

**University of Guelph  
Numeracy Project**

# About Graphing Quadratics



# TABLE OF CONTENTS

About Graphing Quadratics .....	1
What is a CONIC? .....	1
Background .....	1
Circle .....	2
Ellipse .....	3
Parabola .....	5
Hyperbola.....	6
Identification.....	8
Conic Sections .....	8
Glossary.....	9
References .....	11

## About Graphing Quadratics

### *What is a CONIC?*

- A Conic can be thought of either geometrically or algebraically. Geometrically, they are the shapes you get by taking the intersection of a plane and a cone. The shapes you can get are a circle, an ellipse, a parabola, and a hyperbola. Algebraically, conic sections are graphs of quadratic equations.
- Mathematically, a conic is representative of a curve formed with the intersection of a cone and a plane.

### **Background**

- Geometrically, conic sections can be thought of as the shapes you get by taking the intersection of a plane and a cone. The shapes you can get are a circle, an ellipse, a parabola, and a hyperbola.
- There are also three degenerate cases, where the plane passes through the apex of the cone: a point, a line, and two intersecting lines.
- Algebraically, conic sections are graphs of quadratic equations in two variables of the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0;$$

where A, B, and C are not all equal to zero:

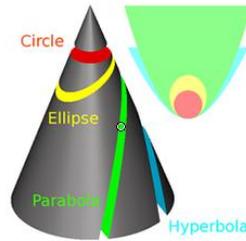
If  $B = 0$  and  $A = C$ , then the equation is a circle or a degenerate case.

If  $B^2 - 4AC < 0$ , then the equation is an ellipse or a degenerate case.

If  $B^2 - 4AC > 0$ , then the equation is a hyperbola or a degenerate case.

If  $B^2 - 4AC = 0$ , then the equation is a parabola or a degenerate case.

- To further help you understand the difference between each type of conic section, here is a diagram:



## Circle

- A circle is a shape consisting of all the points a set distance from a fixed point, where the distance is known as the radius, and the fixed point is known as the centre.

Although a special case of the ellipse, the circle is worth considering on its own.

- There are two forms in which equations of circles are commonly given: Standard Form and General Form.

The equation for a circle in Standard Form looks like:

$$(x - h)^2 + (y - k)^2 = r^2$$

where (h, k) is the centre of the circle and r is the radius.

- Sometimes, the equation of a circle is given to you in what is called General Form, which looks like:

$$Ax^2 + Ay^2 + Dx + Ey + F = 0 \quad (A \neq 0)$$

To find the centre and radius, you need to transform it into Standard Form.

- Occasionally, instead of a circle, you are given a degenerate case.
- To change from Standard Form to General Form, simply multiply out the terms, collecting like terms and putting them in the correct order.

## **Ellipse**

- To define the rest of the conic sections, we need some additional terminology: the focus, the directrix, and the eccentricity. The focus is a fixed point. The directrix is a line which does not pass through the focus, and the eccentricity is the ratio of the distance of a point from the focus to the distance of a point from the directrix. For any ellipse - in fact, for any conic section - this ratio is the same for every point. For a circle, the focus is the centre; the directrix is considered to be indefinitely far away; and the eccentricity is zero. You can think of the eccentricity as a measure of how far the shape is from being a circle.
- There are two geometric definitions of an ellipse:

Definition 1: An ellipse represents a set of points in which the sum of distances between the points and two foci is constant.

Definition 2: An ellipse represents a set of points, which have a focus and directrix with an eccentricity of less than one.

- In Standard Form, the equation of an ellipse is:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

- The major axis of the ellipse is the line segment connecting two points on the ellipse which goes through the two foci. It is the longest line segment you can draw from one point on the ellipse to another. It is of length  $2a$  or  $2b$ , whichever is larger. To tell whether the ellipse is oriented vertically or horizontally, compare the denominators of the  $x$  and  $y$  terms. If the denominator of the  $x$  term is larger, the ellipse is horizontal; otherwise, it is vertical.
- The Standard Form equation of an ellipse only specifies ellipses whose major axis is parallel either to the  $x$ -axis or the  $y$ -axis. The minor axis is the line segment perpendicular to the major axis which goes through the midpoint of the major axis. The midpoint of the major axis is also the midpoint of the minor axis and is called the centre of the ellipse. The minor axis is of length  $2a$  or  $2b$ , whichever is smaller. The centre of the ellipse is the point  $(h, k)$ . The vertices are the points on the ellipse which are on the major and minor axes. They are  $(h \pm a, k)$  and  $(h, k \pm b)$ .
- The eccentricity  $e$  can be calculated using one of the following formulas:

$$\sqrt{1 - \frac{b^2}{a^2}} \text{ or } \sqrt{1 - \frac{a^2}{b^2}}.$$

Choose the formula which does not require you to take the square root of a negative number. The distance between the foci is  $2ae$  or  $2be$  (whichever is larger).

- In its most General Form, the equation of an ellipse is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where  $B^2 - 4AC < 0$ .

As with the circle, there are some degenerate cases in which an equation in this form does not represent an ellipse, but all ellipses can be represented by an equation in this form. It is always possible to get rid of the Bxy term by choosing a different set of xy axes. You need to rotate the axes so that the major axis of the ellipse is parallel to either the x-axis or the y-axis. So, you can think of the General Form of an ellipse as:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where  $AC > 0$ , i.e., A and C are both positive or both negative. Converting from General Form to Standard Form is done by completing the square in the same way as for circles.

- To convert from Standard Form to General Form, just multiply out and collect like terms.

## Parabola

- The path followed by a thrown ball is a parabola. The mathematical definition of a parabola is the locus of points which are equidistant from a focus and a directrix (i.e. a conic section with eccentricity equal to one).
- The Standard Form equation for a parabola is:

$$(y - k)^2 = 4a(x - h)$$

or

$$(x - h)^2 = 4a(y - k)$$

- As with ellipses, the Standard Form only works for parabolas which are orientated so that their axis is parallel to either the x-axis or the y-axis. The point (h, k) is called the vertex of the parabola. It is the point on the parabola which lies on the line which passes through the focus and which is perpendicular to the directrix. The focus is either (h, k + a) or (h + a, k) depending on the orientation of the parabola. The directrix is either the line  $y = k - a$  or  $x = h - a$ , again depending on the orientation.
- The General Form of a parabola is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where  $B^2 - 4AC = 0$ . But, again, it is always possible to get rid of the Bxy term by rotating the axes. So, you can think of the General Form as:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where either  $A = 0$  or  $C = 0$ , but not both.

## Hyperbola

- A hyperbola is similar in appearance to a parabola. The main difference is that a hyperbola is made up of two disconnected curves while the parabola is a single curve. Like an ellipse, a hyperbola has two directrices and two foci. Geometrically, a hyperbola is representative of the points, whose difference in distance from the foci (taken absolutely) is constant.

The Standard Form for a hyperbola is:

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

or

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

- Like the Standard Forms for the ellipse and parabola, Standard Form works for hyperbolas that are orientated with the coordinate axes.

The eccentricity  $(e) = \sqrt{1 + b^2/a^2}$ .

The foci are either  $(h + \sqrt{a^2 + b^2}, k)$  and  $(h - \sqrt{a^2 + b^2}, k)$   
 or  $(h, k + \sqrt{a^2 + b^2})$  and  $(h, k - \sqrt{a^2 + b^2})$ .

Hyperbolas have asymptotes: two lines which cross at the centre of the hyperbola and approximate it far from the centre. The slope of the asymptotes is either  $\pm a/b$  (if the hyperbola opens up and down) or  $\pm b/a$  (if the hyperbola opens to the left and right). To find the equations of the asymptotes, you need to use the point-slope formula for a line.

- The General Form of a hyperbola is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where  $B^2 - 4AC > 0$ . Again, we can always rotate the axes to eliminate the  $Bxy$  term, so the General Form can be thought of as:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where either  $A$  is positive and  $C$  is negative, or vice versa.

## Identification

### *Conic Sections*

- Provided below are some excellent sites on Conic Sections.

[http://en.wikipedia.org/wiki/Conic section](http://en.wikipedia.org/wiki/Conic_section)

<http://britton.disted.camosun.bc.ca/jbconics.htm>

<http://www.mathacademy.com/pr/prime/articles/conics/index.asp>

<http://www.sparknotes.com/math/precalc/conicsections/>

<http://www.mathwarehouse.com/geometry/circle/interactive-circle-equation.php>

<http://www.mathopenref.com/ellipseoptics.html>

<http://www.mathopenref.com/ellipse.html>

<http://www.geocities.com/Area51/Quadrant/3864/sketchunifiedconstr.htm>

[http://www.miterclamp.com/Woodworking video index.htm](http://www.miterclamp.com/Woodworking_video_index.htm)

<http://www.falstad.com/ripple/ex-parabola.html>

## Glossary

Apex of a Cone:	the point at the top.
Asymptote:	line which a curve approaches but never reaches.
Base of a Cone:	the circle at the bottom.
Centre of a Circle:	the unique point which is the same distance from every point on the circle.
Centre of an Ellipse:	the midpoint of the major axis.
Centre of a Hyperbola:	midpoint of the line joining the two foci.
Cone (informal definition):	a pointed figure with a circular base.
Cone (mathematical definition):	a three-dimensional figure consisting of the line segments connecting a circle to a point not in the plane of the circle which is on a line which passes through the center of the circle and forms a right angle with a diameter of the circle.
Directrix:	a line used to define a conic section.
Eccentricity:	the ratio of the distance to the directrix and the distance to the focus; this is the same for every point on any given conic section.
Focus:	a point used to define a conic section.

Major axis of an Ellipse:	the longest line segment which connects two points on the ellipse.
Major radius of an Ellipse:	half the length of the major axis, often symbolized $a$ .
Minor axis of an Ellipse:	the line segment connecting two points on an ellipse which passes through the centre of the ellipse and is perpendicular to the major axis.
Minor radius of an Ellipse:	half the length of the minor axis, often symbolized $b$ .
Plane:	a two-dimensional flat surface.
Quadratic Equation:	polynomial equation of degree 2.
Radius:	the distance from the centre of a circle to a point on the circle.
Unit Circle:	circle with centre $(0,0)$ and radius 1.
Vertex of a Parabola:	the point on the parabola which is also on the line joining the focus to the directrix which is perpendicular to the directrix.

## References

<http://en.wikipedia.org/>