About Limits and Continuity: Examples
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Rational Functions

Continuity
Introduction to Limits

Take the function \( f(x) = 3x - 7 \). We know this function as a linear function (it has the form of \( y = mx + b \)), now let us see what happens when \( x \) approaches 3.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.9</td>
<td>1.7</td>
</tr>
<tr>
<td>2.99</td>
<td>1.97</td>
</tr>
<tr>
<td>2.999</td>
<td>1.997</td>
</tr>
<tr>
<td>2.9999</td>
<td>1.9997</td>
</tr>
<tr>
<td>3.1</td>
<td>2.3</td>
</tr>
<tr>
<td>3.01</td>
<td>2.03</td>
</tr>
<tr>
<td>3.001</td>
<td>2.003</td>
</tr>
<tr>
<td>3.0001</td>
<td>2.0003</td>
</tr>
</tbody>
</table>

It looks like our function approaches 2 when \( x \) approaches 3. Of course, this makes sense, as \( f(3) = 2 \). Notice that we approached 3 from the left (plugging in values less than 3) and from the right (plugging in values greater than 3).
As a limit, the previous example would be written as:

\[ \lim_{x \to 3} (3x - 7) = 2 \]

**Limits**

**Limits**

In our initial example, the right-hand and left-hand limits were the same. Now, we will look at an example where these limits are not the same.

Consider the function below:

\[ f(x) = \begin{cases} 
-5x & \text{if } x < 6 \\
 x + 2 & \text{if } x \geq 6 
\end{cases} \]

Let us find the limit of \( f(x) \) as \( x \) approaches 6. Taking \( x \) as it approaches 6 from the left:

\[ f(5.99) = -5(5.99) = -27.95 \]

Now, taking \( x \) as it approaches 6 from the right:

\[ f(6.01) = 6.01 + 2 = 8.01 \]

So, having calculated the limits, we now know the following:

\[ \lim_{x \to 6^-} f(x) = -28 \]

\[ \lim_{x \to 6^+} f(x) = 8 \]

Because \( \lim_{x \to 6^-} f(x) \neq \lim_{x \to 6^+} f(x) \), we say this limit d.n.e.
Properties of Limits

Property 1

\[ \lim_{x \to a} 121 = 121 \]

Property 2

\[ \lim_{x \to 7} (x + 5) = \lim_{x \to 7} x + \lim_{x \to 7} 5 \]

\[ = 7 + 5 \]

\[ = 12 \]

Property 3

\[ \lim_{x \to 0} (x - 1) = \lim_{x \to 0} x - \lim_{x \to 0} 1 \]

\[ = 0 - 1 \]

\[ = -1 \]

Property 4

\[ \lim_{x \to 5} ((2x^2)(8x)) = (\lim_{x \to 5} 2x^3) (\lim_{x \to 5} 8x) \]

\[ = 2(125) * 8(5) \]

\[ = 250 * 40 \]

\[ = 10,000 \]

Property 5

\[ \lim_{x \to 3} ((x^4)/(2x)) = (\lim_{x \to 3} x^4)/(\lim_{x \to 3} 2x) \quad (\text{assume } x \neq 0) \]

\[ = 81 / 6 \]

\[ = 13.5 \]

As is evident in the examples given, properties can – and often are necessarily - combined. To further emphasize this point, consider:

\[ \lim_{x \to a} (7x^2 - x + 11) = \lim_{x \to a} 7x^2 - \lim_{x \to a} x + \lim_{x \to a} 11 \]
Rational Functions

Consider the following function:

\[ f(x) = \frac{1}{x-7} \]

We know that this function has a denominator of zero when \( x = 7 \), giving us a vertical asymptote. So what happens when we try taking the limit?

\[
\lim_{x \to 7} f(x)
\]

To solve this, we approach 7 from the left-hand and right-hand sides.

Notice that as we approach 7 from the left-hand side, we get smaller and smaller values. Here, we say \( f(x) \) approaches negative infinity \((-\infty)\).

Conversely, when approaching 7 from the right-hand side, we get larger and larger values. Here, we say \( f(x) \) approaches positive infinity \((+\infty)\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.9</td>
<td>-10</td>
</tr>
<tr>
<td>6.99</td>
<td>-100</td>
</tr>
<tr>
<td>6.999</td>
<td>-1000</td>
</tr>
</tbody>
</table>

\[
\lim_{x \to 7^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 7^+} f(x) = +\infty
\]

Since the right-hand and left-hand limits are unequal, we conclude the limit d.n.e.
Now, let us consider a more complex example, where \( f(x) = \frac{x-3}{x^3 - 27} \).

Notice that \( x = 3 \) produces a denominator of zero, where \( f(3) \) yields an indeterminate result (i.e. \( 0/0 \)). As before, let us investigate the limit by approaching it from both the left-hand and right-hand sides.

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
2.9 & 0.03830 \\
2.99 & 0.03716 \\
2.999 & 0.03705 \\
\hline
x & f(x) \\
\hline
3.1 & 0.03583 \\
3.01 & 0.03691 \\
3.001 & 0.03702 \\
\hline
\end{array}
\]

It appears that the function approaches 0.037. What is causing this finding?

Let us consider the function again, by first factoring it.

\[
f(x) = \frac{x-3}{x^3 - 27}
\]

We notice here that the denominator is factorable.

\[
f(x) = \frac{x-3}{(x-3)(x^2 + 3x + 9)}
\]

\( (x^3 - 27) = (x-3)(x^2 + 3x + 9) \)

By simplification:

\[
f(x) = \frac{1}{x^2 + 3x + 9}
\]

Now, \( f(3) \) is no longer indeterminate, and equals \( 1/27 \) (or 0.037). As expected, this is precisely what our limit approached.
Here, we see the graph. Because the limit existed at 3, but was undefined in the original function, we say there is a "hole" at (3, 1/27).

**Continuity**

► Below is an example of a "removable discontinuity:"

► Below is an example of a "jump discontinuity:"
Below is an example of "displaced point jump:"