Nonunique Equilibria of Projected Dynamical Systems and Their Applications

by
Fatima Etbaigha

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Projected dynamical systems were formulated in the 1990s by Dupuis and Nagurney. In contrast to classical dynamical systems, projected dynamical systems have discontinuous right hand sides that are associated with a projection operator. By the projection mechanism, the whole Hilbert space $H$ is projected onto a non-empty, closed convex set $K \subset H$. Critical points of a projected dynamical system coincide with the solutions of a corresponding variational inequality problem; therefore, the applications of projected dynamical systems have been found in various fields such as economics, operations research and engineering. In this thesis, we study applications of equilibria of projected dynamical systems for market equilibrium problems, games and compartmental population models. First, the well-known market disequilibrium model with excess supply and demand is investigated to determine if it exhibits changes in the structure and the number of equilibrium states for specific choices of parameter values. We study the bifurcation problem (i.e., a qualitative change in equilibrium states) as a parameterized variational inequality problem. We conduct our analysis by modeling the markets via a projected dynamical system. Second, we present a combination
of theoretical and computational results meant to give insights into the question of the existence of nonunique Nash equilibria for N-player nonlinear games. Our inquiries make use of the theory of variational inequalities and projected systems to classify cases where multiplayer Nash games with parameterized payoffs exhibit changes in the number of Nash equilibria, depending on given parameter values. Finally, we use the compartmental population model, namely, the deterministic Susceptible-Exposed-Infectious-Recovered model, to analyze the dynamics of influenza infection of a farrow-to-finish swine farm, and we explore the reinfection at the farm level. We further examine the effectiveness of two control strategies: vaccination and reduction of indirect contact. In this case, we show that the model is a projected dynamical system but the projection does not add any relevant applied meaning to the results. Therefore, we show that the problem can be studied using classical dynamical systems.
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Chapter 1

Introduction

A dynamical system captures the progress of a particular problem over time [42]. It describes the processes and behaviours which change over time [19]. This description is done using a mathematical model [89].

Dynamical systems are classified according to the equations used to describe the evolution of the applied problem. Dynamical systems are called continuous if they are governed by differential equations [21]. They model applications undergoing changes usually described by $C^1$ functions [1]. Unlike typical continuous dynamical systems, projected dynamical systems (PDSs) are non-smooth systems, i.e., their right hand sides are governed by a differential equation involving the projection operator applied to a $C^0$ function or a Lipschitz function [45]. PDSs have been characterized as discontinuous systems. This discontinuity takes place due to the projection of the function on the boundary of the constraint set [86]. Since there are constraints in several types of equilibrium problems, we can use a PDS to model their time dependent evolution. For instance, there may be constraints because of budgets, flow conservation and non-negativity, and classical dynamical systems theory may not be sufficient to model and solve such problems [31].

The stationary points of a PDS are the same as the solutions of a variational inequality
Therefore, it can be said that PDSs give a dynamic extension of VIs [57]. PDSs have been applied in various fields such as economics, operations research, finance and network analysis to study the behaviour of equilibrium problems, for example traffic network equilibrium [85] and spatial price equilibrium [83].

The phenomenon known as bifurcations, which describes qualitative change to a dynamical system due to variations of the parameters, is considered an essential part of dynamical systems theory [69]. While bifurcations have been well studied for classical dynamical systems (see for instance [94] and the references therein), this issue has not been studied as much for constrained dynamics because non-smooth systems have discontinuities, while the classical dynamical systems are smooth; thus PDS do not follow the classical bifurcations theory [13]. For instance, it has been shown that for a monotone vector field, the dynamic structure and behaviour of the PDS changes [23].

In this thesis, we study applications of equilibria of PDS for market equilibrium problems, N-player nonlinear games and compartmental population models. We show that in order to study and analyze some problems, the projection mechanism is necessary, which in turn leads to interesting dynamics, such as the market equilibrium problems and games. However, in other problems, such as the compartmental population models, the projection mechanism is not necessary. In this case, classical dynamical systems are used to study the problem because the projection is not adding any relevant applied meaning to the results.

Objectives

We have completed three projects in different fields, all of which are applications of PDS. Namely, these fields are economics, game theory, and mathematical biology. For the economics and games problems, we conduct our analyses based on PDS. For the mathematical biology problem, we conduct our analyses based on classical dynamical systems. Our main
objectives in each chapter are as follows:

- Examine the possible structural changes of the behavior of the market equilibrium and disequilibrium problems under changing parameters (Chapter 2)

- Classify cases where multiplayer Nash games with parametrized payoffs exhibit changes in the number of Nash equilibria, depending on given parameter values (Chapter 3)

- Analyze the dynamics of influenza infection of a farrow-to-finish swine farm, explore the reinfection at the farm level, and examine the effectiveness of different control strategies (Chapter 4)

Below, we present the materials, frameworks, definitions and tools that are used in the thesis and show how they are related.

**Literature Review**

1.1 Convex and variational analysis

Convex analysis is a part of mathematics that deals with convex sets and functions [73]. The study of convex sets has influenced various area of mathematics: functional analysis, complex analysis, calculus of variations and many other fields (see for example [14, 48, 68]). Throughout this thesis, convex sets and their properties play an essential role for the existence of solutions of variational inequality problems and projected dynamical systems, as can be seen later.

In this section, we recall a couple of basic concepts and results of convexity. We present the projection on a closed convex set and a closed convex cone, as well as the relation between two convex cones that are polar to each other. Classical references are [28, 29, 43, 98].

We start this section by recalling the definition of a closed set [43].
**Definition 1.** The set $K$ is said to be closed if all limit points of any sequence in $K$ are in $K$.

Next we recall the definition of a convex set and convex function (see [98]).

**Definition 2.** Let $K \subset \mathbb{R}^n$, $K$ is called convex if, for any $x, y \in K$ and any $\lambda \in [0, 1]$ the point

$$x\lambda + (1 - \lambda)y \in K.$$

**Definition 3.** A mapping $F : K \rightarrow \mathbb{R}$ is said to be convex if, for any $x, y \in K$ and any $\lambda \in [0, 1]$, we have

$$F(x\lambda + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda)F(y).$$

Basic concepts for the cone, convex cone and polar cone [23] are recalled below.

**Definition 4.** Let $H$ be a Hilbert space. A set $C \subset H$ is called a cone if $ax \in C$ for all $x \in C$ and $a \geq 0$.

**Definition 5.** Let $H$ be a Hilbert space. A set $C \subset H$ is called a convex cone if, for any $x, y \in C$ and any scalar $a \geq 0$ we have

1. $x + y \in C$,
2. $ax \in C$.

**Definition 6.** If $C$ is a cone, then the set

$$C^0 := \{ p \in H \mid \langle p, v \rangle \leq 0, \forall v \in C \},$$

is called the polar cone of $C$.

We now define two cones which are polar to each other.
Definition 7. Let $H$ be a Hilbert space and $K \subset H$ be a closed convex non-empty set. For each $x \in K$, the set

$$T_K(x) = \bigcup_{h > 0} \frac{1}{h}(K - x),$$

is the tangent cone to $K$ at the point $x$.

Definition 8. The normal cone to the set $K \subset H$ at a point $x \in K$ is the polar cone of $T_K(x)$, given by

$$N_K(x) := \{ p \in H \mid \langle p, x - y \rangle \geq 0, \forall y \in K \}.$$

For more details regarding the cone and tangent cone see [6, 23, 76].

The projection operator onto a closed convex set is defined as follows:

Definition 9. Let $H$ be a Hilbert space and $K \subset H$ be a closed convex non-empty set. The projection operator of $H$ onto $K$ denoted by $P_K : H \rightarrow K$ is given by $z \mapsto P_K(z)$ where $P_K(z)$ satisfies $\|P_K(z) - z\| = \inf_{y \in K} \|y - z\|$.

The operator $P_K(z)$ is also called the closest element mapping, i.e. the minimum distance between the vector $z$ and the set $K$. For more information regarding the properties of the projection operator, readers are referred to [28, 29] and the references therein.

The following result shows that any arbitrary vector in a Hilbert space $H$ can be decomposed as a sum of projections onto a pair of polar cones (see [54], Theorem 2.23).

Theorem 1. (Moreau’s theorem). Let $C \subset H$ be a closed convex cone, $C^0 \subset H$ its polar cone and $x_1, x_2, x \in H$; then the following statements are equivalent:

- $x = x_1 + x_2$, with $x_1 \in C, x_2 \in C^0$ and $\langle x_1, x_2 \rangle = 0$.
- $x_1 = P_C(x)$ and $x_2 = P_{C^0}(x)$.

In terms of the tangent cones and normal cones, Moreau’s Theorem can be formulated as: any vector $v \in H$ can be projected onto $T_K(x)$ or $N_K(x)$ respectively, such that $v =$
$$P_{T_K(x)}(v) + P_{N_K(x)}(v) \text{ and } \langle P_{T_K(x)}(v), P_{N_K(x)}(v) \rangle = 0 \text{ (see [23]).}$$

This connection will provide a greater understanding of the theory of PDS stated below in Section 1.3.

### 1.2 Variational Inequalities

#### 1.2.1 Classic variational problems

The theory of variational inequalities (VIs) has been used to solve and formulate various equilibrium problems related to optimization, economics, physics, etc. For instance, Nash equilibrium [26, 32] and market equilibrium [9, 87] have been shown to be equivalent to VI problems. Due to the diverse applicability of VI, it has been the focus of study for many years. The study of VIs began with Hartman and Stampacchia's research in the early 1960s [56, 71, 103]. They studied classes of partial differential equations and boundary value problems using VIs. Stampacchia [74] was the first to prove the existence and uniqueness of the solutions to VIs. Dafersmos [35] was the first to realize that a traffic network equilibrium pattern (Smith, [102]) could be formulated as a VI problem.

In recent years, classical VI problems have been generalized and extended for use in the qualitative study of equilibrium patterns arising in various fields. For details, we refer to [24, 61, 66, 90, 91, 101] and references therein. The advancement in this area of research has enabled connections to be made between mathematics and other areas such as engineering, transportation, operations research and economics (see for instance [24, 81, 87]).

Below, we present a few definitions and important results regarding VI problems and the existence and uniqueness of their solutions (see [30, 86, 113]). These results will be used later on in this thesis.

**Definition 10.** Let $K$ be a non-empty closed and convex subset of a Hilbert space $H$ and let
$F : K \to H$ be a continuous mapping, the VI problem is to determine a vector $x \in K$, such that

$$\langle F(x), y - x \rangle \geq 0, \forall y \in K$$  \hspace{1cm} (1.1)

The following result for the existence of a solution to a VI problem (1.1) was found by Kinderlehrer and Stampacchia [66]. As stated below, a solution can be found under the condition that the set $K$ is compact.

**Theorem 2.** Let $H$ be a Hilbert space and $K \subset H$ be a compact and convex subset and let $F : K \to H$ be a continuous mapping. Then a VI problem (1.1) admits at least one solution.

In the study of VI, monotonicity has been widely used in optimization and in nonlinear analysis (see for instance [67, 77, 86]). Recently, several generalized classes of monotonicity have been proposed and applied in the context of VI problems (see [72, 112] and references therein). The class of generalized monotone mappings that we consider consists of pseudo-monotone mappings. As monotonicity and pseudo-monotonicity are fundamental to guaranteeing the existence of solutions to VIs, we must first define monotone and pseudo-monotone mappings (as given in [25, 30, 66, 80, 112]).

**Definition 11.** Let $H$ be a Hilbert space and $K \subset H$ be a closed convex non-empty set. Then $F : K \to H$ is called locally monotone at $x^* \in K$ if there is a neighborhood $N(x^*)$ contained in $K$ such that

$$\langle F(x^*) - F(x), x^* - x \rangle \geq 0, \forall x \in N(x^*).$$

$F(x)$ is monotone at $x^*$ if this form holds for all $x \in K$.

**Definition 12.** Let $H$ be a Hilbert space and $K \subset H$ be a closed convex non-empty set. Then $F : K \to H$ is called locally strictly monotone at $x^* \in K$ if there is a neighborhood


\( N(x^*) \) contained in \( K \) such that

\[ \langle F(x^*) - F(x), x^* - x \rangle > 0, \forall x \in N(x^*), x \neq x^*. \]

\( F(x) \) is strictly monotone at \( x^* \) if this form holds for all \( x \in K, x \neq x^*. \)

**Definition 13.** Let \( H \) be a Hilbert space and \( K \subset H \) be a closed convex non-empty set. Then \( F : K \to H \) is called locally \( r \)-strongly monotone at \( x^* \in K \) if there is a neighborhood \( N(x^*) \) contained in \( K \) and \( r > 0 \) such that

\[ \langle F(x^*) - F(x), x^* - x \rangle \geq r\|x - x^*\|^2, \forall x \in N(x^*), x \neq x^*. \]

\( F(x) \) is \( r \)-strongly monotone at \( x^* \) if this form holds for all \( x \in K, x \neq x^*. \)

**Definition 14.** Let \( K \) be a closed convex non-empty set of Hilbert space \( H \). Then:

(1) \( F : K \to H \) is called locally pseudo-monotone at \( x^* \in K \) if there is a neighborhood \( N(x^*) \) contained in \( K \) such that

\[ \langle F(x), x^* - x \rangle \geq 0 \Rightarrow \langle F(x^*), x^* - x \rangle \geq 0, \forall x \in N(x^*). \]

(2) \( F : K \to H \) is called locally strictly pseudo-monotone at \( x^* \in K \) if there is a neighborhood \( N(x^*) \subseteq K \) such that

\[ \langle F(x), x^* - x \rangle \geq 0 \Rightarrow \langle F(x^*), x^* - x \rangle > 0, \forall x \in N(x^*). \]

(3) \( F : K \to H \) is called locally strongly pseudo-monotone at \( x^* \in K \) if there is a neighborhood \( N(x^*) \subseteq K \) and \( r > 0 \) such that

\[ \langle F(x), x^* - x \rangle \geq 0 \Rightarrow \langle F(x^*), x^* - x \rangle \geq r\|x^* - x\|^2, \forall x \in N(x^*), x \neq x^*. \]
(4) $F : K \to H$ is called locally strongly pseudo-monotone with degree $\alpha$ on $K$ at $x^* \in K$ if there is a neighborhood $N(x^*) \subseteq K$ and $r > 0$ such that

$$\langle F(x), x^* - x \rangle \geq 0 \Rightarrow \langle F(x^*), x^* - x \rangle \geq r \|x^* - x\|^\alpha, \forall x \in N(x^*), x \neq x^*.$$

**Remark 1.** It is known [62, 63] that monotonicity implies pseudo-monotonicity and the inverse of this implication is not true. For comprehensive material about monotone maps, the reader is referred to [34, 47, 55].

The results below indicate the existence and uniqueness of the solutions to a VI problem under the monotonicity and pseudo-monotonicity conditions. We refer the reader to [30, 46]).

**Theorem 3.** In [46], it is shown that if $F : K \to H$ is a continuous mapping, and $F(x)$ is strictly monotone, then the solution of a VI problem (1.1) is unique, if one exists. Moreover, in [30] it is shown that if $F : K \to H$ a continuous mapping, and $F(x)$ is strictly pseudo-monotone, then the solution of a VI problem (1.1) is unique.

**Theorem 4.** In [46], it is shown that if $F : K \to H$ a continuous mapping, and $F(x)$ is strongly monotone, then there exists precisely one solution to a VI problem (1.1) if a solution exists. Moreover, in [30] it is shown that if $F : K \to H$ a continuous mapping, and $F(x)$ is strongly pseudo-monotone, then the solution of a VI problem (1.1) is unique.

Moreover, in Chapter 3, we will show a new result of the uniqueness of the solution to a VI problem (1.1) related to relaxed cocoercive operators. Relaxed cocoercive mapping is generalized from monotone mappings and defined by Tseng [109] and Zhu and Marcotte [117]. It is known that cocoercive mappings play an important role in studying the existence of solutions to VI problems. Various VIs and their generalizations have been solved using relaxed cocoercive operators (for instance see [2, 18, 20, 111, 113]).

Now, we recall the definition ([44, 113]).
Definition 15. Let $H$ be a Hilbert space. A mapping $F : H \to H$ is said to be:

$(s)$-cocoercive if there exists a constant $s > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geq s\|F(x) - F(y)\|^2, \quad \forall x, y \in H.$$ 

It is called $(m, \gamma)$-relaxed cocoercive if there exists constants $m, \gamma > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geq (-m)\|F(x) - F(y)\|^2 + \gamma\|x - y\|^2, \quad \forall x, y \in H.$$ 

Remark 2. If $m = 0$, then the mapping $F$ in Definition 15 is $\gamma$-strongly monotone.

We display the relation between relaxed cocoercive mappings and strongly monotone mappings using some examples (see [113]).

Example 1. Each $r$-strongly monotone and $\beta$-Lipschitz continuous mapping is an $(r/\beta^2)$-cocoercive mapping for $r$ and $\beta > 0$.

Proof. Since $F$ is $r$-strongly monotone and $\beta$-Lipschitz,

$$\langle F(x) - F(y), x - y \rangle \geq r\|x - y\|^2,$$

$$\|F(x) - F(y)\|^2 \leq \beta^2\|x - y\|^2.$$

Therefore,

$$\langle F(x) - F(y), x - y \rangle \geq (r/\beta^2)\|F(x) - F(y)\|^2,$$

Example 2. Each $r$-strongly monotone mapping is $(1, r + r^2)$-relaxed cocoercive mapping for $r > 0$. 

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Proof. Since $F$ is $r$-strongly monotone,

$$\langle F(x) - F(y), x - y \rangle \geq r \|x - y\|^2,$$

This implies

$$\|F(x) - F(y)\|^2 \geq r^2 \|x - y\|^2.$$

So,

$$\langle F(x) - F(y), x - y \rangle \geq r \|x - y\|^2 + \|F(x) - F(y)\|^2 - \|F(x) - F(y)\|^2,$$

$$\geq r \|x - y\|^2 + r^2 \|x - y\|^2 - \|F(x) - F(y)\|^2,$$

Therefore,

$$\langle F(x) - F(y), x - y \rangle \geq -\|F(x) - F(y)\|^2 + (r + r^2) \|x - y\|^2.$$

\[\square\]

1.2.2 Evolutionary variational inequalities

Evolutionary variational inequalities (EVIs), which are part of the general VIs theory, have enormous applicability across diverse fields, including operation research, economics theory and optimization (see [26, 29, 38] and reference therein ). EVIs are used to study the equilibrium states of problems with constraints varying with a parameter, usually representing time [26]. However, in our study as shown in Chapters 2 and 3, constraints of equilibria were varied with a parameter other than time.

EVIs were first investigated by Lions, Stampacchia and Brezis [15, 29, 71], who use them as a tool to solve problems coming from the area of mechanics. The most significant results come from Steinbach [104] in terms of existence and uniqueness of the solution to such problems. Daniele, Maugeri and Oettli [39] used EVI to model the traffic network
problem with feasible path flows and they demonstrate that the equilibrium conditions of such a problem could be formulated as EVI. Additionally, Cojocaru et al. [24, 29] studied traffic network problems and human migration problems based on the form of EVI presented below. The same framework has also been derived for the study of spatial price equilibrium problems (see for instance [36]). Key contributions, including developing the existence and uniqueness theory, determining the conditions for the existence of the solutions and designing a computational procedure to find an approximate solution for EVI have been made by several authors in these papers: [7, 31, 37, 66].

We now provide the definition of EVI that arises in a time dependent traffic network problem as proposed in [29].

**Definition 16.** Consider the nonempty, convex, closed, bounded subset $K$ of Hilbert space $K \subset L^p([0,T], \mathbb{R}^q)$ defined as:

$$K = \{ u \in L^p([0,T], \mathbb{R}^q) | \lambda(t) \leq u(t) \leq \mu(t), \ a.e. \ t \in [0,T];$$

$$\sum_{i=1}^{q} \xi_i u_i(t) = \rho_j(t) \ a.e. \ t \in [0,T], \ \xi_i \in \{0,1\}, \ i \in \{1,...,q\}, \ j \in \{1,...,l\} \},$$

where $\mu, \lambda, \rho \in L^p([0,T], \mathbb{R}^q)$.

Let $\langle .,. \rangle$ be the duality map between $L^p([0,T], \mathbb{R}^q)$ and $(L^p([0,T], \mathbb{R}^q))^*$ given by:

$$\langle \phi, u \rangle := \int_0^T \langle \phi(t), u(t) \rangle dt, \ \text{with} \ \phi \in (L^p([0,T], \mathbb{R}^q))^* \ \text{and} \ u \in (L^p([0,T], \mathbb{R}^q)).$$

Let $F : K \rightarrow L^p([0,T], \mathbb{R}^q)^*$, then the standard form of the EVI:

$$\text{find} \ u \in K \ \text{such that} \ \langle (F(u), v - u) \rangle \geq 0, \ \forall v \in K,$$

(1.2)

Cojocaru et al. [32] used another equivalent form of the problem (1.2) as shown below.
Definition 17. Find $u(t) \in K(t)$ such that

$$\langle F(u(t)), v(t) - u(t) \rangle \geq 0 \forall v(t) \in K(t) \text{ for a.a. } t \in [0, T],$$

where the sets $K(t)$ in $\mathbb{R}^q$ given by images of functions $u$ in $K$.

The existence and uniqueness of the solution to an EVI problem has been shown in several works (see for instance [8, 75] and references therein).

1.3 Projected Dynamical Systems

A key methodology that is used in this thesis in order to formulate and explore the dynamic behavior of equilibrium problems is PDS theory. PDSs are necessary because of the constraints found in many equilibrium problems such as those arising from budgets, conservation of flow and non-negativity variables, make classical dynamical systems not effective to solve such problems [31]. PDSs are different than classical dynamical systems since their right-hand sides are associated with a projection operator which is no longer continuous [84].

The methodology of PDS was first introduced by Dupuis and Nagurney [46]. They established under certain conditions the basic theory in regard to the existence and uniqueness solutions for PDS. Further, they proved that the stationary points of a PDS are the solutions to the corresponding finite-dimensional variational inequality problem. This connection has been used to study the dynamics of solutions in a wide range of equilibrium problems like spatial price equilibrium [83], Nash equilibrium [28] and traffic network equilibrium [85]. Following this, Isac and Cojocaru [62] started systematically studying PDS on infinite-dimensional Hilbert spaces, and Cojocaru and Jonker worked on the solutions to such problems in an infinite dimensional setting in [33].

Below, we give the definitions of a projected differential equation, PDS and the main
results of existence and uniqueness of its solutions. We refer to [29, 30] and references therein.

**Definition 18.** Let $H$ be a Hilbert space and $K \subset H$ be a closed convex non-empty set. Let $F : K \rightarrow H$ be a Lipschitz continuous mapping on $K$. Then the ordinary differential equation

$$\frac{dx(t)}{dt} = P_{T_K(x(t))}(-F(x(t)))$$

is called the projected differential equation (PrDE) associated with $F$ and $K$.

An initial value problem for PrDE defined as:

$$\frac{dx(t)}{dt} = P_{T_K(x(t))}(-F(x(t))), \quad x(0) = x_0 \in K \quad (1.4)$$

According to Theorem 1 the right-hand side of (1.4) can be expressed as follows:

$$P_{T_K(x)}(-F(x)) = \begin{cases} -F(x), & \text{if } -F(x) \in T_K(x), \\ -F(x) - P_{N_K(x)}(-F(x)), & \text{if } -F(x) \notin T_K(x), \end{cases}$$

where $T_K(x)$ is the tangent cone to the set $K$ at $x$ and $N_K(x)$ is the normal cone to the set $K$ at the same point $x$.

**Remark 3.** It is well known that:

1. A PrDE is a type of ordinary differential equation with a discontinuous and non-linear right-hand side. The vector field $-F(x(t))$ is projected onto the tangent cone of the set $K$ at $x(t)$. This discontinuity is due to the projection operator on the boundary of the constraint set $K$.

2. The trajectories of any initial value problem involving a PrDE evolve within the set
Because of this fact, PrDE are valuable for modeling the time evolution of equilibrium problems.

Here we define the main tool throughout this thesis which is PDS (see [23]).

**Definition 19.** A PDS is given by a mapping \( \psi : [0, \infty) \times K \rightarrow K \) that solves the initial value problem:

\[
\frac{d \psi(t, x)}{dt} = P_{T_K(\psi(t, x))}(-F(\psi(t, x))), \quad \psi(0, x) = x(0) \in K
\]  \hspace{1cm} (1.5)

Another definition of PDS is:

**Definition 20.** A PDS is the dynamical system given by the set of trajectories of a PrDE.

Critical point or an equilibrium of PDS is defined below (see [86]).

**Definition 21.** A point \( x^* \in K \) is called a critical point for PDS (1.5) if

\[
P_{T_K(\psi(t, x^*))}(-F(\psi(t, x^*))) = 0.
\]

The next result shows the relationship between PDS and VI. This result was first presented in \( \mathbb{R}^n \) by Dupuis and Nagurney [46], then by Cojocaru and Jonker [33] on a Hilbert space.

**Theorem 5.** The critical (equilibrium) points of a PDS (1.5) coincide with the solutions to a VI problem (1.1).

Cojocaru et al. in [29] showed a very important result regarding the linkage between a PDS and an EVI problem as follows:

**Theorem 6.** Let \( H \) be a Hilbert space and let \( K \subset H \) be a nonempty closed and convex subset. Let \( F : K \rightarrow H \) be a Lipschitz continuous vector field. Then the solutions to EVI
(1.2) are the same as the critical points of the following PDS and vice versa:

\[
\frac{du(t, \tau)}{d\tau} = P_{T_K(u(t, \tau))}(-F(u(t, \tau))), \ u(t, 0) = u(t) \in K, \tag{1.6}
\]

such that its critical points are given by \(P_{T_K(u(t, \tau))}(-F(u(t, \tau))) = 0\), and \(\tau \in [0, \infty)\).

This result is employed in Chapters 2 and 3.

The following results show that the uniqueness of solutions to an EVI problem and a PDS can hold under monotonicity conditions (see [22, 30]).

**Theorem 7.** Assume that \(F\) is strictly monotone (or strictly pseudo-monotone) and Lipschitz continuous on \(K\). Then the PDS (1.6) and EVI problem (1.2) have a unique solution.

**Theorem 8.** Assume that \(F\) is strongly monotone (or strongly pseudo-monotone, strongly pseudo-monotone of a given degree) and Lipschitz continuous on \(K\). Then the PDS (1.6) and EVI problem (1.2) have a unique solution.

Before stating the main results of the stability of the equilibrium points for PDS, we need to recall the following definition (see [30]).

**Definition 22.** Let \(H\) be a Hilbert space, \(K \subset H\) a nonempty closed and convex subset.

1. A point \(x^* \in K\) is called a local monotone attractor for the PDS if there exists a neighborhood \(N\) of \(x^*\) such that the function

\[
d(t) = \|x(t) - x^*\|
\]

is non-increasing function of \(t\) for any solution \(x(t)\) of PDS starting in the neighborhood \(N\).

2. A point \(x^* \in K\) is called a local strict monotone attractor if the function \(d(t)\) is decreasing.
3. A point \( x^* \in K \) is called exponentially stable if there exists a neighborhood \( N \) of \( x^* \), \( x_0(t) \) is a solution of PDS start at \( x_0 \) when \( t = 0 \), and constants \( b > 0 \) and \( \mu > 0 \) such that

\[
\|x_0(t) - x^*\| \leq b\|x_0 - x^*\| e^{-\mu t}, \quad \forall x_0 \in N(x^*), \ t \geq 0.
\]

4. A point \( x^* \in K \) is called a local finite-time attractor if there exists some \( T(x_0) < \infty \) such that \( x_0(t) = x^* \), for \( x_0 \in N(x^*) \) and \( t \geq T(x_0) \).

Now we give the local stability results of the PDS under various monotonicity conditions. We refer to [30, 116].

**Theorem 9.** Let \( H \) be a Hilbert space and \( K \subset H \) be a closed convex non-empty set. Let \( F : K \rightarrow H \) be Lipschitz continuous on \( K \). Then the following hold:

1. If \( F \) is locally monotone (or locally pseudo monotone) on \( K \), then the equilibrium of PDS (1.6) is a local monotone attractor.

2. If \( F \) is locally strictly monotone (or locally strictly pseudo-monotone) on \( K \), then the unique equilibrium of PDS (1.6) is a local strictly monotone attractor.

3. If \( F \) is locally strongly monotone (or locally strongly pseudo-monotone) on \( K \), then the unique equilibrium of PDS (1.6) is exponentially stable and a local attractor.

4. If \( F \) is locally strongly monotone (or strongly pseudo-monotone) of a given degree on \( K \), then the unique equilibrium of PDS (1.6) is a local finite-time attractor.
1.4 Equilibrium problems and compartmental models

1.4.1 Market equilibrium models as a PDS

The market equilibrium model which consists of supply prices, demand prices and shipments is one of the economic models where the price is governed by the market. Generally the supply presents the total quantity produced by the suppliers, and the demand is the total quantity consumed by the customers. These quantities in the demand and the supply will directly influence the price [41]. Market equilibrium analysis was introduced by Walras in the last half of the nineteenth century [114]. Subsequently, in the twentieth century, a number of formal and important models were created from additional work in this area (see for example [51, 52, 106]).

The market equilibrium models considered in this work are spatial price equilibrium and market disequilibrium models in which the supply price depends on the supplies at all markets, likewise, the demand price depends on the demands at all markets [87, 80]. Samuelson [100] first created the spatial price equilibrium model and it was worked on further by Takayama and Judge [106]. Their work has created the main context for various studies in energy markets, agriculture and mineral economics. The extensive work on the spatial price equilibrium problems induced various improvements in the formulation and the computation of these problems [79, 83]. In the formulation, they evolved from formulation as optimization problems [106] to formulation via variational inequality problems [49] and then to projected dynamical systems as in [82]. In the computation, numerical methods and approximation algorithms are utilized to exploit the solutions at different settings and configurations (see [79, 82]).

According to Nagurney et al. [81], perfect competition governs most models that deal with the theory of market equilibrium. In perfect competition, no one in the economy can control the prices of products. The price of a product is seen as a variable and the actions of
all participants together decide its value. In perfect competition, all participants are equal, have access to all transactions, and are assumed to have perfect information on existing products, price and bids. The supply and demand functions of each product govern the mechanism that controls the amounts traded by a participant and product prices [81].

When the total supply and demand are equal, the equilibrium is reached. The equilibrium price of the product is the price when the equilibrium occurs [80]. Below is the dynamic model of spatial price equilibrium as proposed in [80, 82].

This model is based on various producers and consumers of a homogeneous product under perfect competition. The model presented $m$ supply markets and $n$ demand markets. The supply at market $i$ is expressed as $s_i$ and demand at market $j$ as $d_j$. Further, the supply price at each supply market $i$ is denoted by $\pi_i$, and $\rho_j$ denotes the demand price at demand market $j$. Moreover, the nonnegative commodity shipment attached with a supply and demand market pair $(i,j)$ is denoted by $Q_{ij}$. Each trade between the pair $(i,j)$ is associated with a unit transaction cost $c_{ij}$. It is also considered that the supply price at any supply market $i$ depends on the supplies at all the markets, that is $\pi_i = \pi_i(s)$. Similarly, the demand price at any demand market $j$ depends on the demands at all the markets, that is $\rho_j = \rho_j(d)$. In the same context, the transaction cost associated with the pair of supply and demand markets depends on the commodity shipments between every pair of markets, that is $c_{ij} = c_{ij}(Q)$, where all $\pi_j, \rho_j$ and $c_{ij}$ are smooth functions. Then, The pattern $(s^*, d^*, Q^*) \in K$ where $K = \{(s, d, Q) | s \in \mathbb{R}^m_+, d \in \mathbb{R}^n_+, Q \in \mathbb{R}^{mn}_+\}$ satisfies certain conditions governing the equilibrium market problem if and only if it satisfies the VI problem:

\[
\langle F(Q^*), Q - Q^* \rangle \geq 0 \quad \forall Q \in K^1, K^1 = \mathbb{R}^{mn}_+.
\tag{1.7}
\]

where $F : K^1 \rightarrow \mathbb{R}^{mn}, F_{ij}(Q) = \pi_i(s) + c_{ij}(Q) - \rho_j(d)$.

Due to the relations between a PDS and a VI as in Theorem 5, the adjustment to spatial price
equilibrium states in a VI problem can be obtained by studying the non-smooth dynamical system:
\[
\frac{dQ(t)}{dt} = P_{T_K(Q(t))}(-F(Q(t))), \quad K^1 = \mathbb{R}_+^m, \quad Q(0) \in K^1.
\] (1.8)

Nagurney et al. [87] have extended this model in the market disequilibrium case and it is seen in Chapter 2. In the model, prices are regulated and there is excess demand and supply. In the same manner, the disequilibrium problem has been formulated as a VI then a PDS.

Literature such as [37, 39, 40] has suggested studying the effect of the time on the equilibrium problems. In this study, the problems are affected by a parameter for a range of values. The market models will be used to examine the occurrence of bifurcations (see [105]) by introducing a parameter into the model. In particular, we parameterize the unit transaction cost functions and the price functions separately to examine the possible structural changes of the behavior of the market equilibrium and disequilibrium problems under a parameter variation. Bifurcation problems treated as EVI and numerical analysis is conducted based on modeling the problems via PDS which is necessary for regarding the non-negativity and flow constrains at every parameter value. Bifurcation theory for non-smooth dynamical systems has not been studied as extensively as smooth dynamical systems (see for instance [94] and the references therein). The authors in [70] have addressed the bifurcation issue in VI, where VI is considered with the completely continuous field only. Unlike the previous studies, this work deals with non smooth systems, where the multiple market equilibria, which are critical points of such systems, exist numerically without depending on monotonicity conditions. We refer to Chapter 2 for more details.

### 1.4.2 Nash games as a PDS

A key mathematical field founded in the 20th century is game theory, which offers a mathematical framework to study the strategies of players or opponents and the decision-making
methods in various competitive settings [10]. In the game, every player has a number of choices or strategies, which are plans that describe which of the possible selections he should make and their probabilities. Each player gets some utility, which is also known as payoff [78]. A game that involves \( n \) players is called an \( n \)-player game. Game theory is now used in various areas such as engineering, economics, operation research and biology (see for instance [11, 50, 53] and references therein). John Nash [88] showed, in 1950, that finite games always have an equilibrium point. This is known as Nash equilibrium now, and it is a game state in which no player can get an improved payoff by unilaterally modifying his strategy [10].

In general, a multiplayer game involves a finite number of players, denoted here by \( N > 0 \). A generic player \( i \in \{1,\ldots,N\} \) is thought to have a strategy set \( S_i \subset \mathbb{R}^{n_i} \), whose strategies are vectors \( x_i \in S_i \), and a payoff function \( u_i : K \rightarrow \mathbb{R} \), where we denote by \( K := S_1 \times \ldots \times S_N \subset \mathbb{R}^{n_1+\ldots+n_N} \), assumed to be closed and convex. A Nash equilibrium of a multiplayer game is then defined as follows (see [28]):

**Definition 23.** Assume each player is rational and wants to minimize their payoff function \( u_i : K \rightarrow \mathbb{R} \). Then a Nash equilibrium is a vector \( x^* \in K := S_1 \times \ldots \times S_N \) which satisfies the inequalities:

\[
\forall i \in \{1, 2, \ldots, N\}, \quad u_i(x_i^*, x_{-i}^*) \leq u_i(x_i, x_{-i}^*), \quad \forall x_i \in S_i,
\]

(1.9)

where \( x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \).

It is known that Nash equilibrium points of a nonlinear \( N \)-player game can be obtained by the equivalent reformulation of such a game into a VI problem defined below (see [53, 86] for a proof):

**Theorem 10.** Assume a game as in Definition 24 above. Then if for each \( i \) we have that \( u_i \) is of class \( \mathcal{C}^1 \) and that \(-u_i\) is concave (meaning \( u_i \) is convex) with respect to the strategy \( x_i \),
then a Nash equilibrium of the game (3.1) is a solution of the variational inequality problem:

$$\text{find } x^* \in K \text{ s.t. } \langle F(x^*), y - x^* \rangle \geq 0, \forall y \in K$$ (1.10)

where the mapping $F := \left( \frac{\partial u_1}{\partial x_1}, ..., \frac{\partial u_N}{\partial x_N} \right)$. The converse also holds.

There are many theoretical results asserting that a Nash equilibrium exists and is unique. They all come as a direct application of the intimate relation between some classes of Nash games and variational inequality problems (see for instance [28, 53]). Furthermore, the Nash equilibria of the game in Definition 24 are also the same as the critical points of the non-smooth dynamical system [27]

$$\frac{dx}{dt} = P_{T_K(x(t))}(-F(x(t))), \quad x(0) \in K,$$ (1.11)

This interpretation of the Nash equilibria as critical points of a differential equation is made possible due to known Theorem 5 showing that critical points of PDS (1.5) are solutions of VI problem (1.1) and vice versa.

In Chapter 3, we present a combination of theoretical and computational results meant to give insights into the question of existence of nonunique Nash equilibria for $N$-player nonlinear games. Our inquiries make use of the theory of variational inequalities and projected dynamical systems to classify cases where multiplayer Nash games with parametrized payoffs exhibit changes in the number of Nash equilibria, depending on given parameter values. Whenever a parameter $a \in [\alpha, \beta] \subseteq \mathbb{R}$ is introduced in the players’ payoffs, the Nash equilibrium points will depend on the parameter $a$, thus to determine them we search for the critical points of the perturbed system:

$$\frac{dx}{dt} = P_{T_K(x(t))}(-F(a, x(t))), \quad a \in [\alpha, \beta], \quad x(0) \in K$$
i.e., we search for critical points $x^* \in K$ s.t. $P_{\gamma_K(x^*)}(-F(a,x^*)) = 0$.

In order to analyze and compute Nash equilibria, we use the computational method that relies on PDS. The advantage of this is that Nash equilibria, which are the critical points of PDS, can exist without exhibiting uniqueness, i.e., we need a computational way to bypass monotonicity conditions. (See Chapter 3 for details).

1.4.3 SEIR models as a PDS

Epidemics have been recorded since centuries ago. Human illnesses and mortality that are caused by infectious pathogens have always been a serious concern for many generations [93]. Many infectious diseases are fatal. For instance, in the 14th century, 75-300 million people (30% of Europe’s population) died from the Bubonic plague [5]. The World Health Organization [93] has said that from 1918 – 1919, about 40 million people around the world died from the influenza pandemic. Forty years after this, only 1 – 2 million deaths have been caused from the influenza around the world [64]. Lately, influenza causes at least 250,000 deaths a year and there are around 5 million cases of serious illness from it [93].

Sixty percent of illnesses that humans experience can spread quickly from humans to animals and vice versa [17]. For instance, in 1918, the swine influenza A virus (IAV) was recognized clinically in the United States and it coincided with human influenza that caused about 40 million deaths around the world [16]. Swine herds, which are recognized as reservoirs of IAV, can contribute to disease outbreak in other species [108]. This zoonotic disease continues to be a public health concern due to the ability of the virus to spread readily and evolve [99]. Currently, IAV has become endemic in the swine population around the world [95, 99, 107, 110], and it causes threats not only to the health but also to the production of swine [92].

Various mathematical models have been developed to assist in fully understanding the
dynamics of disease epidemics; and furthermore to assist in the prediction and controlling the spread of infectious diseases (see \[4, 12, 59, 60, 96\]). Mathematical models began being used in the area of epidemiology as early as the beginning of the 20th century, when models were created for diseases like malaria and measles [115]. Kermack and McKendrick [65] presented work on the Susceptible-Infected-Removed (SIR) model in 1926 and since this time, there have been many variations of this model. Most of these models are essentially based on dividing the population into disjoint sets according to their health status with respect to the diseases. The health statuses that are used for classifying the population may vary from model to model and typically include susceptible, infected, and recovered. Due to this structure of modeling, these models are also known as compartmental models [58].

The compartments of the model are selected based on the disease characteristics and the question to be answered by the model. The SEIR and SIR models are the most used models in epidemiology [58]. The SEIR model represents the individuals that are experiencing the following sequence of health status: “Susceptible” (S), “Exposed” (E), “Infectious” (I), “Recovered” (R). In this SEIR model, S represents the individuals who may get infected, once infected these individuals’ status will change to exposed E. I represents the individuals who are infectious and able to transmit the disease to susceptible individuals. After entering the infectious status the individuals become immune, which is described by recovered status R [3]. A simple version of SEIR model is as follows [3]:

\[
\begin{align*}
\frac{dS}{dt} &= -\beta S \frac{I}{N} - \mu S, \\
\frac{dE}{dt} &= \beta S \frac{I}{N} - (\mu + \sigma)E, \\
\frac{dI}{dt} &= \sigma E - (\mu + \gamma)I, \\
\frac{dR}{dt} &= \gamma I - \mu R,
\end{align*}
\]
where \( \beta \) is the transmission rate between \( S \) and \( I \), \( \sigma \) is the latency rate, \( \gamma \) is the recovery rate and \( \mu \) is the death rate.

This is the constraint \( K = \{(S, E, I, R) \in \mathbb{R}^4 : S + E + I + R = N\} \), where \( N \) is a total population and the PDS of this model is:

\[
\frac{d(S(t), E(t), I(t), R(t))}{dt} = P_{T_K(S(t), E(t), I(t), R(t))}(F(S(t), E(t), I(t), R(t))),
\]

\((S(0), E(0), I(0), R(0)) = (S, E, I, R) \in K\), where \( F : K \to \mathbb{R}^4 \) and \( F(S, E, I, R) = (-\beta SI - \mu S, \beta SI - (\mu + \sigma)E, \sigma E - (\mu + \gamma)I, \gamma I - \mu R) \). The equations are scaled by dividing by \( N \), we denote the new variables

\[
x = \frac{S}{N}, \quad m = \frac{E}{N}, \quad y = \frac{I}{N}, \quad z = \frac{R}{N},
\]

where \( 0 \leq x, m, y, z \leq 1 \). Then

\[
F_1 = -\beta xy - \mu x, \\
F_2 = \beta xy - (\mu + \sigma)m, \\
F_3 = \sigma m - (\mu + \gamma)y, \\
F_4 = \gamma y - \mu z.
\]

To find the equilibrium points of the PDS, we choose the values of \( \beta, \mu, \sigma, \gamma \) to be equal to the parameters governing the most common classes in the farm as shown in Chapter 4. The parameters are \( \beta = 0.285, \mu = 0.0004, \sigma = 0.5, \gamma = 0.2 \). Essentially, the SEIR model is a constrained system but it is not interesting from a projection perspective. The projection is not needed since the boundary of the constraint set is not important because endemic equilibria (e.g., \( y \neq \{0, 1\} \)) are considered epidemiologically relevant. Figure 1.1 shows only
Figure 1.1: The phase portrait for SIR model trajectories starting from 50 different initial conditions (circle). The equilibrium points are shown as filled points. One sample equilibrium point numerical values are shown such that x: S, y: I, z: R

the SIR\(^1\) variables for clarity and it is observed that all the equilibrium points are in the interior of the set. The infection variable (\(y\)), is the closest variable to the boundary as can be clearly noticed from the figure. However, the other two variables (\(x, z\)) are clearly not on the boundaries. Therefore, such system can be studied using a classical dynamical system approach. All values obtained here are the same values obtained in Chapter 4, using numerical simulation of SEIR as a regular differential equations system.

In Chapter 4, we extend the deterministic SEIR model presented in [97] to suit the features of a standard Ontario commercial farrow-to-finish swine farm. The model is structured

\(^1\)E is omitted form SEIR model in this figure to be able to plot the trajectories, and E is chosen since it is the least interesting variable in the SEIR model
to include the weekly progress of all pig growth stages including gilts, breeding sows, farrowing sows, and growing pigs. For a reinfection scenario, we assume that the recovered animals can become susceptible to infection again after an average duration of immunity $1/\omega$ days. To evaluate this scenario, the parameter $\omega$ is introduced into the equations to represent the average rate of the immunity waning after the first infection. For the no reinfection case, we set $\omega = 0$ in all equations (see Chapter 4 for more details).
Bibliography


Chapter 2

Bifurcations in the solution structure of market equilibrium problems


Abstract

In this study, the well-known market disequilibrium model with excess supply and demand is investigated to determine if it exhibits changes in the structure and the number of equilibrium states for specific choices of parameter values. We propose to examine the effects of changing separately each price function and unit transaction cost function. We study the bifurcation problem (i.e., qualitative change in equilibrium states) as a parametrized variational inequality problem (VI). We conduct our analysis based on modeling the markets via a projected dynamical system (PDS), which is a type of constrained ordinary differential equation, whose critical points are the market equilibrium states of the economic model.
Numerical simulation for two examples is carried out to see if and when the behavior of these market steady states exhibits any qualitative change.

2.1 Introduction

The market equilibrium models considered in this work are spatial price equilibrium and market disequilibrium models [10, 13]. The extensive work on the spatial price equilibrium induced various improvements in both the formulation and the computation of this problem. In the formulation, it evolved from the formulation of linear complementarity [1], to the formulation via variational inequality theory as in [8]; then to PDS models as in [11]. In the computation, numerical methods and approximation algorithms are utilized to exploit the solutions at different settings and configurations [1, 11]. It also extended to a market disequilibrium case where the prices have been regulated [15]. The initial market disequilibrium problem appeared in [15]. Nagurney et al. [13] have introduced a new market equilibrium model with excess supply and demand, which extended from the spatial price equilibrium model presented in [10]. In this paper, we examine the possible structural changes of the behavior of the market equilibrium and disequilibrium problems under a parameter variation. Two frameworks are combined: the theory of PDS, and the theory of evolutionary variational inequality problems (EVI) (see for example [2, 5, 7] and the references therein). The rest of the paper is organized as follows: in Section 2.2, we give a brief review of both market equilibrium and disequilibrium models; moreover, their variational inequality forms and corresponding adjustment dynamics are also presented. Section 2.3 provides a study of bifurcation questions using the concept of the EVI problem. Numerical simulations for two market equilibrium examples are presented in Section 2.4. The paper is concluded with a discussion and some suggestions for future work.

We assume the reader is familiar with the definitions of a convex cone, and those of the
tangent and normal cone to a closed convex nonempty set (see for instance [3]).

2.2 Brief review of market equilibrium models

2.2.1 Market equilibrium model with excess supply and demand (MESD).

The disequilibrium market model is presented in [13] and summarized below. It presented \( m \) supply markets and \( n \) demand markets which involved the production of a homogeneous commodity under perfect competition. The supply at market \( i \) is expressed as \( s_i \) and demand at market \( j \) as \( d_j \). Further, the supply price at each supply market \( i \) is denoted by \( \pi_i \), and \( \rho_j \) denotes the demand price at demand market \( j \). All the supplies are grouped into a column vector \( s \in \mathbb{R}^m \), and the supply prices are grouped in a row vector \( \pi \in \mathbb{R}^m \). In the same context, the demands and demand prices are grouped respectively into a column vector \( d \in \mathbb{R}^n \) and a row vector \( \rho \in \mathbb{R}^n \). Moreover, the nonnegative commodity shipment attached with a supply and demand market pair \((i, j)\) is denoted by \( Q_{ij} \). Each trade between the pair \((i, j)\) is associated with a unit transaction cost \( c_{ij} \). The unit transaction costs which include the transportation cost and the tax are grouped into a row vector \( c \in \mathbb{R}^{mn} \), and the commodity shipments are grouped in a column vector \( Q \in \mathbb{R}^{mn} \). It is assumed that there are excess supply and excess demand at each market, denoted as \( u_i \) and \( v_j \). All excess supplies and excess demands are grouped respectively into a column vector \( u \in \mathbb{R}^m \) and a row vector \( v \in \mathbb{R}^n \). Therefore, a feasible model requires the following constraints for all supplies, demands and shipments:

\[
    s_i = \sum_j Q_{ij} + u_i, \quad i = 1, \ldots, m \quad \text{and} \quad d_j = \sum_i Q_{ij} + v_j, \quad j = 1, \ldots, n. \quad (2.1)
\]
Note that in [10] the equilibrium market is given as above with $u_i = 0$ and $v_j = 0$. Furthermore in [13], it is assumed that each supply price at the supply market is regulated by a fixed minimum supply price $\pi_i$, called the price floor at supply market $i$. The fixed maximum demand price at demand market $j$ is denoted by $\bar{\rho}_j$, called the price ceiling at demand market $j$. The supply price floors and the demand price ceilings are grouped into vectors $\pi \in \mathbb{R}^m$ and $\bar{\rho} \in \mathbb{R}^n$, respectively. In addition to the above restrictions, the supply, demand, the commodity shipment, excess supply and excess demand, which constitute the disequilibrium pattern must also satisfy the following conditions at all supply and demand markets:

$$\begin{align*}
\pi_i + c_{ij} &\begin{cases} 
= \rho_j, & \text{if } Q_{ij} > 0, \\
\geq \rho_j, & \text{if } Q_{ij} = 0,
\end{cases} \\
\pi_i &\begin{cases} 
= \pi_i, & \text{if } u_i > 0, \\
\geq \pi_i, & \text{if } u_i = 0
\end{cases}, \\
\rho_j &\begin{cases} 
= \bar{\rho}_j, & \text{if } v_j > 0, \\
\leq \bar{\rho}_j, & \text{if } v_j = 0.
\end{cases}
\end{align*}$$

(2.2)

The conditions in (2.2) are known as equilibrium conditions [10]. The conditions (2.3) are presented in detail in [13]. It is also considered that the supply price depends on the supplies at all the markets, that is $\pi = \pi(s)$. Similarly, the demand price depends on the demands at all the markets, that is $\rho = \rho(d)$. In the same context, the unit transaction cost depends on the commodity shipments, that is $c = c(Q)$. All of the functions $\pi, \rho$ and $c$ are smooth. For later use in establishing results we introduce some additional notations, as in [13]. We introduce the notation $\tilde{\pi} = \pi \in \mathbb{R}^m$ and $\tilde{\rho} = \rho \in \mathbb{R}^n$ as relabellings of $\pi$ and $\rho$ with alternative independent variables; that is, $\pi = \pi(s)$ and $\tilde{\pi} = \tilde{\pi}(Q, u)$ and $\rho = \rho(d)$ and $\tilde{\rho} = \tilde{\rho}(Q, v)$. Given $\pi = (\pi_1, \ldots, \pi_m) \in \mathbb{R}^m$, we define $\tilde{\pi}_i = (\pi_i, \pi_i, \ldots, \pi_i) \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$, and $\tilde{\pi} = (\tilde{\pi}_1, \ldots, \tilde{\pi}_m) \in \mathbb{R}^{mn}$. Given $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$, we define $\tilde{\rho}_j = \rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$, $j = 1, 2, \ldots, m$ and $\tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_m) \in \mathbb{R}^{mn}$. Furthermore,
given \( \hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_m) \in \mathbb{R}^m \), we define \( \tilde{\pi}_i = (\hat{\pi}_i, \ldots, \hat{\pi}_i) \in \mathbb{R}^n \), \( i = 1, 2, \ldots, m \), and \( \tilde{\pi} = (\tilde{\pi}_1, \ldots, \tilde{\pi}_m) \in \mathbb{R}^{mn} \). Given \( \hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_n) \in \mathbb{R}^n \), we define \( \tilde{\rho}_j = (\hat{\rho}_1, \ldots, \hat{\rho}_n) \in \mathbb{R}^n \), \( j = 1, 2, \ldots, m \), and \( \tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_m) \in \mathbb{R}^{mn} \).

### 2.2.2 Variational inequality formulation and adjustment dynamics of MESD.

Given \( \tilde{\pi}, \tilde{\rho}, c, \tilde{\pi}, \) and \( \tilde{\rho} \) the pattern \( (Q^*, u^*, v^*) \) satisfies the conditions (2.2) and (2.3) governing the disequilibrium market problem if and only if it satisfies the VI problem

\[
(\tilde{\pi}(Q^*, u^*) + c(Q^*) - \tilde{\rho}(Q^*, v^*)). (Q - Q^*) + (\tilde{\pi}(Q^*, u^*) - \pi). (u - u^*) + \\
(\tilde{\rho} - \hat{\rho}(Q^*, v^*)). (v - v^*) \geq 0, \quad \forall (Q, u, v) \in K \text{ and } K = \mathbb{R}^{mn} \times \mathbb{R}^m \times \mathbb{R}^n.
\]

(2.4)

**Remark:** Note that in [10] the market equilibrium model without the excess supply and demand is formulated as a VI as follows:

\[
\langle F(Q^*), Q - Q^* \rangle \geq 0 \quad \forall Q \in K^1, \quad K^1 = \mathbb{R}^{mn}.
\]

(2.5)

Thus, according to [11], it is known that the adjustment to spatial price equilibrium states in a VI problem can be obtained by studying the nonsmooth dynamical system:

\[
\frac{dQ}{dt} = P_{T_K(Q)}(-F(Q)), \quad Q(0) \in K^1, \quad K^1 = \mathbb{R}^{mn},
\]

(2.6)

where \( F : K^1 \longrightarrow \mathbb{R}^{mn} \), \( F_{ij}(Q) = \pi_i(s) + c_{ij}(Q) - \rho_j(d) \), and the projection \( P_{T_K(Q)} \) is defined as in Definition 9.

In the same manner, we can write the adjustment process of the MESD [7] as

\[
\frac{d(Q, u, v)}{dt} = P_{T_K(Q, u, v)}(-F(Q, u, v)), \quad K = \mathbb{R}^{mn} \times \mathbb{R}^m \times \mathbb{R}^n.
\]

(2.7)
$F : K \mapsto \mathbb{R}^{mn}$ is defined by $F(Q, u, v) = (A(Q, u, v), G(Q, u), B(Q, v))$, and $A : K \rightarrow \mathbb{R}^{mn}$, $G : \mathbb{R}_+^{mn} \times \mathbb{R}_+^{m} \rightarrow \mathbb{R}^{m}$ and $B : \mathbb{R}_+^{mn} \times \mathbb{R}_+^{n} \rightarrow \mathbb{R}^{n}$ are defined by

$$A = \tilde{\pi}(Q, u) + c(Q) - \tilde{\rho}(Q, v), \quad G = \hat{\pi}(Q, u) - \pi, \quad B = \bar{\rho} - \hat{\rho}(Q, v).$$

Let the vector $x = (Q, u, v) \in K$, and $F(x) = F(Q, u, v)$ then (2.7) can be written in the form

$$\dot{x} = P_{TK(x)}(-F(x)). \quad (2.8)$$

### 2.3 Bifurcations in MESD

#### 2.3.1 Formulation of the bifurcation problem for constrained systems.

Bifurcation theory for nonsmooth dynamical systems has not been studied as extensively as smooth dynamical systems (see for instance [14] and the references therein). However, the work in [9] gives results for the existence of bifurcations in VI. The authors have addressed the bifurcation issue in VI, where VI is considered with the completely continuous field only. Since PDS lies outside of the field of classical dynamical systems [12], one way to formulate and study the bifurcation problem is by using the concept of a parametrized VI problem, sometimes called an EVI.

Let us assume that an MESD is formulated as in Section 2.2, such that its solution(s) are given by equilibrium points of the dynamical system (2.7). Let $\lambda \in \mathbb{R}$ and consider the system:

$$\frac{d(Q, u, v)}{dt} = P_{TK(Q, u, v)}(-F(Q, u, v, \lambda)), \quad K = \mathbb{R}_+^{mn} \times \mathbb{R}_+^{m} \times \mathbb{R}_+^{n}, \quad (2.9)$$

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which is equivalent to \( \dot{x} = P_{T_K(x)}(-F(x), \lambda) \), where \( x = (Q, u, v) \in K \) so that whenever \( \lambda = 0 \) we have the initial system (2.8).

We want to study the problem of finding points \( x^* \in K \) so that

\[
P_{T_K(x^*)}(-F(x^*, \lambda)) = 0 \iff -F(x^*, \lambda) \in N_K(x^*),
\]

(2.10)

for some interval of values of \( \lambda \in [a, b] \). From (2.10), using the definition of a normal cone, we have to:

find points \( x^* \in D \) s.t. \( \langle F(x^*, \lambda), y - x^* \rangle \geq 0, \forall y \in D \).

(2.11)

In (3.7), \( F \) depends on \( x \) and \( \lambda \) and hence, we can reformulate it as a parametrized VI problem, or as an EVI problem (see for example [4, 5] for the current use of EVI models) where the bifurcation parameter \( \lambda \) replaces the “time” parameter \( t \):

\[
\langle F(x(\lambda), \lambda), y(\lambda) - x(\lambda) \rangle \geq 0, \forall y(\lambda) \in D(\lambda).
\]

(2.12)

According to the EVI theory, all the functions will be in the Hilbert space \( L^2([a, b], \mathbb{R}^{n+m+nm}) \), then \( F : D \times [a, b] \rightarrow (L^2([a, b], \mathbb{R}^{n+m+nm}) \) and the feasible set \( D \) can be written as

\[
D = \{ x \in L^2([a, b], \mathbb{R}^{n+m+nm}) \mid 0 \leq x(\lambda) \leq M, \ a.e \ \lambda \in [a, b] \},
\]

(2.13)

where \( 0 \in \mathbb{R}^{n+m+nm} \) is the zero vector, and \( M \in \mathbb{R}^{n+m+nm} \) is a vector with large positive components.

Noting that for a.e. \( \lambda \in [a, b] \), we can write \( x(\lambda) = w \in \mathbb{R}^{n+m+nm} \), we define:

\[
D(\lambda) = K = \{ w \in \mathbb{R}^{n+m+nm} \mid 0 \leq w \leq M, \ x(\lambda) = w, x \in K \},
\]

Essentially, the bifurcation problem becomes an EVI problem where the bounds of the
constraint set $D(\lambda) = K$ are fixed, and do not depend on the parameter $\lambda$. In order to solve such an EVI problem, one can use existing methods in the literature [5]. Cojocaru et al. in [5] showed that the linkage between the projected dynamical system and EVI time dependent problem is the critical points of the PDS are the same as the solutions of EVI problem. Using this approach, the PDS associated to $\lambda$ dependent EVI can be formulated as

$$
\frac{dx(\lambda, \tau)}{d\tau} = P_{T_K(x(\lambda, \tau))}(-F(x(\lambda, \tau), \lambda)),
$$

(2.14)

where the time $\tau$ represents the time evolution to the equilibrium $x^*(\lambda)$. This formulation is employed to solve the problem in (2.12).

### 2.3.2 Bifurcations for MESD.

The equilibrium problems are analyzed using different parameterization strategies. Initially, we introduce a parameter representing the supply price floors; however, the system behavior did not change and the equilibrium remained unique. Secondly we impact the supply prices by introducing a parameter $\lambda$ to represent the variation of the supplies on the supply price functions, the equilibrium is still unique and minor changes are observed only in the trajectories’ convergence for different values of $\lambda$. The same experiments are repeated for the demand prices and the same results are observed. These results suggest that the bifurcations are not likely to occur with such parameterization strategies. Parameterizing the unit transaction cost functions by introducing $\lambda$ to represent the variation of the shipments will likely impact the market equilibrium behavior significantly such that $c = c(Q, \lambda)$. Another direction is to introduce $\lambda$ into both the supply and demand prices functions such that $\pi = \pi(s, \lambda)$, and $\rho = \rho(d, \lambda)$. It is expected that this change will impact the market equilibria since changing both the supply and the demand prices at the same time will impact the shipments. The
next section shows two examples of market equilibrium problems using this parameterization strategy.

It is known that whenever $F$ is strictly monotone or beyond (strongly monotone, etc.)\[10\] as a function of $x$, the solution of the MESD problem is unique \[12\]. Thus, there is only one equilibrium point of (2.8). From the EVI theory \[5, 7\], whenever $F(x, \lambda)$ with $\lambda \in [a, b]$ remains strictly monotone, the parametrized MESD (2.12) still has a unique solution. It is therefore logical to seek values of $\lambda$ that will lead to an $F(x, \lambda)$ that is not strictly monotone. In these ranges of values we may find that the equilibria have distinct natures.

### 2.4 Examples

#### 2.4.1 Example 1.

We consider the example of a market equilibrium problem with two supply and two demand markets, which has a unique equilibrium as shown in [10]. In order to detect if the change on the cost functions impacts the equilibrium, we modify the example and introduce $\lambda$ in the cost function. At $\lambda = 0$ the equilibrium is still unique $(Q_{11}, Q_{12}, Q_{21}, Q_{22}) = (0.8, 2.96, 0, 0)$. Additionally, the variables are adjusted to be $\lambda$ dependent where $\lambda \in [a, b]$.

The supply price functions are

\[
\pi_1(s(\lambda)) = 5s_1(\lambda) + s_2(\lambda) + 2, \quad \pi_2(s(\lambda)) = s_1(\lambda) + 2s_2(\lambda) + 3;
\]

the demand price functions are

\[
\rho_1(d(\lambda)) = -2d_1(\lambda) - d_2(\lambda) + 28.75, \quad \rho_2(d(\lambda)) = -d_1(\lambda) - 4d_2(\lambda) + 41;
\]

and the unit transaction costs are defined by

\[
c_{11}(Q, \lambda) = Q_{11}(\lambda) + (0.5 + \lambda)Q_{12}(\lambda) + 1, \\
c_{12}(Q, \lambda) = (2 + \lambda)Q_{12}(\lambda) + Q_{22}(\lambda) + 1.5, \\
c_{21}(Q, \lambda) = (3 - \lambda)Q_{21}(\lambda) + 2Q_{11}(\lambda) + 25,
\]
$c_{22}(Q, \lambda) = (2 - \lambda)Q_{22}(\lambda) + Q_{12}(\lambda) + 30.$

The feasible set is $D = \{Q \in L^2([a, b], \mathbb{R}^4) \mid 0 \leq Q(\lambda) \leq \bar{Q}, \ a.e. \ \lambda \in [a, b]\}.$

We consider a vector field $F$ defined as $F : [a, b] \times K \rightarrow L^2([a, b], \mathbb{R}^4)$, where $K = \mathbb{R}_+^4$

$$F_{ij}(Q(\lambda), \lambda) = \pi_i(s(\lambda)) + c_{ij}(Q(\lambda), \lambda) - \rho_j(d(\lambda)).$$

Then, $F_{11}(Q(\lambda), \lambda) = \pi_1(s(\lambda)) + c_{11}(Q(\lambda), \lambda) - \rho_1(d(\lambda)).$

From conditions (2.1) for the market equilibrium case when $u_i = 0$ and $v_j = 0$, we have

$s_1 = Q_{11} + Q_{12}$, $s_2 = Q_{21} + Q_{22},$

$d_1 = Q_{11} + Q_{21}$, $d_2 = Q_{12} + Q_{22}$, so

$F_{11}(Q(\lambda), \lambda) = 8Q_{11}(\lambda) + (6.5 + \lambda)Q_{12}(\lambda) + 3Q_{21}(\lambda) + 2Q_{22}(\lambda) - 25.75.$

$F_{12}(Q(\lambda), \lambda) = \pi_1(s(\lambda)) + c_{12}(Q(\lambda), \lambda) - \rho_2(d(\lambda))$

$= 6Q_{11}(\lambda) + (11 + \lambda)Q_{12}(\lambda) + 2Q_{21} + 6Q_{22}(\lambda) - 37.5.$

$F_{21}(Q(\lambda), \lambda) = \pi_2(s(\lambda)) + c_{21}(Q(\lambda), \lambda) - \rho_1(d(\lambda))$

$= 5Q_{11}(\lambda) + 2Q_{12}(\lambda) + (7 - \lambda)Q_{21}(\lambda) + 3Q_{22}(\lambda) - 0.75.$

$F_{22}(Q(\lambda), \lambda) = \pi_2(s(\lambda)) + c_{22}(Q(\lambda), \lambda) - \rho_2(d(\lambda))$

$= 2Q_{11}(\lambda) + 6Q_{12}(\lambda) + 3Q_{21}(\lambda) + (8 - \lambda)Q_{22}(\lambda) - 8.$

The EVI problem is given by $\langle F(Q^*(\lambda), \lambda), Q(\lambda) - Q^*(\lambda) \rangle \geq 0, \ \forall Q(\lambda) \in K,$ and the associated PDS can be written as in (2.14).

We examine the effect of changing $\lambda$ in the cost functions considering $\lambda \in [0, 8]$ to ensure that the costs are always nonnegative. We obtain the equilibrium at each value of $\lambda$ by the method in [5] implemented in Matlab. We compute the equilibrium points of PDS (2.14). These equilibria are the solutions of our EVI problem. For $0 \leq \lambda < 8$, for any arbitrary initial condition, the system shows just one equilibrium corresponding to each value of $\lambda$. At $\lambda = 8$, the shipments converge to three boundary equilibria, which means that $F$ is non-strict monotonic. The numerical simulations of the system at $\lambda = 8$ are presented in the Figures 2.1 and 2.2. Shipments at supply market 1 are balanced at $(Q_{11}, Q_{12}) \in \{(0, 1.44), (1.97, 0), (0, 1.97)\}$, and shipments at supply market 2 are balanced.
at \((Q_{21}, Q_{22}) \in \{(5,0), (0,5), (0,0)\}\) as shown in Figure 2.2. As a result of the change in the number of equilibria, \(\lambda = 8\) is a bifurcation value.

\[Q_{11} Q_{12} Q_{21} Q_{22}\]

\[0.5 \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3 \quad 3.5 \quad 4 \quad 4.5\]

**Figure 2.1:** Values of \((Q_{11}, Q_{12}, Q_{21}, Q_{22})\) at equilibrium are presented as a heatmap. The heatmap shows three equilibrium patterns, the equilibria are color-coded such that each repeated pattern is represented by the same color. The initial points are represented vertically in the heatmap.

### 2.4.2 Example 2.

Here we modify the above example to parameterize the supply and demand price functions. The example is extended to the case of excess supply and excess demand. Furthermore \(\lambda\) is introduced into both price functions. Without introducing \(\lambda\) the system shows one equilibrium \((Q_{11}, Q_{12}, Q_{21}, Q_{22}, u_1, u_2, v_1, v_2) = (1.6, 2.8, 0, 0, 0, 0, 0, 0)\).

The supply price functions are defined by
\[
\pi_1(s(\lambda), \lambda) = (5 + \lambda) s_1(\lambda) + s_2(\lambda) + 12,
\]
\[
\pi_2(s(\lambda), \lambda) = (1 + \lambda) s_1(\lambda) + 2 s_2(\lambda) + 30.
\]
Figure 2.2: Phase portraits for $\lambda = 8$, both figures show trajectories starting from 40 initial conditions (equilibria shown by square markers). There are 3 distinct equilibrium points.

The demand price functions are defined by
$$
\rho_1(d(\lambda), \lambda) = -2d_1(\lambda) - (1 - \lambda)d_2(\lambda) + 45,
$$
$$
\rho_2(d(\lambda), \lambda) = -(1 - \lambda)d_1(\lambda) - 4d_2(\lambda) + 55.
$$

The unit transaction cost functions are as in the Example 1 with $\lambda = 0$.

The feasible set is $D = \{ x \in L^2([a, b], \mathbb{R}^8) \mid 0 \leq x(\lambda) \leq 5, \text{ a.e. } \lambda \in [a, b] \}$, where $x(\lambda) = (Q_{11}(\lambda), Q_{12}(\lambda), Q_{21}(\lambda), Q_{22}(\lambda), u_1(\lambda), u_2(\lambda), v_1(\lambda), v_2(\lambda))$. We consider the supply price floors at the markets are $\pi_1 = 10$ and $\pi_2 = 15$ and the demand price ceilings are $\bar{\rho}_1 = 45$ and $\bar{\rho}_2 = 55$. Then the vector field $F$ is defined as

$$
F : K \times [a, b] \longrightarrow L^2([a, b], \mathbb{R}^8), \text{ where } K = \mathbb{R}_+^8.
$$

since

$$
F((Q, u, v), \lambda) = (A((Q, u, v), \lambda), G((Q, u), \lambda), B((Q, v), \lambda)),
$$

we have

$$
A_{ij}((Q, u, v), \lambda) = \hat{\pi}_i((Q, u), \lambda) + c_{ij}(Q) - \hat{\rho}_j((Q, v), \lambda),
$$

$$
G_i = \hat{\pi}_i((Q, u), \lambda) - \bar{\pi}_i, \quad B_j = \bar{\rho}_j - \hat{\rho}_j((Q, v), \lambda).
$$
For simplicity we assume \(Q(\lambda) = (x_1, x_2, x_3, x_4)\).

From conditions (2.1), we have
\[
s_1 = Q_{11} + Q_{12} + u_1, \quad s_2 = Q_{21} + Q_{22} + u_2,
\]
\[
d_1 = Q_{11} + Q_{21} + v_1, \quad d_2 = Q_{12} + Q_{22} + v_2,
\]
so
\[
A_{11}(Q(\lambda), u(\lambda), v(\lambda), \lambda) = (8 + \lambda)x_1 + (6.5 + \lambda)x_2 + (5 + \lambda)u_1 + 3x_3 + (2 - \lambda)x_4 + 2v_1 + u_2 + (1 - \lambda)v_2 - 32.
\]
\[
A_{12}(Q(\lambda), u(\lambda), v(\lambda), \lambda) = (6 + \lambda)x_1 + (11 + \lambda)x_2 + (5 + \lambda)u_1 + 6x_4 + 4v_2 + u_2 + (1 - \lambda)v_1 + (2 - \lambda)x_3 - 41.5.
\]
\[
A_{21}(Q(\lambda), u(\lambda), v(\lambda), \lambda) = (5 + \lambda)x_1 + (1 + \lambda)u_1 + (4 + \lambda)x_2 + (3 - \lambda)x_4 + 2v_1 + (1 - \lambda)v_2 + 7x_3 + 2u_2.
\]
\[
A_{22}(Q(\lambda), u(\lambda), v(\lambda), \lambda)) = 2x_1 + (5 + \lambda)x_2 + (3 - \lambda)x_3 + (1 - \lambda)v_1 + 4v_2 + 6x_4 + (1 + \lambda)u_1 + 2u_2 - 15,
\]
\[
G = (\hat{\pi}_1 - \bar{\pi}_1, \hat{\pi}_2 - \bar{\pi}_2) = (5 + \lambda)x_1 + (5 + \lambda)x_2 + (5 + \lambda)u_1 + x_3 + x_4 + u_2 + 2, (1 + \lambda)x_1 + (1 + \lambda)x_2 + (1 + \lambda)u_1 + 2x_2 + 2x_4 + 2u_2 + 15).
\]
\[
B = (\hat{\rho}_1 - \hat{\rho}_1, \hat{\rho}_2 - \hat{\rho}_2) = (2x_1 + 2x_3 + 2v_1 + (1 - \lambda)x_2 + (1 - \lambda)x_4 + (1 - \lambda)v_2, (1 - \lambda)x_1 + (1 - \lambda)x_3 + (1 - \lambda)v_1 + 4x_2 + 4x_4 + v_2).
\]

Then the EVI problem can be written as
\[
\langle F((Q^*(\lambda), u^*(\lambda), v^*(\lambda), \lambda)), (Q(\lambda), u(\lambda), v(\lambda)) - (Q^*(\lambda), u^*(\lambda), v^*(\lambda)) \rangle \geq 0.
\]

We set \(\lambda \in [-7, 8]\) to ensure the prices are nonnegative. The experiments is carried out as discussed in Example 1. For each value of \(\lambda\) there exists only one equilibrium except for the value of \(\lambda = -6\) at which two equilibria occur. Then at \(\lambda = -6\), \(F\) is non-strict monotonic, and \(\lambda = -6\) is a bifurcation value. Figure 2.3 shows the two equilibria. Shipments at supply market 1 \((Q_{11}, Q_{12}) \in \{(0, 5), (5, 0)\}\) and shipments at supply market 2 \((Q_{21}, Q_{22}) \in \{(0, 0.83), (0.25)\}\) are shown in Figure 2.4. The excess supply and demand at market 1 \((u_1, v_1) \in \{(0, 0), (4.9, 0)\}\) and the excess supply and demand at market 2 \((u_2, v_2) \in \{(4.18, 0), (5, 0)\}\) are presented in Figure 2.5.
Figure 2.3: The heatmap shows the equilibrium pattern ($Q_{11}, Q_{12}, Q_{21}, Q_{22}, u_1, u_2, v_1, v_2$). Each value is represented by singular color, the number of equilibria in this case two which is indicated by switch in color. The initial points are represented vertically in the heatmap.

2.5 Conclusion

In this study we considered the bifurcation problem for the market equilibrium model as an EVI problem. Furthermore, we studied the impact of changing the supply price, demand price and the cost functions on the market equilibrium states. With both the cost functions and price functions, the effect of the variations of the parameter is seen on the number of equilibria occurring at specific values. The equilibrium states were obtained using trajectories of the associated projected dynamics. The empirical results on two examples showed that bifurcations occur in such systems. Further investigation is required to extend the applicability of the proposed method to other equilibrium models. Moreover, additional future work is to study the usage of nonlinear parameter dependencies to explore existence of bifurcations in the solution structure of market equilibrium problems.
(a) \((Q_{11}, Q_{12})\) convergence

(b) \((Q_{21}, Q_{22})\) convergence

**Figure 2.4:** Phase portraits for \(\lambda = -6\) show the convergence of the shipments at each market.

(a) \((u_1, v_1)\) convergence

(b) \((u_2, v_2)\) convergence

**Figure 2.5:** Phase portraits for \(\lambda = -6\) show the convergence of the excess supply and excess demand at each market.
Bibliography


Chapter 3

Equilibria of parametrized $N$-player nonlinear games using inequalities and nonsmooth dynamics

Monica Gabriela Cojocaru and Fatima Etbaigha; submitted to Games, 2nd round of reviews.

Abstract

In this paper we present a combination of theoretical and computational results meant to give insights into the question of existence of nonunique Nash equilibria for $N$-player nonlinear games. Our inquiries make use of the theory of variational inequalities and projected systems to classify cases where multiplayer Nash games with parameterized payoffs exhibit changes in the number of Nash equilibria, depending on given parameter values.
3.1 Introduction

The question of identifying existence and uniqueness results for equilibrium strategies of Nash games dates back to the last century, with the works [28] and [24]. Game theory is a vast area of research to date, where, depending on the type of game, number of players, payoff characteristics, and strategy set characteristics, a large body of existence and uniqueness results are available (see for instance [2, 17, 4, 16, 15] and the references therein).

Theoretical results asserting uniqueness of a Nash equilibrium exist and some come from the direct relation between some classes of Nash games and variational inequality problems (see for instance [14]), i.e., the Nash equilibria coincide with the solution set of a variational inequality problem, thus uniqueness of solutions to the latter leads directly to a singleton set of Nash equilibria. Variational inequalities have been introduced in the last century as well, in relation to studying boundary value problems in partial differential equations. As with game theory, the variational inequality literature is vast (see [21, 18, 8] and the references therein). Some of the application areas of variational inequalities are equilibrium problems, which, apart from games, consist of market (Wardrop, Walras) equilibrium problems, network equilibrium problems and generalized Nash games ([23, 13, 18, 8, 10, 12] and references therein).

In this work we introduce some theoretical results, paired with a computational method, to identify types of Nash games where the players’ payoffs may be dependent on a parameter and where, due to the presence of this parameter, the game’s set of Nash equilibria changes as the parameter takes on values across a given interval of interest. Payoff parametrization is interpreted as an adjustment of player’s strategies to changing conditions in the game (as in [10]). In the current literature, there are results (and we reference a few of them below in detail) which are concerned with uniqueness of Nash equilibria, even in the presence of a parameter variation. There are no particular strategies, however, that practitioners may
employ in situations where the payoff parameterization does not fall into any of the known theoretical results which may guarantee uniqueness. In this case we propose a fairly simple computational method which can be employed to identify ranges of parameter values that lead to the presence of multiple Nash equilibria for the parametrized game.

Our method does not rely on variational inequalities anymore, but rather on projected dynamics ([13, 18, 11]), where, in order to identify possible multiple Nash equilibria, we sweep the set of initial conditions of a projected equation for each sample value of the given parameter. We choose this approach as conditions for existence of critical points of projected equations do not depend on monotonicity properties of the equation’s vector field [11], whereas existence results for solutions of variational inequality problems do ([21]).

Generally speaking, from a dynamical systems perspective, to each Nash game in our context we can associate a dynamical system. If the game is parameterized, then we obtain a family of dynamical systems whose members may display one or multiple equilibria, depending on the parameter values under consideration. Thus in essence we look at a bifurcation-type problem for the projected dynamical system associated to the parameterized Nash game. The issue of bifurcations in constraint dynamics, along the lines of bifurcation theory in classical dynamical systems, is not well studied. The issue of bifurcations in variational inequalities has been tackled in [22], however our results are new, as the authors in [22] consider variational problems with completely continuous fields only.

Last but not least, we would like to highlight that knowing whether a Nash game has unique equilibria is of interest in applied problems: once multiple equilibria are present, the question of selection comes forth: can the equilibria be compared in some meaningful fashion, and if they can be compared, is one “better” than others (see for instance [12] for examples of games with equilibrium sets).

The paper is structured as follows: in Section 2 we provide some mathematical background from Nash games, variational inequalities and projected dynamics. In Section 3 we
present our new theoretical results leading to conditions for uniqueness and nonuniqueness of Nash equilibria for parametrized Nash games, together with a computational method to gain insights into parameter specific values giving rise to multiple Nash equilibria. We illustrate all our results in Section 4 on three examples, two theoretical and one applied. We finish with some conclusions and future ideas.

## 3.2 \( N \)-player games, inequalities and nonsmooth dynamics

In general, a multiplayer game involves a finite number of players, denoted here by \( N > 0 \). A generic player \( i \in \{1, \ldots, N\} \) is thought to have a strategy set \( S_i \subset \mathbb{R}^{n_i} \), whose strategies are vectors \( x_i \in S_i \), and a payoff function \( \theta_i : K \to \mathbb{R} \), where we denote by \( K := S_1 \times \ldots \times S_N \subset \mathbb{R}^{n_1 + \ldots + n_N} \), assumed to be closed and convex. A Nash equilibrium of a multiplayer game is then defined as follows:

**Definition 24.** Assume each player is rational and wants to minimize their payoff function \( \theta_i : K \to \mathbb{R} \). Then a Nash equilibrium is a vector \( x^* \in K := S_1 \times \ldots \times S_N \) which satisfies the inequalities:

\[
\forall i \in \{1, 2, \ldots, N\}, \quad \theta_i(x^*_i, x^*_{-i}) \leq \theta_i(x_i, x^*_{-i}), \quad \forall x_i \in S_i, \tag{3.1}
\]

where in general we denote by \( x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \).

It is known that Nash equilibrium points of a nonlinear \( N \)-player game can be obtained by the equivalent reformulation of such a game into a variational inequality (VI) problem defined below (see [14, 23] for a proof):

**Theorem 11.** Assume a game as in Definition 24 above. Then if for each \( i \in \{1, \ldots, N\} \) we have that \( \theta_i \) is of class \( C^1 \) and that \( -\theta_i \) is concave with respect to the strategy \( x_i \), then a
Nash equilibrium of the game (24) is a solution of the variational inequality problem:

\[
\text{find } x^* \in K \text{ s.t. } \langle F(x^*), y - x^* \rangle \geq 0, \forall y \in K, \tag{3.2}
\]

where the mapping \( F := (\nabla_{x_1}\theta_1, ..., \nabla_{x_N}\theta_N) \). The converse also holds.

Furthermore, the Nash equilibria of the game in Definition 24 are also the same as the critical points of the nonsmooth dynamical system

\[
\frac{dx(t)}{dt} = P_{T_K(x(t))}(-F(x(t))), \ x(0) \in K, \tag{3.3}
\]

where \( P_K : \mathbb{R}^n \rightarrow K \) is the closest element mapping of \( v \in \mathbb{R}^n \) to \( K \), for any \( K \) closed and convex subset in \( \mathbb{R}^n \), and where \( T_K(x) \) is the tangent\(^1\) cone to \( K \) and a point \( x \in K \).

This interpretation of the Nash equilibria as critical points of a differential equation is made possible due to known results (see [23, 18, 11]) showing that critical points of the projected equation (3.3) are solutions of VI problem (3.2) and vice versa.

In this work, we consider the case of a parameter being introduced in both players’ payoffs. We denote this parameter by \( a \in [\alpha, \beta] \subseteq \mathbb{R} \), where \([\alpha, \beta]\) is a given interval in \( \mathbb{R} \). In this case, the equivalent variational inequality problem (3.2) becomes dependent on the payoffs’ parameters. Furthermore, system (3.3) associated to such a VI problem becomes a nonsmooth dynamical system whose right hand side will depend on the payoffs’ parameters.

Whenever a parameter \( a \in [\alpha, \beta] \subseteq \mathbb{R} \) is introduced in the players’ payoffs, the Nash equilibrium points will depend on the parameter \( a \), thus to determine them we search for the critical points of the perturbed system:

\[
\frac{dx(t)}{dt} = P_{T_K(x(t))}(-F(a, x(t))), \ a \in [\alpha, \beta], \ x(0) \in K, \tag{3.4}
\]

\(^1\)We assume the reader is familiar with the definitions of tangent and normal cones to a closed, convex subset of the Euclidean space at a given point in the set (see [21])
i.e., we search for critical points \( x^* \in K \) so that \( P_{T_K(x^*)}(-F(a, x^*)) = 0 \), or equivalently

\[
\text{find } x^* \in K, \ a \in [\alpha, \beta] \text{ s.t. } \langle F(a, x^*), y - x^* \rangle \geq 0, \ \forall y \in K. \tag{3.5}
\]

We note at this point that the values of the possible critical points \( x^* \) will depend on the parameter \( a \); this further implies that the formulation of problem (3.5) is not complete. Since \( x^* := x^*(a) \), then we should formulate the search of \( x^*(\cdot) \) on a space of functions:

\[
\text{find } x^* \in D \text{ s.t. } \langle\langle F(a, x^*(\cdot)), y(\cdot) - x^*(\cdot) \rangle\rangle \geq 0, \ \forall y \in D, \tag{3.6}
\]

where \( D := \left\{ u \in L^2([\alpha, \beta], \mathbb{R}^n) \mid u(a) \in K, \ \text{a.e. } a \in [\alpha, \beta] \right\} \), \( \langle\langle \cdot, \cdot \rangle\rangle \) is the inner product on \( L^2([\alpha, \beta], \mathbb{R}^n) \), and where \( K \) is given above in Definition 24. Problem (3.6) is known in the literature as an evolutionary variational inequality problem (EVI) (see [8]). An equivalent formulation is the so-called pointwise EVI problem (see [9]):

\[
\text{find } x^*(a) \in D(a) \text{ s.t. } \langle F(a, x^*(a)), y(a) - x^*(a) \rangle \geq 0, \ \forall y(a) \in D(a), \tag{3.7}
\]

where \( D(a) := K \) in our case. Results on existence, uniqueness and regularity of solutions \( x^*(\cdot) \) for an EVI problem as above exist in the literature (see [3, 8] and the references therein).

In this context, we are not strictly interested in whether or not the solutions \( x^*(a) \) are unique for a given parameter value \( a \). We are interested in the dynamic behaviour of system (3.4), and thus we want to derive conditions under which the solutions of problem (3.7) are not unique, at least for some parameter values \( a \in [\alpha, \beta] \).

Based on current literature on VI problems and dynamical systems in general, we know that there are clear instances where the answer to our question can be obtained by means of known results. First, if \( x^*(a) \) belongs to the interior of \( K \) for some \( a \in [\alpha, \beta] \), then the discussion on the effects of the payoff parameter takes place exactly as in the classical theory
of dynamical systems whenever $-F$ is a function of class $C^1$ (see [26]). Second, it is known from previous work [22] that certain conditions applied to the variational inequality problem of Definition 11 leads to bifurcation points for the VI solution set. Last but not least, it is also known from [5, 19, 10] that whenever $F$ is monotone (as recalled below in Definition 25) but not strictly so (i.e., $\langle F(x^*) - F(x), x^* - x \rangle = 0$), the dynamics structure and behaviour of (3.3) changes, for instance periodic cycles may appear around a stable unique critical point.

### 3.3 Changes in game equilibria using variational inequalities

In this section we propose new theoretical and computational results to test and visualize the behaviour of Nash equilibria of a parametrized Nash game of the type introduced above.

#### 3.3.1 Theoretical approach

We further our investigations into game equilibria by highlighting how the lack of known, widely used monotonicity conditions on the vector field $F$ in (3.3) may lead to the emergence of multiple Nash equilibria. We recall first (see for instance [23, 18]) that uniqueness and stability of critical points of equations (3.3) are based on monotonicity-type properties of the function $F$ on neighbourhoods of the constraint set $K$.

For the ease of reading, we recall first monotonicity-type conditions for mappings (see also [20] and [18]).

**Definition 25.** Let $K$ be a closed, convex non-empty set of $\mathbb{R}^n$. Then $F : K \rightarrow \mathbb{R}^n$ is called locally monotone at $x^* \in K$ if for every $x \in \mathcal{N}(x^*)$ a neighbourhood of $x^*$ in $K$, we have that $\langle F(x^*) - F(x), x^* - x \rangle \geq 0$. 
It is locally strictly monotone at \( x^* \) if for every \( x \neq x^* \in \mathcal{N}(x^*) \) we have \( \langle F(x) - F(x), x^* - x \rangle > 0 \).

It is locally \( r \)-strongly monotone if for every \( x \neq x^* \in \mathcal{N}(x^*) \) we have \( \exists r > 0 \) so that \( \langle F(x^*) - F(x), x^* - x \rangle > r \| x^* - x \|^2 \).

Finally, it is called locally pseudo-monotone at \( x^* \in K \) if for every \( x \in \mathcal{N}(x^*) \) a neighbourhood of \( x^* \) in \( K \), we have that \( \langle F(x), x^* - x \rangle \geq 0 \Rightarrow \langle F(x^*), x^* - x \rangle \geq 0 \).

It is known that if \( F \) is strictly pseudo-monotone (or strongly pseudo-monotone, strongly pseudo-monotone of a given degree [18]) on a certain neighbourhood of \( F \) around a critical point \( x^* \in K \), then \( x^* \) is unique and is a locally monotone attractor (respectively locally exponentially stable, local finite-time attractor) on that neighbourhood.

Here, we use relaxed cocoercive mappings to prove a new result regarding the uniqueness of solutions to a VI problem of type (3.2). To do this, we need to recall the definition of a relaxed cocoercive mapping, and its relation with monotone mappings (see [25],[27], [1]).

**Definition 26.** Let \( H \) be a Hilbert space. A mapping \( F : H \rightarrow H \) is said to be:

\((s)\)-cocoercive if there exists a constant \( s > 0 \) such that

\[ \langle F(x) - F(y), x - y \rangle \geq s \| F(x) - F(y) \|^2, \quad \forall x, y \in H. \]

It is called \((m, \gamma)\)-relaxed cocoercive if there exists constants \( m, \gamma > 0 \) such that

\[ \langle F(x) - F(y), x - y \rangle \geq (-m) \| F(x) - F(y) \|^2 + \gamma \| x - y \|^2, \quad \forall x, y \in H. \]

The following links between cocoercive and monotone-type mappings are known (see [27]): Each \( r \)-strongly monotone and \( b \)-Lipschitz continuous mapping is \( (r/b^2)\)-cocoercive for \( r \) and \( b > 0 \); Each \( r \)-strongly monotone mapping is \( (1, r + r^2)\)-relaxed cocoercive for \( r > 0 \).

We are now ready to state the main theoretical results of our paper.
Lemma 1. Let $K$ be a closed, convex subset of a Hilbert space $H$ and let $F : K \to H$ be a continuous mapping. If the VI($F,K$) problem:

$$\langle F(x), y - x \rangle \geq 0, \forall y \in K,$$  \hspace{1cm} (3.8)

has two distinct solutions $x_1 \neq x_2 \in K$, then $F$ satisfies: $\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \leq 0$.

Proof. In the context of a known existence result in [21], we have that our VI has at least one solution, say $x$. Thus we have that: $\langle F(x), y - x \rangle \geq 0, \forall y \in K$. If we assume that this VI has 2 solutions $x_1 \neq x_2 \in K$, then the following statements hold;

$$\langle F(x_1), y - x_1 \rangle \geq 0, \forall y \in K, \langle F(x_2), y - x_2 \rangle \geq 0, \forall y \in K.$$  \hspace{1cm} (3.9)

Let $y = x_2$ in the first inequality in (3.9) and $y = x_1$ in the second inequality in (3.9). This implies

$$\langle F(x_1), x_2 - x_1 \rangle \geq 0 \text{ and } \langle F(x_2), x_1 - x_2 \rangle \geq 0 \iff \langle F(x_2), x_2 - x_1 \rangle \leq 0.$$  \hspace{1cm} (3.10)

Now by (3.10) we obtain $\langle F(x_2), x_2 - x_1 \rangle \leq 0 \leq \langle F(x_1), x_2 - x_1 \rangle$. Hence this implies

$$\langle F(x_2), x_2 - x_1 \rangle \leq \langle F(x_1), x_2 - x_1 \rangle \implies \langle F(x_1) - F(x_2), x_1 - x_2 \rangle \leq 0.$$

$\square$

Theorem 12. Let $K$ be a closed, convex subset of a Hilbert space $H$ and $F : K \to H$ a Lipschitz continuous mapping with Lipschitz constant $b > 0$. If $F$ is $(m, \gamma)$-relaxed cocoercive, then

$$\langle F(x) - F(y), x - y \rangle > 0 \forall x \neq y \in K \text{ whenever } \gamma > mb^2.$$
Moreover, the \( \text{VI}(F,K) \) problem (3.8) has at most one solution whenever \( \gamma > \frac{mb}{2} \).

**Proof.** If \( F \) is \((m,\gamma)\)-relaxed cocoercive, this implies \( \exists \gamma, m > 0 \) s.t.

\[
\langle F(x) - F(y), x - y \rangle \geq (-m)\|F(x) - F(y)\|^2 + \gamma\|x - y\|^2, \forall x, y \in K.
\]

This is equivalent to

\[
\langle F(x) - F(y), x - y \rangle \geq (-mb^2)\|x - y\|^2 + \gamma\|x - y\|^2, \forall x, y \in K, \tag{3.11}
\]

since \( F \) is \( b \)-Lipschitz continuous. Then we have that

\[
\langle F(x) - F(y), x - y \rangle \geq \gamma \|x - y\|^2, \forall x, y \in K. \tag{3.12}
\]

Our result follows for \( \gamma - mb^2 > 0 \iff \gamma > \frac{mb}{2} \) and \( x \neq y \). Assume now that there are at least 2 distinct solutions of \( \text{VI}(F,K) \), \( x_1 \neq x_2 \in K \). Then we have Lemma 1 for \( x = x_1 \) and \( y = x_2 \). But we also have that (3.12) holds at \( x_1 \neq x_2 \), which is a contradiction. Thus the \( \text{VI}(F,K) \) can have at most one solution.

**3.3.2 Computational approach for nonunique Nash equilibria**

Returning now to our problem formulation, we note that if there are qualitative changes in the game dynamics due to the presence of the parameter \( a \) (as in 3.7), then we propose here to solve for the critical points of (3.4) using an approach tailored to solve the pointwise VI problems (3.7) at some fixed \( a \) values, regardless of whether or not they are unique. We present below a computational approach for determining parameter values where the game dynamics may give rise to more than one Nash equilibrium.

Let \( a \in [\alpha, \beta] \subseteq \mathbb{R} \) as before.
1. Let $\Delta_k$ be a division of the interval $[\alpha, \beta]$ given by

$$a_0 := \alpha < a_1 < a_2 < ... < a_k := \beta$$

2. Solve the pointwise VI problem (3.7) at each $a_i, i \in \{0, ..., k\}$ as follows:

(a) For a given $a_i$, we generate a uniform distribution of initial conditions $x(0, a_i)$ for the system (3.4).

(b) We numerically integrate to obtain trajectories of the perturbed system (3.4) using a projection type method, based on the constructive proof in [11]; we integrate all trajectories starting from all the initial points generated in Step 2 (2a).

(c) We stop integration along a trajectory when a critical point is reached (with a tolerance of $10^{-6}$)

3. Collect the solutions $x^*(a_i), i \in \{0, ..., k\}$ found in Step 2.

4. Identify the values $\{a_i, a_{i+1}\}$ for which a change in dynamics behaviour (change in the number of critical points) takes place as the system (3.4) evolves between two consecutive sample parameter values.

### 3.4 Examples and discussion of results

In this section we show how all our results can be used to study changes in Nash equilibria for three games proposed below. The first two games are theoretical; the third one comes from previous applied work.

---

2This stage is very important to our approach and is different than other algorithmic solutions for VI problems.
3.4.1 Example 1

Let us consider a 2-player game where each player has a fixed 1-dimensional strategy vector, denoted by $x$, respectively $y$, so that $x \in [0, 10]$ and $y \in [0, 10]$. Let the payoff functions be denoted by

$$\theta_1(x, y) = \frac{ax^2}{4} + 3x \quad \text{and} \quad \theta_2(x, y) = \frac{ay^2}{4} - 7y, \ a \in \mathbb{R}.$$ 

We assume that players want to minimize their payoffs, subject to the other player’s choices. Thus a vector of strategies $(x^*, y^*) \in [0, 10]^2$ is a Nash equilibrium of the game if the following are satisfied:

$$\theta_1(x^*, y^*) \leq \theta_1(x, y^*), \ \forall x \in [0, 10] \quad \text{and} \quad \theta_2(x^*, y^*) \leq \theta_2(x^*, y), \ \forall y \in [0, 10]$$

This game gives rise to the projected system:

$$\frac{d(x, y)(t)}{dt} = P_{T_{K(t)}}(-F(x(t), y(t)))$$

with initial conditions $(x(0), y(0)) \in [0, 10]^2$. The vector field associated to this game is as follows:

$$F(x, y) = (ax/2 + 3, ay/2 - 7).$$

We now check whether this example satisfies the assumptions of Theorem 12.

For any elements $(x_1, y_1), (x_2, y_2) \in [0, 10]^2$, we have

$$F(x_1, y_1) = (ax_1/2 + 3, ay_1/2 - 7),$$

$$F(x_2, y_2) = (ax_2/2 + 3, ay_2/2 - 7).$$
It follows that

\[
\|F(x_1, y_1) - F(x_2, y_2)\| = \|(ax_1/2 + 3, ay_1/2 - 7) - (ax_2/2 + 3, ay_2/2 - 7)\|
\]

\[
= \sqrt{(ax_1 - ax_2)^2 + (ay_1 - ay_2)^2}
\]

\[
= (|a|/2)\|(x_1, y_1) - (x_2, y_2)\|.
\]

Then \(F\) is \(b\)-Lipschitz continuous with \(b = (|a|/2)\) where \(a \neq 0\). As a result, we can apply our theory when \(a \in \mathbb{R}, a \neq 0\), in other words for \(a > 0\) and \(a < 0\).

Further,

\[
\langle F(x_1, y_1) - F(x_2, y_2), (x_1, y_1) - (x_2, y_2) \rangle
\]

\[
= ((ax_1/2 + 3, ay_1/2 - 7) - (ax_2/2 + 3, ay_2/2 - 7), (x_1, y_1) - (x_2, y_2))
\]

\[
= ((a/2(x_1 - x_2), a/2(y_1 - y_2)), (x_1 - x_2, y_1 - y_2))
\]

\[
= a/2(x_1 - x_2)^2 + a/2(y_1 - y_2)^2
\]

\[
= (a/2)\|(x_1, y_1) - (x_2, y_2)\|^2.
\] (3.13)

Case 1: let \(a > 0\). By (3.13), \(F\) is \((a/2)\)-strongly monotone, which implies that \(F\) is \((1, a/2 + a^2/4)\)-relaxed cocoercive. Since \(\gamma = (a/2 + a^2/4)\) and \(m = 1 \Rightarrow (a/2 + a^2/4) > 1 \cdot (a^2/4)\), therefore, according to our Theorem, we expect a unique solution of the \(VI(F, K)\) when \(a > 0\), thus a unique Nash equilibrium for the game.

Case 2: let \(a < 0\). From (3.13), we have

\[
\langle F(x_1, y_1) - F(x_2, y_2), (x_1, y_1) - (x_2, y_2) \rangle
\]

\[
= (a/2)\|(x_1, y_1) - (x_2, y_2)\|^2 + \|F(x_1, y_1) - F(x_2, y_2)\|^2 - \|F(x_1, y_1) - F(x_2, y_2)\|^2
\]

\[
= (a/2)\|(x_1, y_1) - (x_2, y_2)\|^2 + (a^2/4)\|(x_1, y_1) - (x_2, y_2)\|^2 - \|F(x_1, y_1) - F(x_2, y_2)\|^2
\]
\[(a/2 + a^2/4)\| (x_1, y_1) - (x_2, y_2)\|^2 - \| F(x_1, y_1) - F(x_2, y_2)\|^2.\]

Since \((a/2 + a^2/4) > 0 \forall a \in (-\infty, -2)\), we can say that \(F\) is \((1, a/2 + a^2/4)\)-relaxed cocoercive, where \(a \in (-\infty, -2)\). In this case, we know that \(a/2 < 0 \Rightarrow a/2 + a^2/4 < a^2/4\). Thus \(\gamma < mb^2\) for all \(a \in (-\infty, -2)\). In this case the numerical method showed that for each value of \(a\) there is a unique Nash equilibrium.

The last case to investigate would be \(a \in [-2, 0)\). For this case, we now use our computational approach. Let \(\Delta\) be a division of \([-2, 0)\) with 20 equally spaced points. Implementing our method gives rise to the following cases (in Figure 3.1 below).

**Figure 3.1:** Nash equilibria values for the game with \(a = -2\) (upper right panel), \(a = -1.25\) (lower left panel) and \(a = -0.5\) (lower right panel) presented in heatmap format and computed using the projected dynamics starting at each value of the 50 initial points, illustrated in upper left panel.
3.4.2 Example 2

Let us consider a 2-player game, where each player has a fixed 1-dimensional strategy vector, denoted by \( x \), respectively \( y \), so that \( x \in [0, 10] \) and \( y \in [0, 10] \). Let the payoff functions be denoted by \( \theta_1(x, y) = x^2 + (a - y)x \) for player 1, and \( \theta_2(x, y) = -axy + \frac{y^2}{2} + 3 \) for player 2, where \( a \in [1, 5] \). We assume that players want to maximize their payoffs, thus a vector of strategies \((x^*, y^*)\) is a Nash equilibrium of the game if the following are satisfied:

\[
\theta_1(x^*, y^*) \geq \theta_1(x, y^*), \quad \forall x \in [0, 10] \text{ and } \theta_2(x^*, y^*) \geq \theta_2(x^*, y), \quad \forall y \in [0, 10]
\]

The VI form of this game gives rise to the VI problem

\[
\text{find } z \in [0, 10]^2 \text{ s. t. } \langle F(z), w - z \rangle \geq 0, \quad \forall w \in [0, 10]^2
\]

and the associated projected system:

\[
\frac{d(x, y)}{dt} = P_{T_{x(t),y(t)}}(-F(x(t), y(t))), \text{ with initial conditions } (x(0), y(0)) \in [0, 10]^2, \text{ where } F := (-\nabla_x \theta_1, -\nabla_y \theta_2). \text{ We study the parametrized game according to conditions in Theorem 12.}
\]

First we can show that \( F(x, y) = (-2x + a - y), -(ax + y) \) is Lipschitz continuous for any \( a \in [1, 5] \). This is clearly the case as \( F \) is linear and is defined on a compact set \([0, 10]^2\).

For a fixed \( a \in [1, 5] \), we can find a Lipschitz constant as follows:

\[
||F(x_1, y_1) - F(x_2, y_2)||^2 = || -2(x_1 - x_2) + (y_1 - y_2), a(x_1 - x_2) - (y_1 - y_2)||^2 =
\]

\[
(4 + a^2)(x_1 - x_2)^2 - 2(a + 2)(x_1 - x_2)(y_1 - y_2) + 2(y_1 - y_2)^2 \leq 4 + a^2 \leq 4 + a^2 + 4a
\]

\[
(4 + a^2 + 4a)(x_1 - x_2)^2 - 2(a + 2)(x_1 - x_2)(y_1 - y_2) + (y_1 - y_2)^2 \leq
\]

\[
2(4 + a^2 + 4a)(x_1 - x_2)^2 + 2(y_1 - y_2)^2 =
\]

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\[
2(2 + a)^2(x_1 - x_2)^2 + 3(y_1 - y_2)^2 \leq (2 + a)^2 \Rightarrow 3 < (2 + a)^2
\]

\[
2(2 + a)^2((x_1 - x_2)^2 + (y_1 - y_2)^2) = 2(2 + a)^2 ||(x_1, y_1) - (x_2, y_2)||^2 \Rightarrow (3.14)
\]

\[
b = (2 + a)\sqrt{2} \text{ for any } a \in [1, 5].
\]

Further, we want to show that \( F \) is \((m, \gamma)\)-relaxed cocoercive for appropriate \( m, \gamma > 0 \) for any \( a \in [1, 5] \). We first compute the lefthand side:

\[
\langle F(x_1, y_1) - F(x_2, y_2), (x_1, y_1) - (x_2, y_2) \rangle = -2Z_1^2 + (a + 1)Z_1Z_2 - Z_2^2, \tag{3.15}
\]

where \( Z_1 = (x_1 - x_2) \) and \( Z_2 = (y_1 - y_2) \).

Further, using (3.14), we have that

\[
(-m)||F(x_1, y_1) - F(x_2, y_2)||^2 > -2m(a + 2)^2 Z_1^2 + Z_2^2 \Rightarrow
\]

\[
(-m)||F(x_1, y_1) - F(x_2, y_2)||^2 + \gamma||F(x_1, y_1) - F(x_2, y_2)||^2 > (\gamma - 2m(a + 2)^2)(Z_1^2 + Z_2^2). \tag{3.16}
\]

Now if we show that for any \( a \in [1, 5] \) there exist \( m, \gamma > 0 \) so that (3.15) > (3.16), then \( F \) is \((m, \gamma)\)-relaxed cocoercive. Thus we want to find \( m, \gamma > 0 \) so that

\[-2Z_1^2 + (a + 1)Z_1Z_2 - Z_2^2 > (\gamma - 2m(a + 2)^2)(Z_1^2 + Z_2^2) \Leftrightarrow
\]

\[(2m(a + 2)^2 - 2 - \gamma)Z_1^2 + (a + 1)Z_1Z_2 + (2m(a + 2)^2 - 1 - \gamma)Z_2^2 > 0. \tag{3.17}
\]

Now let us rewrite the middle term above as: \((a + 1)Z_1Z_2 = 2^{a+1}Z_1Z_2\) and let us show that, for all \( a \in [1, 5] \), there exist \( m, \gamma > 0 \) so that (3.17) is a sum of squares. For this
purpose, take: \( m := \frac{C}{(2+a)^2} \), \( C > 0 \) and find \( C \) so that

\[
(2m(a + 2)^2 - 2 - \gamma) = \frac{(a + 1)^2}{4} \quad \text{and} \quad (2m(a + 2)^2 - 1 - \gamma) > 1.
\]

Using our value of \( m \) from above, we search to find \( C > 0 \) so that:

\[
\gamma > 0 \quad \text{and} \quad \gamma = 2C - 2 - \frac{(a + 1)^2}{4}, \quad \gamma < 2C - 2.
\]

Since the maximal value of \( \frac{(a+1)^2}{4} = 9 \) for \( a \in [1,5] \), then let us choose \( C = 6 \). Then we have that there exist: \( m = \frac{6}{(2+a)^2} > 0 \) and \( \gamma = 10 - \frac{(a+1)^2}{4} > 0 \) which imply the following in (3.17):

\[
(2m(a + 2)^2 - 2 - \gamma)Z_1^2 + (a + 1)Z_1Z_2 + (2m(a + 2)^2 - 1 - \gamma)Z_2^2 = \left( \frac{a + 1}{2}Z_1 + Z_2 \right)^2 + \frac{(a + 1)^2}{4}Z_2^2 > 0.
\]

The last condition of Theorem 12 states that the \( VI(F, K) \) (and thus the game) will have a unique solution if \( \gamma > mb^2 \), which in our case amounts to checking that

\[
10 - \frac{(a + 1)^2}{4} > \frac{6}{2+a}(2+a)^2\sqrt{2}, \quad a \in [1,5]
\]

\[-(a + 1)^2 > 8, \quad \forall a \in [1,5],\]

which is false.

To conclude our analysis, we showed that our example falls outside of the scope of Theorem 12, as \( F \) is \((m, \gamma)\)-relaxed cocoercive but without \( \gamma > mb^2 \). To identify values of \( a \in [1,5] \) which give rise to non-unique Nash pairs, we now employ our numerical method on the interval \([1,5]\).
We choose a division $\Delta_{k=17}$ values given by

$$a_0 = 1 < a_1 = 1.25 < a_2 = 1.5 < \ldots < a_{17} = 5.$$  

As in Example 1 above, we include in Figure 3.2 a few instances of our results for parameter values $a = 1$, $a = 3$, $a = 5$, however in all cases we found either 2 or 3 distinct Nash strategy pairs $(x^*, y^*)$. To clearly see the projected dynamics trajectories leading to some of the above cases we present our results in Figure 3.3 below.

Figure 3.2: Two or three Nash equilibrium pairs of the parametrized game. Values were obtained based on 30 randomly selected initial points of the projected dynamics (3.3) depicted in the upper left panel. Note that when $a = 1$ (upper right panel), we have 3 distinct Nash pairs: $(yellow, yellow) \mapsto (10,10)$, $(yellow, blue) \mapsto (10,0)$ and $(blue, yellow) \mapsto (0,10)$.
3.4.3 Example 3

In [6, 7], Cojocaru and colleagues introduced a simple Nash vaccinating game played by cohorts of parents of babies who consider whether or not to vaccinate their offspring against pediatric diseases such as measles, mumps, rubella, polio etc.. The game we proposed is played among a finite number of groups of parents \((k > 1)\) where parents in a group are considered to hold the same risk perceptions regarding both vaccinating their offspring, as well as not vaccinating. We consider each group to represent a fixed proportion \(\epsilon_i \in (0, 1)\) in the population, and we consider the population to be fixed, i.e. \(\sum \epsilon_i = 1\). In general, each group has a mixed strategy given by their probability of vaccinating \(P_i \in [0, 1]\), where the coverage in the population is considered to be (excluding time lags between vaccination and uptake of the vaccine) \(p = \sum_{i=1}^{k} \epsilon_i P_i\). Each group is given a utility of vaccinating, defined as:

\[
u_i(P) = -r_i P_i - \pi_p^i (1 - P_i),
\]

where by \(r_i := \frac{r^i_v}{r^i_{inf}}\) we denote the relative perceived risk of vaccination versus infection in group \(i\), by \(r^i_v\) we denote the perceived probability of significant morbidity due to vaccination and by \(r^i_{inf}\) we denote the perceived probability of significant morbidity upon infection.
The perceived probability in group $i$ of becoming infected given that a proportion $p$ of the population is vaccinated, is denoted by $\pi_i^p$. The overall probability of experiencing significant morbidity because of not vaccinating is thus $r_{inf}^i \pi^p_i$. In [6], we assume $\pi^i_p$ are group dependent and distinct among groups: $\pi^i_p = e^{ap}$, $a_i \in [1, 10]$, whereas in [7] we considered all groups having the same $\pi := \frac{b}{\alpha + p}$, where $\alpha, b$ are parameter values dependent on the type of infection considered (in case of measles, we took $b = 0.09$ and $\alpha = 0.1$). In both instances of these previous models, the vaccinating games were transformed into their respective variational inequality problems, whose $F$ fields were strongly monotone. The strong monotonicity guaranteed in turn the uniqueness of the Nash equilibrium strategies.

Here we consider the case where the groups have distinct and functionally different $\pi_i$ expressions. Specifically, we consider

$$\pi^1_p = e^{-ap}, \quad \pi^2_p := \frac{0.09}{0.1 + p}, \quad \pi^3_p = e^{\frac{1}{2}p}, \quad a \in [0, 1]$$

with further values of $r_1 = 1$, $r_2 = 2$, $r_3 = 0.5$, $\epsilon_1 = 0.5$, $\epsilon_2 = 0.1$, $\epsilon_3 = 0.4$. The risk values highlight that group 1 is indifferent between risks ($r_1 = 1$), group 2 is a vaccine adverse group ($r_2 > 1$) and group 3 is a vaccine inclined group ($r_3 < 1$). The VI problem associated to this game (where players wish to maximize their utility) is driven by the vector field $F(P_1, P_2, P_3) := \left( -\frac{\partial u_1}{\partial P_1}, -\frac{\partial u_2}{\partial P_2}, -\frac{\partial u_3}{\partial P_3} \right)$, where

$$F(P_1, P_2, P_3) = \begin{cases} 
-\frac{\partial u_1}{\partial P_1} = -e^{-ap} - (1 - P_1)\epsilon_1 a e^{-ap} + 1 \\
-\frac{\partial u_2}{\partial P_2} = -\frac{0.09}{0.1 + p} - \frac{0.09 \epsilon_2 (1 - P_2)}{(0.1 + p)^2} + 2 \\
-\frac{\partial u_3}{\partial P_3} = -e^{\frac{1}{2}p} - (1 - P_3)\epsilon_3 a e^{-\frac{1}{2}p} + 0.5 
\end{cases}$$

It is clear that the functional form of $F$ in this example is a lot more complicated than previous Examples 1 and 2, which leads to higher complexities when testing for conditions
of Theorem 12. However, we can apply our computational method and explore the interval $a \in [0, 1]$. Our findings in Figure 3.4 below showed that the case $a = 0.01$ resulted in three equilibria, where $P_2$ and $P_3$ had the same values unlike $P_1$; case $a = 1$ resulted in only one equilibrium.

![Heatmap of initial points and Nash equilibria](image)

**Figure 3.4:** Nash equilibrium triplets of the game for $a = 0.01$ (upper right panel) and for $a = 1$ (lower panel). The 60 randomly selected initial feasible points for the dynamics (3.3) are shown in the upper left panel.

### 3.5 Conclusions and future work

In this paper we introduced a larger class of monotone-like conditions giving rise to unique solutions of variational inequality problems, which in turn give rise to unique Nash equilibria of a differential game. We also showed that there are theoretical conditions leading to
conclude that uniqueness may not be present, hence a computational method was introduced to identify parameter values giving rise to multiple Nash equilibria in a parameterized Nash game. We showed that our theoretical results can be tested for and used in concrete examples, however that sometimes, due to the complexity of the game’s payoffs, the easier way to tackle the issue is computational in nature. As future ideas, our work could be expanded to bridge our results with known results in [22], and to see to what extent the presence of periodic solutions of the projected dynamics (as in [5]) can be related, meaningfully, with a game’s equilibria set.


Chapter 4

An SEIR Model of Influenza A Virus Infection and Reinfection within a Farrow-to-Finish Swine Farm


Abstract

Influenza A virus (IAV) in swine is a pathogen that causes a threat to the health as well as to the production of swine. Moreover, swine can spread this virus to other species including humans. The virus persists in different types of swine farms as evident in a number of studies. The core objectives of this study are (i) to analyze the dynamics of influenza infection of a farrow-to-finish swine farm, (ii) to explore the reinfection at the farm level, and finally (iii) to examine the effectiveness of two control strategies: vaccination and reduction of indirect
contact. The analyses are conducted using a deterministic Susceptible-Exposed-Infectious-Recovered (SEIR) model. Simulation results show that the disease is maintained in gilts and piglets because of new susceptible pigs entering the population on a weekly basis. A sensitivity analysis shows that the results are not sensitive to variation in the parameters. The results of the reinfection simulation indicate that the virus persists in the entire farm. The control strategies studied in this work are not successful in eliminating the virus within the farm.

4.1 Introduction

In 1918, the swine influenza A virus was recognized clinically in the United States which coincided with human influenza that caused about 20 million deaths around the world [3]. This zoonotic disease continues to be a public health concern due to the ability of the virus to spread readily and evolve [23]. Swine herds, which are recognized as reservoirs of IAV, can contribute to disease outbreak in other species [26]. This virus causes a respiratory disease in pigs with clinical signs including lethargy, coughing and nasal discharge [11, 1]. Currently, IAV has become endemic in the swine population around the world [23, 27, 25, 20], and it causes threats not only to the health but also to the production of swine [17].

Several factors affect the transmission of influenza virus in pigs including age, vaccination, and immunity levels [29]. Vaccination has often been used to minimize the spreading of IAV in pigs [2, 22]. However, it is still not clear whether vaccination is an effective strategy to reduce the virus in an entire swine herd [15, 22]. Additional studies have shown that maternally derived immunity can not only reduce clinical symptoms but also can be beneficial to reduce the spread of IAV in pigs. However, maternally derived immunity is effective only for a limited period of time [1, 12]. In spite of this research into IAV transmission, it is still not well understood how the dynamics of transmission operates at the level of the pig
Many modelling approaches have been carried out to improve the understanding of disease dynamics in swine for infectious diseases, such as Salmonella \cite{10}, Pseudorabies \cite{8} and Nipah virus \cite{21}. In the context of influenza, although IAV has been frequently recorded in swine herds with risks to other animals and public health, there are still gaps in the information regarding the evaluation of IAV, and limited modelling studies conducted on IAV at the pig farm level \cite{5}.

Recently, a few articles on mathematical models of IAV spread in swine herds have been published. Pitzer et al. \cite{18}, developed a stochastic model of IAV in swine that showed a relation between the finishing herd size and seroprevalence but not between farrow-to-finish farm herd size and seroprevalence. They also examined the persistence of IAV in differently sized farms. Their findings indicated that as long as there is an inflow of new susceptible pigs to the farm, the virus persists even in small populations. Another stochastic approach of IAV in swine has been proposed by Cador et al. \cite{4}. This work focuses on the effect of maternally derived immunity on IAV persistence in a farrow-to-finish farm. The results indicated that IAV in piglets can last a long time if maternal immunity is present. Additionally, Reynolds et al. \cite{22} created a deterministic model to address the dynamics of IAV and the vaccination efficacy in USA breeding and wean-to-finish farms. Results showed that the disease is maintained in the breeding farm, while it becomes extinct in the wean-to-finish farm. Furthermore, the most common vaccination strategies did not prevent the spread of infection across the breeding farm. More recently, White et al.\cite{31} proposed a stochastic model of IAV in a standard USA breeding farm. The authors tested different vaccination and management strategies and confirmed the finding of the persistence of IAV in the piglets population.

A study conducted by Poljack et al. \cite{20, 19} confirmed that the influenza virus infection level is growing over the years in pig farms in Ontario. In this work we extend the deter-
ministic SEIR model presented in [22] to suit the features of a standard Ontario commercial farrow-to-finish swine farm. This extension allows us to address the infection dynamics issue of IAV in this farm. In particular, our goals are to use this model to explore the persistence of the influenza virus, evaluate the reinfection at the farm level, and examine the effectiveness of vaccination and reduction of indirect contact at reducing the influenza virus infection through the farm. The model is structured to include the weekly progress of all pig growth stages including gilts, breeding sows, farrowing sows, and growing pigs. The assumptions of direct and indirect transmission between these different stages are considered in the model.

4.2 Materials and methods

4.2.1 Population and process

As illustrated in Figure 4.1, the farrow-to-finish farm involves four types of animals: gilts (females that have not given birth yet), sows (females that have reproduced), piglets (young pigs less than 4 weeks old) and growing pigs (pigs from weaning to marketing level). The production in the farm uses a system of rooms that are associated with four stages of a pig’s life cycle: gilt development stage, breeding/gestation stage, farrowing stage and growing stage [30]. The pigs in each of these stages are divided into (weekly) age classes, where the pigs in the last age class of each stage enter the first age class of the next stage as shown in Figure 4.1. Furthermore, pigs of different ages or reproductive status can be grouped into one room, for example the farrowing room contains sows and piglets and the breeding/gestation room contains weaned and pregnant sows.

New gilts enter the gilt development room each week and remain there for 10 weeks (70 days) until they join the sows in the breeding/gestation room. At that time, they are artificially inseminated. The reproductive cycle of sows is 147 days. The pregnant sows
spend 112 days in the breeding/gestation room, then they are moved to the farrowing room a few days before the expected day of birthing. In the farrowing room, sows nurse their piglets for 4 weeks (28 days) until weaning. Then the individual weaned sows are either culled or moved back to the breeding room, where they stay 7 days until insemination and then they start a new cycle again.

Note that the pregnancy period of sows is typically 115 days [24]. In our model, sows get pregnant at the beginning of week class 11, plus or minus a few days. The pregnant sows spend week classes 11 to 26 in the breeding/gestation room (112 days), then they move to the farrowing room and give birth in days 2 to 7 of week class 27.

The sows give birth to an average of 12 piglets per sow. Once the piglets are weaned in piglets week class 4 ($P_4$), they are moved directly to the growing room for 140 days, at which point they are transported for slaughter.

![Diagram](image)

**Figure 4.1:** Standard commercial farrow-to-finish swine farm. This farm includes gilts, sows, piglets and growing pigs. They are housed in four buildings. In building 1, gilts enter the development room then they will go through building 2 (breeding/gestation room). Here, artificial insemination has been used for breeding. Then the pregnant sows will enter building 3 for farrowing (where they give birth through the first week). After culling, the weaned sows return to the breeding room where the cycle starts again. The piglets stay 4 weeks in building 3. After that, weaned piglets will move to the growing pigs room, then they leave the farm after 20 weeks.
4.2.2 Construction of the model

The infection and reinfection process of IAV in the farrow-to-finish farm is represented by an SEIR model. The SEIR model presented in [22] is extended to include the group of growing pigs. This is necessary since the previously proposed model in [22] studied a breeding farm that does not include the growing pigs group.

In this model, the compartments are selected based on the disease characteristics and age status. For gilts and sows, \( S_i(t), E_i(t), I_i(t) \) and \( R_i(t) \) are the number of susceptible, exposed, infectious, and recovered, respectively; \( t \) is the time, which is measured in days, where \( t \geq 0 \), and \( i \) represents the week class. For the piglets and growing pigs, we specify the week class with \( j \), and add a superscript \( p \): \( S_p^j(t), E_p^j(t), I_p^j(t) \) and \( R_p^j(t) \). Furthermore, for piglets and growing pigs with immunity, the superscripts are changed to \( pm \): \( S_{pm}^j(t), E_{pm}^j(t), I_{pm}^j(t) \) and \( R_{pm}^j(t) \). All the individuals within the farm move from the susceptible pigs population to exposed pigs population due to either direct contact or indirect contact. The direct contact comes from the infectious pigs in the same room, while indirect contact comes from infectious pigs in other rooms of the farm. The individuals in the exposed class move to the infectious class at some latency rate \( \sigma \). After entering the infectious class, the individuals recover at some recovery rate \( \gamma \). Lastly, the individuals who have recovered, return back to susceptible class at some immunity rate \( \omega \). Pigs enter the farm as cohort into class 1. Each week that cohort moves to subsequent number class. The sows, after giving birth, move back to class 11. Piglets enter class 5 in the growing room where they have different immunity. Each location in the farm has different classes of pigs (see Table 4.1). This table extends the corresponding table in [22].

Here we describe the model of the reinfection within the farrow-to-finish swine farm. In order to model the reinfection scenario, similar to [22], we assume that the recovered animals can become susceptible to infection again at an average duration of immunity \( 1/\omega \) days. To
Table 4.1: Class of pigs corresponding to each location.

<table>
<thead>
<tr>
<th>Class</th>
<th>Population variables</th>
<th>Animal</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i \in {1, 2, \ldots, 10}$</td>
<td>$S_i, E_i, I_i, R_i$</td>
<td>Gilts</td>
<td>Gilt development room</td>
</tr>
<tr>
<td>$i \in {11, 12, \ldots, 26}$</td>
<td>$-$</td>
<td>Pregnant sows</td>
<td>Breeding/gestation room</td>
</tr>
<tr>
<td>$i \in {27, 28, 29, 30}$</td>
<td>$-$</td>
<td>Farrowing sows</td>
<td>Farrowing room</td>
</tr>
<tr>
<td>$i \in {31}$</td>
<td>$-$</td>
<td>Weaned sows</td>
<td>Breeding/gestation room</td>
</tr>
<tr>
<td>$j \in {1, 2, 3, 4}$</td>
<td>$S_{j}^{p}, E_{j}^{p}, I_{j}^{p}, R_{j}^{p}$</td>
<td>Piglets</td>
<td>Farrowing room</td>
</tr>
<tr>
<td>$j \in {5, 6, \ldots, 24}$</td>
<td>$S_{j}^{p}, E_{j}^{p}, I_{j}^{p}, R_{j}^{p}$</td>
<td>Growing pigs</td>
<td>Growing pigs room</td>
</tr>
</tbody>
</table>

evaluate this scenario, the parameter $\omega$ is introduced into the equations to represent the average rate of the immunity waning after the first infection.

Formally, the SEIR model of sows and gilts is given by the following ordinary differential equations (ODEs) system:

\[
\begin{align*}
\frac{dS_i}{dt} &= -\beta d I_d S_i - \beta_{ind1} I_{ind} S_i - \beta_{ind2} I_{ind}^* S_i - \mu S_i + \omega R_i, \\
\frac{dE_i}{dt} &= \beta d I_d S_i + \beta_{ind1} I_{ind} S_i + \beta_{ind2} I_{ind}^* S_i - (\mu + \sigma) E_i, \\
\frac{dI_i}{dt} &= \sigma E_i - (\mu + \gamma) I_i, \\
\frac{dR_i}{dt} &= \gamma I_i - (\mu + \omega) R_i,
\end{align*}
\]

where $i \in \{1, 2, \ldots, 10\}$ for gilts and $i \in \{11, 12, \ldots, 31\}$ for sows. Furthermore,

\[
\begin{align*}
I_{d_i} &= \sum_{k=1}^{10} I_k \quad \forall \ i \in \{1, 2, \ldots, 10\}, \\
I_{d_i} &= \sum_{k=11}^{26} I_k + I_{31} \quad \forall \ i \in \{11, 12, \ldots, 26, 31\}, \\
I_{d_i} &= \sum_{k=27}^{30} I_k + \sum_{j=1}^{4} I_{j}^p \quad \forall \ i \in \{27, 28, 29, 30\}, \\
I_{ind_i} &= \sum_{k=11}^{31} I_k + \sum_{j=1}^{4} I_{j}^p \quad \forall \ i \in \{1, 2, \ldots, 10\},
\end{align*}
\]
\[
I_{ind_i} = \sum_{k=1}^{10} I_k + \sum_{k=27}^{30} I_k + \sum_{j=1}^{4} I_p^j \quad \forall \ i \in \{11, 12, \ldots, 26, 31\}, \quad (4.9)
\]

\[
I_{ind_i} = \sum_{k=1}^{26} I_k + I_{31} \quad \forall \ i \in \{27, 28, 29, 30\}, \quad (4.10)
\]

\[
I^*_{ind} = \sum_{j=5}^{24} I_p^j \quad \text{for all class of gilt and sows.} \quad (4.11)
\]

The direct and indirect transmission rates are defined respectively as the parameters \(\beta_d\), \(\beta_{ind_1}\) and \(\beta_{ind_2}\). All governing parameters are stated in Table 4.2.

We divide the piglets into two groups: one inherits the maternal immunity and the other does not as in [22]. Sows that are susceptible, exposed, or infectious give birth to piglets without maternal immunity, while sows that are recovered give birth to piglets with maternal immunity. The equations for piglets without maternal immunity are as follows:

\[
\frac{dS^p_j}{dt} = b_j(S_{27} + E_{27} + I_{27}) - \beta_d^p I_d S^p_j - \beta_{ind_1}^p I_{ind_1} S^p_j - \beta_{ind_2}^p I_{ind_2}^* S^p_j - \mu^p S^p_j + \omega R^p_j + \omega R^p_{pm}, \quad (4.12)
\]

\[
\frac{dE^p_j}{dt} = \beta_d^p I_d S^p_j + \beta_{ind_1}^p I_{ind_1} S^p_j + \beta_{ind_2}^p I_{ind_2}^* S^p_j - (\mu^p + \sigma) E^p_j, \quad (4.13)
\]

\[
\frac{dI^p_j}{dt} = \sigma E^p_j - (\mu^p + \gamma) I^p_j, \quad (4.14)
\]

\[
\frac{dR^p_j}{dt} = \gamma I^p_j - (\mu^p + \omega) R^p_j. \quad (4.15)
\]

For piglets with maternal immunity the equations are:

\[
\frac{dS^p_{pm,j}}{dt} = b_j(R_{27}) - \beta_d^{pm} I_d S^p_{pm,j} - \beta_{ind_1}^{pm} I_{ind} S^p_{pm,j} - \beta_{ind_2}^{pm} I_{ind}^* S^p_{pm,j} - \mu^{pm} S^p_{pm,j}, \quad (4.16)
\]

\[
\frac{dE^p_{pm,j}}{dt} = \beta_d^{pm} I_d S^p_{pm,j} + \beta_{ind_1}^{pm} I_{ind} S^p_{pm,j} + \beta_{ind_2}^{pm} I_{ind}^* S^p_{pm,j} - (\mu^{pm} + \sigma) E^p_{pm,j}, \quad (4.17)
\]

\[
\frac{dI^p_{pm,j}}{dt} = \sigma E^p_{pm,j} - (\mu^{pm} + \gamma) I^p_{pm,j}, \quad (4.18)
\]
Table 4.2: Parameters stated in the IAV infection model with descriptions and values. Parameter values are taken from [22], except for $\omega$ which is taken from [4], and $\beta_{\text{ind}2}$, $\beta_{\text{ind}2}^{\text{pm}}$, $\beta_{\text{ind}2}^{\text{p}}$ and $\beta_{\text{ind}2}^{\text{gm}}$ which are an assumption.

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter</th>
<th>Value (Range)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct transmission rate for gilts, sows and growing pigs</td>
<td>$\beta_d$</td>
<td>0.285 (0.091 – 0.9) $\text{day}^{-1}$</td>
</tr>
<tr>
<td>Indirect transmission rate for gilts and sows</td>
<td>$\beta_{\text{ind}1}$</td>
<td>0.0016 $\text{day}^{-1}$ = ($\beta_d$/178)</td>
</tr>
<tr>
<td>Indirect transmission rate for gilts, sows and growing pigs</td>
<td>$\beta_{\text{ind}2}$</td>
<td>0.0057 $\text{day}^{-1}$ = ($\beta_d$/500)</td>
</tr>
<tr>
<td>Natural death rate of sows and gilts</td>
<td>$\mu$</td>
<td>0.0004 $\text{day}^{-1}$</td>
</tr>
<tr>
<td>Direct transmission rate for piglets</td>
<td>$\beta_d^p$</td>
<td>0.218 (0.147 – 0.310) $\text{day}^{-1}$</td>
</tr>
<tr>
<td>Indirect transmission rate for piglets</td>
<td>$\beta_{\text{ind}1}^p$</td>
<td>0.001 $\text{day}^{-1}$ = ($\beta_d^p$/178)</td>
</tr>
<tr>
<td>Direct transmission rate for piglets with maternal immunity</td>
<td>$\beta_d^{\text{pm}}$</td>
<td>0.014 (0.001 – 0.061) $\text{day}^{-1}$</td>
</tr>
<tr>
<td>Indirect transmission rate for piglets with maternal immunity</td>
<td>$\beta_{\text{ind}1}^{\text{pm}}$</td>
<td>0.00008 $\text{day}^{-1}$ = ($\beta_d^{\text{pm}}$/178)</td>
</tr>
<tr>
<td>Indirect transmission rate for piglets</td>
<td>$\beta_{\text{ind}2}^{\text{pm}}$</td>
<td>0.00044 $\text{day}^{-1}$ = ($\beta_d^{\text{pm}}$/500)</td>
</tr>
<tr>
<td>Indirect transmission rate for piglets with maternal immunity</td>
<td>$\beta_{\text{ind}2}^{\text{pm}}$</td>
<td>0.000028 $\text{day}^{-1}$ = ($\beta_d^{\text{pm}}$/500)</td>
</tr>
<tr>
<td>Natural death rate for piglets</td>
<td>$\mu^p$</td>
<td>0.005 $\text{day}^{-1}$</td>
</tr>
<tr>
<td>Average of latent period</td>
<td>$1/\sigma$</td>
<td>$\sigma = 1/2$ $\text{day}^{-1}$</td>
</tr>
<tr>
<td>Average of infectious period</td>
<td>$1/\gamma$</td>
<td>$\gamma = 1/5$ $\text{day}^{-1}$</td>
</tr>
<tr>
<td>Birth rate</td>
<td>$b$</td>
<td>12 births per litter per sow</td>
</tr>
<tr>
<td>Immunity waning after the first infection</td>
<td>$1/\omega$</td>
<td>$\omega = 1/180$ $\text{day}^{-1}$</td>
</tr>
<tr>
<td>Direct transmission rate for growing pigs with maternal immunity (depends on time)</td>
<td>$\beta_d^{\text{gm}} = \beta_d^{\text{gm}}(T)$</td>
<td>$\beta_d \left(1.01 - 0.96 e^{-0.06T}\right)$ $\text{day}^{-1}$</td>
</tr>
<tr>
<td>Indirect transmission rate for growing pigs with maternal immunity</td>
<td>$\beta_{\text{ind}2}^{\text{gm}}$</td>
<td>($\beta_d^{\text{gm}}$/500)</td>
</tr>
<tr>
<td>Natural death rate for growing pigs</td>
<td>$\mu^g$</td>
<td>0.00028 $\text{day}^{-1}$</td>
</tr>
</tbody>
</table>

$$\frac{dR_{j}^{\text{pm}}}{dt} = \gamma I_{j}^{\text{pm}} - (\mu^p + \omega)R_{j}^{\text{pm}},$$ \hspace{1cm} (4.19)
where \( j \in \{1, 2, 3, 4\} \), for both cases. Furthermore,

\[
I_d = \sum_{i=27}^{30} I_i + \sum_{j=1}^{4} I^p_j, \quad (4.20)
\]

\[
I^*_{\text{ind}} = \sum_{j=5}^{24} I^p_j, \quad (4.21)
\]

\[
I_{\text{ind}} = \sum_{i=1}^{26} I_i + I_{31}, \quad (4.22)
\]

and \( b_j \) is the birth rate. Since the average birth number of sows is 12 piglets between days 2 and 7, therefore,

\[
b_1(t) = \begin{cases} 
0 & \text{if } 0 \leq t < 2 \\
12/5 & \text{if } 2 \leq t \leq 7,
\end{cases}
\]

and \( b_j = 0 \ \forall j \in \{2, 3, 4\} \). Eqs. (4.16)–(4.19) represent classes of piglets who have inherited immunity from their mother. In the case of the immunity group, the direct transmission rate \( \beta_{pm} \) is assumed to be the same for all piglets since the maternal immunity started to decay at age 3 weeks [14]. See Table 4.2 for the description of the parameters that are involved in these equations.

Corresponding to this case, the growing pigs are also separated into two groups: one with maternal immunity and another without. The equations for pigs with maternal immunity are:

\[
\frac{dS^\text{pm}_j}{dt} = -\beta_{gm} I^p_d S^\text{pm}_j - \beta_{\text{ind}_2} I^p_{\text{ind}} S^\text{pm}_j - \mu^g S^\text{pm}_j, \quad (4.23)
\]

\[
\frac{dE^\text{pm}_j}{dt} = \beta_{gm} I^p_d S^\text{pm}_j + \beta_{\text{ind}_2} I^p_{\text{ind}} S^\text{pm}_j - (\mu^g + \sigma) E^\text{pm}_j, \quad (4.24)
\]
\[ \frac{dI_{j}^{pm}}{dt} = \sigma E_{j}^{pm} - (\mu^{g} + \gamma)I_{j}^{pm}, \quad (4.25) \]
\[ \frac{dR_{j}^{pm}}{dt} = \gamma I_{j}^{pm} - (\mu^{g} + \omega)R_{j}^{pm}, \quad (4.26) \]

where \( j \in \{5, \ldots, 14\} \). Furthermore,

\[ I_{d}^{p} = \sum_{j=5}^{24} I_{j}^{p}, \quad (4.27) \]
\[ I_{ind}^{p} = \sum_{i=1}^{31} I_{i} + \sum_{j=1}^{4} I_{j}^{p}. \quad (4.28) \]

The weaned piglets with maternal immunity will enter these classes at age approximately 21 days when the maternal immunity starts to wane. Furthermore, the maternal antibodies will decay to zero by age 13 weeks old [14]. Therefore to represent this waning, we consider the direct transmission rate for these ten classes of pigs is depending on time i.e. \( \beta_{d}^{gm} = \beta_{d}^{gm}(T) \), where \( T = \text{age of pig} - 21 \text{ days} \), and we set \( \beta_{ind2}^{gm} = \beta_{d}^{gm}/500 \) (see Table 4.2).

The equations of growing pigs without maternal immunity are:

\[ \frac{dS_{j}^{p}}{dt} = -\beta_{d}^{p}I_{d}^{p}S_{j}^{p} - \beta_{ind2}^{p}I_{ind}^{p}S_{j}^{p} - \mu^{g}S_{j}^{p} + \omega R_{j}^{p} + \omega R_{j}^{pm}, \quad (4.29) \]
\[ \frac{dE_{j}^{p}}{dt} = \beta_{d}I_{d}^{p}S_{j}^{p} + \beta_{ind2}^{p}I_{ind}^{p}S_{j}^{p} - (\mu^{g} + \sigma)E_{j}^{p}, \quad (4.30) \]
\[ \frac{dI_{j}^{p}}{dt} = \sigma E_{j}^{p} - (\mu^{g} + \gamma)I_{j}^{p}, \quad (4.31) \]
\[ \frac{dR_{j}^{p}}{dt} = \gamma I_{j}^{p} - (\mu^{g} + \omega)R_{j}^{p}, \quad (4.32) \]

where \( j \in \{5, \ldots, 24\} \).

Note that for the no reinfection case, we set all \( \omega = 0 \) in the above equations. All these equations are solved together using the ODE45 solver on weekly basis, thus the time span of the solver is set to \( 0 \leq t \leq 7 \). The solutions provided at the end of the time span for each
week represent the number of pigs in every particular class. These solutions are then used to apply the farm dynamics. As illustrated in Figure 4.1, gilts enter the farm each week, and some sows are culled each week. Pigs in class $i$ move to class $i + 1$, $1 \leq i \leq 30$ and pigs in class 31 move to class 11. Pigs in class $j$ move to class $j + 1$, $1 \leq i \leq 23$, and pigs in class $j = 24$ leave the farm. Once all movements of the pigs are completed, the current status of the farm represent the initial conditions for the next week. This cycle is repeated for the desired number of weeks.

4.2.3 Disease management practices

A goal of this paper is to examine the effectiveness of the two control strategies: vaccination, and reduction of indirect contact, by identifying whether these two strategies help in reducing the virus within the farm.

Vaccination strategies

To model the vaccination strategies, the susceptible animals only are moved to a recovered state where reinfection can occur. We test effectiveness of the vaccination when the disease is endemic in the farm, i.e. at some time after the system reaches the equilibrium. We assume that the effect of vaccine wanes at the same rate as natural immunity. We test the effectiveness of four common ways of vaccination: 1) vaccinating only the incoming gilts each week, 2) pre-farrow vaccination of the pregnant sows each weak so that the piglets will obtain passive maternal immunity through colostrum from their mother, 3) vaccinating the piglets at birth, and 4) mass vaccination, where all gilts, sows, piglets and growing pigs are vaccinated once at the same time.
Reduction of indirect contact

To reduce indirect contact, the farm can apply various preventive measures such as reducing the movement of people and equipment between rooms, and cleaning the boots and clothes of the farm workers regularly. These measures can help prevent disease transmission between the rooms. In the model, this scenario is achieved by reducing $\beta_{\text{ind}}$ in the whole farm to the best possible reduction case which is equal to zero.

4.2.4 Model parameters and farm assumptions

All model parameters are stated in Table 4.2. The model parameters and their values presented in [22] are used here except $\omega$ which is taken from [4]. $\beta_{\text{ind2}}$, $\beta_{\text{pmind2}}$, $\beta_{\text{pind2}}$, and $\beta_{\text{gmind2}}$ are new parameters in the SEIR model. In this farm, the room for growing pigs is likely a greater distance away from the rest of the rooms than the other rooms are from each other. For this purpose, we assume the values of these new transmission rates are scaled by 500. (e.g., $\beta_{\text{ind2}} = \beta_d/500$). Scaling these rates by 500 will allow more effective contact between the different age groups than the ones reported by Evans et al. [7] in which they scale these values by $10^{-3}$ and $10^{-4}$.

Regarding the farm population and farm dynamics, in this study, we assume the farm contains 646 sows and 50 gilts. This farm size is about the same size as average farm size in Ontario [30]. In addition, a farm with this number of sows in a sow herd is more likely to have animals of all age groups on the same premises (i.e. site or geographical location). For this sow herd with 100% farrowing rate and 2.48 annual litters per sow (based on 365 days/147 days of reproductive cycle), the weekly starting number of sows that enter the breeding room is 31 (based on total number of sows times the litters per sow per year/farrowing rate/52 weeks). Additionally, we assume that the annual rate of replacement of sows in the farm is 40%. Then the number of gilts to be introduced weekly in the farm is 5 (based on the
weekly starting number of sows times the sow replacement rate/litters per sow per year). We also assume that the population size is constant, therefore it is assumed that the number of culled sows plus the natural death each week is 5, which is equal to the number of gilts introduced weekly. These calculations are based on [30]. The weekly starting number of sows (31) will yield 372 piglets each week. Therefore, the total number of piglets in the farm is approximately 1039 to 1361 (with death rate of 3.4% each week), while the total number of growing pigs is approximately 6640.

The developed SEIR model has been numerically simulated on a farrow-to-finish system. As described above, the movement of the pigs is on a weekly basis. The farm has been initialized as a fully populated farm with all individuals in the susceptible state. For the sows, we set the susceptible individuals’ initial conditions such that every class of them is initialized according to the natural death rate of the class age.

\[ S_i(0) = S_{i-1}(0) e^{-7\mu} \quad \forall \ i \in \{12, 13, ..., 30\} \]

For week class 11, the susceptible initial condition is

\[ S_{11}(0) = S_{31}(0) e^{-7\mu} + S_{10}(0) e^{-7\mu} \]

the susceptible initial condition for week class 31 is

\[ S_{31}(0) = (S_{30}(0) e^{-7\mu}) - \text{cull} \]

where, noting that some of the sows and gilts will die each week, we have cull= total number of sows in the whole farm + number of gilts that becomes sows each week—total number of
sows, cull = \(646 \ e^{-7\mu} + 5 \ e^{-70\mu}\) − 646.

\[E_i(0) = 0, \ I_i(0) = 0 \text{ and } R_i(0) = 0 \ \forall \ i \in \{11, 12, \ldots, 31\}.\]

For piglets with immunity, we start with the initial conditions

\[(S_j^{pm}(0), E_j^{pm}(0), I_j^{pm}(0), R_j^{pm}(0)) = (0, 0, 0, 0) \text{ for all classes.}\]

As for the piglets without immunity, we set the initial condition for \(j = 1\) to

\[(S_1^p(0), E_1^p(0), I_1^p(0), R_1^p(0)) = (0, 0, 0, 0),\]

and the initial conditions for the rest can be formulated as follows:

\[S_j^p(0) = 372 \ e^{-\left(7(j-1) \ \mu_p\right)} \ \forall \ j \in \{2, 3, 4\},\]

\[E_j^p(0) = 0, \ I_j^p(0) = 0 \text{ and } R_j^p(0) = 0 \ \forall \ j \in \{2, 3, 4\}.\]

The initial conditions for the growing pigs are

\[S_j^p(0) = S_{j-1}^p(0) \ e^{-7\mu_g} \ \forall \ j \in \{5, 6, \ldots, 24\},\]

\[E_j^p(0) = 0, \ I_j^p(0) = 0 \text{ and } R_j^p(0) = 0 \ \forall \ j \in \{5, 6, \ldots, 24\}.\]

Finally the gilts are initialized with 5 susceptible for the first week class \(S_1(0) = 5\), and for
next age classes, the initial conditions are

\[ S_i(0) = S_{i-1}(0) e^{-7\mu} \quad \forall \; i \in \{2, 3, \ldots, 10\}, \]

\[ E_i(0) = 0, \; I_i(0) = 0 \quad \text{and} \quad R_i(0) = 0 \quad \forall \; i \in \{2, 3, \ldots, 10\}. \]

To start our simulation, we consider only one infected individual in the gilts entering week class 1, so the initial condition for week class one becomes

\[ (S_1(0), E_1(0), I_1(0), R_1(0)) = (4, 0, 1, 0). \]

The ordinary differential equation solver has been used to solve the system of ODEs for our model (ode45 solver using MATLAB 2016). At the end of each week, movements of the pigs are implemented as described in Figure 4.1: five susceptible gilts enter the farm and the growing pigs at the final stage leave the farm.

### 4.2.5 Sensitivity analysis

To evaluate the sensitivity of the model to the parameters, we vary the values of all direct and indirect transmission rates for the pigs in the farm. The main control transmission parameters are \( \beta_d, \beta_p, \) and \( \beta_{pm} \), and all the other transmission parameters are computed based on these control transmission parameters as shown in Table 4.2. For each control parameter, the values are varied as shown in Table 4.2. To evaluate the effect of each control parameter, we uniformly sample 100 different values from the control parameter range at equal interval. All the other parameters in the model such as \( \gamma \) and \( \sigma \) are fixed as they are determined by the disease and are not related to the farm structure or management.
strategy [22].

4.3 Results

4.3.1 Infection dynamics in the farm

For the no reinfection case, we set $\omega = 0$ in all our equations. We found that a single virus introduced to gilts spreads quickly in the farm as evident in Figure 4.2. For gilts, Figure 4.2a shows rapid reduction in the number of susceptible gilts until none of them are susceptible. After a few days, about 50% of the gilts are infectious, after which a decline in the number of infectious gilts is observed, eventually resulting in all gilts recovering and never getting infectious again. The steady state is reached at week 3 with only 3 infectious gilts (approximately 6% of the gilts), and most of the rest of the gilt population having recovered. The oscillating behavior is related to the introduction of new susceptible gilts every week. In contrast, Figure 4.2b shows that the number of infectious sows diminishes to zero by approximately 14 days after the peak.

For piglets, they become infectious immediately once the virus is spread in the farm (see Figure 4.2c). Then the number of infectious piglets starts to decline until it reaches the steady state. It is reached after approximately 4 weeks; at the steady state, approximately 25% of the piglets are infectious. The susceptible piglets are all from the piglets with immunity group. This is due to the fact that the piglets with immunity do not become infectious immediately.

For growing pigs, as evident in Figure 4.2d, the number of infectious animals decreases to zero after the initial peak. In our study, no difference is observed between the growing pigs with immunity and without immunity (not shown) since most of the incoming pigs had already recovered.
Figure 4.2: Influenza dynamics in a farrow-to-finish swine farm for (a) gilts, (b) sows, (c) piglets and (d) growing pigs. In panel (a) all 10 classes of gilts are combined into one group. In panel (b) all 21 classes of sows are combined into one group. In panel (c) all 4 classes of piglets are combined into one group. In panel (d) all 20 classes of growing pigs are combined into one group.

4.3.2 Reinfection

In the reinfection scenario, our model reflects the condition where the individuals can re-enter the susceptible state once recovered. The animals in the susceptible state include some that have moved from the previous room and some that have moved from the recovered state back to the susceptible state. This movement from recovered to susceptible results in a slight increase in the number of infectious animals in the whole farm compared to that number in
the no reinfection scenario (see Figure 4.3).

Figure 4.3: Infectious levels for (a) gilts, (b) sows, (c) piglets and (d) growing pigs under the no reinfection and reinfection scenarios.

4.3.3 Testing vaccination strategies

For testing the vaccination strategies, we model the case when the vaccination is administered after the disease is already present on the farm. In our model, vaccinating incoming gilts results in a reduction in the number of infectious gilts, as illustrated in Figure 4.4. Prefarrow vaccination, as implemented in our model, does not show any change in the number of infectious animals in the piglets. Vaccinating the piglets at the end of week class 1 also
shows no significant effect, as all the piglets without immunity become infectious as soon as they are born (see Fig 4.5). For the mass vaccination, only the number of infectious gilts is decreased during the vaccination week as can be seen in Figure 4.6, the gilts return to the regular steady state after the vaccination week.

Figure 4.4: The effect of vaccinating incoming gilts each week starting at the end of week 8. The panel shows a reduction in the number of infectious gilts after the vaccination.

4.3.4 Reduction of indirect contact

To test reduction of indirect contact, we set $\beta_{ind}$ to zero in the model. The results show the delay of the spread of the disease in the farm. There is no infection in the growing pigs until 25 days when the infection increases sharply to a spike. This is because no infectious pigs enter the growing stage until 25 days (see Fig 4.7).
Figure 4.5: The panel shows the piglets at week class one where most of them are infectious once they are born in day two.

Figure 4.6: Gilts mass vaccination at week 8.
4.3.5 Sensitivity analysis

Varying all the control parameters, i.e., the direct and indirect transmission rates for pigs, has no significant effect on the model behavior, and the number of infectious animals associated with all of these parameters is almost identical. However, we notice that when we change the direct transmission rate for piglets with immunity $\beta_{pm}^{d}$, many more piglets are susceptible. Figure 4.8 shows the results of 100 uniformly sampled values in the range of $(0.001 - 0.061)$. This larger number of susceptible animals is only noticed when $\beta_{pm}^{d}$ is very small. As a result of the facts that the virus transmission is small, and the piglets are already immune, a larger portion of the newborn piglets stay in the susceptible state for a longer time.

4.4 Discussion

Several studies reported high prevalence of the IAV in pigs in different regions of the world, such as Europe [27] East Asia [25] and North America[20, 16]. Specifically, it is observed
Figure 4.8: Model results by uniformly varying the range of direct transmission rate for piglets with immunity parameter. Most susceptible animals were noticed when $\beta_{pm}$ is very small= 0.001. The blue line represents susceptible animals.

by Poljak et al.[20, 19] that pigs in Ontario are positive to IAV virus and the prevalence is increasing over time. Our study illustrates the transmission of IAV and the reinfection within a farrow-to-finish swine farm in Ontario. A mathematical SEIR model presented in [22] is extended and implemented. Simulation results indicate that in a fully populated farm an IAV outbreak through the farm causes the persistence of the infection within the piglet and gilt populations. The disease was observed at a high level in the piglets even though they had maternal immunity from immune sows. The virus persisted in the piglet population due to the continuous supply of new piglets being born each week, while in the gilt population was due to weekly incoming susceptible gilts. Moreover, as a result of the incoming recovered individuals each week, the disease died out among the sows and growing pigs in the infection scenario but not in the reinfection scenario.

Our finding is in agreement with other experimental studies where the major IAV infec-
tion takes place in the piglets [25, 13]. Furthermore, the same observation was also observed at a breeding farm in the modelling studies performed by Reynolds et al. [22] and White et al. [31]. We conclude that the persistence of the virus in the farrow-to-finish farm is due to the supply of new susceptible pigs. This observation has also been reported by [18], however, we are utilizing different methods and assumptions. Moreover, our results agree with the empirical results of [6], in which they also find out that the progress of the influenza outbreak through the farm is within three weeks.

Furthermore, we also studied the reinfection scenario, in which the recovered pigs could be susceptible to receiving the virus once more. We studied the effect of $\omega$ by changing its value from $1/50$ to $1/200$. We noticed that the number of infectious pigs decreases as this value decreases, but the virus is still in circulation even though the duration of immunity is long ($\omega = 1/200$). Results revealed that the disease was endemic in the entire farm, and unlike the typical infection scenario, the virus persisted among the growing pigs and sows.

The fact that widespread IAV infection was confirmed and the disease was maintained in the farm, raises the question of what are the efficient strategies to control the spread of the disease. Vaccination is the most common strategy that is used to minimize the transmission of a disease. Another strategy is the reduction of contact within the population. Several studies showed that vaccination can reduce the transmission of IAV virus but it does not completely eliminate it [28, 23, 31].

Clinically, veterinarians are using vaccination of gilts to help control influenza circulation in a herd. Although this could result in abortion, it is a relatively common practice. The second vaccination strategy is to vaccinate breeding sows before lactation. With this strategy, farmers are trying to maximize maternal immunity of newly born piglets, and it is done continuously in a herd. Another vaccination strategy is mass vaccination, where all sows, or all sows and piglets, are vaccinated at one time. It is not done frequently, but the goal is to eliminate infection from a herd by creating a high level of immunity in all animals. This
strategy is only applied in breeding herds. The reducing contact strategy, which is called McRebel strategy, is applied to any infectious disease of pigs. The aim of this strategy is to reduce contact between animals and prevent infection. We refer the reader to [9] for further details about these strategies.

In this study, we modify the dynamics of the model to apply these strategies. We investigate the effect of these strategies, and particularly on a farrow-to-finish herd, which is unique and challenging because animals of different ages are at the same location. It is easy to eliminate infection from farms where animals are segregated by age, but in farrow-to-finish facilities, this is a real challenge. The decision whether or not to apply these strategies will be made depending on their cost and their effectiveness, which is the reason for this study.

Based on our model, these vaccination strategies are incapable of reducing the influenza infection on the whole farm. Especially among the piglets, the infection level remains high, as they become infected almost immediately after they are born. In the continuous case, when the vaccination is applied every week, and since naturally all incoming gilts are susceptible, vaccinating of these gilts is effective in the reduction of the infectious gilts. In contrast, the pre-farrow vaccination does not show any change in the number of infectious individuals in the farm. This is because there are almost no susceptible sows to be vaccinated, as most of them are already in the recovered group. Mass vaccination (single discrete case) only reduces the infectious level in the gilts for one week, then it returns back to endemic equilibrium. This is expected since new gilts are entering the farm every week and vaccination has not been applied to the new comers. Furthermore, mass vaccination has no effect to the rest of the pigs, since at any given time most of the animals are already in the recovered state.

Reynolds et al. [22] also suggested using vaccination strategies to reduce the influenza transmission in a breeding and wean-to-finish farm. They found that these strategies are ineffective in reducing the virus in the breeding herd, but caused a small reduction in infectious pigs in the wean-to-finish farm. They modeled these strategies by using the transmission
parameter $\beta$. In our study, this scenario is evaluated by moving susceptible animals to a recovered state where reinfection can occur. In the lack of empirical data about the immunity time of the vaccine, we assume that the immunity time for the vaccine is the same as the natural immunity, which is 180 days.

Likewise, regarding the strategy of reduction in indirect contact, our model indicates that this strategy is also ineffective in reducing the level of infection in the farm. It resulted in only the delay of the spread of the disease in the farm. The disease does not die out due to the weekly continuous movement of the pigs through the farm. It is also observed that there is no delay or change among the gilts (see Figure 4.2a) since the disease starts in the gilts room.

Sensitivity analysis is implemented to test the effect of variation in the direct and indirect transmission rates on the farm. Despite the variation of these parameters the IAV is still persistent in the farm, and particularly between piglets.

A major limitation of this study is that there are no empirical data for many parameters that affect the behavior of the model such as the vaccine immunity time and the indirect transmission rates. However, the exact same model can be applied once such data becomes available. Another limitation of this study is that, for simplicity, we only focus on a single influenza strain and we are not aware of any modeling study of pig farms considering multiple influenza strains.

In conclusion, the dynamics of IAV virus is not fully understood and the disease is maintained in the farm specifically in the piglet population, which is a serious concern for public health. The effectiveness of vaccination strategies is still questionable. Reducing the indirect contact results in delaying the disease, however, it is also not able to reduce the virus to an acceptable level. A high level of infection in the animals could cause high risks to humans and other species. Therefore, public awareness about this virus should be increased. This requires better understanding of how other factors, such as farm management practices
and the interaction of the farm workers with the pigs, can contribute to the persistence of the disease in the swine. We argue that, by fully understanding the dynamics of the IAV virus, most of the limitations can be successfully addressed and resolved. Therefore, more comprehensive experimental studies are required to cover this gap.
Bibliography


Chapter 5

Conclusion and future work

5.1 Conclusion

This thesis focused on the applications of finite dimensional PDS in two different areas, market equilibrium problems and $N$-player Nash games, as well as focusing on the classical dynamics of the SEIR model. We completed that in three different Chapters; specifically, they were about computing equilibrium points of PDS, and showing how these equilibria were relevant to a market equilibrium problem, particular types of Nash games. PDS was not necessary to study the SEIR model, and, thus classical dynamical systems are used to study this problem.

In Chapter 2, we examined the possible structural changes of the behavior of the market equilibrium and disequilibrium problems. Particularly, we studied the impact of changing the supply price, demand price and the cost functions. We studied the bifurcation problem (i.e., qualitative changes in equilibrium states) as a parametrized VI problem. With both the cost functions and price functions, the effect of the variations of parameters is seen on the number of equilibria occurring at specific values. The equilibrium states were obtained using trajectories of the associated projected dynamics. The empirical results on two examples
showed that bifurcations occur in such systems.

In Chapter 3, we examined cases where multiplayer Nash games with parametrized payoffs exhibit changes in the number of Nash equilibria, depending on given parameter values. We parametrized N-player Nash games by introducing the parameter in the player’s payoffs. We introduced some theoretical results using the VI problem to prove the existence of unique Nash equilibria for N-player nonlinear games. When the conditions of the Theorem are not satisfied, we are able to identify cases where nonunique equilibria can arise. Additionally, we introduced a computational method that relies on PDS to identify parameter values giving rise to multiple Nash equilibria in a parameterized Nash game. Due to the presence of the parameter, the game’s set of Nash equilibria exhibited change in its numbers as the parameter takes on values across a given interval of interest. We showed also that our theoretical results can be tested for and used in concrete examples; however, due to the complexity of the game’s payoffs, sometimes the easier way to tackle the issue is computational in nature.

Chapter 4 was cooperation work with researchers in the Department of Population Medicine-Guelph. We modeled the infection and reinfection of IAV within a farrow-to-finish swine farm in Ontario. A mathematical SEIR model, presented in [3] was extended and implemented. The model was structured to include the weekly progress of all pig growth stages including gilts, breeding sows, farrowing sows, and growing pigs. In addition, we investigated the effectiveness of two control strategies: vaccination and reduction of indirect contact.

Simulation results indicated that in a fully populated farm, an IAV outbreak through the farm caused persistence of the infection within the piglet and gilt populations. The virus persisted in the piglet population due to the continuous supply of new piglets being born each week, while in the gilt population, it was due to weekly incoming susceptible gilts. Moreover, as a result of the incoming recovered individuals each week, the disease died out among the sows and growing pigs. For the reinfection scenario, results revealed that the
disease was endemic in the entire farm, and unlike in the typical infection scenario, the virus persisted among the growing pigs and sows.

Based on our model, vaccination strategies were incapable of reducing the influenza infection on the whole farm. Especially among the piglets, the infection level remained high, as they became infected almost immediately after they were born. Regarding the strategy of reduction in indirect contact, our model indicated that this strategy was also ineffective in reducing the level of infection in the farm. It resulted in only the delay of the spread of the disease in the farm. The disease did not die out due to the weekly continuous movement of the pigs through the farm. Sensitivity analysis was implemented to test the effect of the variation of the direct and indirect transmission rates in the farm. Despite the variation of these parameters, the IAV still persisted in the farm, particularly, within the piglets population.

As shown throughout this study, the projection mechanism is meaningful and leads to interesting dynamics in the cases of market equilibrium problems and N-player Nash games, while it is not relevant in the case of the SEIR model. The numerical and computational exploration of the problems is needed and helpful to analyze the results, especially when the theoretical results do not give the complete picture for the problems such as market equilibrium problems and N-player Nash games. In addition, the numerical and computational exploration is also needed for very complex problems, as seen with the SEIR model. Furthermore, as we have shown throughout this work, the multiple equilibria are important to see how the behavior of the systems change under perturbations represented by parameters.

5.2 Future work

There are many other interesting equilibrium problems, such as traffic network and transportation network equilibrium problems, that are reformulated as VI and PDS. One of our
study objectives that we would like to tackle in the near future is applying our work to these problems, especially performing bifurcation analysis for these systems using both our theoretical and numerical results. We have been successful in obtaining some theoretical results for the games problem, which could also be applied to the market equilibrium problem. Therefore, it will be interesting as a future study to further explore the theoretical results for this problem. Although there is an attempt from Khoi et al., [2] to provide a general framework to perform bifurcation analysis directly using the VI problem, they have constrained the framework with strong assumptions, making it inapplicable to many problems and certainly our proposed ones. It will be interesting to see a more comprehensive work similar to Khoi and colleagues’ [2] with weaker assumptions to make the bifurcation analysis more widely applicable. Furthermore, this may lead to the presence of periodic solutions similar to the method used in Cojocaru’s study [1] using PDS.

Our work using a SEIR model for IAV showed that the disease persists in the farm despite the various control strategies we attempted during this study. Therefore, further work is required to establish an effective control strategy. Establishing new control strategies needs better understanding of how other factors, such as farm management practices and the interaction of the farm workers with the pigs, can contribute to the persistence of the disease in swine. This can be achieved by a comprehensive experimental study in different farms and environments.
Bibliography

