A Comparative Study of PID and LQR Control Strategies Applied to Inverted Pendulum Systems

by

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ABSTRACT

A Comparative Study of PID and LQR Control Strategies Applied to Inverted Pendulum Systems

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This study provides a comprehensive comparison of two popular control strategies: the well-known proportional-integral-derivative controller (PID) and the linear-quadratic regulator (LQR). Both strategies are applied to two types of inverted pendulum systems: the linear single inverted pendulum and the linear double inverted pendulum. The PID and LQR controllers were simulated in MATLAB for the linear single inverted pendulum. Both strategies were used for real-time control of a double inverted pendulum; an experimental setup built by Quanser. The comparative results and control performance study are discussed in the report. Future work and recommendations are also provided.
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Chapter 1

Introduction

1.1 Control System

A control system has components which are known as the control subject, controlled object and control media. People can maintain or change any variables in a machine, mechanism or other devices by applying the knowledge of control system; which means the controlled devices could reach a specific predetermined idea state by applying the different control strategies. In other words, a control system causes the controlled object to tend to a specific steady state. As an indispensable cornerstone of modern social development, the control system plays an important role. There is a close relationship between the control system and people’s daily life. We can find that many applications in our daily life are inseparable from control systems, such as the regulation of temperature for air conditioning, the adjustment of the elevator lifting speed, and automated aircraft landings [1].

On the other hand, control systems are not only limited to people’s daily lives, but are also widely used in scientific research and industry. Some examples include the balanced control of a rocket while landing and robotic automation in manufacturing cells. In 2017, R. Lungu and M. Lungu performed research that used different control methods to achieve safe landing of an aircraft. In the research, they mentioned several control methods, which include proportional-integral-derivative control (PID), fuzzy control and linear quadratic optimal control.
These control methods not only contribute to the research and development of the aerospace field but also have a high impact on the speed of industrial development. There are several applications that utilize automatic control, however most favourably are robots used in the manufacturing environment. Most of the automated machining robots utilize PID control, which was introduced by Elmer Sperry in 1911. As one of the critical research in the field of engineering, this report will focus on studying and comparing two popular control algorithms.

1.2 Inverted Pendulum System

An inverted pendulum system has an extremely unstable structure: the center of gravity is at the top of the system, but is attached to the ground (or table) at the bottom. This system has several characteristics, such as a non-linear, multivariable, and strong coupling. The study of an inverted pendulum is a combination of the research of robotics, control theory, computer control and many other fields, which is the classical study of dynamics and control theory. There are several types of inverted pendulums that have been expanded from original straight-line single inverted pendulums, such as a circular inverted pendulum, flat inverted pendulum, and composite inverted pendulum. According to the number of stages of the inverted pendulum, it can be classed into the single inverted pendulum, double inverted pendulum, triple inverted pendulum, and quadruple inverted pendulum. The single inverted pendulum is often used to learn the basic experiments of control theory, and the multi-stages inverted pendulum is commonly used in the research to test the control algorithms. As one of the popular equipment to test the control algorithms, there are several control algorithms have been applied, such as: the classical control theory (PID control theory), modern control theory.
(state feedback, fuzzy control theory, neural network control theory), and variable structure control theory (sliding mode control), among others.

The research of inverted pendulums can help engineers understand and solve several problems, such as: nonlinear problems, robustness problems, stabilization problems, and tracking problems. The methods and techniques found by studying inverted pendulum systems can be used in different industry sectors such as aeronautical docking control technology, verticality control in rocket landing, and precision instrument processing. Hence, the study of the inverted pendulum provides an excellent platform for control theory practice.

1.3 Previous Works and Expected Goals

Previous works can be found in [6], where we performed a comparison on the control methods of a linear double inverted pendulum. Two types of control theory, which are LQR control and LQG control, were implemented to control a linear double inverted pendulum. The results show that both the LQR and LQG controllers can balance the double inverted pendulum; however, the LQG controller demonstrated better performance.

This report will focus on the study and comparison of two different control algorithms applied to a single inverted pendulum and a double inverted pendulum. Both the PID and (LQR) controllers will be applied. The single inverted pendulum system will be simulated and controlled in the Matlab environment. The experimental apparatus of a double inverted pendulum were provided by Quanser. It is experimentally controlled and interfaced using Matlab. This report is organized as follows: related works are briefly summarized in Chapter 2, followed by the mathematical system in Chapter 3, and the results in Chapter 4.
Chapter 2

Related Works

In the past few years, many papers have been published on the research of inverted pendulum systems. There are three different purposes to control an inverted pendulum, which will be discussed next [3].

The first important research is to obtain stabilization of the system and tracking control of the inverted pendulum. A study from Wang in 2011 shows the results of applied PID controller on three different types of inverted pendulums. It was found that PID controllers show good tracking control and stabilization performance for the systems [3]. Another research from Wai and Chang in 2006 shows that they successfully controlled and tracked a dual-axis inverted pendulum by using sliding mode control strategy [4].

Another purpose is to control the inverted pendulum for stabilization. In 2008, controllers were developed by Chaturvedi, Mcclamroch, and Bernstein for controlling a 3D axially symmetric pendulum. The controllers, which they developed, succeeded in helping the system reach a stable situation [5]. Another study included the real-time control of a double inverted pendulum by Xu, Lyu, and Gadsden in 2018. This paper applied the Kalman filter to the system and fed the estimates into the controller. The control result between LQR and linear–quadratic–Gaussian (LQG) controllers were compared [6]. They found that both of these control strategies can balance the double inverted pendulum. In 2013, research from the Song, Song,
Liu, and Zhao applied several control methods for the real-time control for an inverted pendulum [7]. In their study, they found that fuzzy control demonstrated a better control result compared with the PID controller and pole placement method [7].

The third purpose of studying inverted pendulums are to control the pendulum during swing-up or the initial start-up of the system. In 2013, Glück, Eder, and Kugi combined a feedforward controller, a feedback controller, and an extended Kalman filter to control the swing-up of a triple inverted pendulum [8].

The stability and tracking control of an inverted pendulum are most relevant to industry, which can be used for research of robotic control and aircraft tracking. By browsing relevant literature, it was found that most of the papers studied the control a single inverted pendulum [3, 5, 7, 9, 10]. In recent years, research included newer control algorithms, such as fuzzy control, neural network control, and sliding mode control. However, these advance control algorithms are more difficult to apply in real life due to a high requirement to implement [10]. In industry, the PID controller is still widely accepted because of the advantage that it controls a system by the measured process variable instead of the underlying model process. It is also relatively easy to implement and tune for a desired system.
Chapter 3

Methodology and Experimental Results

This section introduces the system modeling and mathematics required to apply different control algorithms to inverted pendulum systems. The system modeling and control of a single inverted pendulum is discussed in Section 3.1. The balance control of a double inverted pendulum will be discussed in Section 3.2.

3.1 Single Inverted Pendulum

This section will focus on the mathematical modelling of a linear single inverted pendulum. Different control algorithms will be applied to this model in order to obtain an appropriate control signal to balance the system. The simulation in Matlab will be used to test the control results.

![Inverted pendulum diagram]

*Figure 1. Inverted pendulum*
3.1.1 System Modelling

A system of an inverted pendulum is shown in Figure 1. In the Figure, M is the mass of the cart that can move horizontally on the x-axis. The length of the pendulum is L, and the m is the mass of the pendulum. \( u \) symbolizes the force on the cart. \( \theta \) is the angle of the inverted pendulum, and will be zero when the system is in a stable or balanced situation. Applying Newton's second law of motion to the system shown in Figure 1, we can define the equation of motion in the horizontal direction as follows:

\[
M \frac{d^2x}{dt^2} + m \frac{d^2}{dt^2} (x + L \sin \theta) = F
\]  
(1)

The equation of motion in the vertical direction is defined as follows:

\[
mL \cos \theta \frac{d^2}{dt^2} (x + L \sin \theta) = mgL \sin \theta
\]  
(2)

The control techniques use in this report only work well for linear systems, as such that equations of motion found above need to be linearized. We can assume small angle motion of the pendulum in this system; such that, after we simplify, we have the following:

\[
(M + m)\ddot{x} + mL\ddot{\theta} = u(t)
\]  
(3)

\[
m\dddot{x} + mL\dddot{\theta} = u(t)
\]  
(4)

Applying the Laplace transform to equations (3) and (4), the state-space equations will be generated as follows:

\[
\dot{x} = Ax + Bu
\]  
(5)

\[
y = Cx + Du
\]  
(6)
In equations (5) and (6), $x$ represents the states in this system: the cart position, the velocity of the cart, the angle of the inverted pendulum, and velocity of the inverted pendulum. $A$ is the system state matrix, $B$ is the input matrix. The output matrix and the feedforward matrices are $C$ and $D$, respectively. $u$ is the input vector. The system state matrix and input matrix are defined (with values used in the study) as equations (7) and (8), respectively.

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -0.2 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & -0.1 & 6 & 0
\end{bmatrix}
\]  

(7)

\[
B = \begin{bmatrix}
0 \\
0.2 \\
0 \\
0.1
\end{bmatrix}
\]  

(8)

After mathematically modelling the system, observability, controllability and stability are studied. The observability and the controllability of the system can be tested by calculating the rank of observability and controllability matrices, respectively. If the value of rank is equal to the number of the states, the system is considered observable or controllable. The stability of a system can be tested by calculating the poles. The controllability and observability matrices are defined as follows:

\[
\text{Observability} = [C', A'C', A'^2C' ... A'^nC']
\]

(9)

\[
\text{Controllability} = [B, A'B, A'^2B ... A'^nB]
\]

(10)

where $n$ is the number of states and $C$ is the measurement matrix defined as follows:

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}
\]

(11)

Using equations (7) through (11), the observability rank and controllability rank are found to be equal to the number of states. This means that the system is both observable and controllable. However, the system is not stable because some of the poles are positive. Therefore, the PID
and LQR controllers will be applied to the system in order to provide a stable control and balance of the inverted pendulum.

3.1.2 PID Control of the Linear Single Inverted Pendulum

Proportional-integral-derivative (PID) controllers have been developed over the last century. It is the most popular controller for industrial control systems, as it is reliable and relatively easy to implement. The three important control terms used in a PID controller are proportional, integral, and derivative, and are illustrated in Figure 2. The principle of the PID controller is to manipulate the system via control signal to reach a steady state using the difference between the desired and measured values. A simple example is the cruise control function of a vehicle, where the engine power will be increased or decreased during an uphill or a downhill run to ensure that the vehicle moves at a constant speed.

![PID controller diagram](image)

*Figure 2. PID controller*
Equation (12) is the standard PID control algorithm. In the PID algorithm, $K_p$ refers to the proportional gain, which has a close relationship with the response of a system; a higher proportional gain causes a faster response of a system. However, if the proportional gain is too large, the system might be more unstable. Turning the integral gain which is $K_i$ can help the proportional controller to reduce the steady-state error, at the same time, it also causes overshoot of the system. The derivative gain $K_d$ can contribute to the system’s stability because it has the function of predicting the behaviour of the system.

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \frac{de(t)}{dt} \quad (12)$$

The tuning of the three control terms is the essential part of designing and applying a PID controller for a system. There are several ways to help engineers tune the PID controller. The first method of choosing PID control terms is manual tuning. Firstly, both the value of integral gain and the derivative gain are set into zero; and increase the value of proportional gain. The second step to tune a PID controller is to increase the integral gain when the system output reaches a loop oscillates. After the residual steady-state error is corrected by tuning value of integral gain, increase the value of derivative gain to help the system reaching the setpoint and keeping stable. Figure 3 illustrates the main PID parameters and their effects [11].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Rise time</th>
<th>Overshoot</th>
<th>Settling time</th>
<th>Steady-state error</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_p$</td>
<td>Decrease</td>
<td>Increase</td>
<td>Small change</td>
<td>Decrease</td>
<td>Degrade</td>
</tr>
<tr>
<td>$K_i$</td>
<td>Decrease</td>
<td>Increase</td>
<td>Increase</td>
<td>Eliminate</td>
<td>Degrade</td>
</tr>
<tr>
<td>$K_d$</td>
<td>Minor change</td>
<td>Decrease</td>
<td>Decrease</td>
<td>No effect in theory</td>
<td>Improve if $K_d$ small</td>
</tr>
</tbody>
</table>

*Figure 3. Parameters effects independently [11]*
The PID controller can control only one object at the same time. However, in the studied system there are two objects: the angle of the pendulum and the position of the cart. To solve this problem, the double PID controller is used in this report to control the single inverted pendulum. The control design is built in Matlab’s Simulink environment, which is shown in Figure 4. In Figure 4, the single inverted pendulum is built by the S-Function. The PID controller one is used to control the position of the cart; the angle of the pendulum is controlled by the PID controller 2. The scope, scope 1, scope2, and scope 3 are tracking the output of the single inverted pendulum system, which represents the displacement of the cart, the angle of the inverted pendulum, the velocity of the cart and velocity of the inverted pendulum respectively.

![Double PID control block diagram of an inverted pendulum](image)

**Figure 4. Double PID control block diagram of an inverted pendulum**

3.1.3 PID control result of the linear single inverted pendulum

The control result of a single inverted pendulum is shown in Figure 5 and Figure 6. According to the analysis of the results, we can find that the single inverted pendulum system spends about 7 seconds to reach an up-balance position. Also, the pendulum reaches the up-
balance position. The overshoot and low steady-state error are shown in the results. The good performance of control the stabilization for the single inverted pendulum is reached by applying the double PID controller.

*Figure 5. Cart position of PID control in an inverted pendulum*
3.1.4 LQR Control of the Single Inverted Pendulum

This section will focus on the simulation and applying a linear quadratic regulator (LQR) controller for a single inverted pendulum system.

The LQR control algorithm is one of the modern optimal control, which applies the knowledge from the optimization theory into feedback control method. The control target of LQR is a linear system in the form of state-space. In a linear state-space equation (5), the states are shown below:

\[ x = [x, \dot{x}, \theta, \dot{\theta}] \]  \hspace{1cm} (13)

According to control law, we know that:
\[ u = -Kx, \]  

(14)

Applying the equation (5) and (14), we can get a new state-space equation:

\[ \dot{x} = (A - BK)x, \]  

(15)

Where the \( K \) is to make the quadratic cost function to take the minimum value. The quadratic cost function is shown as:

\[ J = \int_{0}^{\infty} x(t)'^{T}Qx(t) + u(t)'Ru(t) dt \]  

(16)

Where the \( Q \) and \( R \) is system and measurement noise covariances [6]. The LQR control gain \( K \) can be calculated by the equation as follows:

\[ K = R^{-1}B'P, \]  

(17)

Where \( P \) can be found from the solution of the Riccati equation, which is as follows [12]:

\[ A'P + PA - PB R^{-1}B'P + Q = 0 \]  

(18)

According to the previous equations, it is clear that the LQR control gain \( K \) has a close relationship with \( Q \) and \( R \). In this report, the \( Q \) and \( R \) is selected as follows:

\[ Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 1 \\ 0 & 0 & 0 & 100 \end{bmatrix} \]  

(19)

\[ R = 0.0001 \]  

(20)

The LQR controller is applied in the Matlab Simulink to control the single inverted pendulum. The Matlab Simulink block of the single inverted pendulum is shown in Figure 7.
The LQR control gain is calculated as follows:

\[ K = [-100 \quad -249.59 \quad 26500 \quad 1540.8] \]  

(21)

3.1.5 LQR control results of the linear single inverted pendulum

The results of LQR control of the single inverted pendulum is shown in Figure 8 and Figure 9. In Figure 8, the blue curve is the position setting, and the cart output is the yellow curve. The cart output shows a reasonable overshoot; on the other hand, there still is a small steady-state error can be found in the yellow curve. After a small movement of the cart, the system can reach a stable situation in a short time.

In Figure 9, the pendulum reaches the up-balance position at around the 18th second, which spends almost 8 seconds to reach a stable situation after the cart move from one side to another side.
The results show that the LQR controller can provide a good control performance of the linear single inverted pendulum.

Figure 8. Cart position of LQR control in an inverted pendulum
3.2 Double Inverted Pendulum

In Section 3.1, a PID controller and an LQR controller are successfully applied to control a linear single inverted pendulum. However, it is an ideal model of the linear single inverted pendulum that is used. In order to learn more about the PID controller and LQR controller, this section will use these two controllers for a linear double inverted pendulum to achieve real-time control.

3.2.1 System Modeling

As a complex dynamic system, the linear double inverted have three important part, which are the cart, a short pendulum, and a long pendulum. The double inverted pendulum
system, which is used in this report, was built by Quanser (Markham, Ontario). The system model is shown in Figure 10.

In Figure 10, the length of the long pendulum and the short pendulum are symbolized by \( l_{p2} \) and \( l_{p1} \), respectively. The mass of cart, short pendulum and long pendulum is symbolized by \( m_c \), \( m_{p1} \) and \( m_{p2} \), respectively. The position of the cart is shown as \( X_c \). The vector \((X_{p1}, Y_{p1})\) is the position of the short pendulum; the long pendulum’s position is shown as vector \((X_{p2}, Y_{p2})\). The angle of the short pendulum is \( \alpha \); \( \theta \) is the angle of the long pendulum. The force that is given by the motor drives is \( F_c \). If the value of the two angles are close to zero.
when the cart is moving under the force that is generated the motor, the system reaches a up-
balance position.

In order to find the motion equations for the double inverted pendulum, the Euler-
Lagrange equation can be used; the Euler-Lagrange equation can be found as follows [6]:

\[
Q_i = \left( \frac{\partial^2 L}{\partial t \partial q_i} \right) - \frac{\partial L}{\partial q_i} \tag{22}
\]

The generalized coordinates \( q_i \) of this double inverted pendulum is found as follows:

\[
q(t)' = \begin{bmatrix} x_c(t) & \alpha(t) & \theta(t) \end{bmatrix} \tag{23}
\]

So that:

\[
\dot{q}(t)' = \begin{bmatrix} \frac{\partial x_c(t)}{\partial t} & \frac{\partial \alpha(t)}{\partial t} & \frac{\partial \theta(t)}{\partial t} \end{bmatrix} \tag{24}
\]

Applying the equation (24) and equation (23) into (22):

\[
Q_1 = \left( \frac{\partial^2 L}{\partial t \partial x_c} \right) - \frac{\partial L}{\partial x_c} \tag{25}
\]

\[
Q_2 = \left( \frac{\partial^2 L}{\partial t \partial \alpha} \right) - \frac{\partial L}{\partial \alpha} \tag{26}
\]

\[
Q_3 = \left( \frac{\partial^2 L}{\partial t \partial \theta} \right) - \frac{\partial L}{\partial \theta} \tag{27}
\]

To find the motion equation by using the Lagrangian equation, as follows:

\[
L = T - V \tag{28}
\]

So, the motion equation of the double inverted pendulum is shown as below:

\[
Q_1 = F_c - B_{eq} \dot{x}_c \tag{29}
\]

\[
Q_2 = -B_{p1} \dot{\alpha} \tag{30}
\]

\[
Q_3 = -B_{p2} \dot{\theta} \tag{31}
\]
Where the $B_{eq}, B_{p_1}, B_{p_2}$ is viscous damping of the cart, short pendulum and long pendulum respectively. The force from the servo motor can be described by the follow equation [13]:

$$
F_c = \frac{\eta_m K_g K_t}{r_m r_{mp}} \left( -\frac{K_g K_m \dot{x}_c}{r_{mp}} + \eta_m V_m \right)
$$

(32)

Applying the Jacobian matrix for this non-linear system to linearize the system; the Laplace transform function can be used to find the state-space equation as follows:

$$
\dot{x} = Ax + Bu
$$

(33)

$$
y = Cx + Du
$$

(34)

Where we have the following:

$$
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 4.8 & -0.16 & -35.47 & -0.0175 & 0.0192 \\
0 & 76.6004 & -31.9057 & -185.0961 & -0.3767 & 0.7224 \\
0 & -84.3013 & 123.7709 & 203.7045 & 0.7224 & -2.0634
\end{bmatrix}
$$

(35)

$$
B = \begin{bmatrix}
0 \\
0 \\
0 \\
4.1609 \\
21.7083 \\
-23.8907
\end{bmatrix}
$$

(36)

$$
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

(37)

In the system of the double inverted pendulum, there are six states. According to calculating the rank of observability matrix and controllability matrix, the results show that this system is observable and can be controlled because the value of the rank is the same as the number of the states.
3.2.2 PID Control of the Linear Double Inverted Pendulum

In the linear double inverted pendulum system, there are three states need to control which are the position of the cart, the angle of the two pendulums. The theory of a PID controller has been mentioned in Section 3.1.2. In order to control the linear double Inverted pendulum, three PID controls are implemented due to the PID controller's characteristic that a PID controller can only control one value. The control of a double inverted pendulum diagram block is shown as below:

![PID controller block diagram of the double inverted pendulum](image)

In Figure 11, the PID controller 1 is to control the value of the cart, and the value of short pendulum and the long pendulum is controlled by controller 2 and controller 3, respectively. The value of the control gain is shown as follow:

<table>
<thead>
<tr>
<th>Value</th>
<th>Controller 1</th>
<th>Controller 2</th>
<th>Controller 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_p$</td>
<td>-54</td>
<td>190</td>
<td>422</td>
</tr>
<tr>
<td>$K_i$</td>
<td>-28</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_d$</td>
<td>-10</td>
<td>43</td>
<td>45</td>
</tr>
</tbody>
</table>

*Figure 12. Value of PID Control gain for control double inverted pendulum*
3.2.3 PID Control Results of the Linear Double Inverted Pendulum

The PID control results of the linear double inverted pendulum are shown as below. According to the control results, the double inverted pendulum system is successfully controlled by the three PID controllers; the system is steady. Figure 13 shows the movement of the cart; the distance of the cart in the horizontal direction does not exceed 2 cm. The swing angles of the two pendulums are small, which is shown in Figure 14 and 15. The input voltage changed of the system is reasonable, which is shown in Figure 16.
Figure 13. Cart position of a PID controls in a double inverted pendulum system

Figure 14. Long pendulum position of a PID control in a double inverted pendulum system
Figure 15. Short pendulum position of PID control in a double inverted pendulum system

Figure 16. The voltage value in the PID control of the double inverted pendulum
Figure 17 shows the result the value of short pendulum after a external force applied. It is clear that value, after the system is disturbed, is greatly deviated from the value in the previous steady state.

3.2.4 LQR Control of the Linear Single Inverted Pendulum

The LQR control theory has been introduced in Section 3.1.4. The important parameters of control the double inverted pendulum are the system noise covariance matrix $Q$ and the measurement noise covariance matrix $R$, which are selected as follows:
\[
Q = \begin{bmatrix}
10 & 0 & 0 & 0 & 0 & 0 \\
0 & 50 & 0 & 0 & 0 & 0 \\
0 & 0 & 50 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.9 \\
0 & 0 & 0 & 0 & 0 & 0.9
\end{bmatrix}
\] (38)

\[
R = 0.002
\] (39)

The value of control gain is selected as follows:

<table>
<thead>
<tr>
<th>Value</th>
<th>K_1</th>
<th>K_2</th>
<th>K_3</th>
<th>K_4</th>
<th>K_5</th>
<th>K_6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>42</td>
<td>-180</td>
<td>-417</td>
<td>31</td>
<td>-48</td>
<td>-43</td>
</tr>
</tbody>
</table>

Figure 18 LQR control gain of the double inverted pendulum

The control diagram of control the double inverted pendulum is shown as follows:

Figure 19. LQR controller block diagram of the double inverted pendulum
3.2.5 LQR Control Results of the Linear Double Inverted Pendulum

The LQR controller is succeeded to control the balance of the double inverted pendulum. Figure 20 shows the cart position result; Figure 21 and Figure 22 is the angles of the long pendulum and short pendulum, respectively. Figure 23 is the result of the voltage value. The control result is shown as follow:

![Cart position of LQR control in a double inverted pendulum system](image)

*Figure 20. Cart position of LQR control in a double inverted pendulum system*
Figure 21. Long pendulum position of LQR control in a double inverted pendulum system

Figure 22. Short pendulum position of LQR control in a double inverted pendulum system
According to observe the voltage value results, the system starts function at 9 seconds. In Figure 20, the cart moves quickly at 9th second; the movement of the cart becomes more stable after a few seconds. By observing Figure 21 and Figure 22, we can find that there is a small oscillation of the two angles after the system works. The double inverted pendulum can reach an up-balance state. The motion state of a double inverted pendulum in the 40 seconds is shown as follows (next page).
Figure 24. Cart position of LQR control in a double inverted pendulum system in 40 seconds

Figure 25. Short pendulum position of LQR control in a double inverted pendulum system in 40 seconds
Figure 26. Long pendulum position of LQR control a double inverted pendulum system in 40 seconds

Figure 27. The voltage value of the LQR control of the double inverted pendulum in 40 seconds

We find the system reaches steady-state at 40 seconds, which means the LQR controller successfully helps the system to remain balanced upwards. Figures 28 to 31 show the LQR control results of the double inverted pendulum system from 10 to 30 seconds.
Figure 28. Cart position of LQR control a double inverted pendulum system from 10 seconds to 30 seconds

Figure 29. Short pendulum position of LQR control a double inverted pendulum system from 10 seconds to 30 seconds
In Figure 28, the blue curve is the cart position and the yellow curve is the setpoint. The position of the cart is very close to the setpoint. There are small values of overshoot and steady-error in the system. In order to prove the theory that the LQR has a good control effect
for the double inverted pendulum, an external force is added in this system. The results are shown in the following figures.

Figure 32. Cart position under the force applied

Figure 33. Short pendulum position under the force applied
According to the results, we find that the system starts working at 10 seconds; then the system stays at a balance condition. The value of the cart position is close to the setpoint value; small values of the overshoot and steady state error still exist in the system.
After the pendulums are balanced, an external force is applied to the double inverted system at 25 seconds. It is clear that the system has a more significant fluctuation when the force applied to the pendulums. However, the system can return to a balanced condition in a short time.

Overall, the LQR controller exhibits excellent performance of the double inverted pendulum stability control.
Chapter 4

Brief Discussion and Concluding Remarks

In Section 3.1, the linear single inverted pendulum system was simulated in the Matlab environment. Both the LQR and PID controllers are applied to this ideal model, which did not consider disturbances from noise. The control results demonstrate that the PID and LQR controller have good performance and are able to balance the linear single inverted pendulum. According to the PID control results, the overshooting is 12.8 percent; the rising time is 3.332 seconds, the root mean square error is 0.137, compared with 0.521 percent overshooting, 2.108 seconds of rising time and 0.37 of root mean square error by LQR control. The PID has a faster settling time compared with LQR, but the LQR shows better performance in terms of accuracy. The smaller values of overshoot and steady-state error are shown by using the LQR controller compared with the PID controller.

In Section 3.2, the goal of real-time control of a linear double inverted pendulum system was successful by using LQR and PID controllers. Both LQR and PID control can help the system reach a steady state in a short time. However, when there is an external force applied to the system, steady-state error becomes more prevalent using a PID controller compared with the LQR controller. The results of the real-time double inverted pendulum system control demonstrates that the LQR controller has a better balance control performance than the tune PID controller.
Overall, according to the control test of the ideal model of single inverted pendulum and physical model of a double inverted pendulum, we can find that the LQR control strategy provides a better control performance for the linear inverted pendulum system; which can overcome the external forces and instabilities.

The research of inverted pendulums can help engineers understand and solve several problems, such as: nonlinear problems, robustness problems, stabilization problems, and tracking problems. The methods and techniques found by studying inverted pendulum systems can be used in different industry sectors such as aeronautical docking control technology, verticality control in rocket landing, and precision instrument processing. Hence, the study of the inverted pendulum provides an excellent platform for control theory practice. Suggested future work for this project involves studying more external disturbances and implementing adaptive forms of the PID and LQR controllers.
References


