d) 15 (e) 2  
2011: (a) 135 (b) 168 (c) 32 (d) 24 (e) 4  

TECHNOLOGY

7. The Maple demonstrations in class helped me learn.

(a) Strongly agree (b) Agree (c) No opinion (d) Disagree (e) Strongly Disagree  

Results  
2010: (a) 46 (b) 155 (c) 84 (d) 34 (e) 5  
2011: (a) 86 (b) 188 (c) 61 (d) 25 (e) 4
8. The TA online tests helped me learn.

(a) Strongly agree (b) Agree (c) No opinion (d) Disagree (e) Strongly Disagree

Results 2010: (a) 161 (b) 128 (c) 22 (d) 12 (e) 1

2011: (a) 207 (b) 119 (c) 25 (d) 11 (e) 2

9. The Desire To Learn discussions site helped me learn.

(a) Strongly agree (b) Agree (c) No opinion (d) Disagree (e) Strongly Disagree

Results 2010: (a) 24 (b) 76 (c) 164 (d) 43 (e) 17

2011: (a) 16 (b) 89 (c) 178 (d) 54 (e) 27

CLASS NOTES

10. The class notes in the course manuals are 60 to 80% complete, with the remainder to be completed together with the professor in class. This was an effective way to keep interested and participate in class.

(a) Strongly agree (b) Agree (c) No opinion (d) Disagree (e) Strongly Disagree

Results 2010: (a) 198 (b) 105 (c) 16 (d) 3 (e) 2

2011: (a) 219 (b) 113 (c) 20 (d) 9 (e) 3

11. I used Maple instructions in the class notes, modifying them as needed, in my own investigations after class.

(a) Strongly agree (b) Agree (c) No opinion (d) Disagree (e) Strongly Disagree

Results 2010: (a) 11 (b) 47 (c) 96 (d) 85 (e) 85

2011: (a) 15 (b) 34 (c) 92 (d) 138 (e) 85
TA ONLINE TESTS

12. In each course, there were ten or eleven weekly online tests, each worth 2%, that consolidated that week’s material and occasionally reviewed some topics from earlier courses. Each test was designed to take about 45 minutes and repeated attempts—with different tests each time—were allowed. Choose one of the following statements for future courses.

(a) One test per week is a good idea.

(b) There should be a test every second week.

(c) There should be a test per week but tests should be optional.

(d) The tests should be a study resource only and not count for marks.

**Results**

2010: (a) 292  (b) 23  (c) 3  (d) 5

2011: (a) 235  (b) 107  (c) 8  (d) 12

13. I followed the TA protocol recommended by the professor.

(a) course manual + practice + homework

(b) only practice and homework

(c) only homework

**Results**

2010: Question did not appear on survey

2011: (a) 163  (b) 129  (c) 59
14. If the online tests were voluntary and did not count for marks, I still would make time to write them.

(a) Strongly agree (b) Agree (c) No opinion (d) Disagree (e) Strongly Disagree

**Results**

<table>
<thead>
<tr>
<th></th>
<th>2010</th>
<th></th>
<th>2011</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>24</td>
<td>(d)</td>
<td>46</td>
<td>(d)</td>
</tr>
<tr>
<td>(b)</td>
<td>76</td>
<td>(c)</td>
<td>111</td>
<td>(c)</td>
</tr>
<tr>
<td>(c)</td>
<td>40</td>
<td>(e)</td>
<td>50</td>
<td>(d)</td>
</tr>
<tr>
<td>(d)</td>
<td>116</td>
<td></td>
<td>106</td>
<td></td>
</tr>
<tr>
<td>(e)</td>
<td>66</td>
<td></td>
<td>51</td>
<td></td>
</tr>
</tbody>
</table>

15. This is how I approached each online test.

(a) as a test to be done on my own

(b) as a group assignment with classmates

(c) as a group assignment where I may accept an answer from a classmate even if I don’t understand it

(d) as a test I may arrange for someone else to write for me

**Results**

<table>
<thead>
<tr>
<th></th>
<th>2010</th>
<th></th>
<th>2011</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>264</td>
<td>(d)</td>
<td>285</td>
<td>(d)</td>
</tr>
<tr>
<td>(b)</td>
<td>38</td>
<td>(c)</td>
<td>62</td>
<td>(c)</td>
</tr>
<tr>
<td>(c)</td>
<td>19</td>
<td>(b)</td>
<td>17</td>
<td>(b)</td>
</tr>
<tr>
<td>(d)</td>
<td>1</td>
<td></td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

16. I believe my midterm and final exam marks were better because of my work on the TA test.

(a) Yes (b) No (c) TA helped me learn but I don’t believe this was reflected on my midterm and final exam grades.

**Results**

<table>
<thead>
<tr>
<th></th>
<th>2010</th>
<th></th>
<th>2011</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>212</td>
<td>(c)</td>
<td>262</td>
<td>(c)</td>
</tr>
<tr>
<td>(b)</td>
<td>27</td>
<td>(a)</td>
<td>26</td>
<td>(a)</td>
</tr>
<tr>
<td>(c)</td>
<td>80</td>
<td>(b)</td>
<td>76</td>
<td>(b)</td>
</tr>
</tbody>
</table>
17. These courses (math 1200 and math 1210) should
(a) continue to use the weekly TA tests and have three midterms.
(b) use fewer TA tests and have three midterms.
(c) omit the TA tests and have three midterms.
(d) use the TA tests as practice only and have three midterms.
(e) use the TA tests as practice only and have one or two midterms.

**Results** 2010: (a) 292 (b) 18 (c) 2 (d) 5 (e) 3
2011: (a) 289 (b) 40 (c) 2 (d) 15 (e) 14

18. Students should be allowed to write each TA homework test
(a) once as long as practice tests are available beforehand. A grad student would proctor this to minimize cheating.
(b) up to three times.
(c) up to five times.
(d) an unlimited number of times.

**Results** 2010: (a) 8 (b) 3 (c) 14 (d) 296
2011: (a) 8 (b) 57 (c) 233 (d) 64

19. For the most part, I found the TA tests
(a) very easy (b) easy (c) average (d) difficult (e) very difficult

**Results** 2010: (a) 5 (b) 47 (c) 230 (d) 40 (e) 1
2011: (a) 8 (b) 62 (c) 232 (d) 57 (e) 4
OVERALL

20. In THE CURRENT COURSE (Math 1210), I estimate my mark will be

(a) 80 or better (b) from 70 to 79 (c) from 60 to 69 (d) from 50 to 59 (e) under 50

**Results**  
2010: (a) 198 (b) 94 (c) 22 (d) 7 (e) 1  
2011: (a) 232 (b) 86 (c) 38 (d) 6 (e) 1

21. Overall, I benefited from the inclusion of technology (TA, Maple, D2L) in the courses.

(a) Strongly agree (b) Agree (c) No opinion (d) Disagree (e) Strongly Disagree

**Results**  
2010: (a) 111 (b) 158 (c) 36 (d) 15 (e) 2  
2011: (a) 146 (b) 167 (c) 30 (d) 14 (e) 5

22. *For those who took both Math 1200 and Math 1210*

I found the TA online testing

(a) more useful in Math 1200.

(b) more useful in Math 1210.

(c) Equally useful in both courses.

(d) Not useful in either course.

(e) This question does not apply to me as I did not take Math 1200 in the Fall, 2008 semester.

**Results**  
2010: (a) 56 (b) 115 (c) 114 (d) 8 (e) 29  
2011: (a) 43 (b) 39 (c) 241 (d) 9 (e) 19
23. The course manual was
(a) indispensible (b) very useful (c) useful (d) not very useful (e) not at all useful.

Results 2010: Question did not appear on survey
2011: (a) 187 (b) 125 (c) 42 (d) 8 (e) 0

24. The textbook by Stewart was
(a) indispensible (b) very useful (c) useful (d) not very useful (e) not at all useful.

Results 2010: Question did not appear on survey
2011: (a) 3 (b) 5 (c) 47 (d) 122 (e) 183

25. The text solutions manual was
(a) indispensible (b) very useful (c) useful (d) not very useful (e) not at all useful.

Results 2010: Question did not appear on survey
2011: (a) 16 (b) 22 (c) 69 (d) 97 (e) 156

26. For Math 1200 students only
The Mathematics Survival Kit was
(a) indispensible (b) very useful (c) useful (d) not very useful (e) not at all useful.

Results 2010: Question did not appear on survey
2011: (a) 18 (b) 59 (c) 142 (d) 73 (e) 34
Appendix C

Class 20: Graph Sketching

Here is a graph of $y = f(x)$

The graph of $y = f'(x)$ looks like:

The graph of $y = f''(x)$ looks like:
### $y$ vs $y'$ vs $y''$

<table>
<thead>
<tr>
<th>$y$ increases</th>
<th>$y$ decreases</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$ is concave up</td>
<td>$y$ is concave down</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$y'$ vs $y''$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$y' &gt; 0$, $y'' &gt; 0$</td>
<td>$y' &gt; 0$, $y'' &lt; 0$</td>
</tr>
<tr>
<td>$y$: slopes:</td>
<td>$y$: slopes:</td>
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</table>

### First Derivative Test for Relative Extremes

<table>
<thead>
<tr>
<th>MIN’s</th>
<th></th>
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<tbody>
<tr>
<td>MAX’s</td>
<td></td>
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</table>
STEPS FOR GRAPH SKETCHING

1. Look very carefully at the equation. Are there any vertical or horizontal asymptotes? Is there any symmetry about the $x$-axis, $y$-axis, and/or the origin? Try to get a general picture of the graph. Then use the calculus techniques to determine the details.

2. Find the $x$ and $y$ intercepts.

3. Check for any (i) vertical asymptotes ($x = \text{number}$) (ii) horizontal asymptotes ($y = \text{number}$)

4. Find the fully factored form of $y'$. Do a ”$+/−$” chart analysis. State the $x$ intervals for which $y$ increases and those for which $y$ decreases. State the max and min points.

5. Find the fully factored form for $y''$. Do a ”$+/−$” chart analysis. State the $x$ intervals for which the graph is concave up and those which it is concave down. State the points of inflection

6. Construct an ”appropriate” table of values.
7. Set up an \( xy \)-axis using an APPROPRIATE SCALE!!! Using dotted lines, draw in the asymptotes.

8. Accurately, draw the graph of \( y = f(x) \)!

\[ \text{eg 1)} \quad y = 3 \cdot x^4 + 2 \cdot x^3 = 3 \cdot x^3 \cdot (x + \frac{2}{3}) \]

\[ y' = 12 \cdot x^2 \cdot (x + \frac{1}{2}) \quad y'' = 36 \cdot x \cdot (x + \frac{1}{3}) \]

Intercepts:

Asymptotes:

\( y' \) chart:

\( y'' \) chart:
<table>
<thead>
<tr>
<th>interval</th>
<th>$y'$</th>
<th>$y''$</th>
<th>shape</th>
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</tbody>
</table>
In Maple:

Define the function

\[ f := x \rightarrow 3 \cdot x^4 + 2 \cdot x^3 \]

\[ x \rightarrow 3x^4 + 2x^3 \]

Find function values

\[ f\left(-\frac{1}{2}\right) \]

\[ -\frac{1}{16} \]

Get first and second derivatives

\[ fp := \text{unapply}\left(\frac{d}{dx}f(x), x\right) \]

\[ x \rightarrow 12x^3 + 6x^2 \]

\[ fpp := \text{unapply}\left(\frac{d}{dx}fp(x), x\right) \]

\[ x \rightarrow 36x^2 + 12x \]

Find slope values

\[ fp\left(-\frac{1}{3}\right) \]

\[ \frac{2}{9} \]
Finally, graph it!

plot(f(x), x = -1..1, y = -0.1..0.1, color=black)

Note: This class contains two more example graph sketching problems, which we have omitted. The full class is available on the enclosed disc.
Appendix D

Maple Code for MVT Procedure

#Title: MVT

#Author: Gord Clement, May 2011

#Description: For a given function and interval, this procedure
# determines if the conditions of the mean value theorem hold. If so,
# all possible values c on the interval which satisfy the conclusion of the
# theorem are found. An animation shows the function plotted on the
# interval with the tangent line moving across the function, highlighting
# the c values when they are reached

#Usage:

#Call:MVT(function, interval)

# function: function to be used for the theorem, note: must be
# continuous on the given interval [a, b] and differentiable on (a, b)
# interval: interval to be used in the theorem, entered in
# standard Maple notation, ie., [a, b] would be entered x = a..b

MVT:= proc(expr, range)
    # local variable declarations
    local slope, dir, clist, fullList, temp, i, j, found,
    output, var, a, b, step, tanslope, tempx, tany, tanline, aplot,
    bplot, secplot, funcplot, fAta, tanplot, fullplot, miny, maxy,
tempc,multiplec,previousStop,nextStop, length,maxdir,ctanplot;

# extract variable and end-points of interval
var := op( 1, range);
a:= evalf(op(1, op(2, range ) ));
b:= evalf(op(2, op(2, range) ));
miny:=minimize(expr, var=a..b);
maxy:=maximize(expr, var=a..b);

#calculate length of interval, used for scaling in final plots
length:=evalf((b-a)/(5));

#calculate slope of the line joining (a,f(a)) to (b,f(b))
slope:=(subs(var=b,expr)-subs(var=a,expr))/(b-a);

#calculate f(a)
fAta:=subs(var=a,expr);

# obtain derivative
dir:=diff(expr,var);

# obtain highest magnitude of derivative on interval
maxdir:=maximize(|dir|,var=a..b);

# dummy variable to indicate if c value is found
found:=false;
output:=-1;

#find all potential c values
fullList:=evalf(solve(dir=slope, var, dropmultiplicity=true ));

#dummy variable to indicate if multiple c values were found
multiplec :=false;

#confirm interval given is valid
if (a>=b) then

   output:="Error: a must be less than b";

#confirm function does not have vertical tangent lines
elif (is(type(maxdir,infinity))) then

   output:= “Error: Function does not satisfy the conditions of the

       Mean Value Theorem since it is not differentiable on (a,b)”;

#confirm function is continuous on given range
elif (not(iscont(expr,var=a..b,'closed'))) then

   output:= “Error: Function does not satisfy the conditions of the

       Mean Value Theorem since it is not continuous on [a,b]”;

#confirm function is not linear, which would be a trivial case

       of the theorem.
elif (diff(expr,var,var)=0) then

   output:= “Error: Do not use a linear function, as all values

       between a and b satisfy the Mean Value Theorem”;

else

   #if only one potential c, confirm it is in the interval
if (nops([fullList])=1) then
    if (is(fullList<b) and is(fullList>a)) then
        clist:=fullList;
    else
        output:="Error: Function does not satisfy the conditions of the Mean Value Theorem on the given range";
    end if;
else
    #find all of the potential $c$ values that are in the interval
    multiplec:=true;
    clist:=[ ];
    for i from 1 to nops([fullList]) do
        temp:=fullList[i];
        if (is(temp<b) and is(temp>a)) then
            found:=true;
            clist:=[op(clist),temp];
        end if;
    end do;
    if (not(found)) then
        output:="Error: Function does not satisfy the conditions of the Mean Value Theorem on the given range";
    end if;
end if;
end if;

end if;

end if;

# if there was an error, display error message

if (not (output=-1)) then

output;

else

# animation code for single c value

if (not(multiplec)) then

# animate up to c value

step:=evalf((clist-a)/(25));

output:=[ ];

tempx:=a;

# plot a, b the function and the secant line

aplot:=plot([a,t,t=miny-0.2..maxy+0.2],linestyle=dash,color=black):

bplot:=plot([b,t,t=miny-0.2..maxy+0.2],linestyle=dash,color=black):

funcplot:=plot([var,expr,var=a-0.2..b+0.2], thickness=2, color=black):

secplot:=plot([t, slope*(t-a)+fAta,t=a..b],thickness=2,color=red):

# plot tangent lines

for i from 1 to 25 do

    tempx:=tempx+step;

end do;


tanslope:=evalf(subs(var=tempx,dir));

tany:=evalf(subs(var=tempx,expr));

tanline:=tanslope*(t-tempx)+tany;

tanplot:=plot([t,tanline,t=tempx-length..tempx+length],
               color=black,thickness=2);

fullplot:=plots[display]([aplot,bplot,tanplot,funcplot,
                           secplot],title=typeset(var,"=",evalf(tempx)), labels=[" "," "]):

output:=[op(output),fullplot]:

end do;

# plot tangent line for c value

tanslope:=evalf(subs(var=clist,dir));

tany:=evalf(subs(var=clist,expr));

tanline:=tanslope*(t-clist)+tany;

catanplot:=plot([t,tanline,t=clist-length..clist+length],color=red,thickness=2):

# freeze plot for 10 frames

fullplot:=plots[display]([aplot,bplot,catanplot,funcplot,secplot],
                           title=typeset("c=",clist), labels=[" "," "]):

for i from 1 to 10 do

    output:=[op(output),fullplot]:

end do;

# plot tangent lines after c value
step := evalf((b-clist)/(25));

tempx := clist;

for i from 1 to 25 do

    tempx := tempx + step;

    tanslope := evalf(subs(var=tempx,dir));

    tany := evalf(subs(var=tempx,expr));

    tanline := tanslope*(t-tempx)+tany;

    tanplot := plot([t, tanline, t=tempx-length..tempx+length],
                    color=black, thickness=2);

    fullplot := plots[display]([aplot, bplot, tanplot, funcplot,
                                 secplot, ctanplot], title=typeset(var,"=",evalf(tempx)),
                               labels=[" ", " "]);

    output := [op(output), fullplot];

end do;

# add final frame and animate

fullplot := plots[display]([aplot, bplot, funcplot, secplot,
                           ctanplot], title=typeset(var,"=",evalf(tempx)),
                        labels=[" ", " "]);

output := [op(output), fullplot];

plots[display]([op(output)], insequence=true, view=[a-0.2..b+0.2,
                                                    miny-0.2..maxy+0.2]);

else

    # multiple c value animation code
# order c values from low to high
clist := sort(clist);

# plots for a, b, f and the secant
aplot := plot([a, t, t = miny - 0.2..maxy + 0.2], linestyle = dash, color = black):
bplot := plot([b, t, t = miny - 0.2..maxy + 0.2], linestyle = dash, color = black):
funcplot := plot([var, expr, var = a - 0.2..b + 0.2], thickness = 2, color = black):
secplot := plot([t, slope*(t-a)+fAta, t = a..b], thickness = 2, color = red):
output := [ ];
tempx := a;

# variables to determine which to values the tangents are plotting between
previousStop := a;
nextStop := evalf(clist[1]);
ctanplot := [ ];

# loop plots tangents between stopping values (endpoints or c values) and pauses at c values
for i from 1 to nops(clist) do
    tempc := evalf(clist[i]);
    step := evalf((nextStop - previousStop)/(25));
    # plot tangents
    for j from 1 to 25 do
        tempx := tempx + step;
    end do;
end do;
tanslope:=evalf(subs(var=tempx,dir));

tany:=evalf(subs(var=tempx,expr));

tanline:=tanslope*(t-tempx)+tany;

tanplot:=plot([t,tanline,t=tempx-length..tempx+length],
            color=black,thickness=2):

fullplot:=plots[display]([aplot,bplot,tanplot,funcplot,secplot
            op(ctanplot)],title=typeset(var,"=",evalf(tempx)), labels=[" ", "]):

output:=[op(output),fullplot]:
end do;

# plot tangent for current c value

tanslope:=evalf(subs(var=tempc,dir));

tany:=evalf(subs(var=tempc,expr));

tanline:=tanslope*(t-tempc)+tany;

c tanplot:=[op(ctanplot),plot([t,tanline,t=tempc-length
            ..tempc+length],color=red,thickness=2)]:

fullplot:=plots[display]([aplot,bplot,op(ctanplot),funcplot,
            secplot],title=typeset("c=" tempc), labels=[" ", " "]):

# pause animation for 10 frames

for j from 1 to 10 do

output:=[op(output),fullplot]:
end do;
#determine next stopping value

if (not(i=nops(clist))) then
    previousStop:=nextStop;
    nextStop:=evalf(clist[i+1]);
else
    previousStop:=nextStop;
    nextStop:=b;
end if;
end do;

# plot from final c value to b
step:= evalf((nextStop-previousStop)/(25)) ;

for j from 1 to 25 do
    tempx:=tempx+step;
    tanslope:=evalf(subs(var=tempx,dir));
    tany:=evalf(subs(var=tempx,expr));
    tanline:=tanslope*(t-tempx)+tany;
    tanplot:=plot([t,tanline,t=tempx-length..tempx+length],
                   color=black,thickness=2):
    fullplot:=plots[display]([aplot,bplot,tanplot,funcplot,secplot,
                               op(ctanplot)],title=typeset(var,”=”,evalf(tempx)),
                             labels=[“ ”,” ”]):
    output:=[op(output),fullplot]:
end do;

#add final frame and animate

fullplot:=plots[display]([aplot,bplot,funcplot,secplot,op(ctanplot)],
    title=typeset(var,"=",evalf(tempx)), labels=[“”, “”]):

output:=[op(output),fullplot]:

plots[display]([op(output)],insequence=true, view=[a-0.2..b+0.2,
    miny-0.2..maxy+0.2]);

end if;

end if;

end proc: