About Trigonometry

II
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About Trigonometry II

What is TRIGONOMETRY?

- This module is part two about learning with Trigonometry. The user should be familiar with the ideas and concepts of "About Trigonometry" before proceeding with this module.

Special Triangles

Special Triangles

- In trigonometry, there are two "special" right-angled triangles that make finding sines, cosines, etc simpler.

- The first special triangle can be referred to as the "45-45" Special Triangle, since two of its angles are 45 degrees. The remaining angle, therefore, is 90 degrees.

![45-45 Special Triangle]

The "45-45" Special Triangle is found by taking an isosceles triangle (meaning it has two equal sides and two equal angles) and finding its hypotenuse by using Pythagoras' Theorem. Using SOH CAH TOA, we can find exact values for sin, cos, tan, csc, sec, and cot of 45 degrees. These are as follows:

\[
\begin{align*}
\sin 45 &= \frac{1}{\sqrt{2}} \\
\cos 45 &= \frac{1}{\sqrt{2}} \\
\tan 45 &= 1 \\
\csc 45 &= \sqrt{2} \\
\sec 45 &= \sqrt{2} \\
\cot 45 &= 1
\end{align*}
\]
The second special triangle can be referred to as the "30-60" Special Triangle, as its corresponding angles are 30 and 60 degrees. The remaining angle, again, is 90 degrees. This is found through use of an equilateral triangle (meaning it has three equal sides, and three equal interior angles) of side length 2, as shown below:

Using this triangle, and SOH CAH TOA, we can see the following:

\[
\begin{align*}
\sin 30 &= \frac{1}{2} & \csc 30 &= 2 \\
\cos 30 &= \frac{\sqrt{3}}{2} & \sec 30 &= \frac{2}{\sqrt{3}} \\
\tan 30 &= \frac{1}{\sqrt{3}} & \cot 30 &= \sqrt{3} \\
\sin 60 &= \frac{\sqrt{3}}{2} & \csc 60 &= \frac{2}{\sqrt{3}} \\
\cos 60 &= \frac{1}{2} & \sec 60 &= 2 \\
\tan 60 &= \sqrt{3} & \cot 60 &= \frac{1}{\sqrt{3}}
\end{align*}
\]

**Cosine Law**

Cosine Law

Sometimes, we will not have right-angled triangles, preventing the use of SOH CAH TOA. As such, we require another method. Enter: The Cosine Law.
We are already familiar with right-angled triangles, but what of triangles that are not right-angled? Below are provided two additional types of triangles relative to their interior angles.

### Obtuse Triangle

- The Cosine Law may require knowledge of two side lengths, along with the angle corresponding to the unknown side length. For example, if we wished to find 'b' in the triangle below, we would need 'c,' 'a,' and the angle 'B.'

![Obtuse Triangle](image)

The Cosine Law is:

\[
\begin{align*}
  a^2 &= b^2 + c^2 - 2bc\cos A \\
  b^2 &= a^2 + c^2 - 2ac\cos B \\
  c^2 &= a^2 + b^2 - 2ab\cos C
\end{align*}
\]

Note: the cosine in the equation accounts for why the law is called "The Cosine Law."

- Sometimes, we will be given three side lengths and be asked to find an angle. We can use Cosine Law, again, but this requires manipulation of the equation.
Say we have side lengths 'a,' 'b,' and 'c,' where we want to find angle A. With the Cosine Law, we have:

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ a^2 - b^2 - c^2 = -2bc \cos A \quad \text{Here, we subtracted both } b^2 \text{ and } c^2 \text{ from each side of the equation.} \]
\[ \frac{(a^2 - b^2 - c^2)}{(-2bc)} = \cos A \quad \text{Here, we divided both sides of the equation by } -2bc. \]
\[ \cos^{-1}\left(\frac{(a^2 - b^2 - c^2)}{(-2bc)}\right) = A \quad \text{Finally, we take the inverse Cos to isolate the angle } A. \]

So, \[ A = \cos^{-1}\left(\frac{(a^2 - b^2 - c^2)}{(-2bc)}\right) \]

An approach similar to that of the previous can be used for finding angles B and C. Below is a summary of the resulting equations.

\[ a^2 = b^2 + c^2 - 2bc \cos A \rightarrow A = \cos^{-1}\left(\frac{(a^2 - b^2 - c^2)}{(-2bc)}\right) \]
\[ b^2 = a^2 + c^2 - 2ac \cos B \rightarrow B = \cos^{-1}\left(\frac{(b^2 - a^2 - c^2)}{(-2ac)}\right) \]
\[ c^2 = a^2 + b^2 - 2ab \cos C \rightarrow C = \cos^{-1}\left(\frac{(c^2 - a^2 - b^2)}{(-2ab)}\right) \]

**Sine Law**

- For the Sine Law, we require knowledge of two side lengths and one angle (opposite to one of the known side lengths), or knowledge of two angles and one side length (the side length opposite to one of the known angles). Consider the diagram on the below.
To use Sine Law, we will need angle C and side c (or angle B and side b, or angle A and side a), and one other angle or side. The goal is to find the unknown angle or side length, corresponding to the unpaired "other angle or side." For example, say we have angle A and side a, and we want side 'b.' In this case, we need to know the value of angle B beforehand.

- The Sine Law is written as shown below:

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

It does not matter whether the side lengths or angles are in the numerator, provided you remain consistent (i.e. if one side length is in the numerator, every side length should be in the numerator, etc.).

Note: the Sines in the equation account for why the law is called "The Sine Law."

**The Ambiguous Case**

- There are times when two different triangles can satisfy our solution to the Sine Law. This is called the Ambiguous Case.

- Earlier in this module, we discovered that \( \sin 45 = \frac{1}{\sqrt{2}} \). In a previous module, we found that we can find the same angle by subtracting the given angle from 180, as below:

\[
\Theta = 45 \\
180 - \Theta = 135
\]

\[
\sin 45 = \frac{1}{\sqrt{2}} \\
\sin 135 = \frac{1}{\sqrt{2}}
\]

This causes a problem when using Sine Law, as our angle could be acute or oblique, giving two different triangles.
For the Ambiguous Case, the following criteria must be met:

1) Angle A is acute
2) Side length 'a' is shorter than side length 'b'
3) Side length 'a' is longer than the height from angle B to angle C, which can be represented by \( b \sin A \) through SOH CAH TOA.

With the given criteria, angle B (or C) can be obtuse or acute, with either case being considered correct. The only instance that a distinction between the obtuse and acute cases is required is in real life situations (e.g. architecture).

**Indentities**

**Trigonometric Identities**

- In Trigonometry, there exist a number of identities or 'truths.' Much of the time, these identities are useful to remember, as they can result in the simplification of difficult questions. This section will summarize some important identities.
The first set of identities, called "Pythagorean Identities," involve right-angled triangles and The Pythagorean Theorem.

We are already familiar with the fact that $a^2 + b^2 = c^2$, and we can use SOH CAH TOA to find that $\sin \theta = \frac{a}{c}$ and $\cos \theta = \frac{b}{c}$. From here, it is simply a matter of plugging in.

So, we have the following:

\[
\begin{align*}
\sin \theta &= \frac{a}{c} \quad \rightarrow \quad a = \sin \theta c \\
\cos \theta &= \frac{b}{c} \quad \rightarrow \quad b = \cos \theta c
\end{align*}
\]

Through substitution, we arrive at the equation below:

\[
c^2 \sin^2 \theta + c^2 \cos^2 \theta = c^2
\]

\[
\sin^2 \theta + \cos^2 \theta = 1 \quad \text{(through division by } c^2)\]

This is the first Pythagorean Identity. We can find the other two by similar arguments, resulting in the following:

\[
\begin{align*}
1 + \tan^2 \theta &= \sec^2 \theta \\
1 + \cot^2 \theta &= \csc^2 \theta
\end{align*}
\]
Below are six additional identities, known as the "Sum and Difference Identities."

\[ \cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y) \]
\[ \cos(x-y) = (\cos x)(\cos y) + (\sin x)(\sin y) \]

\[ \sin(x+y) = (\sin x)(\cos y) + (\cos x)(\sin y) \]
\[ \sin(x-y) = (\sin x)(\cos y) - (\cos x)(\sin y) \]

\[ \tan(x+y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)} \]
\[ \tan(x-y) = \frac{\tan x - \tan y}{1 + (\tan x)(\tan y)} \]

We will now consider the three "Product-to-Sum Identities," which can be found by algebraically manipulating the "Sum and Difference Identities." What follows is a sample manipulation, to demonstrate how these equations are derived.

\[ (\sin x)(\sin y) = (1/2)[\cos(x-y) - \cos(x+y)] \] (1)
\[ (\sin x)(\cos y) = (1/2)[\sin(x+y) + \sin(x-y)] \] (2)
\[ (\cos x)(\cos y) = (1/2)[\cos(x+y) + \cos(x-y)] \] (3)

Now, for (1), remember that:

\[ \cos(x-y) = (\cos x)(\cos y) + (\sin x)(\sin y) \] (call this equation 'a')
\[ \cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y) \] (call this equation 'b')

If we subtract equation 'a' from equation 'b,' we get:

\[ \cos(x-y) - \cos(x+y) = (\sin x)(\sin y) - (-\sin x)(\sin y) \]

This can be simplified to our desired equation:

\[ (1/2)[\cos(x-y) - \cos(x+y)] = (\sin x)(\sin y) \]
These final identities are known as the "Double-Angle Identities," which are also derived from the "Sum and Difference Identities." As before, we will cover a motivating derivation.

The two identities are:

\[
\cos(2x) = 1 - (2)\sin^2 x \quad (1)
\]
\[
\sin(2x) = (2)\sin x \cos x \quad (2)
\]

Looking at (1), and remembering that \( \cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y) \), we let \( x = y \):

\[
\cos(x+x) = (\cos x)(\cos x) - (\sin x)(\sin x)
\]
\[
\cos(2x) = \cos^2 x - \sin^2 x
\]
\[
\cos(2x) = \cos^2 x - (1 - \cos^2 x) \quad (\sin^2 x = 1 - \cos^2 x)
\]
\[
\cos(2x) = 2\cos^2 x - 1
\]
\[
\cos(2x) = 2(1 - \sin^2 x) - 1 \quad (\cos^2 x = 1 - \sin^2 x)
\]
\[
\cos(2x) = 2 - 2\sin^2 x - 1
\]
\[
\cos(2x) = 1 - 2\sin^2 x
\]
Glossary

30-60 Special Triangle: A triangle that has a 30° angle and a 60° angle.

45-45 Special Triangle: A triangle that has two 45° angles, with the other angle being 90°. Two of the side lengths are equal, with the hypotenuse being the longest side.

Acute Angle: An angle measuring less than 90°.

Acute Triangle: A triangle in which all three angles are less than 90°.

Equilateral Triangle: A triangle in which all side lengths are equal.

Hypotenuse: For a right-angled triangle, is the side opposite of the right angle.

Isosceles Triangle: A triangle in which two side lengths are equal.

Obtuse Angle: An angle measuring more than 90°.

Obtuse Triangle: A triangle in which one angle is greater than 90°.

Pythagorean Identity: One of three trigonometric identities that are derived from the Pythagorean Theorem.

Pythagorean Theorem: Given a right-angled triangle with sides a and b, and hypotenuse c, \( a^2 + b^2 = c^2 \).

Right Angle: A 90° or \( \pi/2 \) radian angle.

Right Triangle: A triangle containing a right angle.
SOH-CAH-TOA: A useful tool for finding angles in right-angled triangles. It stands for
\[ \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \]
\[ \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \]
\[ \tan \theta = \frac{\text{opposite}}{\text{adjacent}} \]

Special Triangles: Two triangles that are useful in determining the exact value of different angles.
References
