University of Guelph Numeracy Project

About Asymptotes: Examples

UNIVERSITY of GUELPH

TABLE OF CONTENTS

Limits at Infinity

- Consider the following function:
  f(x) = (2x+1)/(x-1)

x	f(x)
1	undefined
10	2.333333
100	2.030303
1000	2.003003
10000	2.000300
100000	2.000030

- It appears that as x gets larger and larger, f(x) gets closer and closer to 2. As a result, we can conclude that:

$$\lim_{x \to \infty} f(x) = 2$$

- Considering the same function as before, let us approach negative infinity.

x	f(x)
-1	0.5
-10	1.7272727
-100	1.9702970
-1000	1.9970029
-10000	1.99970003
-100000	1.99997

- Here, we notice that as x approaches negative infinity, f(x) approaches 2. As such, we conclude:

$$\lim_{x \to -\infty} f(x) = 2$$

- Remarks: It would be tedious to do a table for each function, and there is an easier way.

- Notice that f(-1) = 0.5, f(-10) = 1.727, and f(-100) = 1.97. The difference between f(-10) and f(-1) is much larger than the difference between f(-10) and f(-100). This trend holds true for each x. In other words, as x increases, the difference between f(x) and the previous f(x) becomes increasingly smaller. However, this is not true for all limits, though it will prove useful in further investigations. To convince yourself of the failure of this trend across some functions, consider $f(x) = x^2$ as $x \to \infty$.

- Consider a function of the following form:

  f(x) = N(x)/D(x)

  Where N(x) is the polynomial in the numerator, and D(x) is the polynomial in the denominator.

- We want to find: $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$

- Step 1) Find the degree (highest power of x) of both the numerator (N(x)) and the denominator (D(x)). Let n be the highest of the two values.

  ex: f(x) = (2x+1)/(x-1) n=1
  ex: $g(x) = (x^2 -1)/(x^3+1)$ n=3

- Step 2) Divide both the numerator and denominator by x^n. Notice, the function is not being changed, it is simply being re-expressed.

- ex: f(x) = (2+1/x)/(1-1/x)
  ex: g(x) = $(1/x-1/x^3)/(1+1/x^3)$

  Ex: $\lim_{x \to \infty} f(x)$ = $\lim_{x \to \infty}$ (2+1/x)/(1-1/x) = (2+0)/(1-0) = 2
  Ex: $\lim_{x \to \infty} g(x)$ = $\lim_{x \to \infty}$ (1/x-1/x3)/(1+1/x3) = (0-0)/(1+0) = 0

- Finally, let us consider h(x) = $(x^3-1)/(x^2+1)$.

  $\lim_{x \to \infty} h(x)$ = $(x^3-1)/(x^2+1)$
  $\lim_{x \to \infty} h(x)$ = $(1-1/x^3)/(1/x + 1/x^3)$
  $\lim_{x \to \infty} h(x)$ = ∞ Infinity because the bottom goes to zero, causing the function to "blow up" in a sense.

Horizontal Asymptotes

- Consider the function:

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \ldots + b_1 x + b_0}$$

- Case I: n<m
  $\lim_{x \to \infty} f(x)$ = $\lim_{x \to \infty}$ $x^2/(x^3-4)$ = $\lim_{x \to \infty}$$(1/x)/(1-4/x^3)$ = 0

- Case II: n = m
  f(x) = $(3x^2 + 4x - 1)/(2x^2 + 1)$
  $\lim_{x \to \infty} f(x)$ = $\lim_{x \to \infty}$ $(3x^2 + 4x - 1)/(2x^2 + 1)$

  = $\lim_{x \to \infty}$ $(3 + 4/x - 1/x^2)/(2 + 1/x^2)$ = 3/2

  y = 3/2 is our horizontal asymptote.

- Case III: $n > m$

$f(x) = (x^3 + 1)/x$

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (x^3 + 1)/x = \lim_{x \to \infty} (1 + 1/x^3)/(1/x^2)$

$= \lim_{x \to \infty} x^2 + 1/x = \infty$

Vertical Asymptotes

- $f(x) = (x+2)/(x^2 - 1)$
  Factoring the bottom, we get:
  $x^2 - 1 = 0$
  $x^2 = 1$
  $x = \pm 1$
  Therefore, we have vertical asymptotes at x = 1 and x= -1

- Consider the function $f(x) = x^2/(x^3 + 4x^2)$.
  $f(x) = x^2/(x^2(x+4))$
  Cancelling out the common x^2 factor, we get:
  $f(x) = 1/(x+4)$
  In the original function, the denominator was zero when x was 0 or -4; however, the numerator was also zero when x=0. This means the function has a hole at x=0 and a vertical asymptote at x = -4.

- In the previous example, where $f(x) = x^2/(x^2(x+4))$. The limit of f(x) as x approaches -4 is, in fact, infinity. Therefore, we have a vertical asymptote.

- Consider the following:

$$f(x) = \begin{cases} \frac{1}{x^2} & x > 0 \\ 3 & x \leq 0 \end{cases}$$

- The function is defined at x=0; however, consider the limits as x->0. The limit as x approaches 0 from the left is 3. Now, consider the limit as x approaches 0 from the right.
- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1/x^2 = \infty$

Oblique Asymptote

- Consider the following function:
  $f(x) = (x^2 + 4)/x$
  $= x + (4/x)$
  $\lim_{x \to \infty} f(x) - (mx + b) = \lim_{x \to \infty} x + (4/x) - (mx + b)$
  Setting m=1 and b=0, we get the following line, y=x:
  $\lim_{x \to \infty} x + 4/x - x = \lim_{x \to \infty} 4/x = 0$

  Thus, we conclude that the function is asymptotic to the line y=x.

  Synthetic Division

```
   317.5
2 |635
   6
  ---
   03
    2
  ---
    15
    14
  ---
     10
```

Now, we ask ourselves, how many times does 2 go into 6? The answer being 3. 3 multiplied by 2 is 6, 6 minus 6 is 0, the three comes down, and we repeat. 2 goes into 3 once, 2 goes into 15 seven times, and 2 goes into 10 five times.
In the end, we arrive at the given result.

- Synthetic division is akin to long division of functions.
  Consider the following division:
  f(x)=(x2+4)/x

$$\begin{array}{r|l} & x \\ \hline x & x^2 + 4 \\ & x^2 \\ \hline & \quad\; 4 \end{array}$$

Before, we asked ourselves "how many times does 2 go into 6?" Now, we ask "how many times does x go into x^2?" The answer is "x" times. You might wonder why the answer is not x^2 , but just remember that 6 goes into 6 once, so x^2 goes into x^2 once.

Now, as before, $f(x)=(x2+4)/x = x+ (4/x)$

From here, it can be seen that the limit of $f(x) - x$ as x goes to infinity (positive or negative) will be equal to 0.