University of Guelph Numeracy Project

About Permutations and Combinations

UNIVERSITY of GUELPH

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About Permutations and Combinations

What is COMBINATORICS?

- Combinatorics is the study of the arrangement of objects. It and probability are closely related and has applications in things like gambling and computer science.

- This module will focus on the basics of combinatorics specifically permutations and combintations, as well as an introduction to Pascal's Triangle.

Basics

- Theorem (Product Rule)
  If a procedure can be accomplished with two disjoint subtasks and if there are n_1 ways of doing the first task and n_2 ways of doing the second, then there are:
  $n_1 \cdot n_2$ ways of doing the overall procedure.

- Theorem (Sum Rule): If an event e_1 can be done in n_1 ways and an event e_2 can be done in n_2 ways and e_1 and e_2 don't effect each other, then the number of ways of both events occurring is:
  $n_1 + n_2$

- Much like the Product Rule, the Sum Rule can be generalized for more events and more ways of doing things.

Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.

- We say there are n! ways to arrange n objects. Here the "!" means factorial, so it reads "n factorial"

- We calculate n factorial the following way:
  n! = n*(n-1)*(n-2)*(n-3)*....*2*1

- Definition: An ordered arrangement of r elements of a n-set is called an r-permutation.

- The formula of which is the following: P(n,r) = n!/(n-r)!

Combinations

- With permutations, order matters, we were interested in who came in first in a race, or that ABC was different then BCA when choosing from a list of say ABCD.
  Now we're going to ignore order.

- Definition: An r-combination of an n-set is an unordered selection of r elements from the set.

  The equation, looks similar to our permutation one, it has one slight difference.

- The number of r-combinations of an n-set is:
  C(n,r) = n!/r!(n-r)! = $\binom{n}{r}$

  And is read as "n choose r"

Pascal's Triangle

- Pascal's Triangle is a famous combinatorial result, but before we look into it, we'll need to look at what's called "binomial coefficients"

- In the previous section we said the number of r-combinations from a set with n elements is often denoted by C(n,r). We also call this number a binomial coefficient because these numbers occur as coefficients in the expansion of powers of a binomial expressions such as $(a + b)^n$.

- A binomial expression is simply the sum of two terms, such as x+y.

The Binomial Theorem

- Let x and y be variables, and let n be a nonnegative integer. Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$\sum_{j=0}^{n}$	is called a "summation" symbol (denoted by the greek letter sigma). It represents addition, so:
$\sum_{j=0}^{n} j$	Means that 'j' starts at 0 and goes up to n, in other words this equals 0 + 1 + 2 + 3 + 4 + 5 +...+ n-1 + n

- The Binomial Theorem can thus be expanded to the following:

$$\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Pascal's Identity/Triangle

- Then we construct a "pyramid" in the following manner. Notice that 'n' changes on the row, and 'k' changes depending on the location.

- Pascal's Identity: Let n and k be positive integers with n ≥ k. Then:

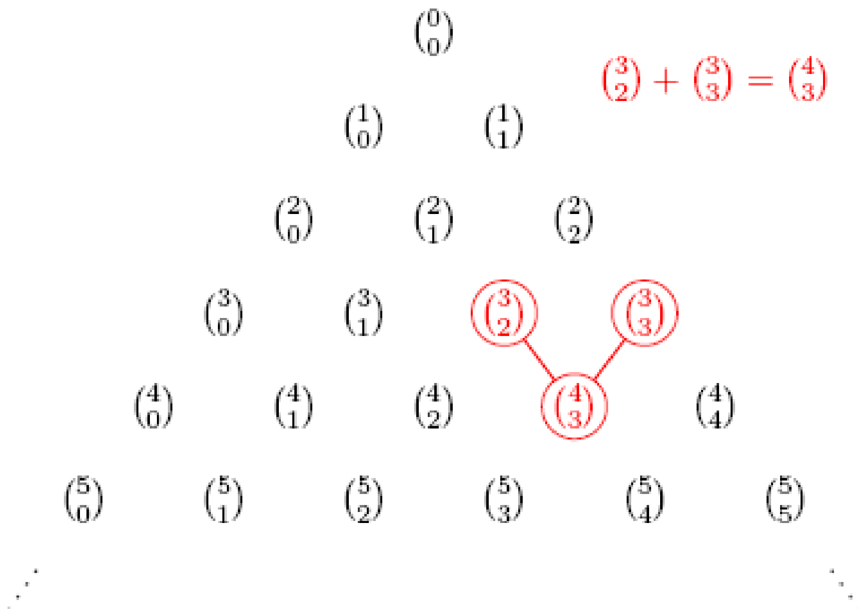
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
\iddots & & & & & & & & & & \ddots
\end{array}
$$

- Solving for each combination, we arrive at the following triangle.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\iddots & & & & & & & & & & \ddots
\end{array}
$$

- We get each row by adding the two top numbers. This is exactly Pascal's Identity. k is the rows number, and n goes in order across the row.



Examples

- We introduced Pascal's Triangle in the following manner.
$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

- However, equivalently we can write in terms of C(n,r)
$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Glossary

Binomial Coefficient: The number of r-combinations from a set with n elements denoted C(n,r). These numbers occur as coefficients in the expansion of powers of binomial expressions.

Binomial Expression: The sum of two terms, for example x + y. Note: The terms can be products of constants and variables.

Combination: And r-combination of elements of a set is an unordered selection of r elements from the set.

Combinatorics: The study of arrangements of objects
Combination: An r-combination of elements of a set is an unordered selection of r elements from the set.

Permutation: An ordered arrangement of r-objects
Factorial: We say n! is "n factorial" and is equal to the product of all positive integers less than or equal to, eg 6! = 6*5*4*3*2*1 = 720

Pascal's Triangle: A triangle constructed by using Pascal's Identity

References

Rosen, Kenneth. Discrete Mathematics and Its Applications, 6th ed. McGraw Hill Higher Education, New York, 2007.