Attractors and Semi-Attractors of IFS

by
Maxwell Fitzsimmons

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ABSTRACT

ATTRACTORS AND SEMI-ATTRACTORS OF IFS

Maxwell Fitzsimmons
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Advisor:
Dr. Herb Kunze

It is well known that a finite set of contractive self maps on a metric space, called an iterated function system (IFS), admits a nonempty compact invariant set called the attractor of the IFS. It is also well known that the chaos game converges to “draw” the attractor. We examine generalized notions of IFSs, attractors and the convergence of the chaos game to these generalized attractors. We focus on IFSs whose Hutchinson-Barnsley operator is a lower semicontinuous (l.s.c) multifunction, this includes infinite and possibly discontinuous IFS. In this case we develop several characterizations of smallest/minimal nonempty closed sub-invariant sets of the IFS. Under the same assumptions, we then give some necessary conditions for the chaos game to converge. Then, under the assumption that the set of all finite compositions of functions in the IFS are equicontinuous and certain compactness assumptions, we establish that the chaos game converges.

Dedication

Firstly, I would like to thank my family, my mother and brother for their stalwart support of all my activities; including but not limited to mathematics, video games and dating foreign women. Speaking of, I would like to thank Ningping Cao for her unending encouragement she has given me throughout our time together. Additionally, I would like to thank Henry Hong, Lucas Janssen, Ryan Godard, Timothy Lam, Nickolas Foster and Jamey Newton for their constant friendship over the last 9 years. I also like to thank all my peers I've had the pleasure of working with throughout my time at the University of Guelph, particularly (and in no particular order), Harry Gabler, Connor Gregor, Erin Hanley, Alvaro Balkowski, Thomas Kielstra, Matthew Kreitzer, Arvind Ravi, Victoria Brott, Abby Das, Mike Yodzis, Nishan Mudalige, Richard Yam, Bryson Boreland and Adrian Lee. All of which deserve there own specific personal thanks from me, however the margins are not large enough to contain such gratitude. I would also like to personally thank Herb Kunze for all his efforts in fostering me both academically and personally as his graduate student, as a undergraduate student and as his friend.

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Chapter 1

Introduction

The subject of iterated function systems (IFS) is well studied and well known in popular culture. Typically, an IFS is defined to be a finite collection of contractive self maps on a metric space. In this case it is well known there is a unique nonempty compact invariant set (or self similar) of the IFS called the attractor. Attractors are often fractals that are interesting to look at. Thus there is some motivation to find out what this attractor looks like. The most common method to draw the attractor is to use random iteration algorithms, such as the chaos game.

In less conventional settings the maps of an IFS are allowed to be possibly non-contractive. In this setting, it is an active area of research to determine whether an IFS possesses an attractor and, if it does possess one then, to determine whether the chaos game can draw the attractor: see [3, 9, 10, 11, 5]. In many of these works attractors are defined via limits in the Hausdorff metric (or Vietoris topology). This is perhaps a natural setting for the chaos game, as any set that is a limit set of a sequence of sets in the Hausdorff metric (or Vietoris topology) will be compact and thus bounded. If the attractor is unbounded it will be impossible to draw on a computer and perhaps defeats the purpose of the chaos game and IFS to begin with.

Nevertheless, in this work, for the sake of generality, we avoid the use of Hausdorff metric (and also the Vietoris topology). Instead we follow Lasota, Andrzej and Myjak [11] and work with IFSs with lower semicontinuous Hutchinson-Barnsley operators. We characterize when such an IFS has a smallest nonempty closed sub-invariant set and to a lesser degree we explore minimal nonempty closed sub-invariant sets of IFSs. Furthermore, under the assumption that the set of all finite compositions of functions of the IFS is an equicontinuous set, we show that the chaos game produces these minimal (and therefore smallest) nonempty closed sub-invariant sets, with initial point starting in a (possibly) large basin of attraction. If in addition to these assumptions the space is compact, then the basin of attraction is as large as possible. Furthermore, we discuss a necessary condition on the point used to initialize the chaos game.

Throughout this work, we assume the reader is comfortable with mathematical analysis and metric spaces, including, open sets, closed sets, continuous functions, uniformly continuous functions, Lipschitz continuous functions, forward/backward images of functions and the composition of functions.

In Chapter 2 we briefly touch on the convergence of sequences of functions in a metric space. We move on to a discussion on the convergence of sets and the various types of continuity of multifunctions with an emphasis on lower semicontinuous multifunctions. Anyone unfamiliar with these subjects on multifunctions should not skip reading section 2.3 of Chapter 2. In Chapter 3 we develop the motivating Theorems of finite contractive IFS. A reader familiar with such topics can skip this section to save time as only certain definitions from this chapter are used in the rest of the thesis. In Chapter 4 we develop the main results of this work. Including aforementioned results on minimal/smallest nonempty closed sub-invariant sets of an IFS, sufficient conditions for the convergence of the chaos game and necessary conditions on the convergence of the chaos game.

In the appendix we define some notions that are used freely throughout this work.

Chapter 2

Collections of Facts and Definitions

In this chapter we will mostly define and state Theorems related to metric spaces and set valued analysis. The Theorems will not be proved unless it is very easy, the author finds the proof insightful, the theorem will be used often or the author feels as though it is not a common result.

Throughout this chapter and in this whole work we will assume some basic understanding of metric spaces including: open sets, closed sets, convergence sequences and continuity of functions.

2.1 Miscellaneous Topics in Analysis

In this section we will present some basic definitions used thorough the work

The first thing we will discuss is the continuity of functions.

Definition 1. *Let* (X, d)*,* (Y, ρ) *be metric spaces and let* $\mathrm{f} : X \to Y$*. Then we say* f *is continuous at* $x \in X$ *if for every* $\epsilon > 0$ *there is a* $\delta_{x,\epsilon} > 0$ *for which*

$$\mathbb{B}^X_{\delta_{x,\epsilon}}(x) \subseteq \mathrm{f}^{-1}\big(\mathbb{B}^Y_\epsilon(\mathrm{f}(x))\big).$$

We say f *is continuous on* X *if* f *is continuous at every point in* X*. We also say that* f *is continuous if* f *is continuous on* X*.*

We say f *is uniformly continuous on* X *if for every* $\epsilon > 0$ *there is a* $\delta_\epsilon > 0$ *such that for all* $x \in X$ *we have*

$$\mathbb{B}^X_{\delta_\epsilon}(x) \subseteq \mathrm{f}^{-1}\big(\mathbb{B}^Y_\epsilon(\mathrm{f}(x))\big).$$

We also say that f *is uniformly continuous if* f *is uniformly continuous on* X*.*

We say that f *is Lipschitz continuous or simply Lipschitz on* X *if there is an* $L \in [0, \infty)$ *such that every* $x_1, x_2 \in X$ *we have*

$$\rho(\mathrm{f}(x_1), \mathrm{f}(x_2)) \leq L\,\mathrm{d}(x_1, x_2)$$

and L *is a (or the) Lipschitz constant of* f*. Furthermore we say that* f *is a contraction or contraction map if* f *is Lipschitz with a Lipschitz constant* $L < 1$*. We say that* f *is non-expansive if it has Lipschitz constant* $L \leq 1$ *and we call* f *expansive if it is Lipschitz but not non-expansive.*

2.2 Convergence of Sequences of Functions

The simplest way for a sequence of functions to "converge" is called pointwise convergence.

Definition 2. *Let* X *be a set and* (Y, d) *be a metric space. A sequence of functions,* $\{\mathrm{f}_n\}_{n\in\mathbb{N}}$ *where* $\mathrm{f}_n : X \to Y$ *for all* $n \in \mathbb{N}$*, is said to converge pointwise to the function* f*,* $\mathrm{f} : X \to Y$*, if for all* $x \in X$ *the point sequence* $\{\mathrm{f}_n(x)\}_{n\in\mathbb{N}} \to \mathrm{f}(x)$*.*

Pointwise convergence is a weak form of function convergence; one obstacle in using pointwise convergence is the limit functions are not necessarily continuous even if all the functions in the sequence are continuous.

Example 1. *Let* $X = Y = [0, 1]$ *with the normal metric. Consider the sequence of functions* $\{x^n\}_{n\in\mathbb{N}}$. *For every* $x_0 \in [0, 1)$ *the point sequence* $\{x_0^n\}_{n\in\mathbb{N}}$ *converges to* 0 *but when* $x_0 = 1$ *the point sequence* $\{1^n\}_{n\in\mathbb{N}}$ *converges to* 1. *Hence the limit function is*

$$\mathrm{f}(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

which is discontinuous despite x^n *being continuous for every* $n \in \mathbb{N}$.

Also note that if we take $X = Y = [-1, 1]$ *then the sequence of functions in question does not converge pointwise because when* $x_0 = -1$ *the point sequence* $\{(-1)^n\}_{n\in\mathbb{N}}$ *does not converge.*

Luckily there are conditions that ensure that the limit of a sequence of continuous functions is continuous. One such condition is the notion of uniform convergence.

Definition 3. *Let* X *be a set and* (Y, d) *be a metric space. A sequence of functions,* $\{\mathrm{f}_n\}_{n\in\mathbb{N}}$ *where* $\mathrm{f}_n : X \to Y$ *for all* $n \in \mathbb{N}$, *is said to converge uniformly to the function* f, $\mathrm{f} : X \to Y$, *if for all* $\epsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that for all* $x \in X$ *we have* $\mathrm{d}(\mathrm{f}_n(x), \mathrm{f}(x)) < \epsilon$. *Or equivalently for all* $\epsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that* $\sup_{x\in X} \mathrm{d}(\mathrm{f}_n(x), \mathrm{f}(x)) < \epsilon$.

The definition of uniform convergence is subtlely different from that of pointwise convergence. However this subtle difference actually makes uniform convergence a much stronger condition. Intuitively, uniform convergence of a sequence of functions requires the sequence to converge pointwise, but the point sequences must all converge at the same rate.

We can actually use the idea of uniform convergence to define a metric space on continuous functions.

Theorem 1. *Let* (X, ρ) *be a compact metric space and* (Y, d) *be a complete metric space. Let* $\mathcal{C}(X, Y)$ *be the set of all continuous functions from* X *to* Y. *Then* $\mathcal{C}(X, Y)$ *is a complete* *metric space with metric*

$$\mathrm{d}_u(\mathrm{f}_1, \mathrm{f}_2) = \sup_{x \in X} \mathrm{d}(\mathrm{f}_1(x), \mathrm{f}_2(x)).$$

Often we write $\mathrm{d}(\mathrm{f}_1, \mathrm{f}_2)$ *instead of* $\mathrm{d}_u(\mathrm{f}_1, \mathrm{f}_2)$ *and we call this the uniform distance or sup-distance.*

The compactness of X guarantees that the uniform distance is finite. One can discuss instead the set of all bounded functions from X to Y in a similar manner.

An immediate corollary of the above is for X compact and Y complete a convergent sequence of $\mathcal{C}(X, Y)$ converges to a continuous function. However we can get a similar result without the compactness on X.

Theorem 2. *Let* (X, ρ) *and* (Y, d) *be metric spaces. Suppose that* $\{f_n\}_{n \in \mathbb{N}}$*,* $f_n : X \to Y$ *continuous for all* $n \in \mathbb{N}$*, converges to* f *uniformly. Then* f *is continuous.*

Proof. First, fix $\epsilon > 0$ and pick an $n \in \mathbb{N}$ large enough so that $\mathrm{d}(\mathrm{f}, \mathrm{f}_n) < \frac{\epsilon}{3}$. Since f_n is continuous for any $x_0 \in X$ we can pick $\delta > 0$ so that $\mathrm{f}_n\big(\mathbb{B}^X_\delta(x_0)\big) \subseteq \mathbb{B}^Y_{\frac{\epsilon}{3}}(\mathrm{f}_n(x_0))$. Now consider an $x \in \mathbb{B}^X_\delta(x_0)$

$$\begin{aligned}
\mathrm{d}(\mathrm{f}(x), \mathrm{f}(x_0)) &\leq \mathrm{d}(\mathrm{f}(x), \mathrm{f}_n(x)) + \mathrm{d}(\mathrm{f}_n(x), \mathrm{f}_n(x_0)) + \mathrm{d}(\mathrm{f}(x_0), \mathrm{f}_n(x_0)) \\
&< \mathrm{d}(\mathrm{f}, \mathrm{f}_n) + \frac{\epsilon}{3} + \mathrm{d}(\mathrm{f}, \mathrm{f}_n) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

Since x_0 was arbitrary, this shows f is continuous. □

The last subject of this chapter explores how equicontinuity and uniform convergence are related.

Definition 4. *Let (X, ρ) and (Y, d) be metric spaces. Let $\mathcal{F}$ be a subset of Y^X then $\mathcal{F}$ is said to be equicontinuous if for all $x \in X$ and for all $\epsilon > 0$ there is a $\delta > 0$ for all $\mathrm{f} \in \mathcal{F}$ such that*

$$\mathrm{f}\left[\mathbb{B}_{\delta}^{X}(x)\right] \subseteq \mathbb{B}_{\epsilon}^{Y}(\mathrm{f}(x)).$$

Similarly $\mathcal{F}$ is said to be uniformly equicontinuous if for all $\epsilon > 0$ there is a $\delta > 0$ for all $\mathrm{f} \in \mathcal{F}$ and all $x_1, x_2 \in X$ with $\rho[x_1, x_2] < \delta$ we have $\mathrm{d}(\mathrm{f}(x_1), \mathrm{f}(x_2)) < \epsilon$.

A set of functions is equicontinuous if all the functions in the set are continuous and with reference to Definition 4, the same "δ" works for all the functions for any given x and $\epsilon > 0$. We can similarly talk about uniformly equicontinuous sets in which the same δ works for all the functions independent of a given x. Notice any finite set of continuous functions is equicontinuous. The simplest example of an infinite equicontinuous set is the collection of all Lipschitz functions with Lipschitz constant less than some fixed $k \in (0, \infty)$.

Equicontinuity turns out to be deeply related to the uniform convergence of a sequence of functions. But first we need a quick preliminary result.

Theorem 3 (A Generalized Heine-Cantor Theorem)**.** *Let (X, ρ) be a compact metric space, (Y, d) be a metric space and $\mathcal{F} \subseteq Y^X$. Then $\mathcal{F}$ is equicontinuous if and only if $\mathcal{F}$ is uniformly equicontinuous.*

Proof. ($\implies$) Pick $\epsilon > 0$ and for all $x \in X$ get the δ_x from the definition of equicontinuity for the given ϵ. Thus $\bigcup_{x \in X} \mathbb{B}^X_{\frac{\delta_x}{2}}(x)$ is an open cover of X. By compactness it has a finite sub-cover; let this cover be $\bigcup_{i=1}^{N} \mathbb{B}^X_{\frac{\delta_{x_i}}{2}}(x_i)$ for some $N \in \mathbb{N}$ and $x_i, i \in [N]$. Now pick $\delta = \min_{i \in [N]} \frac{\delta_{x_i}}{2}$ and pick $x, y \in X$ with $\rho(x, y) < \delta$. Since $x \in X \subseteq \bigcup_{i=1}^{N} \mathbb{B}^X_{\frac{\delta_{x_i}}{2}}(x_i)$ there is $j \in [N]$ such that $\rho(x, x_j) < \frac{\delta_{x_j}}{2}$. Consider

$$\rho(y, x_j) \leq \rho(y, x) + \rho(x, x_j) < \delta + \frac{\delta_{x_j}}{2} \leq \delta_{x_j}$$

meaning that for all $\mathrm{f} \in \mathcal{F}$

$$\mathrm{d}(\mathrm{f}(x), \mathrm{f}(y)) \leq \mathrm{d}(\mathrm{f}(x), \mathrm{f}(x_j)) + \mathrm{d}(\mathrm{f}(x_j), \mathrm{f}(y)) < 2\epsilon.$$

($\Longleftarrow$) follows immediately from the definitions.

$\square$

Note that the proof of Theorem 3 is the almost exactly the same as the normal Heine-Cantor Theorem. If $\mathcal{F} = \{\mathrm{f}\}$ for some $\mathrm{f} \in Y^X$ in the statement of Theorem 3 then we have the statement for the Heine-Cantor Theorem.

We are now ready to discuss the main result related to equicontinuity.

Theorem 4 (Arzelà–Ascoli Theorem)**.** *Let (X, ρ) be a compact metric space, (Y, d) be a complete metric space and let $\mathcal{C}(X, Y)$ be the set of all continuous functions from X to Y endowed with the uniform metric. Then $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is compact if and only if the following conditions hold:*

1. *$\mathcal{F}$ is closed.*
2. *for all $x \in X$ the set $\bigcup_{\mathrm{f} \in \mathcal{F}} \{\mathrm{f}(x)\}$ is totally bounded (or equivalently relatively compact).*
3. *$\mathcal{F}$ is equicontinuous.*

Proof. ($\Longrightarrow$) Suppose that $\mathcal{F}$ is compact. Then it is closed, as all compact sets are closed.

Take any $x \in X$ and any sequence $\{\mathrm{f}_n(x)\}_{n\in\mathbb{N}}$ of $\bigcup_{\mathrm{f} \in \mathcal{F}} \{\mathrm{f}(x)\}$ then the function sequence $\{\mathrm{f}_n\}_{n\in\mathbb{N}}$ has a convergent subsequence say $\{\mathrm{f}_{n_k}\}_{k\in\mathbb{N}} \to \mathrm{f}$. Since the convergence is uniform it must also converge pointwise. So $\{\mathrm{f}_{n_k}(x)\}_{k\in\mathbb{N}} \to \mathrm{f}(x)$, thus every sequence of $\bigcup_{\mathrm{f} \in \mathcal{F}} \{\mathrm{f}(x)\}$ has a convergent subsequence making it relatively compact. Hence $\bigcup_{\mathrm{f} \in \mathcal{F}} \{\mathrm{f}(x)\}$ is totally bounded.

Lastly, $\mathcal{F}$ is totally bounded. So for any $\epsilon > 0$ there is a finite set of functions $\{\mathrm{f}_n\}_{n\in[N]}$ with $\mathcal{F} \subseteq \bigcup_{n=1}^{N} \mathbb{B}_{\frac{\epsilon}{3}}^{\mathcal{C}(X,Y)}(\mathrm{f}_n)$. Now the set $\{\mathrm{f}_n\}_{n\in[N]}$ is equicontinuous because it is finite so for any $x_0 \in X$ take $\delta > 0$ so that for all $n \in [N]$

$$\mathrm{f}_n\left[\mathbb{B}_\delta^X(x_0)\right] \subseteq \mathbb{B}_{\frac{\epsilon}{3}}^Y(\mathrm{f}(x_0)).$$

For $x \in \mathbb{B}_\delta^X(x_0)$ and any $\mathrm{f} \in \mathcal{F}$ we can take an $n \in [N]$ so that

$$\max\{\mathrm{d}(\mathrm{f}(x_0), \mathrm{f}_n(x_0)), \mathrm{d}(\mathrm{f}_n(x), \mathrm{f}(x))\} \leq \mathrm{d}(\mathrm{f}_n, \mathrm{f}) < \frac{\epsilon}{3}.$$

Now consider

$$\mathrm{d}(\mathrm{f}(x_0), \mathrm{f}(x)) \leq \mathrm{d}(\mathrm{f}(x_0), \mathrm{f}_n(x_0)) + \mathrm{d}(\mathrm{f}_n(x_0), \mathrm{f}_n(x)) + \mathrm{d}(\mathrm{f}_n(x), \mathrm{f}(x)) < \epsilon.$$

Thus $\mathcal{F}$ is equicontinuous.

($\Longleftarrow$) By Theorem 1, $\mathcal{C}(X, Y)$ is complete and by assumption $\mathcal{F}$ is closed. Hence $\mathcal{F}$ is complete and we need only show that $\mathcal{F}$ is totally bounded. Pick $\epsilon > 0$ and as, $\mathcal{F}$ is uniformly equicontinuous, (by Theorem 3) get $\delta > 0$, so for all $\mathrm{f} \in \mathcal{F}$ and $x \in X$

$$\mathrm{f}\left[\mathbb{B}_\delta^X(x)\right] \subseteq \mathbb{B}_{\frac{\epsilon}{3}}^Y(\mathrm{f}(x)).$$

As X is compact it is totally bounded so there is a finite set $\{x_i\}_{i\in[M]}$ with $X \subseteq \bigcup_{i\in[M]} \mathbb{B}_\delta^X(x_i)$. Thus for all $x \in X$ there must be a $j \in [M]$ with $x \in \mathbb{B}_\delta^X(x_j)$ and we can see that for all $\mathrm{f} \in \mathcal{F}$, $\mathrm{d}(\mathrm{f}(x), \mathrm{f}(x_j)) < \frac{\epsilon}{3}$.

Now for each $i \in [N]$ the set $\bigcup_{\mathrm{f}\in\mathcal{F}}\{\mathrm{f}(x_i)\}$ is totally bounded. So there is a finite set of $\bigcup_{\mathrm{f}\in\mathcal{F}}\{\mathrm{f}(x_i)\}$, say $\{\mathrm{f}_{ni}(x_i)\}_{n\in[N_i]}$, with $\bigcup_{\mathrm{f}\in\mathcal{F}}\{\mathrm{f}(x_i)\} \subseteq \bigcup_{n=1}^{N_i} \mathbb{B}_{\frac{\epsilon}{3}}^Y(\mathrm{f}_{ni}(x_i))$.

We now claim that $\mathcal{F} \subseteq \bigcup_{i\in[M]} \bigcup_{n\in N_i} \mathbb{B}_\epsilon^{\mathcal{C}(X,Y)}(\mathrm{f}_{ni})$. Pick $\mathrm{f} \in \mathcal{F}$ and $x \in X$ and let x_j be within δ of x (as discussed before); then there is an $n \in [N_i]$ such that $\mathrm{d}(\mathrm{f}_{nj}(x_j), \mathrm{f}(x_j)) < \frac{\epsilon}{3}$ (as $\bigcup_{\mathrm{f} \in \mathcal{F}} \{\mathrm{f}(x_i)\} \subseteq \bigcup_{n=1}^{N_i} \mathbb{B}^Y_{\frac{\epsilon}{3}}(\mathrm{f}_{ni}(x_i))$ for all $i \in [M]$). Now we can see

$$\mathrm{d}(\mathrm{f}(x), \mathrm{f}_{nj}(x)) \leq \mathrm{d}(\mathrm{f}(x), \mathrm{f}(x_j)) + \mathrm{d}(\mathrm{f}(x_j), \mathrm{f}_{nj}(x_j)) + \mathrm{d}(\mathrm{f}_{nj}(x_j), \mathrm{f}_{nj}(x)) < \epsilon.$$

Since this holds for all $x \in X$ we can take the supremum over X and $\mathcal{F} \subseteq \bigcup_{i \in [M]} \bigcup_{n \in N_i} \mathbb{B}^{\mathcal{C}(X,Y)}_{\epsilon}(\mathrm{f}_{ni})$. Hence $\mathcal{F}$ is totally bounded and complete. Therefore $\mathcal{F}$ is compact. □

We can see now that equicontinuity is an essential part of compactness of subsets of $\mathcal{C}(X, Y)$ (when X is compact and Y is complete). This means that equicontinuity must be deeply tied to the uniform convergence of sequences as, if $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ converges uniformly then it must be relatively compact.

Corollary 4.1. *Let (X, ρ) be a compact metric space, (Y, d) be a complete metric space and let $\mathcal{C}(X, Y)$ be the set of all continuous functions from X to Y endowed with the uniform metric. Furthermore, let $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ be a sequence of $\mathcal{C}(X, Y)$. Then $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ converges uniformly to f if and only if $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ converges pointwise to f and the sequence is equicontinuous.*

Proof. ($\Longrightarrow$) The set $\overline{\{\mathrm{f}_n\}_{n \in \mathbb{N}}}$ is compact as the sequence converges uniformly. Thus by the Arzelà–Ascoli Theorem the set is equicontinuous. $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ converges pointwise since it converges uniformly.

($\Longleftarrow$) Since $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ converges pointwise, the set $\bigcup_{n \in \mathbb{N}} \{\mathrm{f}_n(x)\}$ is relatively compact for all $x \in X$. As $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ is equicontinuous this must mean that $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ is relatively compact (in $\mathcal{C}(X, Y)$). This means that every (uniformly) convergent subsequence of $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ must converge to f (as the sequence converges pointwise). Thus $\{\mathrm{f}_n\}_{n \in \mathbb{N}}$ converges uniformly to f. □

2.3 Convergence of Sets and Multifunctions

In this section we state some (relatively) basic things about multifunctions, particularly about their continuity. Multifunctions will be useful in Chapter 4.

Definition 5. *Let X and Y be sets. A function $F : X \to 2^Y$ is said to be a multifunction of X into Y. We write* $\mathrm{F} : X \rightsquigarrow Y$*. Furthermore for $B \subseteq X$ and $x \in X$ we adopt the notation*

$$\mathrm{F}[x] = \mathrm{F}(x)$$

and

$$\mathrm{F}[B] := \bigcup_{x \in B} \mathrm{F}[x].$$

Also define the domain of a multifunction $\mathrm{Dom}\,(\mathrm{F}) = \{x \in X | \, \mathrm{F}[x] \neq \emptyset\}$.

We use [ ] when dealing with multifunctions in order to make two things more clear. One, so there is no confusion whether $\mathrm{F}(x)$ is a set or a point and, two, seeing as the range of F is technically in 2^Y thus $\mathrm{F}(B)$ should be a collection of subsets (i.e a subset of 2^Y). However we almost always want to deal with the points in those subsets rather then the subsets themselves, the [ ] notation helps us get around this.

Example 2. *Let $X = Y = \mathbb{R}$ and define* $\mathrm{F} : \mathbb{R} \rightsquigarrow \mathbb{R}$ *to be for $x \in \mathbb{R}$*

$$\mathrm{F}[x] = \{y \in \mathbb{R} | x^2 + y^2 = 1\}.$$

We can see that $\mathrm{F}\left[\frac{1}{\sqrt{2}}\right] = \{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\}$ *and in fact* $|\,\mathrm{F}[x]| \leq 2$ *for all $x \in \mathbb{R}$. The domain is* $\mathrm{Dom}\,(\mathrm{F}) = [-1, 1]$.

A simpler example is for any $\mathrm{f} : \mathbb{R} \to \mathbb{R}$ *we can make* f *into an multifunction by letting for $x \in \mathbb{R}$*

$$\mathrm{f}[x] = \{\mathrm{f}(x)\}.$$

Additionally the inverse image of f *is also a multifunction. Let* $\mathrm{f}^{-1} : \mathbb{R} \rightsquigarrow \mathbb{R}$ *be for* $x \in \mathbb{R}$

$$\mathrm{f}^{-1}[x] = \{y |\, \mathrm{f}(y) = x\}.$$

Finally, the $\epsilon > 0$ *balls are also multifunctions. Let* $\epsilon > 0$ *is fixed then* $\mathbb{B}_\epsilon : \mathbb{R} \rightsquigarrow \mathbb{R}$ *defined by, for all* $x \in \mathbb{R}$*,*

$$\mathbb{B}_\epsilon[x] = \{y | \epsilon > |x - y|\}.$$

A multifunction is sometimes called a point-to-set function or a multivalued function. Along the same vein a function $\mathrm{f} : X \to Y$ is called a point-to-point function or single valued function.

We wish to delve into the continuity of multifunctions. Naturally, continuity for multifunctions will be inspired by the continuity of single valued functions. Recall that a function is continuous if and only if it preserves convergent sequences. That is, f is continuous at $x \in X$ if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}} \to x$ we have $\{\mathrm{f}(x_n)\}_{n\in\mathbb{N}} \to \mathrm{f}(x)$. If we wish to generalize this notion to multifunctions we will have to figure out what it means for a sequence of sets to converge.

Definition 6. *Let* (X, d) *be a metric space and let* $\{A_n\}_{n\in\mathbb{N}}$ *be a sequence of* $2^X \setminus \emptyset$*. Then we will say that* $\{A_n\}_{n\in\mathbb{N}}$ *is a sequence of sets of* X *or* $\{A_n\}_{n\in\mathbb{N}}$ *is a set sequence of* X*. If* X *is the only metric space in context will we will say* $\{A_n\}_{n\in\mathbb{N}}$ *is a sequence of sets or it is a set sequence.*

Now define the liminf, the lower limit or the inner limit, of a sequence of sets to be

$$\operatorname*{Li}_{n\to\infty} A_n = \{x |\, \limsup_{n\to\infty} \mathrm{d}(x, A_n) = 0\}.$$

Also define the limsup, the upper limit or the outer limit, of a sequence of sets to be

$$\operatorname*{Ls}_{n\to\infty} A_n = \{x | \liminf_{n\to\infty} \mathrm{d}(x, A_n) = 0\}.$$

In the above the notation for $B \subseteq X$, $\mathrm{d}(x, B) = \inf_{b\in B} \mathrm{d}(x, b)$.

Furthermore the sequence is said to converge to $A \subseteq X$ iff

$$A = \operatorname*{Li}_{n\to\infty} A_n = \operatorname*{Ls}_{n\to\infty} A_n.$$

And we write $\{A_n\}_{n\in\mathbb{N}} \to A$ or $A = \lim_{n\to\infty} A_n$.

Definition 6 may seem backwards at first glance, after all the so called "liminf" of a sequence of sets is defined via the lim sup of real numbers and vice versa for the Ls. But defining them as in Definition 6 allows the Ls and Li to behave much more like the lim sup and lim inf of real numbers. For example if $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers and $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of sets then $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ and similarly $\mathrm{Li}_{n\to\infty} A_n \subseteq \mathrm{Ls}_{n\to\infty} A_n$ (see (i) of Proposition 1).

For the most part, we will be using the terminology lower limit and upper limit. Furthermore Definition 6 is not the most useful way to define the upper and lower limit. We give more understandable characterizations of the upper and lower limits in Proposition 1. Particularly items 1, 2 and 4 will be used extensively.

Proposition 1. *Let (X, d) be a metric space and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of 2^X. Define the notation $\{a_n \in A_n\}_{n\in\mathbb{N}}$ to mean a point sequence in X with $\forall n \in \mathbb{N}$ $a_n \in A_n$. Then the following hold.*

1.

$$\operatorname*{Li}_{n\to\infty} A_n = \{x | \exists \{a_n \in A_n\}_{n\in\mathbb{N}} \text{ such that } \{a_n\}_{n\in\mathbb{N}} \to x\}$$

2.

$$\operatorname*{Li}_{n\to\infty} A_n = \bigcap_{\epsilon>0} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \mathbb{B}_\epsilon(A_n)$$

3.

$$\operatorname*{Ls}_{n\to\infty} A_n = \{x|\exists\{a_n \in A_n\}_{n\in\mathbb{N}} \text{ and a subsequence such that } \{a_{n_k}\}_{k\in\mathbb{N}} \to x\}$$

4.

$$\operatorname*{Ls}_{n\to\infty} A_n = \overline{\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n}$$

5.

$$\operatorname*{Ls}_{n\to\infty} A_n = \bigcap_{\epsilon>0} \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \mathbb{B}_\epsilon(A_n)$$

Furthermore, let $A \subseteq X$, $x^* \in X$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points. Then the following also hold

(i) $\operatorname*{Li}_{n\to\infty} A_n \subseteq \operatorname*{Ls}_{n\to\infty} A_n$

(ii) $\{A_n\}_{n\in\mathbb{N}} \to A$ *iff* $A \subseteq \operatorname*{Li}_{n\to\infty} A_n$ *and* $\operatorname*{Ls}_{n\to\infty} A_n \subseteq A$

(iii) if $\{x_n\}_{n\in\mathbb{N}} \to x^*$ *then the sequence of sets* $\{\{x_n\}\}_{n\in\mathbb{N}} \to \{x^*\}$

Proof. 1. ($\subseteq$) Recall the $\limsup$ of a sequence of real numbers $\{r_n\}_{n\in\mathbb{N}}$ (when the $\limsup$ is finite) is the largest limit of all of the convergent subsequences of $\{r_n\}_{n\in\mathbb{N}}$. Here if $x \in \mathrm{Li}_{n\to\infty} A_n$ then the set of real numbers $\{\mathrm{d}(x, A_n)\}_{n\in\mathbb{N}}$, is bounded below by zero and the largest limit of any convergent subsequence is 0. So the set of limit points of the convergent subsequences of $\{\mathrm{d}(x, A_n)\}_{n\in\mathbb{N}}$ is bounded above and below by zero. Thus every convergent subsequence converges to zero and so $\{\mathrm{d}(x, A_n)\}_{n\in\mathbb{N}} \to 0$. This means for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ with $\forall n \geq N$ $\mathrm{d}(x, A_n) < \epsilon$. We claim that for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for every $n \geq N$ we can pick $a_n \in A_n$ with $\mathrm{d}(x, a_n) < \epsilon$. If this is not the case, there is a $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ with every $a_n \in A_n$ $\epsilon \leq \mathrm{d}(x, a_n)$. Taking the

$\inf_{a_n \in A_n}$ yields a contraction. This shows that $\exists \{a_n \in A_n\}_{n\in\mathbb{N}}$ such that $\{a_n\}_{n\in\mathbb{N}} \to x$.
($\supseteq$) Conversely if x satisfies $\exists \{a_n \in A_n\}_{n\in\mathbb{N}}$ such that $\{a_n\}_{n\in\mathbb{N}} \to x$, then $\lim_{n\to\infty} \mathrm{d}(x, a_n) = 0$. As $0 \leq \mathrm{d}(x, A_n) \leq \mathrm{d}(x, a_n)$ for all $n \in \mathbb{N}$ we can take the limit of both sides and conclude that $\lim_{n\to\infty} \mathrm{d}(x, A_n) = 0$. Noting that because the limit exists it must be equal to the lim sup gives us the result.

2. ($\subseteq$) If $x \in \mathrm{Li}_{n\to\infty} A_n$ in the proof of 1 ($\subseteq$) we showed that $\{\mathrm{d}(x, A_n)\}_{n\in\mathbb{N}} \to 0$. Thus we have for all $\epsilon > 0$ there is a $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathrm{d}(x, A_n) < \epsilon$. This is precisely the definition of the set on the RHS. ($\supseteq$) The preceding argument can be made in reverse.

3. ($\subseteq$) Recall the lim inf of a sequence of real numbers $\{r_n\}_{n\in\mathbb{N}}$ (when the lim inf is finite) is the smallest limit of all of the convergent subsequences of $\{r_n\}_{n\in\mathbb{N}}$. Thus if $x \in \mathrm{Ls}_{n\to\infty} A_n$ there is a subsequence of $\{\mathrm{d}(x, A_n)\}_{n\in\mathbb{N}}$ say $\{\mathrm{d}(x, A_{n_k})\}_{k\in\mathbb{N}} \to 0$. From here one could follow a similar argument to that made in 1 to show there is a sequence $\{a_{n_k}\}_{k\in\mathbb{N}} \to x$ that is a subsequence of some $\{a_n \in A_n\}_{n\in\mathbb{N}}$.
($\supseteq$) Let x be such that there is a $\{a_n \in A_n\}_{n\in\mathbb{N}}$ and a subsequence such that $\{a_{n_k}\}_{k\in\mathbb{N}} \to x$. Then again we consider $0 \leq \mathrm{d}(x, A_{n_k}) \leq \mathrm{d}(x, a_{n_k})$ and take the limit. So $\{\mathrm{d}(x, A_{n_k})\}_{k\in\mathbb{N}} \to 0$ and as the $\liminf_{n\to\infty} \mathrm{d}(x, A_n) \geq 0$. Therefore $\liminf_{n\to\infty} \mathrm{d}(x, A_n) = 0$ and $x \in \mathrm{Ls}_{n\to\infty} A_n$.

4. ($\subseteq$) Proceed by contraposition. Suppose $x \notin \overline{\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n}$: then there is $\delta > 0$ such that $\mathbb{B}_\delta(x) \cap (\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n) = \emptyset$ as $X \setminus \overline{\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n}$ is open. Thus there is $m \in \mathbb{N}$ such that for all $n \geq m$, $\delta \leq \mathrm{d}(x, A_n)$. We can interpret this to mean the tail of the sequence $\{\mathrm{d}(x, A_n)\}_{n\in\mathbb{N}}$ is always bounded below by $\delta > 0$ and so the $\liminf_{n\to\infty} \mathrm{d}(x, A_n)$ must be positive. Thus $x \notin \mathrm{Ls}_{n\to\infty} A_n$.
($\supseteq$) Suppose $x \in \overline{\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n}$ then for all $\delta > 0$ $\mathbb{B}_\delta(x) \cap (\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n) \neq \emptyset$. So for any

$\delta > 0$, $\forall m \in \mathbb{N}\ \exists n \geq m$ and $\exists a_n \in A_n$ with

$$0 \leq \mathrm{d}(x, A_n) \leq \mathrm{d}(x, a_n) < \delta.$$

As the above holds for all $\delta > 0$, it is straightforward to pick a subsequence of $\{\mathrm{d}(x, A_n)\}_{n\in\mathbb{N}}$ that converges to zero.

5. ($\subseteq$) Using 3, if $x \in \mathrm{Ls}_{n\to\infty} A_n$ there is an $\{a_n \in A_n\}_{n\in\mathbb{N}}$ and a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}} \to x$. We have for every $\epsilon > 0$ there is $K \in \mathbb{N}$ such that for every $k \geq K$ we have $\mathrm{d}(x, a_{n_k}) < \epsilon$. So for any fixed $\epsilon > 0$ and any $m \in \mathbb{N}$ there is a $k \geq K$ so that $n_k \geq m$ and $\mathrm{d}(x, a_{n_k}) < \epsilon$. Noting that n_k is equal to some n gives us the result.
($\supseteq$) Let $x \in \bigcap_{\epsilon>0} \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \mathbb{B}_\epsilon(A_n)$. Then for $N \in \mathbb{N}$ we can pick $\epsilon = \frac{1}{N}$ and there is a $n_N \geq N$ and an $a_{n_N} \in A_{n_N}$ with $\mathrm{d}(x, a_{n_N}) < \frac{1}{N}$. Thus by picking the a_{n_N}'s we can construct a sequence $\{a_{n_N} \in A_{n_N}\}_{N\in\mathbb{N}} \to x$ which is also a subsequence of some $\{a_n \in A_n\}_{n\in\mathbb{N}}$. So by 3 $x \in \mathrm{Ls}_{n\to\infty} A_n$.

(i) Using 1 and 3, we can see that if x is such that there is a $\{a_n \in A_n\}_{n\in\mathbb{N}} \to x$ then any subsequence of $\{a_n\}_{n\in\mathbb{N}}$ must also converge to x and so $x \in \mathrm{Ls}_{n\to\infty} A_n$, by 3.

(ii)

$$\begin{aligned}
&\{A_n\}_{n\in\mathbb{N}} \to A \\
\iff &A = \operatorname*{Li}_{n\to\infty} A_n = \operatorname*{Ls}_{n\to\infty} A_n = A \\
\iff &A \subseteq \operatorname*{Li}_{n\to\infty} A_n \subseteq \operatorname*{Ls}_{n\to\infty} A_n \subseteq A \\
\iff &A \subseteq \operatorname*{Li}_{n\to\infty} A_n \text{ and } \operatorname*{Ls}_{n\to\infty} A_n \subseteq A \qquad \text{where (i) gives the “upward” implication}
\end{aligned}$$

(iii) Note that there is only one possible sequence satisfying $\{v_n \in \{x_n\}\}_{n\in\mathbb{N}}$ which is the point sequence $\{x_n\}_{n\in\mathbb{N}}$. All of its subsequences converge to x^* so the Ls is the set containing x^*. Similarly the Li must also be $\{x^*\}$. Thus the sequence of sets converges to $\{x^*\}$. □

Items 1 through 5 of the above give us useful characterizations of the upper and lower limits. Items 1 and 3 are typically used as the definitions of Li and Ls respectively; for good reason it is much easier to speak of the set of all limits of convergent point sequences of the set sequence (the lower limit) or the set of limits of the all the convergent point subsequences of the set sequence (the upper limit). From items 1 and 3 one can more easily reason some elementary proprieties of the upper and lower limits; for instance item (i) or the fact both of the sets are closed. Another conceptually pleasing thought is that the lower limit is the set of all points that are "close" to all but finitely many sets of the set sequence (see item 2, the $\bigcap_{n\geq m} \mathbb{B}_\epsilon(A_n)$ part can be read as being ϵ close to all except the first m sets). Similarly, the upper limit is the set of all points that are "close" to infinitely many sets of the set sequence (see item 4, the $\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \mathbb{B}_\epsilon(A_n)$ part can be read as always being ϵ close to one of the "last infinitely many" A_n's).

Items (i) through (iii) are simple results that help serve as sanity checks to show that Definition 6 is a good generalization of convergent (point) sequences and the lim inf and lim sup of real numbers.

One should notice that if $\{x_n\}_{n\in\mathbb{N}}$ is a point sequence then it converges if and only if $\mathrm{Li}_{n\to\infty}\{x_n\}$ is a singleton. In this case the lower limit is always empty or a singleton. Thus the concept of the lower limit is not a very interesting for point sequences. The upper limit however is more interesting in this case. Here the upper limit would contain the limits of all the convergent subsequences of $\{x_n\}_{n\in\mathbb{N}}$.

Example 3. *Let $X = \mathbb{R}$ with the usual metric. Consider the sequence of sets*

$$A_n = \begin{cases} [-n, n] & n \text{ even} \\ (0, n] & n \text{ odd} \end{cases}$$

Then the lower limit is $\mathrm{Li}_{n\to\infty} A_n = [0, \infty)$ and the upper limit is $\mathrm{Ls}_{n\to\infty} = \mathbb{R}$. Thus the set sequence does not converge. We can also see that $\{A_{2n}\}_{n\in\mathbb{N}}$ converges to $\mathbb{R}$. This would be an example of bounded sets converging to an unbounded set.

Along similar lines consider the set sequence

$$B_n = \begin{cases} (-\infty, -n] \cup (0, 1 + \frac{1}{n}) & n \text{ even} \\ (\frac{1}{n}, 1] \cup (n, \infty) & n \text{ odd} \end{cases}$$

converges to $[0, 1]$. This is an example of a sequence of unbounded sets converging to a bounded set.

An important result about real numbers is the monotone convergence theorem. It states every bounded monotone sequence of real numbers converges. We have a similar result for sequences of sets.

Theorem 5. *Let (X, d) be a metric space. Suppose that a sequence of sets $\{A_n\}_{n\in\mathbb{N}}$ is a monotone sequence, that is for all $n \in \mathbb{N}$ $A_n \subseteq A_{n+1}$ or $A_n \supseteq A_{n+1}$. Then $\{A_n\}_{n\in\mathbb{N}}$ converges to $\overline{\bigcup_{n\in\mathbb{N}} A_n}$ or to $\bigcap_{n\in\mathbb{N}} \overline{A_n}$ respectively.*

Proof. Suppose that $n \in \mathbb{N}$ $A_n \subseteq A_{n+1}$. Then by items (i) and 4 of Proposition 1 we can see that $\mathrm{Li}_{n\to\infty} A_n \subseteq \mathrm{Ls}_{n\to\infty} A_n \subseteq \overline{\bigcup_{n\geq 1} A_n} = \overline{\bigcup_{n\in\mathbb{N}} A_n}$. Thus we need only show that $\overline{\bigcup_{n\in\mathbb{N}} A_n} \subseteq \mathrm{Li}_{n\to\infty} A_n$. Let $x \in \bigcup_{n\in\mathbb{N}} A_n$ then there is an $m \in \mathbb{N}$ such that $x \in A_m$ but we know that for all $n \geq m$ that $A_n \supseteq A_m \ni x$. Thus we can define a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x by picking any $x_n \in A_n$ for $n < m$ and for all $n \geq m$ $x_n = x \in A_n$. Therefore

$x \in \mathrm{Li}_{n\to\infty} A_n$ (by item 1 of Proposition 1) and $\bigcup_{n\in\mathbb{N}} A_n \subseteq \mathrm{Li}_{n\to\infty} A_n$. Noting that $\mathrm{Li}_{n\to\infty} A_n$ is closed we can see that $\overline{\bigcup_{n\in\mathbb{N}} A_n} \subseteq \mathrm{Li}_{n\to\infty} A_n$.

Suppose that for all $n \in \mathbb{N}$ $A_n \supseteq A_{n+1}$. Then by items (i) and 2 of Proposition 1 we can see that $\bigcap_{n\in\mathbb{N}} \overline{A_n} = \bigcap_{\epsilon>0} \bigcap_{n\in\mathbb{N}} \mathbb{B}_\epsilon(A_n) \subseteq \mathrm{Li}_{n\to\infty} A_n \subseteq \mathrm{Ls}_{n\to\infty} A_n$ and so we need only show that $\mathrm{Ls}_{n\to\infty} A_n \subseteq \bigcap_{n\in\mathbb{N}} \overline{A_n}$. Because for all $n \in \mathbb{N}$ $A_n \supseteq A_{n+1}$ it means that for all $m \in \mathbb{N}$ $\bigcup_{n\geq m} A_n = A_m$, so by 4 of Proposition 1

$$\mathop{\mathrm{Ls}}_{n\to\infty} A_n = \overline{\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n} = \overline{\bigcap_{m\in\mathbb{N}} \overline{\bigcup_{n\geq m} A_n}} = \overline{\bigcap_{m\in\mathbb{N}} \overline{A_m}} = \bigcap_{n\in\mathbb{N}} \overline{A_n}$$

which proves the result. □

Remark 1. *A key assumption of the monotone convergence Theorem (of real numbers) is that the sequences need to be bounded (above or below depending on whether it is non-decreasing or non-increasing). This contrasts with Theorem 5 where there is no hypothesis even resembling the bounded condition. To see why this is missing: first note that if a set of real numbers is bounded from above then this is equivalent to the* sup *of the set being finite. The* sup *is of course the least upper bound of the set; this begs the question is there a* sup *of a collection of sets, say $\mathcal{B} \subseteq 2^X$? And in fact there is, in some sense: the union of all sets in the collection $\bigcup_{B\in\mathcal{B}} B$ contains every set in $\mathcal{B}$. So it is "greater" in the sense of inclusion. Furthermore one can easily see that it is the "least" upper bound as if for all $B \in \mathcal{B}$ we have $B \subseteq A$ then so must the union of all the B's and $\bigcup_{B\in\mathcal{B}} B \subseteq A$. Thus in the sense of inclusion every collection of subsets has a least upper bound. Similarly, every collection of subsets has a greatest lower bound given by the intersection of all the subsets in the collection.*

Thus there is no hypothesis that the monotone sequence of sets is bounded because every sequence of sets is bounded in the sense of inclusion.

We are now almost ready to resume our quest to understand the continuity of multifunctions. But first we need to discuss ways of generalizing the inverse image of single valued functions to multi-valued functions. The inverse image of a single valued function f is $\mathrm{f}^{-1}[B] = \{x|\, \mathrm{f}(x) \in B\}$. The trouble with generalizing this is with "$\mathrm{f}(x) \in B$", for if F is a multifunction $\mathrm{F}[x]$ can no longer be an element of B. Perhaps the most sensible generalization is to change all statements with $\mathrm{f}(x) \in B$ to $\mathrm{F}[x] \subseteq B$. This means we take $\mathrm{f}(x) \in B$ to mean all of elements of $\mathrm{f}(x)$ are in B, however since $\mathrm{f}(x)$ is a singleton we can also take $\mathrm{f}(x) \in B$ to mean there is at least one point of $\mathrm{f}(x)$ in B. Thus it is also conceivable to instead write $\mathrm{F}[a] \cap B \neq \emptyset$. Thus we can define two different concepts of the inverse image of a multifunction.

Definition 7. *Let X and Y be sets, $B \subseteq Y$ and let $\mathrm{F} : X \rightsquigarrow Y$. The upper pre-image or core of B by the multifunction* F *is*

$$\mathrm{F}^{+}[B] = \{x \in \mathrm{Dom}\,(\mathrm{F}) \,|\, \mathrm{F}[x] \subseteq B\}.$$

The lower pre-image or inverse image of B by the multifunction F *is*

$$\mathrm{F}^{-}[B] = \{x \in \mathrm{Dom}\,(\mathrm{F}) \,|\, \mathrm{F}[x] \cap B \neq \emptyset\}.$$

Both of these notions lead to two different types continuity for multifunctions. A definition of a continuous function in the point to point case is

$$\forall x \in X, \forall \epsilon > 0, \exists \delta > 0, \mathbb{B}^X_\delta(x) \subseteq \mathrm{f}^{-1}\big(\mathbb{B}^Y_\epsilon(\mathrm{f}(x))\big).$$

Replacing the f^{-1} with either the upper or the lower pre-image, (almost) gives us two distinct notions of continuity for multifunctions.

Definition 8. *Let (X, d), (Y, ρ) be metric spaces and let $\mathrm{F} : X \rightsquigarrow Y$.* F *is said to be upper* *semicontinuous (u.s.c) at the point* $x \in \mathrm{Dom}\,(\mathrm{F})$ *if and only if for all* $\epsilon > 0$ *and all* $y \in Y$ *with* $\mathbb{B}^Y_\epsilon(y) \supseteq \mathrm{F}[x]$ *there is a* $\delta > 0$ *such that*

$$\mathbb{B}^X_\delta(x) \subseteq \mathrm{F}^+\left[\mathbb{B}^Y_\epsilon(y)\right].$$

Furthermore F *is said to be u.s.c on* $B \subseteq \mathrm{Dom}\,(\mathrm{F})$ *iff it is u.s.c at every point in* B*.*

Definition 9. *Let* (X, d)*,* (Y, ρ) *be metric spaces and let* $\mathrm{F} : X \rightsquigarrow Y$*.* F *is said to be lower semicontinuous (l.s.c) at the point* $x \in \mathrm{Dom}\,(\mathrm{F})$ *if and only if for all* $\epsilon > 0$ *and all* $y \in Y$ *with* $\mathbb{B}^Y_\epsilon(y) \cap \mathrm{F}[x] \neq \emptyset$ *there is a* $\delta > 0$ *such that*

$$\mathbb{B}^X_\delta(x) \subseteq \mathrm{F}^-\left[\mathbb{B}^Y_\epsilon(y)\right].$$

Furthermore F *is said to be l.s.c on* $B \subseteq \mathrm{Dom}\,(\mathrm{F})$ *iff it is l.s.c at every point in* B*.*

One should not confuse u.s.c and l.s.c of multifunctions with upper and lower semi continuity of functions $\mathrm{f} : \mathbb{R} \to \mathbb{R}$. They are distinct and have little to do with one another. Despite this much like for the real valued point-to-point functions, we say that a multifunction is continuous at x in the domain of the multifunction if it is both u.s.c and l.s.c at x.

Example 4. *If* $\mathrm{f}(x)$ *is a continuous point to point function the set valued function* $\mathrm{f}[x]$ *is both upper and lower semicontinuous. This is because* $\mathrm{F}^+ = \mathrm{F}^-$ *in the single valued case.*

Let $X = Y = \mathbb{R}$ *with the usual metric. And consider the multifunction*

$$\mathrm{F}[x] = \begin{cases} \{0\} & x < 0 \\ [-1, 1] & x \geq 0 \end{cases}.$$

Then F *is u.s.c at* 0 *but not l.s.c at* 0*. To show upper semicontinuity we consider a* $y \in \mathbb{R}$

and $\epsilon > 0$ *with* $\mathrm{F}[0] \subseteq \mathbb{B}_\epsilon(y)$. *Now, for any* $\delta > 0$ *and* $x \in \mathbb{B}_\delta(0)$ *we have two cases* $\mathrm{F}[x] = [-1,1] \subseteq \mathbb{B}_\epsilon(y)$ *and* $\mathrm{F}[x] = \{0\} \subseteq \mathbb{B}_\epsilon(y)$. *In both cases we have that* $x \in \mathrm{F}^+[\mathbb{B}_\epsilon(y)]$. *Thus* F *is u.s.c.*

To show that F *is not l.s.c at* 0 *we consider* $y = 1$ *and* $\epsilon = 0.5$ *then* $\mathbb{B}_{0.5}(1) = (0.5, 1.5)$. *We can see that* $(0.5, 1.5) \cap \mathrm{F}[0] \neq \emptyset$ *but notice for any* $\delta > 0$ *the set* $\mathbb{B}_\delta(0)$ *must contain a negative number say* x *and so* $\mathrm{F}[x] = \{0\}$. *This means that* $\mathrm{F}[x] \cap \mathbb{B}_{0.5}(1) = \{0\} \cap (0.5, 1.5) = \emptyset$. *Therefore* F *in not l.s.c at* 0.

Now consider F_2 *defined as*

$$\mathrm{F}_2[x] = \begin{cases} \{0\} & x \leq 0 \\ [-1,1] & x > 0 \end{cases}.$$

Then F_2 *is l.s.c at* 0 *but not u.s.c at* 0. *To see l.s.c pick any* $y \in \mathbb{R}$ *and* $\epsilon > 0$ *with* $\mathbb{B}_\epsilon(y) \cap \mathrm{F}_2[0] \neq \emptyset$. *Now for any* $\delta > 0$ *and any* $x \in \mathbb{B}_\delta(0)$ *we have* $0 \in \mathrm{F}_2[x]$ *and since* $\mathrm{F}_2[0] = \{0\}$ *we must have* $\mathbb{B}_\epsilon(y) \cap \mathrm{F}_2[x] \neq \emptyset$ *as* $\mathbb{B}_\epsilon(y) \cap \mathrm{F}_2[0] \neq \emptyset$. *So* F_2 *is l.s.c at* 0.

To show that F_2 *is not u.s.c at* 0 *we consider* $y = 0$ *and* $\epsilon = 0.5$ *then* $\mathbb{B}_{0.5}(0) = (-0.5, 0.5)$. *Now for every* $\delta > 0$ *the set* $\mathbb{B}_\delta(0)$ *must contain a positive number say* x *and so* $\mathrm{F}_2[x] = [-1,1] \not\subseteq (-0.5, 0.5) = \mathbb{B}_{0.5}(0)$. *Therefore* F_2 *in not u.s.c at* 0.

Example 4 shows us that u.s.c and l.s.c are distinct concepts (at least on $\mathbb{R}$). Intuitively one may think that both F and F_2 are "discontinuous": in both cases the "values" of the function change drastically at $x = 0$ and in fact both of these functions fail to be both upper and lower semi lower semicontinuous (and thus continuous). The multifunction F fails to be l.s.c as it suddenly becomes a "smaller" set when $x < 0$ and so all the points in $[-1,1] \setminus \{0\}$ are "stranded"; there are neighborhoods of those points that are not close to the multifunction's value when $x < 0$. On the other hand F_2 fails to be u.s.c because it suddenly becomes much "larger" when $x > 0$; there are points near to 0 that map to sets that are nowhere near the points in $\mathrm{F}_2[0]$.

In short l.s.c multifunctions "care" more about the individual points in $\mathrm{F}[x]$ being close to other nearby values in the range, whereas u.s.c multifunctions consider $\mathrm{F}[x]$ to be a collection of points and cares that the sets are "nearby" to each other (in the sense of containment).

There are many other types of continuity (or more commonly "semi continuity") for multifunctions. However, we will almost exclusively be focusing on lower semicontinuity. As it "plays nicely" with convergent sequences, as opposed to upper semicontinuous that works much better with open sets.

Theorem 6. *Let* (X, d) *and* (Y, ρ) *be metric spaces. Let* $F : X \rightsquigarrow Y$ *with* $\mathrm{Dom}\,(\mathrm{F}) = X$ *then the following are equivalent.*

1. *F is l.s.c on X.*
2. *For every $x \in X$, every $y \in \mathrm{F}[x]$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x' \in \mathbb{B}^X_\delta(x)$ then $\mathrm{F}[x'] \cap \mathbb{B}^Y_\epsilon(y) \neq \emptyset$.*
3. *For all $x \in X$, for all sequences $\{x_n\}_{n\in\mathbb{N}} \to x$ and for all $y \in \mathrm{F}[x]$ there exists a sequence $\{y_n \in \mathrm{F}[x_n]\}_{n\in\mathbb{N}}$ that converges to y.*
4. *For every point sequence of X $\{x_n\}_{n\in\mathbb{N}}$ that converges to x we have $F[x] \subseteq \mathrm{Li}_{n\to\infty}\,\mathrm{F}[x_n]$.*
5. *For every point sequence of X $\{x_n\}_{n\in\mathbb{N}}$ that converges to x we have $F[x] \subseteq \mathrm{Ls}_{n\to\infty}\,\mathrm{F}[x_n]$.*
6. *For every set $U \subseteq X$ we have $\mathrm{F}\left[\overline{U}\right] \subseteq \overline{\mathrm{F}[U]}$.*
7. *For every open set $V \subseteq Y$ the set $\mathrm{F}^-[V]$ is open in X.*

Proof. $1 \implies 2$ Pick $x \in X$ then for all $y \in \mathrm{F}[x]$ and all $\epsilon > 0$ we have $\mathrm{F}[x] \cap \mathbb{B}^Y_\epsilon(y) \neq \emptyset$. So by lower semicontinuity there is a $\delta > 0$ such that $\mathbb{B}^X_\delta(x) \subseteq \mathrm{F}^-\left[\mathbb{B}^Y_\epsilon(y)\right]$. Thus, by definition of the lower preimage we have that if $x' \in \mathbb{B}^X_\delta(x)$ then $\mathrm{F}[x'] \cap \mathbb{B}^Y_\epsilon(y) \neq \emptyset$

$2 \implies 6$ Proceed by contraposition. Suppose that there is $U \subseteq X$ with $\mathrm{F}\big[\overline{U}\big] \nsubseteq \overline{\mathrm{F}[U]}$. Thus there is a $y \in \mathrm{F}\big[\overline{U}\big] \cap (Y \setminus \overline{\mathrm{F}[U]})$; so there is an $x \in \overline{U}$ with $y \in \mathrm{F}[x]$ and since $Y \setminus \overline{\mathrm{F}[U]}$ is open in Y, there is an $\epsilon > 0$ with $\mathbb{B}^Y_\epsilon(y) \subseteq Y \setminus \overline{\mathrm{F}[U]}$. Also, as $x \in \overline{U}$, for all $\delta > 0$ there is a $x' \in \mathbb{B}^X_\delta(x) \cap U \neq \emptyset$. Thus

$$\mathrm{F}[x'] \cap \mathbb{B}^Y_\epsilon(y) \subseteq \mathrm{F}[U] \cap \mathbb{B}^Y_\epsilon(y) \subseteq \overline{\mathrm{F}[U]} \cap \mathbb{B}^Y_\epsilon(y) = \emptyset.$$

$6 \implies 3$ Proceed by contraposition. Thus there is an $x \in X$, a $y \in \mathrm{F}[x]$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ for which every sequence $\{y_n \in \mathrm{F}[x_n]\}_{n\in\mathbb{N}}$ does not converge to y. Thus there is an $\epsilon > 0$ for all $N \in \mathbb{N}$ and an $n \geq N$ for which all $y_n \in \mathrm{F}[x_n]$ have $\epsilon \leq \rho(y, y_n)$ so we can choose a subsequence of $\{x_{n_k}\}_{k\in\mathbb{N}} \to x$ that is eventually non-constant such that $\{y_{n_k} \in \mathrm{F}[x_{n_k}]\}_{k\in\mathbb{N}}$ satisfies for all $k \in \mathbb{N}$ $\epsilon \leq \rho(y, y_{n_k})$. Now define $X_N = \bigcup_{k\geq N}\{x_{n_k}\}$ for some N. Now since $x \in \overline{X_N}$ we have $y \in \mathrm{F}[x] \subseteq \mathrm{F}\big[\overline{X_N}\big]$. We show that $\mathbb{B}^Y_\epsilon(y) \cap \mathrm{F}[X_N] = \emptyset$: supposing $y' \in \mathbb{B}^Y_\epsilon(y) \cap \mathrm{F}[X_N]$ means

$$\epsilon \leq \rho(y, y') < \epsilon$$

as $y' = y_{n_k}$ for some k and $y' \in \mathbb{B}^Y_\epsilon(y)$, giving a contradiction. Thus $\mathbb{B}^Y_\epsilon(y) \cap \mathrm{F}[X_N] = \emptyset$ and so $y \notin \overline{\mathrm{F}[X_N]}$. Therefore $\mathrm{F}\big[\overline{X_N}\big] \nsubseteq \overline{\mathrm{F}[X_N]}$.

$3 \implies 1$ Proceed by contraposition. Suppose that there is an $y \in Y$, a $\epsilon > 0$ and an $x \in X$ with $\mathbb{B}^Y_\epsilon(y) \cap \mathrm{F}[x] \neq \emptyset$ such that for every $\delta > 0$ we have $\mathbb{B}^X_\delta(x) \nsubseteq \mathrm{F}^-\big[\mathbb{B}^Y_\epsilon(y)\big]$. Let $y^* \in \mathbb{B}^Y_\epsilon(y) \cap \mathrm{F}[x]$ and so there is an $\eta > 0$ with $\mathbb{B}^Y_\eta(y^*) \subseteq \mathbb{B}^Y_\epsilon(y)$. For each $n \in \mathbb{N}$, set $\delta = \frac{1}{n}$ and pick $x_n \in \mathbb{B}^X_{\frac{1}{n}}(x) \cap \big(X \setminus \mathrm{F}^-\big[\mathbb{B}^Y_\epsilon(y)\big]\big) \subseteq \mathbb{B}^X_{\frac{1}{n}}(x) \cap \big(X \setminus \mathrm{F}^-\big[\mathbb{B}^Y_\eta(y^*)\big]\big)$ as $\mathrm{F}^-\big[\mathbb{B}^Y_\eta(y^*)\big] \subseteq \mathrm{F}^-\big[\mathbb{B}^Y_\epsilon(y)\big]$. So for each $n \in \mathbb{N}$, since $x_n \notin \mathrm{F}^-\big[\mathbb{B}^Y_\eta(y^*)\big]$, we have $\emptyset = \mathrm{F}[x_n] \cap \mathrm{F}^-\big[\mathbb{B}^Y_\eta(y^*)\big]$. Thus for each $n \in \mathbb{N}$ and every $y_n \in \mathrm{F}[x_n]$ $0 < \eta \leq \rho(y_n, y^*)$. So there can be no $\{y_n \in \mathrm{F}[x_n]\}_{n\in\mathbb{N}}$ that converges to y^* and since $\{x_n\}_{n\in\mathbb{N}} \to x$ we can conclude that 3 does not hold.

$3 \implies 4$ Take $\{x_n\}_{n\in\mathbb{N}} \to x$, pick $y \in \mathrm{F}[x]$; then by Item 3 there exists a sequence of points $\{y_n\}_{n\in N} \to y$ with $y_n \in \mathrm{F}[x_n]$ for all $n \in \mathbb{N}$. Thus by item 1 of Proposition 1 $y \in \mathrm{Li}_{n\to\infty} \mathrm{F}[x_n]$.

$4 \implies 5$ Recall Item (i) of Proposition 1 and by 4 we have for all $x \in X$ and every $\{x_n\}_{n\in\mathbb{N}} \to x$

$$\mathrm{F}[x] \subseteq \operatorname*{Li}_{n\to\infty} \mathrm{F}[x_n] \subseteq \operatorname*{Ls}_{n\to\infty} \mathrm{F}[x_n].$$

$5 \implies 3$ Proceed by contraposition. So there is an $x \in X$ a $\{x_n\}_{n\in\mathbb{N}} \to x$ and a $y \in \mathrm{F}[x]$ such that every $\{y_n \in \mathrm{F}[x_n]\}_{n\in\mathbb{N}}$ does not converge to y. Thus y is not in the lower limit. But by Item (i) of Proposition 1 we can see

$$y \notin \operatorname*{Li}_{n\to\infty} \mathrm{F}[x_n] \subseteq \operatorname*{Ls}_{n\to\infty} \mathrm{F}[x_n].$$

Therefore 5 does not hold.

$1 \implies 7$ Suppose that V is open in Y. If $\mathrm{F}^{-}[V]$ is empty then we are done. So suppose that $x \in \mathrm{F}^{-}[V]$, this means that $\mathrm{F}[x] \cap V \neq \emptyset$. So pick $y \in \mathrm{F}[x] \cap V$ and there is $\epsilon > 0$ such that $\mathbb{B}^Y_\epsilon(y) \subseteq V$ (as V is open) and we can see, since $y \in \mathrm{F}[x]$ as well, that $\mathrm{F}[x] \cap \mathbb{B}^Y_\epsilon(y) \neq \emptyset$. By the l.s.c of F there is a $\delta > 0$ with

$$\mathbb{B}^X_\delta(x) \subseteq \mathrm{F}^{-}\left[\mathbb{B}^Y_\epsilon(y)\right] \subseteq \mathrm{F}^{-}[V].$$

However this means that x is an interior point of $\mathrm{F}^{-}[V]$ and since x was an arbitrary element of $\mathrm{F}^{-}[V]$, $\mathrm{F}^{-}[V]$ is open.

$7 \implies 1$ Pick any $x \in X$, $y \in Y$ and $\epsilon > 0$ with $\mathrm{F}[x] \cap \mathbb{B}_\epsilon(y) \neq \emptyset$ then by item 7 we know that $\mathrm{F}^{-}[\mathbb{B}_\epsilon(y)]$ is open in X with $x \in \mathrm{F}^{-}[\mathbb{B}_\epsilon(y)]$. Hence, there is $\delta > 0$ such that $\mathbb{B}^X_\delta(x) \subseteq F^{-}[\mathbb{B}_\epsilon(y)]$.

□

Note how we assumed that $\text{Dom}(\text{F}) = X$ in the above Theorem. This is done mostly for convenience. If one wished to relax this, one would needs to start considering only open sets (or convergence of sequences) in the domain anyway to avoid trivial or undefined cases.

Item 2 of Theorem 6 is typically the easiest to prove that a given multifunction is l.s.c. Item 3 is perhaps the easiest to remember and the most conceptually pleasing. Items 4, 5 and 6 are very useful; one should keep them in mind when dealing with l.s.c multifunctions.

We will see in Chapters 3 and 4 the multifunctions we will be primarily concerned with are the union of a collection of continuous point to point maps form a metric space to itself. That is, suppose that $\mathcal{F}$ is a set of continuous functions from X to X then we are focused on the multifunction

$$\text{F}[x] = \bigcup_{\text{f} \in \mathcal{F}} \text{f}[x].$$

The above multifunction is often called the Hutchinson-Barnsley operator. It is a very well behaved multifunction but perhaps most notably it is always lower semicontinuous.

Proposition 2. *Suppose that* (X, d)*,* (Y, ρ) *are metric spaces and that* $\mathcal{F}$ *is a set of continuous functions from* X *to* Y *then*

$$\text{F}[x] = \bigcup_{\text{f} \in \mathcal{F}} \text{f}[x]$$

is a l.s.c multifunction.

Proof. Recall Item 6 of Theorem 6. Let $U \subseteq X$ and notice for all $\text{f} \in \mathcal{F}$ we have $\text{f}\left[\overline{U}\right] \subseteq \overline{\text{f}[U]}$

by the continuity of f. Thus

$$\begin{aligned}
\mathrm{F}\left[\overline{U}\right] &= \bigcup_{\mathrm{f}\in\mathcal{F}} \mathrm{f}\left[\overline{U}\right] \\
&\subseteq \bigcup_{\mathrm{f}\in\mathcal{F}} \overline{\mathrm{f}[U]} \\
&\subseteq \overline{\bigcup_{\mathrm{f}\in\mathcal{F}} \overline{\mathrm{f}[U]}} \\
&= \overline{\bigcup_{\mathrm{f}\in\mathcal{F}} \mathrm{f}[U]} \\
&= \overline{\mathrm{F}[U]}
\end{aligned}$$

□

For sake of generality, in Chapter 4 we will often deal with the composition of a l.s.c multifunction with itself and the "union" of a collection of l.s.c multifunctions.

Proposition 3. *Suppose that* (X, d),(Y, ρ) *are metric spaces and that* $\mathcal{F}$ *is a set of l.s.c multifunctions from* X *into* Y*. Then the multifunction defined by*

$$\left(\bigcup_{\mathcal{F}} \mathrm{F}\right)[x] = \bigcup_{\mathrm{F}\in\mathcal{F}} \mathrm{F}[x]$$

is l.s.c.

Proof. The proof is identical to that of Proposition 2. □

Proposition 4. *Let* (X, d), (Y, ρ), (Z, s) *be metric spaces and that* $\mathrm{F} : X \rightsquigarrow Y$ *and* $\mathrm{G} : Y \rightsquigarrow Z$ *be l.s.c multifunctions. Then the composition multifunction* $\mathrm{G} \circ \mathrm{F} : X \rightsquigarrow Z$ *defined by*

$$\mathrm{G} \circ \mathrm{F}[x] = \mathrm{G}[\mathrm{F}[x]] = \bigcup_{y\in\mathrm{F}[x]} \mathrm{G}[y]$$

for all $x \in X$*, is l.s.c.*

Proof. Let $U \subseteq X$ and by the lower semicontinuity of F we have

$$\mathrm{F}\left[\overline{U}\right] \subseteq \overline{\mathrm{F}[U]}$$

we can apply G to both sides yielding,

$$\mathrm{G} \circ \mathrm{F}\left[\overline{U}\right] \subseteq \mathrm{G}\left[\overline{\mathrm{F}[U]}\right].$$

But, $HB[G]$ is l.s.c so $\mathrm{G}\left[\overline{\mathrm{F}[U]}\right] \subseteq \overline{\mathrm{G} \circ \mathrm{F}[U]}$. Hence,

$$\mathrm{G} \circ \mathrm{F}\left[\overline{U}\right] \subseteq \overline{\mathrm{G} \circ \mathrm{F}[U]}.$$

So by Item 6 of Theorem 6 $\mathrm{G} \circ \mathrm{F}$ is lower semicontinuous.

□

Let us return our attention to Items 3, 4 and 5 of Theorem 6, particularly Item 4. These characterizations of lower semicontinuous tell us a great deal about how l.s.c multifunctions work with convergent sequences. However at first glance these results might seem incomplete. After all, a single valued function f is continuous if and only if for every convergent sequence $\{x_n\}_{n\in\mathbb{N}} \to x$ the sequence $\{\mathrm{f}(x_n)\}_{n\in\mathbb{N}}$ converges to $\mathrm{f}(x)$. But in the case of a l.s.c multifunction the set sequence $\{\mathrm{F}[x_n]\}_{n\in\mathbb{N}}$ need not converge at all — let alone to $\mathrm{F}[x]$! The multifunction F_2 in Example 2 is an example of this; sequences of $\{x_n\}_{n\in\mathbb{N}} \to 0$ consisting of infinitely many positive and negative numbers make $\{\mathrm{F}_2[x_n]\}_{n\in\mathbb{N}}$ divergent and if $\{x_n\}_{n\in\mathbb{N}} \to 0$ is strictly positive the set sequence converges to $[-1, 1]$. In order for $\{\mathrm{F}[x_n]\}_{n\in\mathbb{N}}$ to converge to $\mathrm{F}[x]$ we need an additional type of continuity for multifunctions.

Definition 10. *Let* (X, d), (Y, ρ) *be metric spaces and let* $\mathrm{F} : X \rightsquigarrow Y$. F *is said to be outer semicontinuous (o.s.c) at the point* $x \in \mathrm{Dom}\,(\mathrm{F})$ *if and only if for every sequence*

$\{x_n\}_{n\in\mathbb{N}} \to x$ *we have*

$$\operatorname*{Ls}_{n\to\infty} \mathrm{F}[x_n] \subseteq \mathrm{F}[x].$$

We say that F *is outer semicontinuous (o.s.c) on* $B \subseteq \mathrm{Dom}\,(\mathrm{F})$ *if it is outer semicontinuous at every point in* B. *Furthermore if* F *is both o.s.c and l.s.c on* B *then we say* F *is sequentially continuous on* B.

In light of the above definition we can see if a multifunction, say F, is sequentially continuous on X then we have for every $\{x_n\}_{n\in\mathbb{N}} \to x$

$$\mathrm{F}[x] \subseteq \operatorname*{Li}_{n\to\infty} \mathrm{F}[x_n] \subseteq \operatorname*{Ls}_{n\to\infty} \mathrm{F}[x_n] \subseteq \mathrm{F}[x]$$

and thus the sequence of sets $\{\mathrm{F}[x_n]\}_{n\in\mathbb{N}}$ converges to $\mathrm{F}[x]$.

We will not be concerning ourselves so much with outer semicontinuity. It turns out for our purposes we do not need our multifunctions to be sequentially continuous. However, the concept of outer semicontinuity is strongly related to a concept we have not discussed.

Definition 11. *Let* X, Y *be sets and let* $\mathrm{F} : X \rightsquigarrow Y$ *be a multifunction. We define the graph of* F *to be*

$$\mathrm{Graph}\,(\mathrm{F}) = \{(x, y) \in X \times Y | y \in \mathrm{F}[x], x \in \mathrm{Dom}\,(\mathrm{F})\}.$$

In some sources one will find multifunctions to be defined via the graph. Essentially, theses sources say a multifunction F is a subset of $X \times Y$ then define $\mathrm{F}[x] = \{y \in Y | (x, y) \in \mathrm{F}\}$. This may seem questionable at first, considering how most of us are used to seeing $\mathrm{f}(x) =$ "expression", but there are two things to remember: plotted graphs are the usual choice of getting intuition for a function (or multifunction) when possible and many things are easier to define or understand when using the graph (for example the inverse multifunction $\mathrm{F}^- = \{(y, x) \in Y \times X | (x, y) \in \mathrm{F}\}$).

We are now ready to return to our discussion on outer semicontinuity.

Theorem 7. *Let* (X, d) *and* (Y, ρ) *be metric spaces. Let* $F : X \rightsquigarrow Y$ *with* $\mathrm{Dom}\,(\mathrm{F}) = X$ *then the following are equivalent.*

1. F *is o.s.c on* X*.*

2. *For all* $x \in X$ *we have*
$$\overline{\bigcap_{\delta>0} \mathrm{F}[\mathbb{B}^X_\delta(x)]} = \mathrm{F}[x]$$

3. $\mathrm{Graph}\,(\mathrm{F})$ *is a closed subset of* $X \times Y$ *of the metric space* $(X \times Y, \mathrm{d} + \rho)$*.*

4. *For every* $x \in X$ *and* $y \in Y \setminus \mathrm{F}[x]$ *there exist* $\epsilon > 0$ *and* $\delta > 0$ *such that*
$$\mathrm{F}\left[\mathbb{B}^X_\delta(x)\right] \cap \mathbb{B}^Y_\epsilon(y) = \emptyset.$$

Proof. $1 \implies 2$ Note that $\overline{\bigcap_{\delta>0} \mathrm{F}[\mathbb{B}^X_\delta(x)]} \supseteq \mathrm{F}[x]$ is always the case as $x \in \mathrm{F}\left[\mathbb{B}^X_\delta(x)\right]$. Thus we need only show the opposite inclusion. So pick $y \in \overline{\bigcap_{\delta>0} \mathrm{F}[\mathbb{B}^X_\delta(x)]}$ and so for all $\epsilon > 0$ we have $\mathbb{B}^Y_\epsilon(y) \cap \left(\bigcap_{\delta>0} \mathrm{F}\left[\mathbb{B}^X_\delta(x)\right]\right) \neq \emptyset$. Thus for all $n \in \mathbb{N}$ we can choose
$$y_n \in \mathbb{B}^Y_{\frac{1}{n}}(y) \cap \left(\bigcap_{\delta>0} \mathrm{F}\left[\mathbb{B}^X_\delta(x)\right]\right) \subseteq \mathbb{B}^Y_{\frac{1}{n}}(y) \cap \mathrm{F}\left[\mathbb{B}^X_{\frac{1}{n}}(x)\right]$$
and for all $n \in \mathbb{N}$ we can also pick $x_n \in \mathbb{B}^X_{\frac{1}{n}}(x)$ such that $y_n \in \mathrm{F}[x_n]$. We can also see that $\{x_n\}_{n\in\mathbb{N}} \to x$ and $\{y_n \in \mathrm{F}[x_n]\}_{n\in\mathbb{N}} \to y$. Thus by Item 1 and (i) of Proposition 1 we have $y \in \mathrm{Li}_{n\to\infty} \mathrm{F}[x_n] \subseteq \mathrm{Li}_{n\to\infty} \mathrm{F}[x_n] \subseteq \mathrm{F}[x]$ where the last inclusion follows from F being o.s.c. Thus $y \in \mathrm{F}[x]$ and 2 holds.

$2 \implies 3$ Pick $(x, y) \in X \times Y \setminus \mathrm{Graph}\,(\mathrm{F})$ and notice by 2 that $\mathrm{F}[x]$ is a closed set. Since $y \notin \mathrm{F}[x]$ y is an interior point of $Y \setminus \mathrm{F}[x]$ (an open set), there is a $\epsilon > 0$ such that

$\mathbb{B}^{Y}_{\epsilon}(y) \subseteq Y \setminus \mathrm{F}[x]$. Now consider $(x', y') \subseteq \mathbb{B}^{X\times Y}_{\epsilon}((x, y))$ then

$$\mathrm{d}(x', x) + \rho(y', y) < \epsilon$$
$$\rho(y', y) \leq \mathrm{d}(x', x) + \rho(y', y) < \epsilon$$

and $y' \in \mathbb{B}^{Y}_{\epsilon}(y) \subseteq Y \setminus \mathrm{F}[x]$. Thus $y' \notin \mathrm{F}[x']$ and $(x', y') \in X \times Y \setminus \mathrm{Graph}\,(\mathrm{F})$. Thus we have shown (x, y) is an interior point of $Y \setminus \mathrm{Graph}\,(\mathrm{F})$ and since (x, y) was arbitrary this means $Y \setminus \mathrm{Graph}\,(\mathrm{F})$ is open. Therefore $\mathrm{Graph}\,(\mathrm{F})$ is closed.

$3 \implies 4$ Let $x \in X$ and $y \in Y \setminus \mathrm{F}[x]$. So $(x, y) \in X \times Y \setminus \mathrm{Graph}\,(\mathrm{F})$, an open set, so there is a $\eta > 0$ such that $\mathbb{B}^{X\times Y}_{\eta}((x, y)) \subseteq X \times Y \setminus \mathrm{Graph}\,(\mathrm{F})$. We pick $\delta, \epsilon > 0$ so that $\delta + \epsilon < \eta$ and we consider $(x', y') \in \mathbb{B}^{X}_{\delta}(x) \times \mathbb{B}^{Y}_{\epsilon}(y) \subseteq X \times Y$ then

$$\mathrm{d}(x', x) + \rho(y', y) < \delta + \epsilon < \eta$$

so $\mathbb{B}^{X}_{\delta}(x) \times \mathbb{B}^{Y}_{\epsilon}(y) \subseteq \mathbb{B}^{X\times Y}_{\eta}((x, y)) \subseteq X \times Y \setminus \mathrm{Graph}\,(\mathrm{F})$. Now for every $x' \in \mathbb{B}^{X}_{\delta}(x)$ and $y' \in \mathbb{B}^{Y}_{\epsilon}(y)$ we have $y' \notin \mathrm{F}[x']$. If there was a $z \in \mathrm{F}\big[\mathbb{B}^{X}_{\delta}(x)\big] \cap \mathbb{B}^{Y}_{\epsilon}(y)$ there would be a $u \in \mathbb{B}^{X}_{\delta}(x)$ with $z \in \mathrm{F}[u] \cap \mathbb{B}^{Y}_{\epsilon}(y)$ but this contradicts the previous statement. Thus $\mathrm{F}\big[\mathbb{B}^{X}_{\delta}(x)\big] \cap \mathbb{B}^{Y}_{\epsilon}(y)$ is empty.

$4 \implies 1$ Let $x \in X$, $\{x_n\}_{n\in\mathbb{N}} \to x$ and $y \in \mathrm{Ls}_{n\to\infty}\,\mathrm{F}[x_n]$. Recalling Item 3 of Proposition 1, there is a sequence $\{y_n \in \mathrm{F}[x_n]\}_{n\in\mathbb{N}}$ with a subsequence $\{y_{n_k} \in \mathrm{F}[x_{n_k}]\}_{k\in\mathbb{N}}$ that converges to y. We now proceed by contradiction; so suppose $y \notin \mathrm{F}[x]$ and so by 4 there are $\epsilon, \delta > 0$ for which $\mathrm{F}\big[\mathbb{B}^{X}_{\delta}(x)\big] \cap \mathbb{B}^{Y}_{\epsilon}(y) = \emptyset$. However since $\{x_{n_k}\}_{k\in\mathbb{N}} \to x$ and $\{y_{n_k}\}_{k\in\mathbb{N}} \to y$ we can pick $K \in \mathbb{N}$ large enough such that $\mathrm{d}(x_{n_K}, x) < \delta$ and $\rho(y_{n_K}, y) < \epsilon$. Recalling that $y_{n_K} \in \mathrm{F}[x_{n_K}]$, we see that this contradicts $\mathrm{F}\big[\mathbb{B}^{X}_{\delta}(x)\big] \cap \mathbb{B}^{Y}_{\epsilon}(y)$ being empty. Thus $y \in \mathrm{F}[x]$ and $\mathrm{Ls}_{n\to\infty}\,\mathrm{F}[x_n] \subseteq \mathrm{F}[x]$. Therefore F is outer semicontinuous. □

One thing we can see from the above theorem is that a o.s.c multifunction, say F, has closed values i.e $\mathrm{F}[x]$ is closed in Y for all x. This immediately discounts many multifunctions that "feel" like they should be continuous. For instance suppose $V \subseteq Y$ is not closed; the constant multifunction $\mathrm{F}[x] = V$ is not o.s.c at any point (but is both u.s.c and l.s.c at every point). However in some sense, this is a reasonable thing for even in the case of $\mathrm{F}[x] = V$ there are points in the codomain of F that the multifunction gets close to but never attains (i.e the boundary of V). In the single valued case this only happens when there is a discontinuity in the function (or there is some long term behavior like in e^x as $x \to -\infty$, but luckily $-\infty$ is normally not taken to be in the domain of the function). In essence o.s.c says "if there is a way to get close to y via some approach in x then y is actually attained". Contrasting to l.s.c which says "if $y \in \mathrm{F}[x]$ then in every approach to x the approach (in the range) gets close to y".

Although the upper and lower pre-images are not a focus of this work, it is still good to keep in mind some associated properties and relations, specially in how the contrast to the single valued case.

Proposition 5. *Let X and Y be sets, $\mathrm{F} : X \rightsquigarrow Y$ be a multifunction, $U \subseteq X$ and $V \subseteq Y$. Then the following hold.*

1. $X \setminus \mathrm{F}^-[V] = \mathrm{F}^+[Y \setminus V]$ *and* $\mathrm{F}^-[Y \setminus V] = X \setminus \mathrm{F}^+[V]$.

2. $\mathrm{F}\big[\mathrm{F}^+[V]\big] \subseteq V$.

3. $\mathrm{F}^+[V] \subseteq \mathrm{F}^-[V]$

4. $U \subseteq \mathrm{F}^+[\mathrm{F}[U]]$ *and* $U \subseteq \mathrm{F}^-[\mathrm{F}[U]]$.

5. *If $\mathcal{G}$ is a collection of multifunctions from X into Y and for $x \in X$ we define*

$$\mathrm{G}^*[x] = \bigcup_{\mathrm{G} \in \mathcal{G}} \mathrm{G}[x]$$

then $(\mathrm{G}^*)^-[V] = \bigcup_{\mathrm{G}\in\mathcal{G}} \mathrm{G}^-[V]$ *and* $(\mathrm{G}^*)^+[V] = \bigcap_{\mathrm{G}\in\mathcal{G}} \mathrm{G}^+[V]$

6. *If* $W \subseteq Y$ *then* $\mathrm{F}^+[W \cap V] = \mathrm{F}^+[W] \cap \mathrm{F}^+[V]$ *and* $\mathrm{F}^+[W \cup V] \supseteq \mathrm{F}^+[W] \cup \mathrm{F}^+[V]$.

7. *If* $W \subseteq Y$ *then* $\mathrm{F}^-[W \cap V] \subseteq \mathrm{F}^-[W] \cap \mathrm{F}^-[V]$ *and* $\mathrm{F}^-[W \cup V] = \mathrm{F}^-[W] \cup \mathrm{F}^-[V]$.

Proof. 1.

$$x \in X \setminus \mathrm{F}^-[V]$$

$$\Updownarrow$$

$$\mathrm{F}[x] \cap V = \emptyset$$

$$\Updownarrow$$

$$\mathrm{F}[x] \subseteq Y \setminus V$$

$$\Updownarrow$$

$$x \in \mathrm{F}^+[Y \setminus V]$$

and

$$x \in \mathrm{F}^-[Y \setminus V]$$

$$\Updownarrow$$

$$F[x] \cap (Y \setminus V) \neq \emptyset$$

$$\Updownarrow$$

$$\mathrm{F}[x] \not\subseteq V$$

$$\Updownarrow$$

$$x \in X \setminus \mathrm{F}^+[V].$$

2.

If $y \in \mathrm{F}\big[\mathrm{F}^+[V]\big]$ then there is an $x \in \mathrm{F}^+[V]$ such that $y \in \mathrm{F}[x]$ but by definition of F^+ we know that $\mathrm{F}[x] \subseteq V$. Hence, $y \in V$ and the result holds.

3.

If $x \in \mathrm{F}^+[V]$ then $\emptyset \neq \mathrm{F}[x] \subseteq V$. But surely then $\mathrm{F}[x] \cap V \neq \emptyset$, so $x \in \mathrm{F}^-[V]$.

4.

If $x \in U$ then it is clear that $\mathrm{F}[x] \subseteq \mathrm{F}[U]$; but this means that $x \in \mathrm{F}^+[\mathrm{F}[U]]$. By 3 we would also have $x \in \mathrm{F}^+[\mathrm{F}[U]] \subseteq \mathrm{F}^-[\mathrm{F}[U]]$.

5.

Suppose that $x \in (\mathrm{G}^*)^-[V]$ then $\mathrm{G}^*[x] \cap V \neq \emptyset$. So there is a $\mathrm{G} \in \mathcal{G}$ such that $\mathrm{G}[x] \cap V \neq \emptyset$ and $x \in \mathrm{G}^-[V] \subseteq \bigcup_{\mathrm{G} \in \mathcal{G}} \mathrm{G}^-[V]$. This argument can be made in reverse for the other inclusion.

Suppose that $x \in (\mathrm{G}^*)^+[V]$ then $\mathrm{G}^*[x] \subseteq V$. So for all $\mathrm{G} \in \mathcal{G}$ we have $\mathrm{G}[x] \subseteq V$. But this means that $x \in \mathrm{G}^+[V]$ for all $\mathrm{G} \in \mathcal{G}$ and $x \in \bigcap_{\mathrm{G} \in \mathcal{G}} \mathrm{G}^+[V]$. Conversely if $x \in \bigcap_{\mathrm{G} \in \mathcal{G}} \mathrm{G}^+[V]$ then for all $\mathrm{G} \in \mathcal{G}$ we have $\mathrm{G}[x] \subseteq V$. Thus we can take the union over $\mathcal{G}$ to yield $\mathrm{G}^*[x] \subseteq V$ and $x \in (\mathrm{G}^*)^+[V]$.

6.

Note that $W \cap V \subseteq W$ and $W \cap V \subseteq V$. Now, if $x \in \mathrm{F}^+[W \cap V]$ then $\mathrm{F}[x] \subseteq W \cap V$ and so $\mathrm{F}[x] \subseteq W$ and $\mathrm{F}[x] \subseteq V$. Thus $x \in \mathrm{F}^+[W] \cap \mathrm{F}^+[V]$. Conversely, suppose that $x \in \mathrm{F}^+[W] \cap \mathrm{F}^+[V]$ so $\mathrm{F}[x] \subseteq W$ and $\mathrm{F}[x] \subseteq V$. Hence, $\mathrm{F}[x] \in W \cap V$ and so $x \in \mathrm{F}^+[W \cap V]$.

If $x \in \mathrm{F}^+[W] \cup \mathrm{F}^+[V]$ then $\mathrm{F}[x] \subseteq W$ or $\mathrm{F}[x] \subseteq V$. In either case $\mathrm{F}[x] \subseteq W \cup V$ and $x \in \mathrm{F}^+[W \cup V]$.

7.

$\mathrm{F}^-[W \cap V] \subseteq \mathrm{F}^-[W] \cap \mathrm{F}^-[V]$ and $\mathrm{F}^-[W \cup V] = \mathrm{F}^-[W] \cup \mathrm{F}^-[V]$ If $x \in \mathrm{F}^-[W \cap V]$ then

$$\emptyset \neq \mathrm{F}[x] \cap (W \cap V) \subseteq \mathrm{F}[x] \cap W.$$

So $x \in \mathrm{F}^{-}[W]$ and

$$\emptyset \neq \mathrm{F}[x] \cap (W \cap V) \subseteq \mathrm{F}[x] \cap V$$

so $x \in \mathrm{F}^{-}[V]$. Hence, $x \in \mathrm{F}^{-}[W] \cap \mathrm{F}^{-}[V]$.

For the other identity we have,

$$x \in \mathrm{F}^{-}[W \cup V]$$
$$\Updownarrow$$
$$\mathrm{F}[x] \cap (W \cup V) \neq \emptyset$$
$$\Updownarrow$$
$$(F[x] \cap W) \cup (F[x] \cap V) \neq \emptyset$$
$$\Updownarrow$$
$$F[x] \cap W \neq \emptyset \text{ or } F[x] \cap V \neq \emptyset$$
$$\Updownarrow$$
$$x \in \mathrm{F}^{-}[W] \cup \mathrm{F}^{-}[V].$$

□

In Chapters 3 and 4 we will see that we are interested in multifunctions that map a set to itself, say F, for which there is a set A such that $\mathrm{F}[A] = A$.

Definition 12. *Let X be a set, $\mathrm{F} : X \rightsquigarrow X$ and $A \subseteq X$. We say that A is sub-invariant with respect to* F *if*

$$\mathrm{F}[A] \subseteq A.$$

We say A is super-invariant with respect to F *if*

$$A \subseteq \mathrm{F}[A].$$

Lastly we say that A is invariant with respect to F *if it is both sub-invariant and super-invariant with respect to* F *that is,*

$$\mathrm{F}[A] = A.$$

We also will use some terminology from the theory of partially ordered sets.

Definition 13. *Let X be a set and let P be a property subsets of X can have. Then $S \subseteq X$ is said to the the smallest (or the minimum) P set if set $A \subseteq X$ with property P then $S \subseteq A$. That is S is contained in* ***every*** *set with property P .*

Similarly, we define $M \subseteq X$ to be a minimal P set if $A \subseteq M$ with property P then $A = M$.

Note that the smallest P set is unique (whenever it exists), as by definition if S_1 and S_2 were both the smallest then $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$. We will generally avoid the use of the term minimum set as it sounds to similar to minimal.

One way to think of smallest sets, is that everything else is larger than it, whereas in the case of minimal sets nothing is smaller than them.

Chapter 3

Classical IFS Theory

In this chapter we develop classical iterated function system theory. Iterated function systems were first considered by Hutchinson in [8]. Additionally, iterated function systems were popularized by Barnsley. The content of this chapter is "well known" to the research area at large.

3.1 Attractors to Contractive IFS

First we must define what an IFS is.

Definition 14. *Let X be a set. An iterated function system (or IFS) $\mathcal{F}$ is a nonempty subset of $X^X = \{\mathrm{f} \,|\, \mathrm{f} : X \to X\}$. We say that $\mathcal{F}$ is: finite if $|\mathcal{F}|$ is finite, P if P is a property of every function in $\mathcal{F}$.*

Furthermore, we define the Hutchinson-Barnsley operator of $\mathcal{F}$ to be $\mathrm{F} : X \rightsquigarrow X$ *for all* $x \in X$

$$\mathrm{F}[x] = \bigcup_{\mathrm{f} \in \mathcal{F}} \mathrm{f}(x).$$

When reading the literature one should pay attention to the given definition of an IFS.

Unlike in Definition 14, many sources define an IFS to be both finite and contractive. This is done because of the spectacular results of Theorem 9 and Theorem 10 which we will prove in this chapter. To do this we need a reminder of one of the most famous theorems in Mathematics.

Theorem 8 (Banach's Fixed Point Theorem)**.** *Let* (X, d) *be a complete metric space and let* $\mathrm{f} : X \to X$ *be a contraction map. Then* f *has a unique fixed point* $\bar{x} \in X$*. Moreover for all* $x \in X$ *the sequence* $\{\mathrm{f}^{\circ n}(x)\}_{n\in\mathbb{N}}$ *converges to* $\bar{x}$

Proof. First we show that for all $x \in X$ the sequence $\{\mathrm{f}^{\circ n}(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let the contraction factor of f be $c \in [0, 1)$. Pick $\epsilon > 0$ and let $N, n, m \in \mathbb{N}$ with $N \leq n \leq m$; then by the triangle inequality

$$\begin{aligned}
\mathrm{d}(\mathrm{f}^{\circ n}(x), \mathrm{f}^{\circ m}(x)) &\leq \mathrm{d}\Big(\mathrm{f}^{\circ n}(x), \mathrm{f}^{\circ(n+1)}(x)\Big) + \mathrm{d}\Big(\mathrm{f}^{\circ(n+1)}(x), \mathrm{f}^{\circ m}(x)\Big) \\
&\leq \mathrm{d}\Big(\mathrm{f}^{\circ n}(x), \mathrm{f}^{\circ(n+1)}(x)\Big) + \mathrm{d}\Big(\mathrm{f}^{\circ(n+1)}(x), \mathrm{f}^{\circ(n+2)}(x)\Big) + \mathrm{d}\Big(\mathrm{f}^{\circ(n+2)}(x), \mathrm{f}^{m}(x)\Big) \\
&\vdots \\
&\leq \sum_{i=n}^{m-1} \mathrm{d}\Big(\mathrm{f}^{\circ i}(x), \mathrm{f}^{\circ(i+1)}(x)\Big) \\
&= \sum_{i=n}^{m-1} \mathrm{d}\big(\mathrm{f}^{\circ i}(x), \mathrm{f}^{\circ i}(\mathrm{f}(x))\big).
\end{aligned}$$

Now $\mathrm{f}^{\circ i}$ is a contraction with contraction factor c^i for each $i \in \mathbb{N}$. Applying this to the above sum we get

$$\begin{aligned}
\mathrm{d}(\mathrm{f}^{\circ n}(x), \mathrm{f}^{\circ m}(x)) &\leq \sum_{i=n}^{m-1} c^i \, \mathrm{d}(x, \mathrm{f}(x)) \\
&= \mathrm{d}(x, \mathrm{f}(x)) \sum_{i=n}^{m-1} c^i \\
&\leq \mathrm{d}(x, \mathrm{f}(x)) \sum_{i=n}^{\infty} c^i \\
&= \mathrm{d}(x, \mathrm{f}(x)) c^n \sum_{i=0}^{\infty} c^i \\
&= \mathrm{d}(x, \mathrm{f}(x)) \frac{c^n}{1-c}. \\
&\leq \mathrm{d}(x, \mathrm{f}(x)) \frac{c^N}{1-c}.
\end{aligned}$$

The series $\sum_{i=n}^{\infty} c^i = c^n \sum_{i=0}^{\infty} c^i$ is a geometric series and thus converges as $|c| < 1$. Now for every x we can pick N large enough so that $\mathrm{d}(x, \mathrm{f}(x)) \frac{c^N}{1-c} < \epsilon$. This can be seen from the fact $\left\{ \mathrm{d}(x, \mathrm{f}(x)) \frac{c^k}{1-c} \right\}_{k \in \mathbb{N}} \to 0$ for all x. Thus for all $x \in X$ the sequence $\{\mathrm{f}^{\circ n}(x)\}_{n \in \mathbb{N}}$ is Cauchy and so converges, to say $\bar{x} \in X$, by completeness.

As f is a contraction it is continuous and so $\{\mathrm{f}^{\circ n}(x)\}_{n \in \mathbb{N}} \to \bar{x} \implies \{\mathrm{f}(\mathrm{f}^{\circ n}(x))\}_{n \in \mathbb{N}} \to \mathrm{f}(\bar{x})$ however $\{\mathrm{f}(\mathrm{f}^{\circ n}(x))\}_{n \in \mathbb{N}} = \{\mathrm{f}^{\circ n}(x)\}_{n=2}^{\infty}$ is a subsequence of $\{\mathrm{f}^{\circ n}(x)\}_{n \in \mathbb{N}}$ thus both sequences must have the same limit and $\mathrm{f}(\bar{x}) = \bar{x}$. So $\bar{x}$ is a fixed point of f.

To show uniqueness, suppose that $x, y \in X$ are both fixed points of f then

$$\mathrm{d}(x, y) = \mathrm{d}(\mathrm{f}(x), \mathrm{f}(y)) \leq c \, \mathrm{d}(x, y).$$

If $\mathrm{d}(x, y) > 0$ this implies $1 < c$ by the above inequality, which is a contradiction. Thus $\mathrm{d}(x, y) = 0$ and $x = y$ thus the fixed point in the above argument must be unique. □

Banach's Fixed Point Theorem is quite remarkable: not only does it give existence and uniqueness of a fixed point of a contractive function (fixed points having many mathematical applications), but it also gives us an algorithm for approximating the fixed point, namely pick some $x \in X$ then start computing the sequence $\{\mathrm{f}^{\circ n}(x)\}_{n\in\mathbb{N}}$.

Banach's fixed point Theorem is also a key theorem in proving the main result of this section. It states given a complete metric space (X, d) and a finite contractive IFS $\mathcal{F}$ there is a unique nonempty compact set $A \subseteq X$ satisfying $A = \bigcup_{\mathrm{f}\in\mathcal{F}} \mathrm{f}(A)$. Furthermore define $\mathrm{F}(B) = \bigcup_{\mathrm{f}\in\mathcal{F}} \mathrm{f}(B)$ for all $B \subseteq X$ and B compact. Then $\{\mathrm{F}^{\circ n}(B)\}_{n\in\mathbb{N}}$ "converges to" A. We will state this more formally later in the section. Now that we know where we are going, the strategy of proving the result is clear: attempt to apply Banach's fixed point theorem to the function F. There are a number of obstacles in doing this. Firstly, F maps sets to sets, so we need some sort of metric on sets that is also complete; secondly, F must be a contraction with respect to this metric.

Definition 15. *Let* (X, d) *be a metric space. Define Hausdorff hyperspace*

$$\mathcal{H}(X) = \{B \subseteq X : B \textit{ is compact and nonempty}\}.$$

Let $A, B \subseteq X$ *and* $x \in X$*. Define the point to set distance*

$$\mathrm{d}(x, B) = \inf_{b\in B} \mathrm{d}(x, b)$$

and the set to set distance

$$\mathrm{d}(A, B) = \sup_{a\in A} \mathrm{d}(a, B) = \sup_{a\in A} \inf_{b\in B} \mathrm{d}(a, b).$$

Finally define the Hausdorff distance

$$d_H(A, B) = \max\{d(A, B), d(B, A)\}.$$

Intuitively, the point to set distance is the shortest distance starting from x ending in B. This makes the set to set distance the "largest shortest distance" starting in A ending in B. Note that $d(B, A)$ can be infinite, but that can only occur when B is unbounded. Furthermore $d(B, A) = 0$ iff $B \subseteq \overline{A}$. So if $d_H(A, B) = 0$ then $\overline{A} = \overline{B}$.

Example 5. *Let* $X = \mathbb{R}$ *and* $d(x, y) = |x - y|$. *Then consider the following.*

$$d([0, 1], [-1, 1]) = 0$$
$$d([-1, 1], [0, 1]) = 1$$
$$d(-1, [0, 1]) = 1$$
$$d_H([-1, 1], [0, 1]) = 1$$

We can see that set to set distance is not symmetric: that is $d(A, B) \neq d(B, A)$. *When the sets are unbounded we can get an infinite set to set distance:*

$$d([0, \infty), [0, 1]) = \infty$$
$$d([0, 1], [0, \infty)) = 0$$

We now seek to show that $(\mathbb{H}(X), d_H)$ is a metric space.

Proposition 6. *Let* (X, d) *be a metric space. Then* $(\mathbb{H}(X), d_H)$ *is a metric space.*

Proof. It can be shown that $f(x) = d(x, A)$ is a continuous function in $f : X \to [0, \infty)$ for all sets $A \subseteq X$. Thus for all sets $A, B \in \mathbb{H}(X)$ by the extreme value theorem f must achieve its maximum value on A so $d(A, B) \in [0, \infty)$ and similarly $d(B, A) \in [0, \infty)$. In fact from this reasoning we can deduce that $d(a, b) = d_H(A, B)$ for some $a \in A$ and $b \in B$. Thus so must $d_H(A, B) \in [0, \infty)$.

d_H is symmetric because

$$d_H(A, B) = \max\{d(A, B), d(B, A)\} = \max\{d(B, A), d(A, B)\} = d_H(B, A).$$

Now suppose $d_H(A, B) = 0$. We claim that $A = B$. Suppose that $A \not\subseteq B$ then take $a \in A \setminus B$ this means that $0 < d(a, B) \leq d(A, B)$ because B is closed so there is a open set disjoint from B containing a. But this is a contradiction, so $A \subseteq B$. By making a mirror argument we can conclude that $B = A$.

Conversely suppose $A = B$ then we have for all $a \in A$ and $b \in B$, $d(a, B) \leq d(a, b)$ but picking b equal to a we have $d(a, B) = 0$ for all $a \in B$. Thus $d_H(B, B) = 0$.

Now we show triangle inequality. Let $C \in \mathbb{H}(X)$, $a \in A$ such that $d(a, B) = d(A, B)$ (we can do this by the first argument in the proof), $c \in C$ be arbitrary and $b \in B$ be arbitrary.

$$\begin{aligned} d(A, B) = d(a, B) &\leq d(a, b) \\ d(A, B) &\leq d(a, c) + d(c, b) \end{aligned}$$

Now we can take the infimum over $b \in B$ of the right hand side of this inequality to yield

$$d(A, B) \leq d(a, c) + d(c, B).$$

Since $\mathrm{d}(c, B) \leq \mathrm{d}(C, B)$ we have

$$\mathrm{d}(A, B) \leq \mathrm{d}(a, c) + \mathrm{d}(C, B).$$

Again take the infimum over $c \in C$ of the right hand side

$$\begin{aligned} \mathrm{d}(A, B) &\leq \mathrm{d}(a, C) + \mathrm{d}(C, B) \\ &\leq \mathrm{d}(A, C) + \mathrm{d}(C, B). \end{aligned}$$

Since $\mathrm{d}(a, C) \leq \mathrm{d}(A, C)$. Now note $\mathrm{d}(A, C) \leq \mathrm{d}_H(A, C)$ and $\mathrm{d}(C, B) \leq \mathrm{d}_H(C, B)$. Thus

$$\mathrm{d}(A, B) \leq \mathrm{d}_H(A, C) + \mathrm{d}_H(C, B).$$

We can make the same argument with $\mathrm{d}(B, A)$ which implies $\mathrm{d}_H(A, B) \leq \mathrm{d}_H(A, C) + \mathrm{d}_H(C, B)$. $\square$

Remark 2. *In actuality* d_H *forms what is called a pseudometric on the nonempty bounded subsets of* X *and a metric on the nonempty closed and bounded subsets of* X*. However we have no need for this more general result.*

Now that we have a metric on sets (well, the nonempty compact ones) we need the metric to be complete. Luckily, $\mathbb{H}(X)$ inherits many of the properties of X, such as completeness and compactness.

Proposition 7. *Let* (X, d) *be a complete metric space. Then* $(\mathbb{H}(X), \mathrm{d}_H)$ *is a complete metric space*

The proof is long and rather technical and so it is omitted; see [6] for a proof.

Remark 3. *The first step in proving Proposition 7 involves having a candidate set for a Cauchy sequence of sets to converge to in the Hausdorff metric. If* $\{A_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence of sets, this "candidate" set is* $\mathrm{Li}_{n\to\infty} A_n$ *the topological lower limit of the sequence. It turns out that if* $\{A_n\}_{n\in\mathbb{N}} \to A$ *with respect to the Hausdorff metric then* $A = \mathrm{Li}_{n\to\infty} A_n = \mathrm{Ls}_{n\to\infty} A_n$ *is the set limit in the sense of Definition 6.*

We have one last result to prove before getting the main result of the section.

Proposition 8. *Let* (X, d) *be a metric space and* $\mathcal{F} = \{\mathrm{f}_i\}_{i\in[N]}$ *be a finite contractive IFS on* X*. Define the set to set map*

$$\mathrm{F}[B] = \bigcup_{i=0}^{N} \mathrm{f}_i[B].$$

Then $\mathrm{F} : \mathbb{H}(X) \to \mathbb{H}(X)$ *and* F *is a contraction on* $(\mathbb{H}(X), \mathrm{d}_H)$*.*

Proof. Let $B \in \mathbb{H}(X)$ then for all $\mathrm{f} \in \mathcal{F}$ $\mathrm{f}[B]$ is nonempty. Furthermore it is also compact as f is continuous. Thus recalling the finite union of compact sets is compact we have $\mathrm{F}[B] \in \mathbb{H}(X)$.

Now let $B, C \in \mathbb{H}(X)$ and WLOG let $\mathrm{d}_H(\mathrm{F}[B], \mathrm{F}[C]) = \mathrm{d}(\mathrm{F}[B], \mathrm{F}[C])$. Recalling the proof of Proposition 6 there is a $b \in B$ and $\mathrm{f} \in \mathcal{F}$ such that

$$\mathrm{d}_H(\mathrm{F}[B], \mathrm{F}[C]) = \mathrm{d}(\mathrm{F}[B], \mathrm{F}[C]) = \mathrm{d}(\mathrm{f}(b), \mathrm{F}[C]).$$

But now that means for any $\mathrm{g} \in \mathcal{F}$ and $c \in C$ we have $\mathrm{d}(\mathrm{f}(b), \mathrm{F}[C]) \leq \mathrm{d}(\mathrm{f}(b), \mathrm{g}(c))$. So pick $\mathrm{g} = \mathrm{f}$ and let k_{f} be the contraction factor for f. Then we have for any $c \in C$

$$\mathrm{d}_H(\mathrm{F}[B], \mathrm{F}[C]) \leq \mathrm{d}(\mathrm{f}(b), \mathrm{f}(c)) \leq k_{\mathrm{f}}\, \mathrm{d}(b, c).$$

Since the above holds for all $c \in C$ we can take the infimum over C and $\mathrm{d}_H(\mathrm{F}[B], \mathrm{F}[C]) \leq k_{\mathrm{f}}\, \mathrm{d}(b, C)$. Finally, as $\mathrm{d}(b, C) \leq \mathrm{d}(B, C) \leq \mathrm{d}_H(B, C)$, we have $\mathrm{d}_H(\mathrm{F}[B], \mathrm{F}[C]) \leq k_{\mathrm{f}}\, \mathrm{d}_H(B, C)$.

Now k_{f} is dependent on C and B, so we note that $k = \max_{\mathrm{g} \in \mathcal{F}} k_{\mathrm{g}} \geq k_{\mathrm{f}}$ and so

$$\mathrm{d}_H(\mathrm{F}[B], \mathrm{F}[C]) \leq k \, \mathrm{d}_H(B, C).$$

□

We are now finally ready to prove the main result of the section, the Hutchinson-Barnsley Theorem.

Theorem 9 (Hutchinson-Barnsley Theorem). *Let (X, d) be a complete metric space and $\mathcal{F} = \{\mathrm{f}_i\}_{i \in [N]}$ be a finite contractive IFS on X. Then there is a unique nonempty compact set $A \subseteq X$*

$$A = \bigcup_{i=0}^{N} \mathrm{f}_i[A]. \tag{3.1}$$

Furthermore for all nonempty compact sets $B \subseteq X$ the sequence $\{\mathrm{F}^{\circ n}[B]\}_{n \in \mathbb{N}}$ converges to A with respect to the Hausdorff metric. Particularly for all $x \in X$ the sequence $\{\mathrm{F}^{\circ n}[x]\}_{n \in \mathbb{N}}$ converges to A with respect to the Hausdorff metric.

Proof. By Propositions 6, 7 and 8 we have that F is a contraction map on the complete metric space $(\mathbb{H}(X), \mathrm{d}_H)$. Thus by Banach's fixed point Theorem (Theorem 8) the result follows. □

The set A in Theorem 9 is called the attractor of the IFS. The attractor is an invariant set (a set satisfying Equation 3.1) with respect to the IFS. Finding sets that are invariant with respect to the IFS are a core topic of this thesis and will be explored more in Chapter 4.

Example 6 (Cantor set). *Consider the finite contractive IFS on $[0, 1]$ that consists of the* *following functions.*

$$f_0(x) = \frac{x}{3}$$
$$f_2(x) = \frac{x}{3} + \frac{2}{3}$$

The attractor of this IFS is the set of all numbers in $[0,1]$ *that have a base-3 representation containing no 1's . To see this pick any* $x \in [0,1]$ *and let* $x = 0.x_1x_2x_3\ldots$ *be a base-3 representation of* x*. Now notice the effect of the functions on this representation. For instance:*

$$f_0(x) = 0.0x_1x_2x_3\ldots$$
$$f_2(x) = 0.2x_1x_2x_3\ldots$$
$$f_2\, f_0(x) = 0.200x_1x_2x_3\ldots$$
$$f_2 \circ f_0 \circ f_0(x) = 0.200x_1x_2x_3\ldots$$

Thus we can see that for any $N \in \mathbb{N}$ *and all* $n \geq N$ *the set* $F^{\circ n}[x]$ *only contains elements that have a base-3 representation containing no 1's in their first* N *digits. This means that in the Hausdorff limit of* $\{F^{\circ n}[x]\}_{n\in\mathbb{N}}$ *contains only elements that have a base-3 representation containing no 1's. To see that the attractor contains all the numbers in question, suppose* $y \in [0,1]$ *with* $y = 0.y_1y_2y_3y_4\ldots$ *with* $y_i = 0,2$ *for all* $i \in \mathbb{N}$ *then the sequence* $\{f_{y_1} \circ f_{y_2} \circ \cdots \circ f_{y_n}(x)\}_{n\in\mathbb{N}}$ *converges to* y *for all* $x \in [0,1]$.

Typically, the attractor is a fractal and is quite interesting to look at. Conveniently, the Hutchinson Barnsley Theorem gives us an algorithm to actually look at the attractor.

Unfortunately, the algorithm is inefficient practically. Suppose that $\mathcal{F}$ contains N functions. Then the set $\mathrm{F}^{\circ n}[x]$ contains at most N^n elements, and to compute $\mathrm{F}^{\circ(n+1)}[x]$ we need to evaluate all N functions on all N^n elements in $\mathrm{F}^{\circ n}[x]$. Thus N^{n+1} function evaluations are needed. Luckily, there are more efficient algorithms for approximating attractors.

Theorem 10 (Chaos Game or Elton's Ergodic Theorem)**.** *Let (X, d) be a metric space, and $\mathcal{F} = \{\mathrm{f}_i\}_{i=1}^N$ be a finite contractive IFS on X and A be the attractor of the IFS. Suppose that an infinite sequence of the numbers $\{i_n\}_{n\in\mathbb{N}}$ $i_n = 1, 2, \ldots, N$ for $n \in \mathbb{N}$ are generated by picking $i_n = j$ with probability $p_j > 0$ and independently of the previous i_n's, with $\sum_{j=1}^N p_j = 1$.*

Then the sequence defined by $x_n = \mathrm{f}_{i_n}(x_{n-1})$ with $x_0 \in A$ is dense on the attractor, that is $\overline{\{x_n\}_{n\in\mathbb{N}}} = A$. Furthermore, the sequence $y_n = \mathrm{f}_{i_n}(y_{n-1})$ with $y_0 \in X$ satisfies

$$\lim_{m\to\infty} \mathrm{d}_H\left(\overline{\{y_n\}_{n=m}^{\infty}}, A\right) = 0.$$

Proof. First let us note something useful about invariant sets with respect to the IFS. Since A is invariant it satisfies

$$\mathrm{F}[A] = A.$$

Applying F to both sides of this equation gives

$$\mathrm{F}^{\circ 2}[A] = \mathrm{F}[A] = A.$$

Applying F $n \in \mathbb{N}$ times gives

$$\mathrm{F}^{\circ n}[A] = A.$$

Thus we can conclude the following: for all $i \in [N]$ and $a \in A$ we have $\mathrm{f}_i(a) \in A$. Furthermore for every $a \in A$ and $K \in \mathbb{N}$ there is finite composition of the functions in $\mathcal{F}$ of length K, say $\mathrm{f}_\sigma = \mathrm{f}_{\sigma_K} \circ \mathrm{f}_{\sigma_{K-1}} \circ \ldots \mathrm{f}_{\sigma_1}$ where $\sigma_k \in [N]$ for $k \in [K]$, and an $a_2 \in A$ such that $a = \mathrm{f}_\sigma(a_2)$. Thus we can say that for all $m \in \mathbb{N}$, $\overline{\{x_n\}_{n=m}^{\infty}} \subseteq A$ as $x_0 \in A$. This tells us that $\mathrm{d}\Big(\overline{\{x_n\}_{n=m}^{\infty}}, A\Big) = 0$ for all $m \in \mathbb{N}$ and that $\overline{\{x_n\}_{n=m}^{\infty}}$ is compact for all $m \in \mathbb{N}$. Define $c = \max_{i \in [N]} c_i$ where c_i is the contraction factor of f_i. Now, pick any $a_1 \in A$ and any $\epsilon > 0$ and pick $K \in \mathbb{N}$ so that $c^K \operatorname{diam}(A) < \epsilon$.

Aside: Due to how the sequence $\{i_n\}_{n \in \mathbb{N}}$ is made it has, with probability 1, the property of containing every finite sequence of numbers from $[N]$ infinitely often (this is sometimes called the infinite monkey theorem). That is for all $J \in \mathbb{N}$ and given any finite sequence of numbers of $[N]$ say, $\{\sigma_k\}_{k=1}^{K}$, $\sigma_k \in [N]$ for $k \in [K]$, then there is a $j \geq J$ such that $i_{j+k} = \sigma_k$ for $k \in [K]$.

Now find a sequence of length K, $\{\sigma_k\}_{k=1}^{K}$, and $a_2 \in A$ such that $a_1 = \mathrm{f}_\sigma(a_2)$ and find $j \geq m$ so that $i_{j+k} = \sigma_k$ for $k \in [K]$. This means that $\mathrm{f}_{\sigma_1}(x_j) = x_{j+1}$, $\mathrm{f}_{\sigma_2} \circ \mathrm{f}_{\sigma_1}(x_j) = x_{j+2}$ and so $\mathrm{f}_\sigma(x_j) = x_{j+K}$. Recall that f_σ is a K-fold composition of contraction maps and so is a contraction map with contraction factor c^K. Now consider

$$\begin{aligned} d(a_1, x_{j+K}) &= d(f_\sigma(a_2), f_\sigma(x_j)) \\ &\le c^K d(a_2, x_j) \\ &\le c^K \operatorname{diam}(A) \\ &< \epsilon. \end{aligned}$$

This shows that $d\left(A, \overline{\{x_n\}_{n=m}^\infty}\right) = 0$. Thus for all $m \in \mathbb{N}$ $d_H\left(A, \overline{\{x_n\}_{n=m}^\infty}\right) = 0$ and $A = \overline{\{x_n\}_{n=m}^\infty}$. Picking $m = 1$ gives the result in the theorem.

Now we show

$$\lim_{m\to\infty} d_H\left(\overline{\{y_n\}_{n=m}^\infty}, A\right) = \lim_{m\to\infty} d_H\left(\overline{\{y_n\}_{n=m}^\infty}, \overline{\{x_n\}_{n=m}^\infty}\right) = 0.$$

Fix $y_0, x_0 \in X$ and pick $\epsilon > 0$ and pick $M \in \mathbb{N}$ so that $c^M d(x_0, y_0) < \epsilon$. Now pick any $m, n \in \mathbb{N}$ with $n \ge m \ge M$ so y_n is an arbitrary member of $\{y_n\}_{n=m}^\infty$. Then

$$\begin{aligned} d(y_n, x_n) = d(f_{i_n}(y_{n-1}), f_{i_n}(x_{n-1})) &\le c\, d(y_{n-1}, x_{n-1}) \\ &\le c^2 d(y_{n-2}, x_{n-2}) \\ &\vdots \\ &\le c^n d(y_0, x_0) \\ &\le c^M d(y_0, x_0) \\ &< \epsilon \end{aligned}$$

This shows that for all $m \ge M$ we have $d\left(\overline{\{y_n\}_{n=m}^\infty}, \overline{\{x_n\}_{n=m}^\infty}\right) < \epsilon$. An identical argument shows that for all $m \ge M$ we have $d\left(\overline{\{x_n\}_{n=m}^\infty}, \overline{\{y_n\}_{n=m}^\infty}\right) < \epsilon$. So the result holds recalling that $\overline{\{x_n\}_{n=m}^{\infty}} = A$ for all $m \in \mathbb{N}$. $\square$

The Chaos Game is played as follows to "draw" the attractor. Given $y_0 \in X$, $\mathcal{F} = \{\mathrm{f}_i\}_{i=1}^{N}$ and probabilities $p_i > 0:$ pick $i \in [N]$ with probability p_i and plot the point $x_1 = \mathrm{f}(x_0)$. Then pick another (possibly, the same) map with probability p_i and plot $x_2 = \mathrm{f}_i(x_1)$. Continue in this fashion until there is very little change in the picture being drawn. It is typical that one does not actually plot the first m points of the chaos game, see example 7 for more details.

Example 7. *Consider an IFS on $\mathbb{R}^2$, with the Euclidean metric, with only the following functions:*

$$\mathrm{f}_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{2} \end{pmatrix}$$
$$\mathrm{f}_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} + \frac{1}{4} \\ \frac{y}{2} + \frac{\sqrt{3}}{4} \end{pmatrix}$$
$$\mathrm{f}_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} + \frac{1}{2} \\ \frac{y}{2} \end{pmatrix}$$

The attractor to $\mathcal{F} = \{\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3\}$ is the Sierpinski Triangle (or Sierpinski Gasket). See Figure 3.1. The points in red are points plotted by f_1, the points in green were plotted by f_2 and the points in blue are plotted by f_3. The chaos game is "shown in motion" via Figure 3.2. In this figure the chaos game is played with the above IFS with all probabilities equal to $\frac{1}{3}$ and initial point $x_0 = (0, 1)$ a point near the top left of the pictures. Notice in (a) there are 100 points of the x_n's plotted; we can already see some things that vaguely look like triangles, and there are a few points near the initial point plotted that are not on or near the attractor. In (b) there are 1000 points of the x_n's plotted; the image looks quite similar to the attractor,

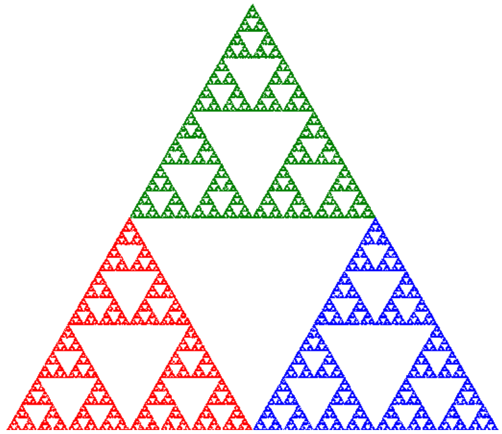

Figure 3.1: The Sierpinski Triangle.

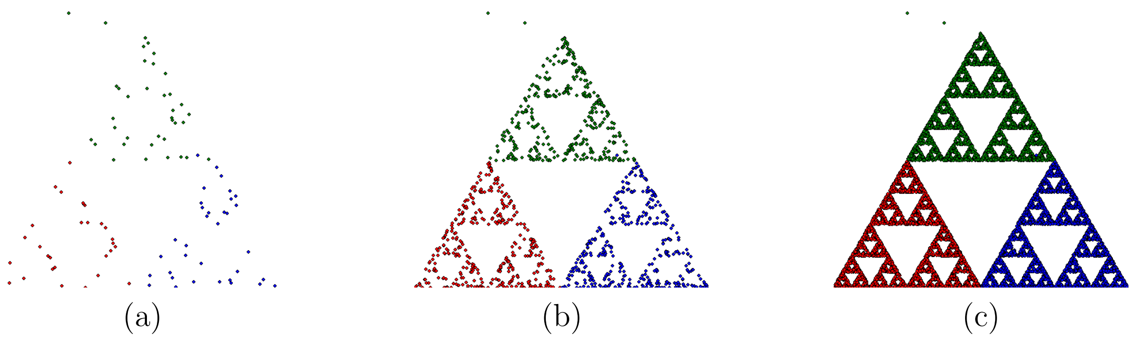

Figure 3.2: The chaos game forming the Sierpinski Triangle at: (a) 100 points plotted, (b) 1000 points plotted and (c) 10000 points plotted.

although it clearly needs some more "filling out" to do. In (c) there are 10000 points of the x_n*'s plotted; we see that the picture looks very simular to that in Figure 3.1 and so we would say the chaos game has "drawn" the attractor.*

As noted above, there are a number of green dots in (a)-(c) of Figure 3.2 that are clearly not on the attractor. This is due to the chaos game picking f_2 *the first 4 times or so and the fact* $x_0 \notin A$*. Usually, programs include a "burn in period", meaning the first* m *points are not plotted, as it takes some time to "get on the attractor". This is what was done in Figure 3.1.*

3.2 Address Maps and Infinite Compositions of Functions

The last topic of this chapter is address maps. We saw in the proof of Theorem 10 two points on the attractor are close together when they are the image points of the same composition of functions. This can be generalized (and indeed can simplify the proof of Theorem 10) to the case of infinitely many maps composed. But first we need to define some useful notations.

Definition 16. *Let I and X be sets and $\mathcal{F}$ be a IFS on X. Then it is said that I indexes $\mathcal{F}$ if and only if there exists a bijective function from I to $\mathcal{F}$.*

In other words, we can label the functions in $\mathcal{F}$ by the members of I. Notationally, we do this by writing for each $i \in I$ $f_i \in \mathcal{F}$ as the function corresponding to i (and vice versa). So we can write $\mathcal{F} = \{f_i : i \in I\}$ for convenience we will often write $\mathcal{F} = \{f_i\}_{i \in I}$ without defining I.

Previously in this chapter we were dealing with finite IFS so it was convenient to index the sets over $[N]$. Of course, this is arbitrary; for instance, in Example 6 we indexed the IFS with the set $\{0, 2\}$ rather than $\{1, 2\}$.

The reason we bother with index sets is so we have a convenient way of expressing compositions of functions in an IFS.

Definition 17. *Let I be an index set and $N \in \mathbb{N}$. An element $\alpha \in I^N$ is called a word or string over I of length N. We write that $\alpha = \alpha_1\alpha_2\alpha_3 \ldots \alpha_N$ instead of $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_N)$.*

Let $I^{\mathbb{N}}$ be the set of all semi-infinite words or strings over I. If $\sigma \in I^{\mathbb{N}}$ then we write $\sigma = \sigma_1\sigma_2\sigma_3 \ldots$.

Let $I^* = \bigcup_{n\in\mathbb{N}} I^n$ be the set of all finite words over I. We define the following operation: if $\alpha \in I^*$ then $|\alpha|$ denotes the length of the string and if $\alpha \in I^{\mathbb{N}}$ then we write $|\alpha| = \infty$.

Furthermore, let $n, m \in \mathbb{N}$ with $n \leq m \leq |\alpha|$ then

$$\alpha_{[n,m]} = \alpha_n \alpha_{n+1} \ldots \alpha_m$$

and if $\alpha \in I^{\mathbb{N}}$ then we may write

$$\alpha_{[n,\infty)} = \alpha_n \alpha_{n+1} \ldots .$$

Let $\alpha, \beta \in I^*$ and let $\sigma \in I^{\mathbb{N}}$ with $\alpha = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_N$ for some $N \in \mathbb{N}$ and $\beta = \beta_1 \beta_2 \beta_3 \ldots \beta_M$ for some $M \in \mathbb{N}$. We define the concatenation of strings denoted by

$$\alpha\beta = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_N \beta_1 \beta_2 \beta_3 \ldots \beta_M$$

and

$$\alpha\sigma = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_N \sigma_1 \sigma_2 \ldots .$$

Finally, let $\mathcal{F} = \{\mathrm{f}_i\}_{i \in I}$ be an IFS over a set X and let $\alpha \in I^*$ with length $N \in \mathbb{N}$. Then define

$$\mathrm{f}_\alpha = \mathrm{f}_{\alpha_1} \circ \mathrm{f}_{\alpha_2} \circ \ldots \mathrm{f}_{\alpha_N} .$$

We saw this notation before in Chapter 3. It is useful when dealing with compositions of functions. However it is introduced mainly to help us get a handle on the infinite compositions of functions.

Definition 18. *Let (X, d) be a metric space, $\mathcal{F} = \{\mathrm{f}_i\}_{i \in I}$ be an IFS on X, $\sigma \in I^{\mathbb{N}}$, $n \in \mathbb{N}$ and $x \in X$. We define the right composition functions: $R : I^{\mathbb{N}} \times \mathbb{N} \times X \to X$*

$$R(\sigma, n, x) = \mathrm{f}_{\sigma_{[1,n]}}(x).$$

(Hopefully) Without confusion we also define $R : I^{\mathbb{N}} \times X \to X$

$$R(\sigma, x) = \lim_{n\to\infty} \mathrm{f}_{\sigma_{[1,n]}}(x)$$

whenever the above limit exists. Further, if the above limit is independent of x, that is for all $x, y \in X$ $R(\sigma, x) = R(\sigma, y)$ we will define $R : I^{\mathbb{N}} \to X$ to be

$$R(\sigma) = \lim_{n\to\infty} \mathrm{f}_{\sigma_{[1,n]}}(x)$$

where x is arbitrary. Finally we let $\mathrm{Dom}\,(R) \subseteq I^{\mathbb{N}}$ be the set on which the limit $\lim_{n\to\infty} \mathrm{f}_{\sigma_{[1,n]}}(x)$ exists for all $x \in X$ and is independent of x.

If the function R is referred to without arguments, to make it clear which of the above functions we are referring to, assume that R is a function that maps $I^{\mathbb{N}}$ to X.

In the context of finite contractive maps the set $I^{\mathbb{N}}$ is called the address space and $\mathrm{Dom}\,(R) = I^{\mathbb{N}}$; meaning for all $\sigma \in I^{\mathbb{N}}$ and $x \in X$ the sequence $\{R(\sigma, n, x)\}_{n\in\mathbb{N}}$ converges to a constant independent of x that we call $R(\sigma)$. In other words the sequence of functions $\{R(\sigma, n, \cdot)\}_{n\in\mathbb{N}}$ converges pointwise to a constant function $R(\sigma)$. Furthermore it turns out that every point on an attractor has an address that is $R(I^{\mathbb{N}}) = A$ if A is the attractor. For this reason the function R is called the address map.

Example 8. *If we recall Example 6 we can now see that there was the implicit use of the address map. In fact in that example the address map has a very clear meaning to the space $[0, 1]$ it represents trinary expansions. In general much like every point in $[0, 1]$ has a trinary expansion every point on an attractor has an address (see Theorem 11). Even better, because of the limit involved in the definition it means the first few finite compositions will give us some information of where the limit is. That is if $\mathrm{f}_{\sigma_{[1,n]}}(x)$ should be close to $R(\sigma)$ much like knowing the first n digits of a number $x = 0.x_1x_2x_3 \cdots \in [0, 1]$ would give us a number close* *to x. Thus one can think of $\sigma \in I^{\mathbb{N}}$ with $R(\sigma) = a \in A$ as an N-ary expansion of a in some sense, where $N \in \mathbb{N}$ is $|I| = N$.*

To solidify this idea, we generalize Example 6 slightly. Let $N \in \mathbb{N}$ be fixed. We define the IFS on $[0,1]$ with the normal metric by $\mathcal{F}_N = \{\mathrm{f}_i\}_{i=0}^{N-1}$ where for $i = 0, 1, 2, \ldots N-1$ and $x \in [0,1]$

$$\mathrm{f}_i(x) = \frac{x}{N} + \frac{i}{N}.$$

If we let $x = 0.x_1x_2x_3\ldots$ be an N-ary representation of x we can see the action of f_i on x is

$$\mathrm{f}_i(x) = 0.ix_1x_2x_3\ldots.$$

If we let $I = \{0, 1, 2, \ldots, N-1\}$ we can see for $\sigma \in I^{\mathbb{N}}$ that $R(\sigma) = 0.\sigma_1\sigma_2\sigma_3\ldots$. Thus we can see that $R\big(I^{\mathbb{N}}\big) = [0,1]$, by picking σ appropriately so that it matches an N-ary expansion of a given $x \in [0,1]$.

Now $[0,1]$ isn't exactly what we would call a fractal. It gets more interesting when we take $\emptyset \neq J \subseteq I$ and consider $\{\mathrm{f}_j\}_{j \in J}$. Then the attractor $R\big(J^{\mathbb{N}}\big)$ is the set of numbers in $[0,1]$ with no N-ary representation containing a member of $I \setminus J$ as an N-git.

If we take $N = 3$ and $J = \{0, 2\}$ we get Example 6.

The infinite composition of contractive functions are very interesting in general.

Proposition 9. *Let (X, d) be a complete metric space, $\mathcal{F} = \{\mathrm{f}_i\}_{i \in I}$ be a contractive IFS on X with $c = \sup_{i \in I} c_i < 1$ where c_i is the contraction factor of f_i and for $i \in I$, for all $x \in X$ we have $r = \sup_{i \in I} \mathrm{d}(x, \mathrm{f}_i(x)) < \infty$. Then for all $\sigma \in I^{\mathbb{N}}$ the function $R(\sigma)$ is defined. Notably if $\mathcal{F}$ is finite then both $c < 1$ and $r < \infty$.*

Proof. Noting that for all $i \in I$ the functions f_i have contraction factor c and we can see that for all $\sigma \in I^{\mathbb{N}}$, $x \in X$ and $n \in \mathbb{N}$ we have

$$\mathrm{d}(R(\sigma,n,x),R(\sigma,n+1,x)) = \mathrm{d}\Big(\mathrm{f}_{\sigma_{[1,n]}}(x),\mathrm{f}_{\sigma_{[1,n]}} \circ \mathrm{f}_{\sigma_{n+1}}(x)\Big) \le c^n\,\mathrm{d}\big(x,\mathrm{f}_{\sigma_{n+1}}(x)\big) \le c^n r.$$

From here we can follow the proof of Banach's Fixed Point Theorem (Theorem 8) to show that $\{R(\sigma,n,x)\}_{n\in\mathbb{N}}$ is Cauchy and therefore converges to, say, $R(\sigma,x)$.

Now pick any $y \in X$ and $\epsilon > 0$ and pick $N \in \mathbb{N}$ large enough so that for all $n \ge N$ $\max\{\mathrm{d}(R(\sigma,x),R(\sigma,n,x)),c^n\,\mathrm{d}(x,y)\} < \frac{\epsilon}{2}$. Then for all $n \ge N$

$$\mathrm{d}(R(\sigma,x),R(\sigma,n,y)) \le \mathrm{d}(R(\sigma,x),R(\sigma,n,x)) + \mathrm{d}(R(\sigma,n,x),R(\sigma,n,y)) < \frac{\epsilon}{2} + c^n\,\mathrm{d}(x,y) < \epsilon.$$

Thus $\{R(\sigma,n,y)\}_{n\in\mathbb{N}}$ converges to $R(\sigma,x)$ and so $R(\sigma,x) = R(\sigma,y)$ for all $x,y \in X$. As σ is arbitrary this completes the proof. □

Traditionally, for finite contractive IFS the well definedness of the addressing function R is proved via a different method. But to do so we need to know a bit more about the address space $I^{\mathbb{N}}$.

Proposition 10. *Let I be a set. Then $I^{\mathbb{N}}$ is a complete metric space with metric*

$$\mathrm{s}(\sigma,\lambda) = \begin{cases} \frac{1}{2^n} & \sigma \neq \lambda \\ 0 & \sigma = \lambda \end{cases}$$

where $\sigma,\lambda \in I^{\mathbb{N}}$ and $n \in \mathbb{N}\cup\{0\}$ is the largest $n \in \mathbb{N}$ with $\sigma_{[1,n]} = \lambda_{[1,n]}$, and if no such natural number exists then $n = 0$. Additionally, s *satisfies, for all $\sigma,\lambda,\gamma \in I^{\mathbb{N}}$*

$$\mathrm{s}(\sigma,\lambda) \le \max\{\mathrm{s}(\sigma,\gamma),\mathrm{s}(\gamma,\lambda)\}.$$

Furthermore, if I is finite then $\big(I^{\mathbb{N}},\mathrm{s}\big)$ is a compact metric space.

Proof. First we show that s is a metric. Firstly we should argue that s is well defined; the case where $\sigma = \lambda$ is well defined. So suppose that $\sigma \neq \lambda$ then there is an $N \in \mathbb{N}$ such that $\sigma_N \neq \lambda_N$ and take N to be the least such $N \in \mathbb{N}$. Thus $N-1 \in \mathbb{N} \cup \{0\}$ is the n in the above definition. It follows from the definition that s maps $I^{\mathbb{N}} \times I^{\mathbb{N}}$ to the non-negative real numbers. To show symmetry we first consider the case where $\sigma = \lambda$ and $\mathrm{s}(\sigma, \lambda) = \mathrm{s}(\sigma, \sigma) = \mathrm{s}(\lambda, \sigma) = 0$. So assume that $\sigma \neq \lambda$ and that $\mathrm{s}(\sigma, \lambda) = \frac{1}{2^{n_\sigma}}$ for some $n_\sigma \in \mathbb{N}$, so $\sigma_{[1,n_\sigma]} = \lambda_{[1,n_\sigma]}$, and because n_σ is the largest such n_σ we have that $\sigma_{[1,n_\sigma+1]} \neq \lambda_{[1,n_\sigma+1]}$. This means that the largest n satisfying $\lambda_{[1,n]} = \sigma_{[1,n]}$ must also be n_σ. If $\sigma_1 \neq \lambda_1$ then we can make a similar argument. Therefore $\mathrm{s}(\sigma, \lambda) = \mathrm{s}(\lambda, \sigma)$.

Now suppose that $\mathrm{s}(\sigma, \lambda) = 0$. If $\sigma \neq \lambda$ then $\mathrm{s}(\sigma, \lambda) > 0$ thus $\sigma = \lambda$. The converse holds by definition.

We now show the "additionally". Firstly, if $\sigma = \lambda$ the inequality holds for all $\gamma \in I^{\mathbb{N}}$. So assume that $\sigma \neq \lambda$. If $\gamma = \lambda$ then the max will be equal to $\mathrm{s}(\sigma, \gamma) = \mathrm{s}(\sigma, \lambda)$ and the inequality holds. Now suppose that σ, λ, γ are three distinct elements of $I^{\mathbb{N}}$ and that $m, n, k \in \mathbb{N} \cup \{0\}$ such that $\mathrm{s}(\sigma, \lambda) = \frac{1}{2^n}$, $\mathrm{s}(\sigma, \gamma) = \frac{1}{2^m}$ and $\mathrm{s}(\gamma, \lambda) = \frac{1}{2^k}$. Now if $m \leq n$ or $k \leq n$ then the inequality holds. So assume that both $m, k > n$ this means that $\gamma_{[1,n+1]} = \sigma_{[1,n+1]}$ and $\gamma_{[1,n+1]} = \lambda_{[1,n+1]}$. But this means that $\lambda_{[1,n+1]} = \sigma_{[1,n+1]}$ which is a contradiction as n was supposed to be the largest such n.

To show the triangle inequality, fix $\sigma, \lambda, \gamma \in I^{\mathbb{N}}$ and without loss of generality let $\mathrm{s}(\sigma, \gamma) \geq \mathrm{s}(\gamma, \lambda)$. Now by the "additionally" we have

$$\mathrm{s}(\sigma, \lambda) \leq \mathrm{s}(\sigma, \gamma) \leq \mathrm{s}(\sigma, \gamma) + \mathrm{s}(\gamma, \lambda).$$

To show completeness, consider a Cauchy sequence $\{\sigma^n\}_{n \in \mathbb{N}}$ of $I^{\mathbb{N}}$. Now define $\sigma \in I^{\mathbb{N}}$ by, for all $N \in \mathbb{N}$, defining $\sigma_N = \sigma_N^{k_N}$, where k_N is the least such $k_N \in \mathbb{N}$ satisfying: for all $m, n \geq k_N$ $\mathrm{s}(\sigma^n, \sigma^m) \leq \frac{1}{2^N}$. We claim that for all $N \in \mathbb{N}$ we have $\mathrm{s}\big(\sigma, \sigma^{k_N}\big) \leq \frac{1}{2^N}$. This follows from the sequence $\{k_M\}_{M\in\mathbb{N}}$ being non-decreasing. Indeed, for all $m \le N$ we have $\sigma_m = \sigma_m^{k_m} = \sigma_m^{k_N}$ and the claim follows. Now pick any $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $\frac{1}{2^{N-1}} < \epsilon$ and get k_N from the definition of σ. Consider $n \ge k_N$

$$\mathrm{s}(\sigma, \sigma^n) \le \mathrm{s}\big(\sigma, \sigma^{k_N}\big) + \mathrm{s}\big(\sigma^{k_N}, \sigma^n\big) \le \frac{1}{2^{N-1}} < \epsilon.$$

Thus $\{\sigma^n\}_{n\in\mathbb{N}}$ converges to σ and $\big(I^{\mathbb{N}}, \mathrm{s}\big)$ is complete.

Finally, to show compactness we show that $I^{\mathbb{N}}$ is totally bounded. Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that $r = \frac{1}{2^{N-1}} < \epsilon$. We claim that for any $\sigma \in I^{\mathbb{N}}$

$$I^{\mathbb{N}} \subseteq \bigcup_{\alpha\in I^N} \mathbb{B}_r(\alpha\sigma) \subseteq \bigcup_{\alpha\in I^N} \mathbb{B}_\epsilon(\alpha\sigma).$$

The second inclusion is from the definition of r. To show the first inclusion consider $\lambda \in I^{\mathbb{N}}$ then $\lambda = \lambda_{[1,N]}\lambda_{[1,\infty)}$ and $\lambda_{[1,N]} \in I^N$. So $\mathrm{d}\big(\lambda_{[1,N]}\sigma, \lambda\big) \le \frac{1}{2^N} < \frac{1}{2^{N-1}} = r$. Therefore, $\lambda \in \bigcup_{\alpha\in I^N} \mathbb{B}_r(\alpha\sigma)$ and $I^{\mathbb{N}} \subseteq \bigcup_{\alpha\in I^N} \mathbb{B}_r(\alpha\sigma)$. Now, since I is finite I^N is finite, and thus $\bigcup_{\alpha\in I^N} \mathbb{B}_\epsilon(\alpha\sigma)$ is a finite ϵ cover of $I^{\mathbb{N}}$, giving that $I^{\mathbb{N}}$ is totally bounded. Therefore $I^{\mathbb{N}}$ is compact as it is complete and totally bounded. □

Remark 4. *If I is infinite then $\big(I^{\mathbb{N}}, \mathrm{s}\big)$ is not compact. To see this, recognize there must be an element of $I^{\mathbb{N}}$, say σ, such that for all $k, n \in \mathbb{N}$ $\sigma_k \neq \sigma_n$ (i.e the letters of σ are all distinct). This mean the sequence $\{\sigma_{[n,\infty)}\}_{n\in\mathbb{N}}$ has no convergent sequences, as for every $n, k \in \mathbb{N}$ we have $\mathrm{s}\big(\sigma_{[n,\infty)}, \sigma_{[k,\infty)}\big) = 1$.*

Additionally, the use of $\frac{1}{2}$ in the definition of s is arbitrary we could use any $c \in (0,1)$ for Proposition 10 to hold.

Note the actual distances given by s don't really matter. What does matter is the n that the distances give us. This n is the length of the longest finite prefix that the two elements share (if one exists); it is this n that tells us how close together the elements are.

It turns out the address map R (in the context of Proposition 9) is a continuous function from $I^{\mathbb{N}}$ to X.

Theorem 11. *Let (X, d) be a complete metric space and $\mathcal{F} = \{\mathrm{f}_i\}_{i\in I}$ be a finite contractive IFS on X. Then there is a unique continuous function $R : I^{\mathbb{N}} \to X$ satisfying for all $i \in I$ and $\sigma \in I^{\mathbb{N}}$*

$$\mathrm{f}_i \circ R(\sigma) = R(i\sigma)$$

and

$$R(\sigma) = \mathrm{f}_{\sigma_1} \circ R\big(\sigma_{[2,\infty)}\big).$$

Furthermore $R\big(I^{\mathbb{N}}\big)$ is a nonempty compact invariant set and so is the attractor of $\mathcal{F}$.

Proof. Fix $x \in X$. Then for each $n \in \mathbb{N}$ $R(\sigma, n, x) = R_n(\sigma)$ is a function from $I^{\mathbb{N}}$ to X that is continuous. If $\lambda \in \mathbb{B}^{I^{\mathbb{N}}}_{\frac{1}{2^{n-1}}}(\sigma)$ then $R_n(\sigma) = R_n(\lambda)$; so $R_n(\lambda)$ is in every open set containing $R_n(\sigma)$. Also note that $R_n(\sigma) = \mathrm{f}_{\sigma_{[1,n]}}(x)$.

We now show that $\{R_n\}_{n\in\mathbb{N}}$ converges uniformly. Much like in the proof of Banach's Fixed Point Theorem, we consider for some $\sigma \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$

$$\begin{aligned}\mathrm{d}(R_{n+1}(\sigma), R_{n+2}(\sigma)) &= \mathrm{d}\Big(\mathrm{f}_{\sigma_{[1,n]}} \circ R_1\big(\sigma_{[n+1,\infty]}\big), \mathrm{f}_{\sigma_{[1,n]}} \circ R_2\big(\sigma_{[n+1,\infty]}\big)\Big)\\ &\leq c^n\, \mathrm{d}\big(R_1\big(\sigma_{[n+1,\infty]}\big), R_2\big(\sigma_{[n+1,\infty]}\big)\big),\end{aligned}$$

where c is the largest contraction factor of the f_i's, taking the supremum over $I^{\mathbb{N}}$ gives $\mathrm{d}(R_{n+1}, R_{n+2}) \leq c^n\, \mathrm{d}(R_1, R_2)$. Now we can see that for all $m, n \geq N \in \mathbb{N}$

$$\mathrm{d}(R_{n+1}, R_{m+1}) \leq \mathrm{d}(R_1, R_2) \sum_{k=n}^{\infty} c^k \leq \mathrm{d}(R_1, R_2) \sum_{k=N}^{\infty} c^k = \mathrm{d}(R_1, R_2) \frac{c^N}{1-c}.$$

So for any given $\epsilon > 0$ we can take N large enough for $\mathrm{d}(R_1, R_2)\frac{c^N}{1-c} < \epsilon$. Thus $\{R_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence (in $\mathcal{C}(X, X)$ with respect to the uniform distance) and so converges to a continuous function as $\mathcal{C}(X, X)$ is complete by Theorem 1.

Let $\{R_n\}_{n\in\mathbb{N}} \to R$ uniformly. So $\{R_n\}_{n\in\mathbb{N}}$ converges pointwise to R. Let $i \in I$, $\sigma \in I^{\mathbb{N}}$ and so $\{\mathrm{f}_i \circ R_n(\sigma)\}_{n\in\mathbb{N}} \to \mathrm{f}_i \circ R(\sigma)$ by continuity. But $\mathrm{f}_i \circ R_n(\sigma) = R_{n+1}(i\sigma)$ for all $n \in \mathbb{N}$ and $\{R_{n+1}(i\sigma)\}_{n\in\mathbb{N}}$ is a subsequence of $\{R_n(i\sigma)\}_{n\in\mathbb{N}} \to R(i\sigma)$. Therefore $R(i\sigma) = \mathrm{f}_i \circ R(\sigma)$.

Now for $\sigma \in I^{\mathbb{N}}$ we know that $\{\mathrm{f}_{\sigma_1} \circ R_n(\sigma_{[2,\infty]})\}_{n\in\mathbb{N}} \to \mathrm{f}_{\sigma_1} \circ R(\sigma_{[2,\infty]})$, and so, by the above, $\mathrm{f}_{\sigma_1} \circ R(\sigma_{[2,\infty]}) = R(\sigma)$.

To show uniqueness suppose R and g both satisfy for all $\sigma \in I^{\mathbb{N}}$ $\mathrm{f}_{\sigma_1} \circ R(\sigma_{[2,\infty]}) = R(\sigma)$ and $\mathrm{f}_{\sigma_1} \circ \mathrm{g}(\sigma_{[2,\infty]}) = \mathrm{g}(\sigma)$. Consider for any $\sigma \in I^{\mathbb{N}}$

$$\mathrm{d}(R(\sigma), \mathrm{g}(\sigma)) = \mathrm{d}\big(\mathrm{f}_{\sigma_1} \circ R(\sigma_{[2,\infty]}), \mathrm{f}_{\sigma_1} \circ \mathrm{g}(\sigma_{[2,\infty]})\big) \leq c\,\mathrm{d}\big(R(\sigma_{[2,\infty]}), \mathrm{g}(\sigma_{[2,\infty]})\big)$$

taking the supremum over $I^{\mathbb{N}}$ of both sides tells us

$$\mathrm{d}(R, \mathrm{g}) \leq c\,\mathrm{d}(R, \mathrm{g}).$$

Since $c < 1$, the only possibility is $\mathrm{d}(R, \mathrm{g}) = 0$ and $R = \mathrm{g}$.

Finally $R(I^{\mathbb{N}})$ is compact, as $I^{\mathbb{N}}$ is compact and R continuous. $R(I^{\mathbb{N}})$ is invariant, as for every $R(\sigma) \in R(I^{\mathbb{N}})$ we have $R(\sigma) = \mathrm{f}_{\sigma_1} \circ R(\sigma_{[2,\infty)})$. Noting $\sigma_1 \in I$ and so $\sigma_{[2,\infty)} \in I^{\mathbb{N}}$ we have $R(I^{\mathbb{N}}) \subseteq \bigcup_{i\in I} \mathrm{f}_i(R(I^{\mathbb{N}}))$. Simulaly $\mathrm{f}_i \circ R(\sigma) = R(i\sigma)$ for all $i \in I$ so $R(I^{\mathbb{N}}) \supseteq \bigcup_{i\in I} \mathrm{f}_i(R(I^{\mathbb{N}}))$. Therefore, $R(I^{\mathbb{N}})$ is compact invariant set and so is the attractor. □

We would like to point out the construction of the limit function in the above proof does in fact coincide with Definition 18.

As one might expect we can use the address map to help us prove that the chaos game converges. This will be a recurring strategy in Chapter 4.

Chapter 4

Generalized Attractors and The Chaos Game

In this Chapter we use well known generalizations of attractors and give sufficient conditions for the chaos game to "draw" these generalized attractors.

4.1 Some Generalizations of Attractors

We will focus on one particular generalization of attractors, something that is known as semi-attractors. They were first introduced by Losota and Myjak [9, 10, 11]. These authors define semi-attractors via set limits introduced in Chapter 2. The author of this thesis had essentially independently discovered these semi-attractors early this year. When he became aware of the results of Losota and Myjak he was able to deduce some interesting proprieties of a mild generalization of semi-attractors. We will now state the definition of this mild generalization.

Definition 19. *Suppose that* (X, d) *is a metric space,* $A \subseteq X$*, and* $\mathrm{F} : X \rightsquigarrow X$ *is a lower semicontinuous multifunction. Then* A *is called the semi-attractor of* F *if it is the smallest* *nonempty closed sub-invariant set of* F*. That is for all* $B \subseteq X$ *with* B *nonempty, closed and satisfying* $\mathrm{F}[B] \subseteq B$ *then* $A \subseteq B$ *where* A *is nonempty, closed, and satisfies* $\mathrm{F}[A] \subseteq A$*.*

Furthermore, we say that A *is the semi-attractor of* $\mathcal{F}$ *on* X *if* A *is the semi-attractor of the Hutchinson-Barnsley operator of the* $\mathcal{F}$*.*

Note that a semi-attractor must be unique as it is "the smallest", see the discussion following Definition 13. It will be easier to see why this is a generalization of the semi-attractors introduced in [11] after we prove the following.

Theorem 12. *Let* (X, d) *be a metric space,* $\mathrm{F} : X \rightsquigarrow X$ *be a lower semicontinuous multi-function on* X *and suppose* A *is the semi-attractor of* F*. Then*

$$A = \bigcap_{x \in X} \operatorname*{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x] = \bigcap_{x \in X} \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]}.$$

Furthermore for all $a \in A$*,*

$$A = \operatorname*{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[a] = \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[a]}.$$

The proof of this Theorem follows in part from the below Proposition.

Proposition 11. *Let* (X, d) *be a metric space, and* $\mathrm{F} : X \rightsquigarrow X$ *be a multifunction on* X*. Let* $A \subseteq X$ *be a non-empty, closed and sub-invariant with respect to* F*. If* F *is lower semicontinuous then*

$$\bigcap_{x \in X} \operatorname*{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x] \subseteq \bigcap_{x \in X} \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]} \subseteq A.$$

Furthermore, both $\bigcap_{x \in X} \mathrm{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x]$ *and* $\bigcap_{x \in X} \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]}$ *are closed and sub-invariant with respect to* F*. Thus if one of these sets is non-empty it is the semi-attractor of* F*.*

Proof. As A is sub-invariant, for all $n \in \mathbb{N}$ and for all $a \in A$, we have $\mathrm{F}^{\circ n}[a] \subseteq A$. Thus, recalling A is closed we have for all $N \in \mathbb{N}$ and for all $a \in A$,

$$\overline{\bigcup_{n\geq N} \mathrm{F}^{\circ n}[a]} \subseteq \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]} \subseteq A.$$

Recalling that $\mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[a] = \bigcap_{N\in\mathbb{N}} \overline{\bigcup_{n\geq N} \mathrm{F}^{\circ n}[a]}$ we conclude that for all $a \in A$, $\mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[a] \subseteq \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]} \subseteq A$. Thus

$$\bigcap_{a\in A} \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[a] \subseteq \bigcap_{a\in A} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]} \subseteq A.$$

It follows from $A \subseteq X$ that $\bigcap_{x\in X} \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x] \subseteq \bigcap_{a\in A} \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[a]$ and $\bigcap_{x\in X} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]} \subseteq \bigcap_{a\in A} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]}$.

Using a similar argument as above, one can show for all $x \in X$ $\mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x] \subseteq \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}$. Therefore

$$\bigcap_{x\in X} \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x] \subseteq \bigcap_{x\in X} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]} \subseteq A.$$

Now I must show that $\bigcap_{x\in X} \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x]$ and $\bigcap_{x\in X} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}$ are sub-invariant.

I claim that the intersection of sub-invariant sets is sub-invariant. Suppose $\mathcal{B}$ is collection of sub-invariant sets with respect to F, then $\mathrm{F}\left[\bigcap_{B\in\mathcal{B}} B\right] \subseteq \bigcap_{B\in\mathcal{B}} \mathrm{F}[B] \subseteq \bigcap_{B\in\mathcal{B}} B$.

Now I claim that for all $N \in \mathbb{N}$ and for all $x \in X$ the set $\overline{\bigcup_{n\geq N} \mathrm{F}^{\circ n}[x]}$ is sub-invariant. Recall that as F is lower semicontinuous for all $B \subseteq X$, $\mathrm{F}\left[\overline{B}\right] \subseteq \overline{\mathrm{F}[B]}$. Thus

$$\mathrm{F}\left[\overline{\bigcup_{n\geq N} \mathrm{F}^{\circ n}[x]}\right] \subseteq \overline{\bigcup_{n\geq N} \mathrm{F}^{\circ(n+1)}[x]} = \overline{\bigcup_{n>N} \mathrm{F}^{\circ n}[x]} \subseteq \overline{\bigcup_{n\geq N} \mathrm{F}^{\circ n}[x]}$$

taking $N = 1$ gives $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}$ is sub-invariant for all $x \in X$. Taking the intersection over all $x \in X$ gives us $\bigcap_{x\in X} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}$ is sub-invariant and closed. Now we can see $\bigcap_{N\in\mathbb{N}} \overline{\bigcup_{n\geq N} \mathrm{F}^{\circ n}[x]} = \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x]$ is sub-invariant for all $x \in X$ and so $\bigcap_{x\in X} \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x]$ is sub-invariant and closed. $\square$

Now we prove Theorem 12.

Proof. (Theorem 12) In the proof of Proposition 11 we show for all $N \in \mathbb{N}$ and $x \in X$ the set $\overline{\bigcup_{n \geq N} \mathrm{F}^{\circ n}[x]}$ is sub-invariant. The set is also closed and non-empty, thus as A is the semi-attractor of F we have for all $N \in \mathbb{N}$ and for all $x \in X$

$$\emptyset \neq A \subseteq \overline{\bigcup_{n \geq N} \mathrm{F}^{\circ n}[x]}.$$

Thus

$$A \subseteq \bigcap_{x \in X} \bigcap_{N \in \mathbb{N}} \overline{\bigcup_{n \geq N} \mathrm{F}^{\circ n}[x]} = \bigcap_{x \in X} \operatorname*{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x] \neq \emptyset$$

and, by Proposition 11, $\bigcap_{x \in X} \mathrm{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x] \subseteq \bigcap_{x \in X} \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]} \subseteq A$. The result follows.

To prove the "furthermore", note that $\mathrm{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x] \neq \emptyset$ for all $x \in X$ by the above argument and it is always sub-invariant and closed. $\overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]}$ is also non-empty closed and sub-invariant for all $x \in X$. Now pick any $a \in A$. As A is sub-invariant, it follows that

$$\operatorname*{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[a] \subseteq \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[a]} \subseteq A$$

But the sets $\mathrm{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[a]$ and $\overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[a]}$ are non-empty, closed and sub-invariant so the opposite inclusions holds, as A is the smallest such set. □

An interesting consequence of Theorem 12 is that a naive iteration algorithm—pick a point in $x \in X$ and keep applying the Hutchinson-Barnsley operator to form the sequence of sets $\{\mathrm{F}^{\circ n}[x]\}$—actually works in some sense whenever $x \in A$. The below Theorem mirrors Theorem 6.2 of [11].

Theorem 13. *Let* (X, d) *be a metric space,* $\mathrm{F} : X \rightsquigarrow X$ *be a lower semicontinuous multifunction on* X *and suppose* A *is the semi-attractor of* F*. Then the following hold:*

(i) $A \subseteq \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[B]$ *for all* $\emptyset \neq B \subseteq X$

(ii) $\overline{\mathrm{F}[A]} = A$

(iii) $\operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[B] = A$ *for all* $\emptyset \neq B \subseteq A$

(iv) $A \subseteq B$ *for all nonempty closed sub-invariant subsets* $B \subseteq X$

Proof. To prove (i), note that using Theorem 12 we have for all $b \in B$ $A \subseteq \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[b]$. Recall that the Ls of a sequence of sets is increasing; that is, if $X_n \subseteq Y_n$ for all $n \in \mathbb{N}$ then the $\mathrm{Ls}_{n\to\infty} X_n \subseteq \mathrm{Ls}_{n\to\infty} Y_n$. We can see

$$A \subseteq \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[b] \subseteq \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[B].$$

(ii) follows from observing that $\overline{\mathrm{F}[A]}$ is nonempty closed sub-invariant set. To see this, first notice that as A is nonempty closed and sub-invariant $\emptyset \neq F^{\circ 2}[A] \subseteq F[A] \subseteq \overline{\mathrm{F}[A]} \subseteq A$. Thus, by the lower semi continuity of F,

$$\mathrm{F}\left[\overline{\mathrm{F}[A]}\right] \subseteq \overline{\mathrm{F}^{\circ 2}[A]} \subseteq \overline{\mathrm{F}[A]} \subseteq A.$$

So $\overline{\mathrm{F}[A]}$ is nonempty closed sub-invariant set but A is the smallest such set, so $A \subseteq \overline{\mathrm{F}[A]}$.
(iii) is due to

$$\operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[B] \subseteq \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[B]} \subseteq A$$

this follows from $\mathrm{F}^{\circ n}[B] \subseteq \mathrm{F}^{\circ n}[A]$ for all $n \in \mathbb{N}$. The other inclusion is given by (i).
(iv) is the definition of being the semi-attractor of F. □

Recalling the discussion at the beginning of this chapter, Definition 19 is not the same definition of semi-attractor given in [11]. They define the semi-attractor of F in the sense of

Losota and Myjak to be

$$A = \bigcap_{x \in X} \operatorname*{Li}_{n \to \infty} \mathrm{F}^{\circ n}[x]$$

whenever A is nonempty. One can see that if A is a semi-attractor of F in the sense of Losota and Myjak, then it is a semi-attractor in the sense of Definition 19. This allows for the Losota and Myjak semi-attractors to have nice interactions with set limits; that is, if A is a Losota and Myjak semi-attractor then $A = \bigcap_{x \in X} \lim_{n \to \infty} \mathrm{F}^{\circ n}[x]$. Unfortunately, it is possible for $\bigcap_{x \in X} \mathrm{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x]$ to be nonempty but for $\bigcap_{x \in X} \mathrm{Li}_{n \to \infty} \mathrm{F}^{\circ n}[x]$ to be empty; see Example 9. Even though semi-attractors in the sense of Definition 19 are not as well behaved, we can still get many similar results to Theorem 6.2 of [11].

Henceforth, one should always assume that semi-attractor is meant in the sense of Definition 19.

Example 9. *Let* $X = \{z \in \mathbb{C} : |z| = 1\}$. *For all* $z \in X$, $z = e^{i\theta}$ *with* $\theta \in \mathbb{R}$ *let* $\mathrm{f} : X \to X$ *be given by,*

$$\mathrm{f}(z) = e^{i\sqrt{2}\pi + i\theta}$$

and $\mathrm{f}[z] = \{\mathrm{f}(z)\}$. *Then* f *(as a multifunction) is lower semicontinuous as it is continuous (as a single valued function). Then for all* $z \in X$, $\mathrm{Ls}_{n \to \infty} \mathrm{f}^{\circ n}[z] = X$ *and* $\mathrm{Li}_{n \to \infty} \mathrm{f}^{\circ n}[z] = \emptyset$.

Returning our attention back to Theorem 12 and considering the case that F is the Hutchinson Barnsley operator of an IFS. Then $a \in A$ if and only if for all $x \in X$, for all $\epsilon > 0$ and for all $N \in \mathbb{N}$ there is a composition of functions of the IFS, f_α, such that it is the composition of more then N functions of the IFS for which $\mathrm{d}(\mathrm{f}_\alpha(x), a) < \epsilon$. Speaking more informally this means no matter where you are in X you can get anywhere (within ϵ) in the semi-attractor by applying a composition of the functions of the IFS (in fact you can take this compositions to be longer then any prescribed $N \in \mathbb{N}$).

Also notice that if there is an $x \in X$ for which $\bigcap_{N \in \mathbb{N}} \overline{\bigcup_{n \geq N} \mathrm{F}^{\circ n}[x]}$ or $\mathrm{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x]$ is compact then the semi-attractor of F is also compact. Of course, the previous sentence holds if we replace "compact" with "bounded".

If we consider $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}$ to be a multifunction, then F has a semi-attractor if and only if the graph of the multifunction $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}}$ contains a horizontal line, that is there is a $y \in X$ such that

$$\left\{(x,y) \in X^2 | x \in X\right\} \subseteq \left\{(x,z) \in X^2 | x \in X, z \in \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}\right\}.$$

Of course, the semi-attractor of F is the union of all such y. Furthermore for all such y, $y \in \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[y]}$; that is y, is a (multi?) fixed point of $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}}$.

The following result is mildly convenient to refer to.

Lemma 1. *Let (X, d) be a metric space, $\mathrm{F} : X \rightsquigarrow X$ be a lower semicontinuous multifunction on X and suppose for all $x \in X$ there is a set $B_x \subseteq \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}$ that is sub-invariant with respect to* F*.*

If $\bigcap_{x\in X} \overline{B_x} \neq \emptyset$ then $\bigcap_{x\in X} \overline{B_x}$ is the semi-attractor of F*.*

Proof. It's clear that

$$\emptyset \neq \bigcap_{x\in X} \overline{B_x} \subseteq \bigcap_{x\in X} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}.$$

But by Theorem 12 $\bigcap_{x\in X} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]} = A$ is the semi-attractor of F whenever it is nonempty. So let A be the semi-attractor of F. Now I claim that $\bigcap_{x\in X} \overline{B_x}$ is sub-invariant. Indeed, by lower semi continuity and sub-invariance of the B_x we have

$$\mathrm{F}\left[\bigcap_{x\in X} \overline{B_x}\right] \subseteq \bigcap_{x\in X} \mathrm{F}\left[\overline{B_x}\right] \subseteq \bigcap_{x\in X} \overline{\mathrm{F}[B_x]} \subseteq \bigcap_{x\in X} \overline{B_x}.$$

But A is the smallest nonempty closed sub-invariant set of F, so $A \subseteq \bigcap_{x\in X} \overline{B_x}$. This completes the proof. □

Remark 5. *At this point, due to the rich nature of semi-attractors (the smallest nonempty* *closed sub-invariant set), it may be natural to ask about the largest nonempty closed sub-invariant set. This of course would be the space itself. But a closely related notion is the largest super-invariant set, see [1]. In [1] they show that (among other things) if*

1. *Given a finite IFS with functions that are finitely fibered, that is for all $x \in X$ and $\mathrm{f} \in \mathcal{F}$ the set $\mathrm{f}^{-1}(x)$ is finite (such as when the functions are injective).*

2. *The space X is compact and the IFS is finite and continuous.*

Then the set

$$\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[X]$$

is the greatest nonempty invariant set of $\mathcal{F}$.

It is rather straightforward to show that $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[X]$ contains every super-invariant set of a multifunction F. Indeed, suppose that $B \subseteq X$ satisfies $B \subseteq \mathrm{F}[B]$. Then by applying F to this relation we see

$$\begin{aligned} &B \subseteq \mathrm{F}[B] \\ &B \subseteq \mathrm{F}[B] \subseteq \mathrm{F}^{\circ 2}[B] \\ &\quad\vdots \\ &B \subseteq \mathrm{F}^{\circ n}[B] \end{aligned}$$

for all $n \in \mathbb{N}$. Thus $B \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[B] \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[X]$. Therefore $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[X]$ contains every super-invariant set and, so, if it is super-invariant then it is the largest super-invariant set with respect to F. Noting that in the above we show that for any super-invariant set B the set $\mathrm{F}[B]$ is also super-invariant, we have that $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[X]$ is also invariant.

Thus the smallest nonempty closed sub-invariant set and the largest super-invariant set *are*

$$\bigcap_{x\in X}\overline{\bigcup_{n\in\mathbb{N}}\mathrm{F}^{\circ n}[x]}$$

$$\bigcap_{n\in\mathbb{N}}\bigcup_{x\in X}\mathrm{F}^{\circ n}[x]$$

respectively, whenever they exist. These expressions are strikingly similar.

As alluded to before, there are more generalizations of attractors. However, our focus will be on "attractors" that can be "drawn" by the chaos game. Essentially we are interested in sets A that somehow equal the set of limit points of the sequence generated by playing the chaos game ($A = \bigcap_{m\in\mathbb{N}}\overline{\{x_n\}_{n=m}^{\infty}} = \mathrm{Ls}_{n\to\infty}\{x_n\}$ where $\{x_n\}_{n=m}^{\infty}$ is the set of points generated by the chaos game). Due to this limiting nature it makes sense to define generalized attractors via some limiting process, particularity in some neighborhood of the "attractors".

Definition 20. *Let* (X, d) *be a metric space,* $\mathrm{F} : X \rightsquigarrow X$ *be lower semicontinuous and* $A \subseteq X$*. We define the pointwise basin of weak limsup attraction of* A *under* F *to be*

$$\text{w-}\mathcal{B}(A) = \Big\{x \in X : A \subseteq \operatorname*{Ls}_{n\to\infty}\mathrm{F}^{\circ n}[x]\Big\}.$$

We define the pointwise basin of limsup attraction of A *under* F *to be*

$$\text{Ls-}\mathcal{B}(A) = \Big\{x \in X : A = \operatorname*{Ls}_{n\to\infty}\mathrm{F}^{\circ n}[x]\Big\}.$$

We define the pointwise basin of Hausdorff attraction of A *under* F *to be*

$$\text{PWH-}\mathcal{B}(A) = \Big\{x \in X : A = \lim_{n\to\infty}\mathrm{F}^{\circ n}[x] \textit{ where the limit is taken with respect to the Hausdorff metric}\Big\}.$$

We define the strict basin of Hausdorff attraction of A under F, strict-$\mathcal{B}(A)$, *to be the union of all open sets U with the following properties: $A \subseteq U$, and, for all $K \subseteq U$, with K nonempty and compact we have that the sequence of sets $\{\mathrm{F}^{\circ n}[K]\}_{n\in\mathbb{N}}$ converges to A in the Hausdorff metric.*

Note that for either PWH-$\mathcal{B}(A)$ *or* strict-$\mathcal{B}(A)$ *to be nonempty A must be compact.*

Finally we define the set of compact full orbits under F *to be*

$$\mathcal{O} = \left\{ x \in X : \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]} \text{ is compact} \right\}.$$

If $\mathcal{F}$ is an IFS on X with Hutchinson-Barnsley operator, F, *is lower semicontinuous the we will replace then "under* F*" with "under $\mathcal{F}$" in the above definitions.*

The most common type of generalized attractor discussed concerning the topic of the chaos game is the strict attractor and the point wise attractor. For instance, see [3, 4].

Definition 21. *Let (X, d) be a metric space, $\mathcal{F}$ be an IFS on X and $A \in \mathcal{H}(X)$. Then we say that A is a point wise attractor if there is an open set $U \subseteq X$ with $A \subseteq U \subseteq$* PWH-$\mathcal{B}(A)$.

Similarly, we say that A is a strict attractor if strict-$\mathcal{B}(A) \neq \emptyset$.

The last type of attractor we will be identifying are quasi attractors.

Definition 22. *Let (X, d) be a metric space, $\mathcal{F}$ be an IFS on X and $A \subseteq X$. Then we say that A is a quasi attractor if for all $a \in A$ we have $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]} = A \neq \emptyset$.*

Example 10. *Let $X = \mathbb{C} \setminus \{0\}$ with the normal metric and consider the functions: for all $z = re^{i\theta}$, $\theta \in \mathbb{R}$ and $r \in [0, \infty)$*

$$\mathrm{f}_1(z) = re^{i\left(\theta + \sqrt{2}\pi\right)}$$

$$\mathrm{f}_2(z) = e^{i\theta}.$$

Now, consider the IFS $\{\mathrm{f}_1\}$*. We can see that for every* $r \in \mathbb{R}$ *the set* $\{z \in \mathbb{C} \mid |z| = r\}$ *is a quasi attractor of this IFS. Thus, this is an example of a IFS that has a quasi attractor but no semi-attractor.*

The IFS $\{\mathrm{f}_1, f_2\} = \mathcal{F}$ *has a semi-attractor that is the set* $A = \{z \in \mathbb{C} \mid |z| = 1\}$*. One can see this by recognizing that* $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[z]} \supseteq \mathrm{Ls}_{n\to\infty} \mathrm{f}_{12}^{\circ n}(z) = A$ *for all* $z \in X$*. But* A *is not a point wise attractor of* $\mathcal{F}$ *as for all* $z \in X$ *we have* $z \in \mathrm{Ls}_{n\to\infty} \mathrm{f}_1^{\circ n}[z] \subseteq \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[z]$*, so even if* $\{\mathrm{F}^{\circ n}[z]\}_{n\in\mathbb{N}}$ *converges in the Hausdorff metric this limit must contain* z*, by picking any open set* $U \supseteq A$ *and* $z \in U \setminus A$ *we see that* A *is not a point wise attractor.*

Remark 6. *A question that is typically explored by the literature is: Given an IFS,* $\mathcal{F}$*, with an attractor* A *(possibly any of the attractors mentioned so far) on a metric space* (X, d) *does there exists a set* B *containing* A *(*B *is usually the space or* A *itself) and a metric* d' *on* B *equivalent to the original metric on* (B, d) *such that either: for all* $\mathrm{f} \in \mathcal{F}$*,* f *is a contraction on* $\left(B, \mathrm{d}'\right)$ *or* F *is a contraction on* $\left(\mathcal{H}(B), \mathrm{d}'_H\right)$*. For example see [13]. In general the answer to this question is no. Particularly,* F *being a contraction on* $\left(\mathcal{H}(B), \mathrm{d}'_H\right)$ *is a very strong condition, as if* B *is complete then Banuach's Fixed Point Theorem applies and for every* $K \in \mathcal{H}(B)$ *the set sequence* $\{\mathrm{F}^{\circ n}[K]\}_{n\in\mathbb{N}}$ *converges (realistically to* A*) in the Hausdorff metric. But, we saw in Example 10 and Example 9 that there are IFS for which* $\mathrm{Li}_{n\to\infty} \mathrm{F}^{\circ n}[x] = \emptyset$ *for all* $x \in X$ *so in these cases* $\{\mathrm{F}^{\circ n}[K]\}_{n\in\mathbb{N}}$ *cannot converge with respect to the Hausdorff metric and hence* F *is not a contraction on* $\mathcal{H}(B)$ *for any metric equivalent to* d_H*. Thus this question is far more natural for strict attractors and point wise attractors.*

My first exposure into this field of research was trying to determine whether a particular IFS had a attractor and, if it does have one, when does the chaos draw the attractor starting in X*? I was able to answer this question to a near full extent, see [7]. It turns out that in for this particular IFS, if* x *is not in the attractor the sequence of sets* $\{\mathrm{F}^{\circ n}[x]\}_{n\in\mathbb{N}}$ *for* $x \in X$ *would never converge in the Hausdorff metric to the attractor (whenever the attractor exists).*

Unfortunately, most of the results concerning the chaos game assume that the attractor is a strict attractor or a point wise attractor (not that I understood the subtleties at the time). I also spent a considerable amount of time trying to define a metric for which the F *was a contraction; which of course was forlorn. This particular problem still shapes my current work and is a large reason for why I seek to avoid the use of the Hausdorff metric and assuming compactness (of the attractor or the space). Although, I am unable to completely get rid of compactness assumptions as we will see later.*

We will note that in [5] and [14] quasi attractors are defined to be nonempty compact, invariant with respect to the $\mathcal{F}$ and $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]} = A$ for all $a \in A$. We will show that quasi attractors as defined above are invariant whenever A is compact, the IFS is finite and continuous.

Additionally, it is not clear that strict attractors or point wise attractors are invariant. However it turns out that this is nearly the case whenever F is l.s.c.

Lemma 2. *Let* (X, d) *be a metric space,* $\mathrm{F} : X \rightsquigarrow X$ *be l.s.c and* $A \subseteq X$*. Then we have*

1.

$$\text{strict-}\mathcal{B}(A) \subseteq \text{PWH-}\mathcal{B}(A) \subseteq \text{Ls-}\mathcal{B}(A) \subseteq \text{w-}\mathcal{B}(A).$$

2. If $B \subseteq X$ *is a nonempty, closed, sub-invariant set contained in any of the above basins then* $A \subseteq B$*. Thus if* A *is nonempty closed sub-invariant and contained in the basin then it is the smallest such set in any of the basins and satisfies* $A = \overline{\mathrm{F}[A]}$*.*

3. If $\emptyset \neq \text{Ls-}\mathcal{B}(A)$ *then* A *is closed and sub-invariant. Thus, if* A *is nonempty and is contained in* $\text{strict-}\mathcal{B}(A), \text{PWH-}\mathcal{B}(A)$ *or* $\text{Ls-}\mathcal{B}(A)$ *then* A *is nonempty closed and satisfies* $A = \overline{\mathrm{F}[A]}$*.*

Proof. 1.

Suppose that $x \in$ strict-$\mathcal{B}(A)$ then $\{x\}$ is compact and so $\lim_{n\to\infty} \mathrm{F}^{\circ n}[x] = A$ in the Hausdorff metric. Thus $x \in$ PWH-$\mathcal{B}(A)$.

Suppose that $x \in$ PWH-$\mathcal{B}(A)$ then $\lim_{n\to\infty} \mathrm{F}^{\circ n}[x] = A$ in the Hausdorff metric meaning that $\{\mathrm{F}^{\circ n}[x]\}_{n\in\mathbb{N}}$ converges as a sequence of sets in the sense of Definition 6. Thus $A = \lim_{n\to\infty} \mathrm{F}^{\circ n}[x] = \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x]$ and $x \in$ Ls-$\mathcal{B}(A)$.

Let $x \in$ Ls-$\mathcal{B}(A)$; then $A = \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x]$ and, thus, $A \subseteq \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x]$.

2.

Suppose that $B \subseteq X$ is a nonempty, closed, sub-invariant set contained in any of the basins in 1. Then, by 1, B is a sub-invariant subset of w-$\mathcal{B}(A)$. Now, for all $b \in B$ one can show that $\mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[b] \subseteq B$, as B is closed and sub-invariant. But now since $b \in$ w-$\mathcal{B}(A)$

$$A \subseteq \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[b] \subseteq B$$

and so $A \subseteq B$.

Now if A is nonempty, closed, sub-invariant and contained in the basin it is the smallest such set, by the above. To show $A = \overline{\mathrm{F}[A]}$, we can notice that $\overline{\mathrm{F}[A]}$ is closed, nonempty, sub-invariant and $\overline{\mathrm{F}[A]} \subseteq A \subseteq$ strict-$\mathcal{B}(A)$. Indeed,

$$\emptyset \neq \overline{\mathrm{F}[A]} \subseteq A.$$

Recalling that F is lower semicontinuous yields

$$\mathrm{F}\left[\overline{\mathrm{F}[A]}\right] \subseteq \overline{\mathrm{F}[\mathrm{F}[A]]}.$$

Since $\mathrm{F}[A] \subseteq A$, we obtain

$$\mathrm{F}\left[\overline{\mathrm{F}[A]}\right] \subseteq \overline{\mathrm{F}[A]} \subseteq A \subseteq \text{w-}\mathcal{B}(A),$$

and $\overline{\mathrm{F}[A]}$ is closed nonempty sub-invariant and contained in w-$\mathcal{B}(A)$. But A is the smallest such set, so $A \supseteq \overline{\mathrm{F}[A]}$ and $\overline{\mathrm{F}[A]} = A$.

3.

Suppose that $x \in$ Ls-$\mathcal{B}(A)$; then $A = \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x]$. But in the proof of Proposition 11, we showed that $\mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x]$ is always sub-invariant; hence, A must be as well. Thus, if A is nonempty and is contained in strict-$\mathcal{B}(A)$, PWH-$\mathcal{B}(A)$ or Ls-$\mathcal{B}(A)$ it is contained in Ls-$\mathcal{B}(A)$ and by item 2 we have that A is nonempty closed and satisfies $A = \overline{\mathrm{F}[A]}$.

□

Lemma 2 tells us that we can deduce a great deal about a set A just by assuming that A is a subset of one of the basins and F is l.s.c. Thus, interestingly, this results applies to possibly infinite or discontinuous IFS. But naturally we are still the most concerned with finite continuous IFS.

Theorem 14. *Let (X, d) be a metric space, $\mathcal{F}$ be an IFS on X with Hutchinson-Barnsley operator* F *being l.s.c and $A \subseteq X$. Then the following are equivalent*

1. *A is a quasi attractor of $\mathcal{F}$.*

2. *A is a minimal closed nonempty sub-invariant set of* F.

3. *$\emptyset \neq A \subseteq$ Ls-$\mathcal{B}(A)$.*

Hence, strict attractors, point wise attractors and semi-attractors of $\mathcal{F}$ are quasi attractors of $\mathcal{F}$. Thus if A is a quasi attractor of $\mathcal{F}$ we have for $\mathcal{B} =$ w-$\mathcal{B}(A)$, Ls-$\mathcal{B}(A)$

$$A = \bigcap_{x\in\mathcal{B}} \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x] = \bigcap_{x\in\mathcal{B}} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}$$

and $\overline{\mathrm{F}[A]} = A$*. Additionally, if* A *is a compact quasi attractor of* $\mathcal{F}$ *and* F *maps compact sets to compact sets, such as when* $\mathcal{F}$ *is finite and continuous, then* $\mathrm{F}[A] = A$

Proof. $1 \implies 2$

Suppose that A is a quasi attractor of $\mathcal{F}$. Then, by definition, for every $a \in A$ we have $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]} = A$. Now suppose that B is a nonempty, closed, and sub-invariant subset of A. So for all $b \in B \subseteq A$ one can show that (see the proof of Proposition 11) $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[b]} \subseteq B$. Indeed, by applying F to both sides of

$$\mathrm{F}[b] \subseteq B$$

$$\mathrm{F}^{\circ 2}[b] \subseteq \mathrm{F}[b] \subseteq B$$

$$\vdots$$

$$\mathrm{F}^{\circ n}[b] \subseteq B$$

for all $n \in \mathbb{N}$. Thus we can take the union over $n \in \mathbb{N}$ yielding $\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x] \subseteq B$. Recalling B is closed, we take the closure giving us $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[b]} \subseteq B$. But $b \in B \subseteq A$, so,

$$\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[b]} \subseteq B \subseteq A = \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[b]}.$$

Thus $A = B$ and A is a minimal closed, nonempty, and sub-invariant set.

$2 \implies 1$ Suppose that A is a minimal, closed, nonempty, and sub-invariant set. Then, for all $a \in A$ we have $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]}$ is a closed, nonempty, and sub-invariant set (it is sub-invariant by the proof of Proposition 11). The set $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]}$ is also contained in A by the above argument with "B". Thus, for all $a \in A$ we have $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]} = A$ as A is a minimal closed, nonempty, and sub-invariant. Therefore A is a quasi attractor of $\mathcal{F}$.

$2 \implies 3$

Following the proof of Theorem 12, we can say that for all $N \in \mathbb{N}$ and $a \in A$ the set

$\overline{\bigcup_{n\geq N} \mathrm{F}^{\circ n}[a]} \subseteq A$ is closed nonempty and sub-invariant. Hence, for all $N \in \mathbb{N}$ and $a \in A$

$$A = \overline{\bigcup_{n\geq N} \mathrm{F}^{\circ n}[a]}$$

as A is a minimal closed nonempty and sub-invariant set. But now we can take the intersection over all $N \in \mathbb{N}$ of the right hand side and we have $A = \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[a]$. Therefore, $A \subseteq \text{Ls-}\mathcal{B}(A)$ and A is nonempty by assumption.

$3 \implies 2$

By 2 and 3 of Lemma 2 A is the smallest, nonempty, closed, and sub-invariant set contained in $\text{Ls-}\mathcal{B}(A)$. Thus any nonempty, closed, and sub-invariant set contained in A must be equal to A. So A is a minimal nonempty, closed, and sub-invariant of F.

If A is a strict attractor, by definition, it is contained in $\text{strict-}\mathcal{B}(A) \subseteq \text{Ls-}\mathcal{B}(A)$ so by 3 it is a quasi attractor. Similarly, if A is a point wise attractor by definition it is contained in $\text{PWH-}\mathcal{B}(A) \subseteq \text{Ls-}\mathcal{B}(A)$ so by 3 it is a quasi attractor. If A is a semi-attractor by definition it is the smallest closed, nonempty, and sub-invariant set of F. Thus, it is also a minimal closed, nonempty, and sub-invariant set of F. So, by 2, A is a quasi attractor.

Recalling that $\mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x] \subseteq \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}$ for all $x \in X$ and the definition of $\mathcal{B} = \text{w-}\mathcal{B}(A)$, we have for all $x \in \text{w-}\mathcal{B}(A)$ that

$$A \subseteq \mathop{\mathrm{Ls}}_{n\to\infty} \mathrm{F}^{\circ n}[x] \subseteq \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}.$$

Thus, $A \subseteq \bigcap_{x\in\mathcal{B}} \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x] \subseteq \bigcap_{x\in\mathcal{B}} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]}$. But since A is a quasi attractor and $A \subseteq \text{Ls-}\mathcal{B}(A) \subseteq \text{w-}\mathcal{B}(A)$ there is an $a \in A \subseteq \text{w-}\mathcal{B}(A)$ such that $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[a]} = A$. Therefore, $\bigcap_{x\in\mathcal{B}} \overline{\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]} \subseteq A$ and the result holds. The case where $\mathcal{B} = \text{Ls-}\mathcal{B}(A)$ is the same.

The fact that $\overline{\mathrm{F}[A]} = A$ follows from applying 2 of Lemma 2.

Lastly, if A is a compact quasi attractor of $\mathcal{F}$ and F maps compact sets to compact sets then $\mathrm{F}[A]$ is compact. Thus, $A = \overline{\mathrm{F}[A]} = \mathrm{F}[A]$. □

We should note that most of the above Theorem was proved in a slightly different setting as Theorem 1.2, in [5].

Theorem 14 shows us that of the four kinds of attractors discussed so far quasi attractors are the most general (when the Hutchinson-Barnsley operator is l.s.c). On the other hand items 2 and 3 of Lemma 2 and item 3 of Theorem 14 tell us that if A is a quasi attractor of an IFS it is the smallest nonempty, closed, and sub-invariant set in w-$\mathcal{B}(A)$, which makes it sound a lot like a semi-attractor. In fact in the case of strict attractors and point wise attractors the strict basin and pointwise basin are sub-invariant with respect to the Hutchinson-Barnsley operator. Thus, in these cases, we can take the space equal to the basin and the attractor becomes a semi-attractor.

Remark 7. *Recently, the literature has been exploring topological IFS, see [4, 2, 3, 5], and playing the chaos game in these topological spaces. Topological spaces are more general than metric spaces (every metric space is a topological space but some topological spaces cannot be made into a metric space). Thus, it is unsurprising that some results about the chaos game that are easy to show in metric spaces are much more difficult to show in topological spaces.*

One such hurdle is showing that strict attractors and point wise attractors are invariant, as they remark in [5]. As far as the Author of this work can tell, the proofs of Lemma 2 and Theorem 14 (and indeed this chapter so far) only rely on the Hutchinson-Barnsley operator, say F*, having the following property*

$$\mathrm{F}\left[\overline{U}\right] \subseteq \overline{\mathrm{F}[U]}$$

for all $U \subseteq X$ *(which is item 6 of Theorem 6). Since, even in topological spaces, a function* f *is continuous on* X *if and only if* $\mathrm{f}\left(\overline{U}\right) \subseteq \overline{\mathrm{f}(U)}$ *for all* $U \subseteq X$*, item 6 of Theorem 6 holds for*

F *whenever the* $\mathcal{F}$ *is continuous (see Proposition 2). Thus this Author expects Lemma 2 and Theorem 14 (and all the major results in this chapter so far) to hold in a general topological context.*

We would like to note again that the results of this chapter so far are due in large part in assuming that the Hutchinson-Barnsley operator is l.s.c. This raises the question of when the Hutchinson-Barnsley operator is l.s.c. We know by Proposition 2 that it is l.s.c whenever the IFS is continuous. Thus we can refine this question down to just IFSs with possible discontinuous functions. The converse of this question is also interesting; that is, if $\mathcal{F}$ is an IFS and F is the Hutchinson-Barnsley operator of $\mathcal{F}$ that is l.s.c, does there exist an IFS, say $\mathcal{H}$, with Hutchinson-Barnsley operator H such that $\mathrm{F} = \mathrm{H}$, and, if $\mathcal{F}$ is finite, can we take $\mathcal{H}$ to be finite as well?

We conclude the chapter with some sufficient conditions for an $\mathcal{F}$ to possess a semi-attractor.

Proposition 12. *Let* (X, d) *be a metric space and let* $\mathcal{F} = \{\mathrm{f}_i\}_{i\in I}$ *be an IFS on* X *with Hutchinson-Barnsley operator l.s.c. Additionally let,* $\mathcal{F}^* = \left\{\mathrm{f}_\alpha \mid \alpha \in \bigcup_{n\in\mathbb{N}} I^n\right\}$.

Then if any of the following hold, $\mathcal{F}$ *has a semi-attractor.*

1. *There is a* $\mathrm{f} \in \mathcal{F}$ *such that* f *is a contraction and* X *is complete.*

2. *There is a* $\mathrm{f} \in \mathcal{F}^*$ *such that* f *is a contraction and* X *is complete.*

3. *There is a* $\mathrm{f} \in \mathcal{F}$ *such that there is an* $\bar{x} \in X$ *such that* $\lim_{n\to\infty} \mathrm{f}^{\circ n}(x) = \bar{x}$*, for every* $x \in X$.

4. *There is a* $\mathrm{f} \in \mathcal{F}^*$ *such that there is an* $\bar{x} \in X$ *such that* $\lim_{n\to\infty} \mathrm{f}^{\circ n}(x) = \bar{x}$*, for every* $x \in X$.

5. *There is a* $\mathrm{f} \in \mathcal{F}$ *for which there is a set* $B \subseteq X$ *such that* $\mathrm{Ls}_{n\to\infty} \mathrm{f}^{\circ n}[x] = B$*, for every* $x \in X$.

6. There is a $\mathrm{f} \in \mathcal{F}^*$ *for which there is a set* $B \subseteq X$ *such that* $\mathrm{Ls}_{n\to\infty} \mathrm{f}^{\circ n}[x] = B$*, for every* $x \in X$*.*

4.2 The Chaos Game

In this section we extend Theorem 10 to some of the more general attractors discussed thus far. There are two versions of the chaos game discussed in the literature: the "probabilistic chaos game" and the "deterministic chaos game". Our focus will be on the deterministic chaos game. First we should refresh some notation we established in Chapter 3. Namely, Definitions 14, 16 and 17.

Definition 23. *Let* (X, d) *be a metric space,* $\mathcal{F} = \{\mathrm{f}_i\}_{i\in I}$ *be an IFS on* X*,* $\sigma \in I^{\mathbb{N}}$*,* $n \in \mathbb{N}$ *and* $x \in X$*. For every* $\alpha \in I^*$*,* $\alpha = \alpha_1\alpha_2\alpha_3\ldots\sigma_{|\alpha|}$*, we define*

$$\rho(\alpha) = \alpha_{|\alpha|}\alpha_{|\alpha|-1}\ldots\alpha_1.$$

Now, define the left composition function $\mathrm{L} : I^{\mathbb{N}} \times \mathbb{N} \times X \to X$

$$\mathrm{L}(\sigma, n, x) = \mathrm{f}_{\rho\left(\sigma_{[1,n]}\right)}(x) = \mathrm{f}_{\sigma_n\sigma_{n-1}\ldots\sigma_1}(x).$$

Furthermore, (hopefully) without confusion we will define the multi function $\mathrm{L} : I^{\mathbb{N}} \times X \rightsquigarrow X$ *to be*

$$\mathrm{L}[\sigma, x] = \operatorname*{Ls}_{n\to\infty} \mathrm{L}(\sigma, n, x).$$

If one carefully reduces Theorem 10 and its proof into its essence one might realize that the random generation of the numbers from 1 to N end up defining an element of $[N]^{\mathbb{N}}$, say σ, and the sequence $\{y_n\}_{n\in\mathbb{N}}$ is of the form $y_n = \mathrm{f}_{\sigma_n\sigma_{n-1}\ldots\sigma_2\sigma_1}(y_0) = \mathrm{L}(\sigma, n, y_0)$. Furthermore, the set $\{y_n\}_{n=m}^{\infty}$ as $m \to \infty$ would only consist of the limit points of $\{y_n\}_{n\in\mathbb{N}}$. Thus the statement $\lim_{m\to\infty} \mathrm{d}_H\left(\overline{\{y_n\}_{n=m}^{\infty}}, A\right) = 0$ and $A = \mathrm{Ls}_{n\to\infty}\, y_n$ are equivalent. Lastly, σ has a very interesting property: it contains every finite string as a substring infinitely often, with probability 1.

Putting all these ideas together motivates the following definitions.

Definition 24. *Let I be a set. Then $\sigma \in I^{\mathbb{N}}$ is said to be disjunctive if for all $N \in \mathbb{N}$ and all $\alpha \in I^*$ then there are $n, m \in \mathbb{N}$ with $N \le n \le m$ such that*

$$\sigma_{[n,m]} = \alpha.$$

In other words σ contains every finite string infinitely often.

Definition 25. *Let (X, d) be a metric space, $\mathcal{F} = \{\mathrm{f}_i\}_{i\in I}$ be an IFS on X and $\mathcal{B}, A \subseteq X$. We say that the chaos game draws A starting in $\mathcal{B}$ if for all $x \in \mathcal{B}$ and for all $\sigma \in I^{\mathbb{N}}$ such that σ is disjunctive we have*

$$A = \mathrm{L}(\sigma, x).$$

If A is a attractor in some sense then we will say that the chaos game draws the attractor.

We note that by this definition of the chaos game, only countable IFS can have attractors drawn by the chaos game. This is because there are no disjunctive sequences in $I^{\mathbb{N}}$ when I is uncountable.

Typically, the set $\mathcal{B}$ in the above definition is one of the basins discussed in the previous section. One should always pay careful attention to the definitions used in the literature for the chaos game. It is typical that authors embed their own basin directly into the definition. This, among other definitional inconsistencies between papers, makes it somewhat difficult to keep track of the cases in which the chaos game draws attractors. Sometimes authors use a weaker definition such as $A \subseteq \overline{\{\mathrm{L}(\sigma, n, x)\}_{n\in\mathbb{N}}}$, but because of the basins and/or the type of attractor these weaker definitions are equivalent to Definition 25.

Remark 8. *Perhaps a quasi attractor is the most general type of attractor on which to play the chaos game. I am of the opinion that the chaos game should always draw the attractor starting in the attractor otherwise I don't think we capture the essence of Theorem 10. Furthermore, an attractor should have some kind of self similarity condition such as* $\overline{\mathrm{F}[A]} = A$*, but much more preferably invariance. Consider a set* A *that has a proper nonempty, closed, and sub-invariant subset* B*. I claim that* A *cannot be drawn by the chaos game starting in* A*, indeed, we have for all* $\sigma \in I^{\mathbb{N}}$ *and* $b \in B$*,* $\mathrm{L}[\sigma, b] \subseteq B \subset A$ *and so* $\mathrm{L}[\sigma, b] \neq A$ *for some* $b \in A$ *and all* $\sigma \in I^{\mathbb{N}}$*. Naturally the sets with no proper nonempty, closed, and sub-invariant subsets satisfying some kind of self similarly condition are quasi attractors.*

Suppose A is a quasi attractor of an IFS. Then for any $x \in \text{Ls-}\mathcal{B}(A)$ and for all $\sigma \in I^{\mathbb{N}}$ we have that

$$A = \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x] \supseteq \mathrm{L}[\sigma, x].$$

This can be easily seen from the following observation: for all $n \in \mathbb{N}$ $\mathrm{F}^{\circ n}[x] = \bigcup_{\lambda \in I^{\mathbb{N}}} \mathrm{L}[\lambda, n, x]$. Thus, if we play the chaos game starting in $\text{Ls-}\mathcal{B}(A), \text{PWH-}\mathcal{B}(A)$ or $\text{strict-}\mathcal{B}(A)$ we only need show that $A \subseteq \mathrm{L}[\sigma, x]$. This inclusion intuitively means that the sequence $\{\mathrm{L}(\sigma, n, x)\}_{n\in\mathbb{N}}$ "gets everywhere". If $x \notin \text{Ls-}\mathcal{B}(A)$ then the inclusion $A \supseteq \mathrm{L}[\sigma, x]$ is actually problematic. One way to deal with this inclusion would be to observe, that if $A \cap \{\mathrm{L}(\sigma, n, x)\}_{n\in\mathbb{N}} \neq \emptyset$ (i.e $\mathrm{L}(\sigma, m, x) \in A$ for some $m \in \mathbb{N}$) then $\{\mathrm{L}(\sigma, n, x)\}_{n=m}^{\infty} \subseteq \overline{\{\mathrm{L}(\sigma, n, x)\}_{n=m}^{\infty}} \subseteq A$ for some $m \in \mathbb{N}$. This follows from A being closed and sub-invariant. Hence, in this case, we have $\mathrm{L}[\sigma, x] \subseteq A$. Thus, perhaps one may think that if $A \cap \mathrm{L}[\sigma, x] \neq \emptyset$ then $\mathrm{L}[\sigma, x] \subseteq A$. Generally, this is not true. However we can achieve this result with an additional assumption on the functions of an IFS.

Definition 26. *Let* (X, d) *be a metric space and* $\mathcal{F} = \{\mathrm{f}_i\}_{i\in I}$ *be a IFS on* X*. Then we say that* $\mathcal{F}$ *is compositionally equicontinuous (c.e.c) if the set of all finite compositions of* $\mathcal{F}$*,* *denoted*

$$\mathcal{F}^* = \{\mathrm{f}_\alpha \,|\alpha \in I^*\},$$

is equicontinuous.

Furthermore, we define for all $x \in X$

$$\mathrm{F}^*[x] = \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x].$$

Sequences of equicontinuous functions are very well behaved: much like for a continuous function, a sequence of equicontinuous functions (essentially) preserves convergence of the sequence.

Proposition 13. *Let* (X, d) *be a metric space,* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* X*, and let* $\{\mathrm{f}_n\}_{n\in\mathbb{N}}$ *be a sequence of equicontinuous functions from* X *to* X*. We have that*

1. *if* $\{\mathrm{f}_n\}_{n\in\mathbb{N}}$ *converges pointwise to* f *and* $\{x_{n_k}\}_{k\in\mathbb{N}}$ *converges to* x *then*

$$\{\mathrm{f}_{n_k}(x_{n_k})\}_{k\in\mathbb{N}} \to \mathrm{f}(x)$$

or equivalently

$$\mathrm{f}(x) \in \operatorname*{Ls}_{n\to\infty} \mathrm{f}_n[x_n].$$

2. *if* $\{\mathrm{f}_{n_k}\}_{k\in\mathbb{N}}$ *converges pointwise to* f *and* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x *then*

$$\{\mathrm{f}_{n_k}(x_{n_k})\}_{k\in\mathbb{N}} \to \mathrm{f}(x)$$

or equivalently

$$\mathrm{f}(x) \in \operatorname*{Ls}_{n\to\infty} \mathrm{f}_n[x_n].$$

.

If $\{x_n\}_{n\in\mathbb{N}}$ *has a convergent subsequence converging to* $x \in X$ *then*

$$\mathrm{f}(x) \in \operatorname*{Ls}_{n\to\infty} \mathrm{f}_n[x_n].$$

Proof. 1.

Suppose $\{x_{n_k}\}_{k\in\mathbb{N}} \to x$ and $\epsilon > 0$. Since $\{\mathrm{f}_n\}_{n\in\mathbb{N}}$ is equicontinuous, for the point x and $\frac{\epsilon}{2}$ we can get a $\delta > 0$ so that for all $n \in \mathbb{N}$, $\mathrm{f}_n(\mathbb{B}_\delta(x)) \subseteq \mathbb{B}_\epsilon(\mathrm{f}_n(x))$. Also $\{\mathrm{f}_n(x)\}_{n\in\mathbb{N}} \to \mathrm{f}(x)$, so there is a $K \in \mathbb{N}$ large enough so that for all $k \geq K$ both $\mathrm{d}(\mathrm{f}(x), \mathrm{f}_{n_k}(x)) < \frac{\epsilon}{2}$ and $\mathrm{d}(x_{n_k}, x) < \delta$. Now consider

$$\mathrm{d}(\mathrm{f}(x), \mathrm{f}_{n_k}(x_{n_k})) \leq \mathrm{d}(\mathrm{f}(x), \mathrm{f}_{n_k}(x)) + \mathrm{d}(\mathrm{f}_{n_k}(x), \mathrm{f}_{n_k}(x_{n_k})) < \epsilon.$$

Thus $\{\mathrm{f}_{n_k}(x_{n_k})\}_{k\in\mathbb{N}} \to \mathrm{f}(x)$; Therefore $\mathrm{f}(x) \in \mathrm{Ls}_{n\to\infty} \mathrm{f}_n[x_n]$.

2.

This case is very similar to the proof of 1. □

Although the above Proposition is of little practical importance to us, it aids our intuition on how a c.e.c IFS would allow the set $\mathrm{L}[\sigma, x]$ to be more well behaved.

Lemma 3. *Let* (X, d) *be a metric space,* $\mathcal{F} = \{\mathrm{f}_i\}_{i\in I}$ *be a c.e.c IFS on* X, $x \in X$, $\sigma \in I^{\mathbb{N}}$ *and* $A \subseteq X$ *be a sub-invariant set with respect to* F.

If $\mathrm{L}(\sigma, x) \cap \overline{A} \neq \emptyset$ *then* $\mathrm{L}(\sigma, x) \subseteq \overline{A}$.

Proof. Let $a \in \mathrm{L}(\sigma, x) \cap \overline{A}$ so there is a subsequence of $\{\mathrm{L}(\sigma, n, x)\}_{n\in\mathbb{N}}$ converging to a. Now, take $\epsilon > 0$ and by $\mathcal{F}$ being *c.e.c* there is a δ such that for all $\mathrm{f} \in \mathcal{F}^*$, $\mathrm{f}(\mathbb{B}_\delta(a)) \subseteq \mathbb{B}_\epsilon(\mathrm{f}(a))$. Now, there is $N \in \mathbb{N}$ large enough for $\mathrm{d}(\mathrm{L}(\sigma, N, x), a) < \delta$.

Note that for all $n \geq N$

$$\mathrm{L}(\sigma, n, x) = \mathrm{f}_{\sigma_n \sigma_{n-1} \ldots \sigma_{N+1}} \circ \mathrm{L}(\sigma, N, x) = \mathrm{f}_{\rho\left(\sigma_{[N+1,n]}\right)} \circ \mathrm{L}(\sigma, N, x)$$

and $\overline{A}$ is sub-invariant with respect to F.

So by equicontinuity and $\mathrm{f}_{\sigma_n \sigma_{n-1} \ldots \sigma_{N+1}}(a) \in \overline{A}$ we have

$$\epsilon > \mathrm{d}\big(\mathrm{L}(\sigma, n, x), \mathrm{f}_{\sigma_n \sigma_{n-1} \ldots \sigma_{N+1}}(a)\big) > \mathrm{d}(\mathrm{L}(\sigma, n, x), A).$$

This shows $\lim_{n \to \infty} \mathrm{d}(\mathrm{L}(\sigma, n, x), A) = 0$. However this means that $\mathrm{L}(\sigma, x) \subseteq \overline{A}$, indeed take $y \in \mathrm{L}(\sigma, x)$ and consider for some $n \in \mathbb{N}$

$$\mathrm{d}(y, A) \leq \mathrm{d}(y, \mathrm{L}(\sigma, n, x)) + \mathrm{d}(\mathrm{L}(\sigma, n, x), A).$$

By the above, for every $\epsilon > 0$ and every $N \in \mathbb{N}$ we can pick an $n \geq N$ such that $\mathrm{d}(y, \mathrm{L}(\sigma, n, x)) < \frac{\epsilon}{2}$ and $\mathrm{d}(\mathrm{L}(\sigma, n, x), A) < \frac{\epsilon}{2}$. Thus $\mathrm{d}(y, A) = 0$ and $\mathrm{L}(\sigma, x) \subseteq \overline{A}$. □

Example 11. *Let X be the unit circle in the complex plane ($\mathbb{C}$) and consider the IFS, $\mathcal{F} = \{\mathrm{f}, \mathrm{f}^{-1}\} = \{\mathrm{f}_1, \mathrm{f}_{-1}\}$ where for $x \in X$ with $x = e^{2\pi i \alpha}$ $\alpha \in [0, 1)$*

$$\mathrm{f}(x) = \mathrm{f}_1(x) = e^{2\pi i \alpha^2}$$

and

$$\mathrm{f}^{-1}(x) = \mathrm{f}_{-1}(x) = e^{2\pi i \sqrt{\alpha}}.$$

We can see that for $\mathrm{g} \in \mathcal{F}$ and all $x \in X$ we have $\lim_{n \to \infty} \mathrm{g}^{\circ n}(x) = e^{2\pi i 0} = 1$. This means that $\mathcal{F}$ has a semi-attractor, say A, that contains 1 and noting that 1 is a fixed point of both functions in $\mathcal{F}$ that $A = \{1\}$. However, I claim there is a disjunctive sequence of

$\{1,-1\}^{\mathbb{N}} \ni \sigma$ and $x \in X$ such that $x \in \mathrm{L}[\sigma, x]$. Pick $\sigma = 1{-}1\;111{-}1{-}11{-}1{-}1\;11111{-}1\ldots$. Explicitly for every $n \in \mathbb{N}$, the string $\sigma_{[1+\sum_{m=1}^{n-1} m2^m, \sum_{m=1}^{n} m2^m]}$ contains every string of length n as a substring exactly once. Since functions commute with themselves and a function commutes with its inverse, we can see $\mathrm{L}(\sigma, \sum_{m=1}^{n} m2^m, x) = x$ for all $n \in \mathbb{N}$ as $\sigma_{[1, \sum_{m=1}^{n} m2^m]}$ must contain the same number of 1's and -1's. Therefore, $x \in \mathrm{L}[\sigma, x]$ and if we take $x \notin \{1\} = A$ then $\mathrm{L}[\sigma, x] \nsubseteq A$.

This is an example of a non-equicontinuous IFS for which the result of Lemma 3 does not hold.

For now, we will return to a more general commentary on IFS. Soon, we will be focusing on achieving the inclusion $A \subseteq \mathrm{L}[\sigma, x]$. We will first study the basin w-$\mathcal{B}(A)$; it turns out that this basin is strongly related to the largest set in which we can start the chaos game. First we will define some convenient notations.

Definition 27. *Let (X, d) be a metric space and $\mathrm{F} : X \rightsquigarrow X$ be a multifunction. Then we define for all $N \in \mathbb{N}$ and $x \in X$*

$$\mathrm{F}_N^*[x] = \bigcup_{n \geq N} \mathrm{F}^{\circ n}[x]$$

and

$$\mathrm{F}^*[x] = \mathrm{F}_1^*[x] = \bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x].$$

Furthermore, define for all $N \in \mathbb{N}$ and $x \in X$

$$\mathrm{F}^{\circ -N}[x] = \left(\mathrm{F}^{\circ N}\right)^{-}[x]$$

and

$$\mathrm{F}_N^{-*} = (\mathrm{F}_N^*)^{-}[x] = \bigcup_{n \geq N} \mathrm{F}^{\circ -n}[x].$$

Proposition 14. *Let* (X, d) *be a metric space,* $\mathrm{F} : X \rightsquigarrow X$ *be a l.s.c multifunction ,* $A \subseteq X$ *with* $\overline{\mathrm{F}[A]} = A$ *and* $x \in X$*.*

Then the following are equivalent

1. $A \subseteq \overline{\mathrm{F}_1^*[x]}$

2. *There is an* $N \in \mathbb{N}$ *such that* $A \subseteq \overline{\mathrm{F}_N^*[x]}$

3. *For all* $N \in \mathbb{N}$*,* $A \subseteq \overline{\mathrm{F}_N^*[x]}$

4. $A \subseteq \bigcap_{N \in \mathbb{N}} \overline{\mathrm{F}_N^*[x]} = \operatorname*{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x]$

Proof. $1 \implies 2$

Take $N = 1$.

$2 \implies 3$

Let $M \in \mathbb{N}$ be arbitrary, if $M < N$ then

$$A \subseteq \overline{\mathrm{F}_N^*[x]} \subseteq \overline{\mathrm{F}_N^*[x] \cup \left(\bigcup_{n=M}^{N-1} F^{\circ n}[x] \right)} = \overline{\mathrm{F}_M^*[x]}.$$

If $M \geq N$, we consider the multifunction $\mathrm{F}^{\circ(M-N)}$, it is l.s.c and $\overline{\mathrm{F}^{\circ(M-N)}[A]} = A$. Thus, we can apply $\mathrm{F}^{\circ(M-N)}$ to both sides of $A \subseteq \overline{\mathrm{F}_N^*[x]}$ yielding,

$$\begin{aligned} A &\subseteq \overline{\mathrm{F}_N^*[x]} \\ \mathrm{F}^{\circ(M-N)}[A] &\subseteq \mathrm{F}^{\circ(M-N)}\left[\overline{\mathrm{F}_N^*[x]}\right] \\ \overline{\mathrm{F}^{\circ(M-N)}[A]} &\subseteq \overline{\mathrm{F}^{\circ(M-N)}\left[\overline{\mathrm{F}_N^*[x]}\right]} \\ A &\subseteq \overline{\mathrm{F}^{\circ(M-N)}[\mathrm{F}_N^*[x]]} \\ A &\subseteq \overline{\mathrm{F}_M^*[x]}. \end{aligned}$$

$3 \implies 4$

For all $N \in \mathbb{N}$ we have

$$A \subseteq \overline{\mathrm{F}^*_N[x]}.$$

Thus, by taking the intersection over $\mathbb{N}$ we get $A \subseteq \bigcap_{N\in\mathbb{N}} \overline{\mathrm{F}^*_N[x]} = \mathrm{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x]$.

$4 \implies 1$

For all $n \in \mathbb{N}$ we have

$$A \subseteq \bigcap_{N\in\mathbb{N}} \overline{\mathrm{F}^*_N[x]} \subseteq \overline{\mathrm{F}^*_n[x]}$$

so take $n = 1$. □

Notably, $\overline{\mathrm{F}[A]} = A$ whenever A is a quasi attractor of F.

Theorem 15. *Let (X, d) be a metric space, Then the following hold.*

1. *For all $N \in \mathbb{N}$,* $\text{w-}\mathcal{B}(A) = \left\{x \in X \mid A \subseteq \overline{\mathrm{F}^*_N[x]}\right\}$
2. *For all $N \in \mathbb{N}$,*

$$\text{w-}\mathcal{B}(A) = \bigcap_{a\in A} \bigcap_{\epsilon>0} \mathrm{F}^{-*}_N[\mathbb{B}_\epsilon(a)]$$

3. *For all $N \in \mathbb{N}$ and any $a \in A$,*

$$\text{w-}\mathcal{B}(A) = \bigcap_{\epsilon>0} \mathrm{F}^{-*}_N[\mathbb{B}_\epsilon(a)]$$

4. *Given any $N \in \mathbb{N}$, the set $\mathcal{B} = \text{w-}\mathcal{B}(A)$ is the largest set satisfying*

$$A = \bigcap_{x\in\mathcal{B}} \overline{\mathrm{F}^*_N[x]}$$

5. For all $N \in \mathbb{N}$,

$$\mathrm{F}_N^{-*}[\text{w-}\mathcal{B}(A)] = \text{w-}\mathcal{B}(A)$$

and

$$(\mathrm{F}_N^*)^+[X \setminus \text{w-}\mathcal{B}(A)] = X \setminus \text{w-}\mathcal{B}(A).$$

Proof. 1.

Since A is a minimal closed, sub-invariant set of F we know that $\overline{\mathrm{F}[A]} = A$ and so we can use the equivalence of item 4 and item 3 of Proposition 14 to give the result.

2. $\subseteq$

Let $N \in \mathbb{N}$ and $x \in \text{w-}\mathcal{B}(A)$ and so by 1 we have $A \subseteq \overline{\mathrm{F}_N^*[x]}$. So for all $a \in A$, $a \in \overline{\mathrm{F}_N^*[x]}$, and thus for all $\epsilon > 0$ we have

$$\mathbb{B}_\epsilon(a) \cap (\mathrm{F}_N^*[x]) \neq \emptyset.$$

Hence there is a $n \geq N$ such that

$$\mathbb{B}_\epsilon(a) \cap (\mathrm{F}^{\circ n}[x]) \neq \emptyset.$$

Thus by definition of the inverse of a multifunction $x \in \mathrm{F}^{\circ -n}[\mathbb{B}_\epsilon(a)] \subseteq \mathrm{F}_N^{-*}[\mathbb{B}_\epsilon(a)]$. Putting all of this together yields $x \in \bigcap_{a \in A} \bigcap_{\epsilon > 0} \mathrm{F}_N^{-*}[\mathbb{B}_\epsilon(a)]$.

$\supseteq$

Conversely, suppose that $x \in \bigcap_{a \in A} \bigcap_{\epsilon > 0} \mathrm{F}_N^{-*}[\mathbb{B}_\epsilon(a)]$. Pick any $a \in A$ and any $\epsilon > 0$ then $x \in \mathrm{F}_N^{-*}[\mathbb{B}_\epsilon(a)]$ and so there is an $n \geq N$ such that $x \in \mathrm{F}^{\circ -n}[\mathbb{B}_\epsilon(a)]$. Thus, again by definition of the inverse of a multifunction

$$\emptyset \neq \mathbb{B}_\epsilon(a) \cap (\mathrm{F}^{\circ n}[x]) \subseteq \mathbb{B}_\epsilon(a) \cap (\mathrm{F}_N^*[x]) \neq \emptyset.$$

As this holds for all $\epsilon > 0$, $a \in \overline{\mathrm{F}_N^*[x]}$. But this holds for all $a \in A$ as well, so $A \subseteq \overline{\mathrm{F}_N^*[x]}$ and

$x \in \text{w-}\mathcal{B}(A)$ by 1.

3.

By 2, we need only show for every $a \in A$ that $\bigcap_{\epsilon>0} \mathrm{F}_N^{-*}[\mathbb{B}_\epsilon(a)] \subseteq \bigcap_{b\in A} \bigcap_{\epsilon>0} \mathrm{F}_N^{-*}[\mathbb{B}_\epsilon(b)]$.

So suppose that $a \in A$ and $x \in \bigcap_{\epsilon>0} \mathrm{F}_N^{-*}[\mathbb{B}_\epsilon(a)]$. So for all $\epsilon > 0$ we have $\mathrm{F}_N^*[x] \cap \mathbb{B}_\epsilon(a) \neq \emptyset$, giving $a \in \overline{\mathrm{F}_N^*[x]}$. But A is a minimal closed, nonempty, and sub-invariant set of F, so $\overline{\mathrm{F}^*[a]} = A$. Thus

$$\overline{\mathrm{F}^*[a]} \subseteq \overline{\mathrm{F}^*\left[\overline{\mathrm{F}_N^*[x]}\right]} \subseteq \overline{\mathrm{F}^*[\mathrm{F}_N^*[x]]} = \overline{\mathrm{F}_{N+1}^*[x]}$$

and so $A \subseteq \overline{\mathrm{F}_{N+1}^*[x]}$. Therefore, by 1, $x \in \text{w-}\mathcal{B}(A)$.

4.

Suppose that $\mathcal{B}$ satisfies

$$A = \bigcap_{x\in\mathcal{B}} \overline{\mathrm{F}_N^*[x]}$$

for some given $N \in \mathbb{N}$. Then we can see that for any $y \in \mathcal{B}$

$$A = \bigcap_{x\in\mathcal{B}} \overline{\mathrm{F}_N^*[x]} \subseteq \overline{\mathrm{F}_N^*[y]}$$

and so, by 1, we have $y \in \text{w-}\mathcal{B}(A)$. Therefore $\mathcal{B} \subseteq \text{w-}\mathcal{B}(A)$ and by Theorem 14 $\text{w-}\mathcal{B}(A)$ satisfies the identity.

5.

Pick $x \in \mathrm{F}_N^{-*}[\text{w-}\mathcal{B}(A)]$; for some $N \in \mathbb{N}$ then $\mathrm{F}_N^*[x] \cap \text{w-}\mathcal{B}(A) \neq \emptyset$. So pick $y \in \mathrm{F}_N^*[x] \cap \text{w-}\mathcal{B}(A)$ and consider

$$A \subseteq \overline{\mathrm{F}_1^*[y]} \subseteq \overline{\mathrm{F}_1^*[\mathrm{F}_N^*[x]]} = \overline{\mathrm{F}_{N+1}^*[x]}$$

So $x \in \text{w-}\mathcal{B}(A)$ by 1.

For the converse we proceed by contraposition. Suppose that $x \notin \mathrm{F}_N^{-*}[\text{w-}\mathcal{B}(A)]$ for some

$N \in \mathbb{N}$ then recalling that $A \subseteq \text{w-}\mathcal{B}(A)$ we have

$$\emptyset = \mathrm{F}_N^*[x] \cap \text{w-}\mathcal{B}(A) \supseteq \mathrm{F}_N^*[x] \cap A.$$

So we see that $\emptyset = \overline{\mathrm{F}_N^*[x]} \cap A$ by taking the closure of the above and recalling A is closed. Thus it must be the case that $A \not\subseteq \overline{\mathrm{F}_N^*[x]}$ and $x \notin \text{w-}\mathcal{B}(A)$. Therefore $\text{w-}\mathcal{B}(A) \subseteq \mathrm{F}_N^{-*}[\text{w-}\mathcal{B}(A)]$. By Proposition 5, the identities in question are equivalent. □

Theorem 15 gives us many different ways of expressing the set $\text{w-}\mathcal{B}(A)$. Some of which strike the author as bizarre, particularity Item 5. Item 4 suggests that every minimal closed, nonempty, and sub-invariant set of a l.s.c multifunction behaves like a semi-attractor on its weak limsup basin of attraction. Thus it would very interesting if $\text{w-}\mathcal{B}(A)$ is a sub-invariant set of a l.s.c multifunction. Unfortunately this does not seem to be the case in general. However (perhaps frustratingly), the complement is sub-invariant.

Corollary 15.1. *Let (X, d) be a metric space, $\mathrm{F} : X \rightsquigarrow X$ be a l.s.c multifunction and A be minimal closed, nonempty, and sub-invariant set of* F.

Then for all $N \in \mathbb{N}$

$$\mathrm{F}_N^*[X \setminus \text{w-}\mathcal{B}(A)] \subseteq X \setminus \text{w-}\mathcal{B}(A).$$

Proof. By Theorem 15 we have $(\mathrm{F}_N^*)^+[X \setminus \text{w-}\mathcal{B}(A)] = X \setminus \text{w-}\mathcal{B}(A)$. So we can apply F_N^* to this equation and recall Item 2 of Proposition 5, yielding

$$\mathrm{F}_N^*[X \setminus \text{w-}\mathcal{B}(A)] = \mathrm{F}_N^*\left[(\mathrm{F}_N^*)^+[X \setminus \text{w-}\mathcal{B}(A)]\right] \subseteq X \setminus \text{w-}\mathcal{B}(A).$$

□

We will finally show why $\text{w-}\mathcal{B}(A)$ is of any interest.

Theorem 16. *Let* (X, d) *be a metric space,* $\mathcal{F} = \{\mathrm{f}_i\}_{i\in I}$ *be a IFS with Hutchinson-Barnsley operator being l.s.c multifunction and* $A \subseteq X$ *be a quasi attractor of* $\mathcal{F}$*. Further, let* $\sigma \in I^{\mathbb{N}}$ *and* $x \in X$*.*

If $A \cap \mathrm{L}[\sigma, x] \neq \emptyset$ *then for all* $n \in \mathbb{N}$ *we have* $\mathrm{L}(\sigma, n, x) \in \text{w-}\mathcal{B}(A)$*. Furthermore, suppose that* x *has the property: for every disjunctive* $\lambda \in I^{\mathbb{N}}$ *we have that*

$$A \cap \mathrm{L}[\lambda, x] \neq \emptyset.$$

Then $x \in (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$*.*

Proof. Proceed by contraposition. Suppose that for some $n \in \mathbb{N}$, $\mathrm{L}(\sigma, n, x) \in X \setminus \text{w-}\mathcal{B}(A)$; then by Item 3 of Theorem 15, for all $a \in A$ and some $\epsilon > 0$, for all $\mathrm{f} \in \mathcal{F}^*$ we have $\mathrm{d}(a, \mathrm{f}(\mathrm{L}(\sigma, n, x))) \geq \epsilon$. But for every $m \geq n$ there is an $\mathrm{f}_m \in \mathcal{F}^*$ such that $\mathrm{f}_m \circ \mathrm{L}(\sigma, n, x) = \mathrm{L}(\sigma, m, x)$. But this means that $a \notin \mathrm{L}[\sigma, x]$ for all $a \in A$, so $A \cap \mathrm{L}[\sigma, x] = \emptyset$.

To prove the "furthermore", first define

$$\mathrm{normal}\big(I^{\mathbb{N}}\big) = \big\{\lambda \in I^{\mathbb{N}} \mid \lambda \text{ is disjunctive}\big\}$$

and observe that for any $\alpha \in I^*$ the set

$$\alpha\, \mathrm{normal}\big(I^{\mathbb{N}}\big) = \big\{\alpha\lambda \in I^{\mathbb{N}} \mid \lambda \text{ is disjunctive}\big\} \subseteq \mathrm{normal}\big(I^{\mathbb{N}}\big)$$

as $\alpha\lambda$ still contains every finite substring infinitely often whenever λ does. Now pick any $\alpha \in I^*$. I claim that $\mathrm{f}_\alpha(x) \in \bigcup_{n\in\mathbb{N}} \bigcup_{\lambda\in\mathrm{normal}(I^{\mathbb{N}})} \mathrm{L}(\lambda, n, x)$. Indeed, pick any $\lambda \in \mathrm{normal}\big(I^{\mathbb{N}}\big)$ and $\alpha\lambda \in \mathrm{normal}\big(I^{\mathbb{N}}\big)$. This means that $\mathrm{L}(\alpha\lambda, |\alpha|, x) \in \bigcup_{n\in\mathbb{N}} \bigcup_{\lambda\in\mathrm{normal}(I^{\mathbb{N}})} \mathrm{L}(\lambda, n, x)$. But $\mathrm{L}(\alpha\lambda, |\alpha|, x) = \mathrm{f}_\alpha(x) \in \bigcup_{n\in\mathbb{N}} \bigcup_{\lambda\in\mathrm{normal}(I^{\mathbb{N}})} \mathrm{L}(\lambda, n, x)$.

This shows that

$$\mathrm{F}^*[x] = \bigcup_{\alpha \in I^*} \mathrm{f}_\alpha(x) \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{\lambda \in \mathrm{normal}(I^{\mathbb{N}})} \mathrm{L}(\lambda, n, x).$$

The other inclusion holds as well: observe that $\mathrm{L}(\lambda, n, x) = \mathrm{f}_{\rho\left(\lambda_{[1,n]}\right)}(x)$ for any $\lambda \in I^{\mathbb{N}}$.

Now, assuming that for every disjunctive $\lambda \in I^{\mathbb{N}}$ we have that

$$A \cap \mathrm{L}[\lambda, x] \neq \emptyset,$$

we see that, by the first part of this Theorem, for all $n \in \mathbb{N}$ and $\lambda \in \mathrm{normal}\left(I^{\mathbb{N}}\right)$ we have $\mathrm{L}(\lambda, n, x) \in \text{w-}\mathcal{B}(A)$. Thus

$$\mathrm{F}^*[x] = \bigcup_{n \in \mathbb{N}} \bigcup_{\lambda \in \mathrm{normal}(I^{\mathbb{N}})} \mathrm{L}(\lambda, n, x) \subseteq \text{w-}\mathcal{B}(A).$$

Therefore, $x \in (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$.

□

Theorem 16 tells us that $(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ is the largest set in which we can start the chaos game and expect it to always draw the attractor.

Remark 9. *The fact of* $(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ *being the largest starting set for the chaos game can be seen probabilistically as well. Suppose that when playing the chaos game, with a finite IFS, where we select the each function with probability no less then* $p \in (0,1]$ *to construct* $\sigma \in I^{\mathbb{N}}$*. This means that probability of* σ *having prefix* $\alpha \in I^*$ *is no less then* $p^{|\alpha|} > 0$*.*

Now if $x \notin (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ *then* $x \in X \setminus \mathrm{F}^{*+}[\text{w-}\mathcal{B}(A)] = \mathrm{F}^{*-}[X \setminus \text{w-}\mathcal{B}(A)]$*. This means there is an* $\alpha \in I^*$ *with* $\mathrm{f}_\alpha(x) \in X \setminus \text{w-}\mathcal{B}(A)$ *and by the above there is a nonzero probability that we have* $\mathrm{L}(\sigma, |\alpha|, x) = \mathrm{f}_\alpha(x) \in X \setminus \text{w-}\mathcal{B}(A)$*. So, by the contrapositive of the first part of Theorem 16,* $A \cap \mathrm{L}[\sigma, x] = \emptyset$*. Therefore, if the maps of the chaos game are picked with* *probability no less then $p \in (0,1]$ and $x \notin (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$, then there is a nonzero chance of chaos game failing to draw the attractor (in a most spectacular fashion as $\mathrm{L}[\sigma, x] \cap A = \emptyset$ will occur with nonzero probability).*

Although, we have no reason to believe that $\text{w-}\mathcal{B}(A)$ is sub-invariant we can show that (the arguably more important set taking into account Theorem 16) $(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ is sub-invariant.

Proposition 15. *Let (X, d) be a metric space, $\mathrm{F} : X \rightsquigarrow X$ be a multifunction and $A \subseteq X$. Then*

$$\mathrm{F}\big[(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]\big] \subseteq (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$$

and

$$\mathrm{F}^*\big[(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]\big] \subseteq (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)].$$

Proof. Suppose that $x \in (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$, and consider a $y \in \mathrm{F}^*[x] \subseteq \text{w-}\mathcal{B}(A)$. Since $\mathrm{F}^*[x]$ is sub-invariant with respect to F it is also sub-invariant with respect to F^*. Thus we can see,

$$\mathrm{F}^*[y] \subseteq \mathrm{F}^*[x] \subseteq \text{w-}\mathcal{B}(A)$$

and so $y \in (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$. Therefore, $\mathrm{F}^*\big[(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]\big] \subseteq (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ and recalling that $\mathrm{F} \subseteq \mathrm{F}^*$ we have $\mathrm{F}\big[(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]\big] \subseteq (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$. □

Now we know the sequence of points generated by the chaos game must be contained in $\text{w-}\mathcal{B}(A)$. This fact, along with equicontinuity, will aid us in our quest of achieving the inclusion $A \subseteq \mathrm{L}[\sigma, x]$. We would like to mention that the author was inspired by Lemma 3.15 of [5] (see also Proposition 1.2 of [14]) and Theorem A.2 of [5], in the creation of the following result.

Lemma 4. *Let (X,d) be a metric space, $\mathcal{F}=\{\mathrm{f}_i\}_{i\in I}$ be a c.e.c IFS on X, $x\in X$, and A be a quasi attractor of $\mathcal{F}$.*

If $K\subseteq(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$, where $\overline{K}$ is compact, then for every $\epsilon>0$ and every $a\in A$ there is an $\alpha\in I^*$ such that for every $x\in K$ there is an $n\leq|\alpha|$ with

$$\mathrm{f}_{\rho\left(\alpha_{[1,n]}\right)}(x)\in\mathbb{B}_\epsilon(a)$$

or equivalently

$$x\in\bigcup_{m=1}^{|\alpha|}\mathrm{f}^{-1}_{\rho\left(\alpha_{[1,m]}\right)}[\mathbb{B}_\epsilon(a)]$$

Proof. Pick $a\in A$ and $\epsilon>0$; then, by Theorem 3, $\mathcal{F}^*$ is uniformly equicontinuous on $\overline{K}$, so for $\frac{\epsilon}{2}$ there is a $\delta>0$ such that for all $\mathrm{f}\in\mathcal{F}^*$ and all $x\in\overline{K}$, $\mathrm{f}(\mathbb{B}_\delta(x))\subseteq\mathbb{B}_{0.5\epsilon}(\mathrm{f}(x))$. Now K is totally bounded, so there is a finite set $\{x_k\}_{k=1}^M$ of K such that $K\subseteq\bigcup_{k=1}^M\mathbb{B}_\delta(x_k)$.

We now recursively define $\alpha\in I^*$, we pick $\alpha_1\in I$ such that

$$\mathrm{f}_{\alpha_1}(x_1)\in\mathbb{B}_{0.5\epsilon}(a)$$

we can do this because $x_1\in K\subseteq(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]\subseteq\text{w-}\mathcal{B}(A)$. Also note that $\mathrm{f}_{\alpha_1}(x_2)\in\mathrm{F}^*[x_2]\subseteq\text{w-}\mathcal{B}(A)$, and so we can pick $\alpha_2\in I$ such that

$$\mathrm{f}_{\alpha_2\alpha_1}(x_2)\in\mathbb{B}_{0.5\epsilon}(a).$$

But again we see that $\mathrm{f}_{\alpha_2\alpha_1}(x_3)\in\text{w-}\mathcal{B}(A)$.

Thus we can continue to pick $\alpha_k\in I$ for $k\leq M$, such that

$$\mathrm{f}_{\alpha_k\alpha_{k-1}\ldots\alpha_2\alpha_1}(x_k)\in\mathbb{B}_{0.5\epsilon}(a),$$

provided $\mathrm{f}_{\alpha_{k-1}\ldots\alpha_2\alpha_1}(x_k) \in$ w-$\mathcal{B}(A)$, which is the case since $x_k \in K \subseteq (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ so for all $\mathrm{f} \in \mathcal{F}^*$ we have $\mathrm{f}(x_k) \in \mathrm{F}^*[K] \subseteq$ w-$\mathcal{B}(A)$.

Now pick an $x \in K$; then there is a $k \in [M]$ such that $x \in \mathbb{B}_\delta(x_k)$, and, by equicontinuity, we have that $\mathrm{d}\big(\mathrm{f}_{\alpha_k\alpha_{k-1}\ldots\alpha_2\alpha_1}(x_k), \mathrm{f}_{\alpha_k\alpha_{k-1}\ldots\alpha_2\alpha_1}(x)\big) < \frac{\epsilon}{2}$. Thus, we see,

$$\mathrm{d}\big(\mathrm{f}_{\alpha_k\alpha_{k-1}\ldots\alpha_2\alpha_1}(x), a\big) \leq \mathrm{d}\big(\mathrm{f}_{\alpha_k\alpha_{k-1}\ldots\alpha_2\alpha_1}(x), \mathrm{f}_{\alpha_k\alpha_{k-1}\ldots\alpha_2\alpha_1}(x_k)\big) + \mathrm{d}\big(\mathrm{f}_{\alpha_k\alpha_{k-1}\ldots\alpha_2\alpha_1}(x_k), a\big) < \epsilon$$

by recalling that $\mathrm{f}_{\alpha_k\alpha_{k-1}\ldots\alpha_2\alpha_1}(x_k) \in \mathbb{B}_{0.5\epsilon}(a)$.

This concludes the proof. □

Lemma 4 is a key result in showing the inclusion $A \subseteq \mathrm{L}[\sigma, x]$. It allows us to get as close as we need to a point in A from anywhere in some compact subset, say K, of w-$\mathcal{B}(A)$ using only one map from $\mathcal{F}^*$. The fact this map works for all $x \in K$ is what allows us to be able to say anything definitive about the chaos game.

We can now state a result concerning the chaos game with c.e.c IFS.

Theorem 17. *Let (X, d) be a metric space, $\mathcal{F} = \{\mathrm{f}_i\}_{i\in I}$ be a c.e.c IFS on X, $x \in X$ and A be a compact quasi attractor of $\mathcal{F}$. Then A is drawn by the chaos game starting in $(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)] \cap \mathcal{O}$.*

Proof. Suppose that $x \in (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ and $\sigma \in I^{\mathbb{N}}$ be normal. By Lemma 3 if we show that $A \cap \mathrm{L}[\sigma, x] \neq \emptyset$ then $\mathrm{L}[\sigma, x] \subseteq A$. Thus, we need only show that $A \subseteq \mathrm{L}[\sigma, x]$. To achieve this we apply Lemma 4. First note that $x \in \mathcal{O}$, so $\overline{\mathrm{F}^*[x]}$ is compact. Thus for all $\lambda \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$, $\mathrm{L}(\lambda, n, x) \in \overline{\mathrm{F}^*[x]}$ so the set $\overline{\{\mathrm{L}(\lambda, n, x)\}_{n\in\mathbb{N}}}$ is compact. We must also show that $\{\mathrm{L}(\sigma, n, x)\}_{n\in\mathbb{N}} \subseteq (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$. Pick $n \in \mathbb{N}$ and suppose that $\mathrm{L}(\sigma, n, x) \notin (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$. This means there is an $\mathrm{f} \in \mathcal{F}^*$ such that $\mathrm{f}(\mathrm{L}(\sigma, n, x)) \notin$ w-$\mathcal{B}(A)$. But we can see that $\mathrm{f}(\mathrm{L}(\sigma, n, x)) \in \mathrm{F}^*[x] \subseteq$ w-$\mathcal{B}(A)$, which is a contradiction. So $\{\mathrm{L}(\sigma, n, x)\}_{n\in\mathbb{N}} \subseteq (\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ and we can apply Lemma 4.

Pick any $a \in A$. I claim that there is a subsequence of $\{\mathrm{L}(\sigma, n, x)\}_{n\in\mathbb{N}}$ converging to a. Indeed, by picking $\epsilon = \frac{1}{k}$ for $k \in \mathbb{N}$ we can find $\alpha^k \in I^*$ such that for every $n \in \mathbb{N}$ there is an $\ell \in \mathbb{N}$ for which

$$\mathrm{f}_{\rho\left(\alpha^k_{[1,\ell]}\right)} \circ \mathrm{L}(\sigma, n, x) \in \mathbb{B}_{\frac{1}{k}}(a)$$

by Lemma 4. Now σ is disjunctive so there is an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\sigma_{[n_k, n_k+|\alpha^k|]} = \alpha^k.$$

This means that for all $k \in \mathbb{N}$ and all $m \leq |\alpha^k|$ that

$$\mathrm{f}_{\rho\left(\alpha^k_{[1,m]}\right)} \circ \mathrm{L}(\sigma, n_k, x) = \mathrm{L}(\sigma, n_k + m, x)$$

and from before there is an $m_k \leq |\alpha^k|$ for which

$$\mathrm{f}_{\rho\left(\alpha^k_{[1,m_k]}\right)} \circ \mathrm{L}(\sigma, n_k, x) = \mathrm{L}(\sigma, n_k + m_k, x) \in \mathbb{B}_{\frac{1}{k}}(a).$$

So the sequence $\{\mathrm{L}(\sigma, n_k + m_k, x)\}_{k\in\mathbb{N}} \to a$. Therefore $A \subseteq \mathrm{L}[\sigma, x]$ and, by previous discussion, $A = \mathrm{L}[\sigma, x]$. □

Note that Theorem 17 does not assume the IFS is finite. Furthermore, if the space X is compact then the chaos game draws A starting in $(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$, which by Theorem 16 is the largest set in which we could possibly start. We can also see that if A is a semi-attractor then $(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)] = (\mathrm{F}^*)^+[X]$ (by Item 3 of Theorem 15); but $(\mathrm{F}^*)^+[X] = X$, as for all $x \in X$ we have $\mathrm{F}^*[x] \subseteq X$. So we can conclude the following.

Corollary 17.1. *Let (X, d) be a compact metric space, $\mathcal{F} = \{\mathrm{f}_i\}_{i\in I}$ be a c.e.c IFS on X, $x \in X$ and A be a semi-attractor of $\mathcal{F}$. Then the chaos game draws A starting in X.*

Proof. See preceding discussion. □

If we assume some additional properties of the IFS we can weaken the requirement for compactness of the space in Corollary 17.1.

Definition 28. *Let* (X, d) *be a metric space and* $\mathcal{F} = \{\mathrm{f}_i\}_{i \in I}$ *be a continuous IFS on* X*. Define*

$$C_{\mathcal{F}} = \left\{ x \mid \textit{there is sequence of functions } \{\mathrm{f}_k\}_{k \in \mathbb{N}} \to x \textit{ uniformly, where } \forall k \in \mathbb{N}, \mathrm{f}_k \in \mathcal{F}^* \right\}$$

Theorem 18. *Let* (X, d) *be a metric space and* $\mathcal{F} = \{\mathrm{f}_i\}_{i \in I}$ *be a uniformly continuous IFS on* X*. Then*

$$\mathrm{F}[C_{\mathcal{F}}] \subseteq C_{\mathcal{F}}.$$

Furthermore, if $C_{\mathcal{F}} \neq \emptyset$ *then* $\overline{C_{\mathcal{F}}}$ *is the semi-attractor of* F*.*

Proof. If $C_{\mathcal{F}}$ is empty then it is sub-invariant. Suppose that $c \in C_{\mathcal{F}}$ then there is a $\{\mathrm{f}_{\alpha^k}\}_{k \in \mathbb{N}} \to c$. Thus, for any $x \in X$, $\{\mathrm{f}_k(x)\}_{k \in \mathbb{N}} \to c$. To show sub-invariance, consider any $i \in I$ and, by uniform continuity, $\{\mathrm{f}_i \circ \mathrm{f}_{\alpha^k}\}_{k \in \mathbb{N}}$ converges uniformly to the constant $\mathrm{f}_i(c)$. This proves $C_{\mathcal{F}}$ is sub-invariant.

Suppose now that $c \in C_{\mathcal{F}}$. Observe that for all $k \in \mathbb{N}$ and for all $x \in X$ we have $\mathrm{f}_k(x) \in \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]}$. As $\overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]}$ is closed $c \in \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]}$ for all $x \in X$. Thus, for all $x \in X$ we have $\emptyset \neq C_{\mathcal{F}} \subseteq \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]}$ and,

$$\emptyset \neq \bigcap_{x \in X} \overline{C_{\mathcal{F}}} \subseteq \bigcap_{x \in X} \overline{\bigcup_{n \in \mathbb{N}} \mathrm{F}^{\circ n}[x]}.$$

So, by Lemma 1 $\overline{C_{\mathcal{F}}}$ is the semi-attractor of F. □

Theorem 19. *Let* (X, d) *be a metric space,* $\mathcal{F} = \{\mathrm{f}_i\}_{i \in I}$ *be a c.e.c and uniformly continuous IFS on* X *with* $C_{\mathcal{F}} \neq \emptyset$*. Then the chaos game draws* A *starting in* X*.*

Proof. Let $c \in C_{\mathcal{F}}$ so there is a sequence of $\mathcal{F}^*$, say $\{\mathrm{f}_{\alpha^n}\}_{n\in\mathbb{N}}$, with $\alpha^n \in I^*$ for all $n \in \mathbb{N}$, converging uniformly to c.

We claim that for every sequence $\{x_n\}_{n\in\mathbb{N}}$ of X that the sequence $\{\mathrm{f}_{\alpha^n}(x_n)\}_{n\in\mathbb{N}}$ converges to c. Indeed, since the convergence of $\{\mathrm{f}_{\alpha^n}\}_{n\in\mathbb{N}}$ to c is uniform, for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\mathrm{d}(\mathrm{f}_{\alpha^n}(x_n), c) \leq \sup_{x\in X} \mathrm{d}(\mathrm{f}_{\alpha^n}(x), c) < \epsilon.$$

Thus, the claim holds.

Let $\sigma \in I^{\mathbb{N}}$ be normal, for all $n, N \in \mathbb{N}$ there is an $m_n \geq N$ such that

$$\rho\big(\sigma_{[1,m_n]}\big)_{[1,|\alpha^n|]} = \alpha^n.$$

Hence, there is subsequence of $\{\mathrm{L}(\sigma, n, x)\}_{n\in\mathbb{N}}$ for any $x \in X$ satisfying, for all $n \in \mathbb{N}$,

$$\mathrm{f}_{\alpha^n} \circ \mathrm{L}(\sigma, m_n, x) = \mathrm{L}(\sigma, m_n + |\alpha^n|, x).$$

But $\{\mathrm{L}(\sigma, m_n, x)\}_{n\in\mathbb{N}}$is just some sequence of X, so by the claim $\{\mathrm{L}(\sigma, m_n + |\alpha^n|, x)\}_{n\in\mathbb{N}}$ converges to c.

Thus, $C_{\mathcal{F}} \subseteq \mathrm{L}(\sigma, x)$, but $C_{\mathcal{F}}$ is sub-invariant and $C_{\mathcal{F}} \cap \mathrm{L}(\sigma, x) \neq \emptyset$. So, by Lemma 3, $\overline{C_{\mathcal{F}}} \supseteq \mathrm{L}(\sigma, x)$. Recalling that $\mathrm{L}(\sigma, x)$ is closed yields $\overline{C_{\mathcal{F}}} = \mathrm{L}(\sigma, x)$.

□

Remark 10. *The concept of c.e.c IFS was studied in [12] and to a lesser extent in [5]. Proposition 8 of [12] relates c.e.c IFS to non-expansive IFS. It states that for a c.e.c IFS, $\mathcal{F}$, and for every compact sub-invariant subset K of a metric space (X, d) there exists a metric ρ on K equivalent to* d *such that $\mathcal{F}$ is a non-expansive IFS on (K, ρ). The proof of this Proposition is essentially omitted and is effectively found in Lemma 3.1 of [13].*

This strongly suggests that the case of c.e.c IFS and non-expansive IFS are essentially the same thing. However the proof of this result is truly reliant on compactness and the sub-invariance of K. And so there they should be considered in separate cases.

We end this section with a miscellaneous result the author always thinks is useful but never is.

Proposition 16. *Let (X, d) be a metric space, $\mathcal{F} = \{\mathrm{f}_i\}_{i \in I}$ be a finite continuous IFS on X, $x \in X$ and $\sigma \in I^{\mathbb{N}}$. Then*

$$\mathrm{L}[\sigma, x] \subseteq \mathrm{F}^{-}[\mathrm{L}[\sigma, x]].$$

Furthermore, if $\overline{\{\mathrm{L}[\sigma, n, x]\}_{n \in \mathbb{N}}}$ is compact then

$$\mathrm{L}[\sigma, x] \subseteq \mathrm{F}[\mathrm{L}[\sigma, x]].$$

Proof. Suppose that $a \in \mathrm{L}[\sigma, x]$. We must show there is an $i \in I$ such that $\mathrm{f}_i(a) \in \mathrm{L}[\sigma, x]$. There is a sequence $\{\mathrm{L}(\sigma, n_k, x)\}_{k \in \mathbb{N}} \to a$, and of course we know that

$$\mathrm{f}_{\sigma_{n_k+1}} \circ \mathrm{L}(\sigma, n_k, x) = \mathrm{L}(\sigma, n_k + 1, x).$$

But I is finite, so there must be an $i \in I$ such that $i = \sigma_{n_k+1}$ for infinitely many $k \in \mathbb{N}$. Thus for one choice of such an i we can pick a subsequence of $\{\mathrm{L}(\sigma, n_k, x)\}_{k \in \mathbb{N}}$, say $\{\mathrm{L}(\sigma, n_{k_m}, x)\}_{m \in \mathbb{N}} \to a$, with $\mathrm{f}_i \circ \mathrm{L}(\sigma, n_{k_m}, x) = \mathrm{L}(\sigma, n_{k_m} + 1, x)$ for all $m \in \mathbb{N}$. But by continuity of f_i we have $\{\mathrm{L}(\sigma, n_{k_m} + 1, x)\}_{m \in \mathbb{N}} \to \mathrm{f}_i(a)$. Hence, $\mathrm{f}_i(a) \in \mathrm{L}[\sigma, x]$.

To prove the furthermore, for $a \in \mathrm{L}[\sigma, x]$ and $\{\mathrm{L}(\sigma, n_k, x)\}_{k \in \mathbb{N}} \to a$ we observe instead that

$$\mathrm{f}_{\sigma_{n_k}} \circ \mathrm{L}(\sigma, n_k - 1, x) = \mathrm{L}(\sigma, n_k, x).$$

and much like before we can take a subsequence of $\{\mathrm{L}(\sigma, n_k, x)\}_{k \in \mathbb{N}}$, say $\{\mathrm{L}(\sigma, n_{k_m}, x)\}_{m \in \mathbb{N}}$, with $\mathrm{f}_i \circ \mathrm{L}(\sigma, n_{k_m} - 1, x) = \mathrm{L}(\sigma, n_{k_m}, x)$ for all $m \in \mathbb{N}$ and some fixed $i \in I$. Additionally, we can take $\{\mathrm{L}(\sigma, n_{k_m} - 1, x)\}_{m\in\mathbb{N}}$ to converge, to say a_2, by compactness. This means that $\{\mathrm{f}_i \circ \mathrm{L}(\sigma, n_{k_m} - 1, x) = \mathrm{L}(\sigma, n_{k_m}, x)\}_{m\in\mathbb{N}} \to \mathrm{f}_i(a_2)$ and $\{\mathrm{L}(\sigma, n_{k_m}, x)\}_{m\in\mathbb{N}}$ is a subsequence of $\{\mathrm{L}(\sigma, n_k, x)\}_{k\in\mathbb{N}} \to a$. Thus, $\mathrm{f}_i(a_2) = a$ and $a \in \mathrm{F}[\mathrm{L}[\sigma, x]]$. $\square$

Chapter 5

Conclusion And Further Work

The main contributions of this work are: Theorem 12, where we characterize when a l.s.c multifunction has a semi-attractor and, following that, provide several sufficient conditions for an IFS (possibly discontinuous and infinite) with its Hutchinson-Barnsley operator being l.s.c. Theorem 16 shows that for any $\mathcal{F}$, with its Hutchinson-Barnsley operator being l.s.c, the initial point of the chaos game must start in $(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ in order for the chaos game to be guaranteed to draw a quasi attractor A. Theorem 17 shows that for every c.e.c IFS with compact quasi attractor A, the chaos game draws A with initial point stating in $(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)] \cap \mathcal{O}$.

There are many avenues of future research:

1. Extend the results of Theorems 12 and 16 to more general topological spaces.

2. Extend the results of Section 4.2 to topological spaces, using the concept of evenly continuous sets of functions instead of equicontinuity.

3. Carefully examine how the sets $\text{w-}\mathcal{B}(A)$ and $(\mathrm{F}^*)^+[\text{w-}\mathcal{B}(A)]$ relate to the "probabilistic chaos game" or other random iteration algorithms.

4. Find characterizations of largest/maximal super-invariant sets of a lower semicontinuous or outer semicontinuous multifunction.

5. Prove or disprove that for every IFS $\mathcal{F}$ with lower semicontinuous Hutchinson-Barnsley operator there is a continuous IFS $\mathcal{G}$ for which G = F. Additionally, if $\mathcal{F}$ is finite then can we take $\mathcal{G}$ to be. Further, is playing the chaos game with $\mathcal{F}$ equivalent to playing it with $\mathcal{G}$?

Bibliography

[1] Andres, J., and Fišer, J. Metric and topological multivalued fractals. *International Journal of Bifurcation and Chaos 14*, 04 (2004), 1277–1289.

[2] Barnsley, M. F., and Leśniak, K. On the continuity of the hutchinson operator. *Symmetry 7*, 4 (2015), 1831–1840.

[3] Barnsley, M. F., Leśniak, K., and Rypka, M. Chaos game for ifss on topological spaces. *Journal of Mathematical Analysis and Applications 435*, 2 (2016), 1458–1466.

[4] Barnsley, M. F., and Vince, A. The chaos game on a general iterated function system. *Ergodic Theory and Dynamical Systems 31*, 4 (2011), 1073–1079.

[5] Barrientos, P. G., Ghane, F. H., Malicet, D., Sarizadeh, A., et al. On the chaos game of iterated function systems. *Topological methods in nonlinear analysis 49*, 1 (2017), 105–132.

[6] Edgar, G. *Measure, topology, and fractal geometry.* Springer Science & Business Media, 2007.

[7] Fitzsimmons, M., and Kunze, H. Circle inversion ifs. *Recent Advances in Mathematical and Statistical Methods* (2018), 81.

[8] Hutchinson, J. E. Fractals and self similarity. *Indiana University Mathematics Journal 30*, 5 (1981), 713–747.

[9] Lasota, A., and Myjak, J. Semifractals on polish spaces. *Bulletin of the Polish Academy of Sciences. Mathematics 46* (1998), 179–196.

[10] Lasota, A., and Myjak, J. Fractals, semifractals and markov operators. *International Journal of Bifurcation and Chaos 9*, 02 (1999), 307–325.

[11] Lasota, A. A., and Myjak, J. W. Attractors of multifunctions.

[12] Leśniak, K. Random iteration for infinite nonexpansive iterated function systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science 25*, 8 (2015), 083117.

[13] Miculescu, R., and Mihail, A. On a question of a. kameyama concerning self-similar metrics. *Journal of Mathematical Analysis and Applications 422*, 1 (2015), 265–271.

[14] Sarizadeh, A. A new example of a deterministic chaos game. *Bulletin of the Australian Mathematical Society 94*, 3 (2016), 464–470.

Appendix A

Notation Appendix

Here for reference we define some notion used throughout this work.
Let (X, d) be a metric space and (Y, ρ) be a metric space. Let A and B be sets.

- $B^A = \{\mathrm{f} \,|\, \mathrm{f} : A \to B\}$
- $2^A = \{A_1 | A_1 \subseteq A\}$
- $\mathbb{B}_r^X(x_0) = \{x \in X | \, \mathrm{d}(x, x_0) < r\}$ for $r \in (0, \infty)$ and $x_0 \in X$
- If $\mathrm{f} : A \to B$ and $A_1 \subseteq A$ then $\mathrm{f}(A_1) = \{\mathrm{f}(a) \mid a \in A_1\}$
- If $\mathrm{f} : A \to B$ and $B_1 \subseteq B$ then $\mathrm{f}^{-1}(B_1) = \{a \in A \mid \mathrm{f}(a) \in B_1\}$