ABSTRACT

EXPLORATION OF A NON-COOPERATIVE DUAL MANAGEMENT FISHERY USING A SOCIO-ECOLOGICAL MODEL

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University of Guelph, 2018

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We investigate the effect that two distinctly different fisheries’ management strategies have on a single population of Lake Whitefish in Lake Huron, Ontario, Canada. A system of three differential equations, with stochasticity, are presented. They are an extension from a single management system into a dual management system. We analyzed the existence, and, stability of solutions to this system of differential equations. Numerical simulations are given for a case study between the Saugeen-Ojibway, and, the Ontario Ministry of Natural Resources and Forestry. Results indicate that it is possible for coexistence between these two fisheries, as well as the Lake Whitefish, over a time span of 100 years.
To my loving parents, Laurie and Jeff, who have always been there for me, and to
Wrappers, the cat.
ACKNOWLEDGEMENTS

Thank you to Dr. Dan Gillis for allowing me to work under him, and for sticking it through with me for this incredibly long journey.

Thank you to Dr. Herb Kunze and Dr. Kimberly Levere whose edits helped direct this flightless bird back into the skies.

We acknowledge that the University of Guelph resides on the ancestral lands of the Attawandaron people and the treaty lands and territory of the Mississaugas of the Credit. We recognize the significance of the Dish with One Spoon Covenant to this land and offer our respect to our Anishinaabe, Haudenosaunee and Métis neighbours as we strive to strengthen our relationships with them.

We acknowledge that our thesis focus, (Lake Huron), resides on the ancestral lands of the Anishinabek, Odawa, Huron-Wendat, Haudenosauneega Confederacy, and, the Métis.
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Chapter 1

Introduction

Presented in this thesis is a blend of complex social, economic, and resource dynamics. It consists of an emergent problem on the Canadian portion of Lake Huron, Ontario. In this thesis we find conflicting opinions and stances on the strategic approach to managing the fish stock of Lake Whitefish. We will introduce the reader to the basic outline of these managements in this chapter. In Chapter 2 we will provide our reader with the evolution of the model we have chosen. Chapter 3 will present the base model in whole, as well as the extensions and concessions made that feed directly into our analysis in Chapter 4. Chapter 5 will provide a brief segment on numerical simulations directed by our analysis. Finally, Chapter 6 will conclude results, and outline potential future directions for our work.

Lake Whitefish or *Coregonus clupeaformis* are part of the focus of our research. They are the resource that we consider throughout the rest of the thesis. In this
thesis we encounter two fisheries, the Saugeen-Ojibway Nation (known as the SON), and, the Ontario Ministry of Natural Resources (otherwise known as the OMNRF). The SON consider Lake Whitefish as being a staple to their livelihood as well as an important link to their past. The OMNRF consider Lake Whitefish a great sport fish and a solid economic stock (OMNRF 2014, 2015b, 2016, 2017). In either fishery, Lake Whitefish are just one of the many available commercial and recreational fish stocks, but, because of their traditional and commercial links to both sides of this management situation, we have considered them a primary interest. The management of this fishery in correspondence to both the SON and OMNRF is the major concern of this thesis, we seek to understand how the lack of cooperation between both of these managements can affect the population of Lake Whitefish in Lake Huron.

Total management of Lake Whitefish in Lake Huron has gone in-part to both of these parties, the SON, and OMNRF (SON and MNR 2013). This dual management began in March of 2013 with the signing of the Substantive Commercial Fishing Agreement (SON and MNR 2013). Prior to this signing, and after its signing there has been quite a bit of tension between both sides (Lee and Kahgee 2013; OMNRF 2005a; Walker and Miller 2013).

Estimates of the zone coverage of SON fishing waters can be seen in Figure 1.1 as well as the OMNRF fishing waters. Specifically, the zones defined by the black lines and the red (dark) overlay are considered the SON waters. Conversely, the zones
depicted by the white lines and yellow (light) overlay are considered the OMNRF waters. Everything to the left of the middle lines in the lake, that is centered at the largest portion of the lake, is considered to be owned and operated by the United States of America, and their fisheries (SON 2016). Neither side may fish within
The SON have thought that they employ fishing strategies that consider and enact long term sustainability of Lake Whitefish (OMNRF 2005a; SON 2016). In contrast, the OMNRF, via the Great Lakes Fishery Commission have a very limited public presentation of how management of their lakes, including Huron, operates. If we look at Table 1.1 we see a quick management summary of both the SON and OMNRF. The largest claims to note are the adaptation method, as well as the management method for each of these governing/managing bodies. In the instance of the SON (and, as we will see in Chapter 3), they are harvest reactionary as the SON can and will close their waters in the same year that they reach harvest quota, or in their case, a sustainable quota derived from total allowable catch (known as TAC) based on maximum sustainable yield. The OMNRF are labeled as semi-reactionary, as they will alter a quota based upon previous yearly harvests, and, this is only if quota has not been achieved in the prior year. In this way, the OMNRF can accidentally over

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Table 1.1: A quick summary of each management: SON and OMNRF based on information from (OMNRF 2005a), (OMNRF 2005b), (SON 2016), (Hartford et al. 2007), and (OMNRF 2007).
fish, as they do not react during the year, but only rather after the fishing season has concluded. Although it might seem like there is a stark contrast between these two conflicting communities, there has been next to zero attempt to understand how their management differences affect the abundance of the Lake Whitefish. The following sections will outline how we came to decide on this system and the enigma surrounding its dual non-cooperative management situation. The lack of specific research given necessitates additional mathematics instead of a direct application of any pre-existing models. Following this, we will present the math required to tackle the questions at hand and provide an analysis of the results.

This was a brief introduction to the problem we consider for this thesis. There is a rich history between these two managements however, and their developments help frame this situation with better context. If the reader requires greater information about the historical management of these two fisheries, as well as specifics on Lake Whitefish they should consult their respective sections in Appendix A.
Chapter 2

Literature Review

Sustainability, and its applications have at the least started in the 1930’s and con-
tinued on to our nearly present year, albeit in different forms (Allee 1931; Yodzis et al. 2016). It is highly possible that research has existed much earlier, however, this range of 1930 and onward is where we focus our attention. We address a new concern in regards to the sustainability of fish as resource. Our focus for this thesis is primarily on long term commercial gain and food availability; specifically concerning Lake Whitefish in the context of non-cooperation between two fisheries with respect to this fish. Despite being a seemingly narrow topic, there is still an immense measure of research and discussion that focuses heavily on these items. Overall, the end goal is the introduction of a novel addition to a combined social and economic model. We focused on topics surrounding evolutionary game based models, sustainability philosophy, stock-effort models, and metapopulation models. Although these afore-
mentioned categories also cover an incredible range of research, the intent and focus is within the domain of sustainable fishery management.

2.1 Growth of a Resource

In the search for models that may or may not frame this unique management situation, one must start with a few building blocks. Our problem involves the evolution of a fish stock, so it is apparent that population dynamics be considered. In our case, we began by narrowing our search effort to the population dynamics of fish stocks. We begin with the growth of a resource, taking a look at Figure 2.1 we see the most basic consideration for this concept.

![Figure 2.1: Basic depiction of the growth of a resource, internal growth can be represented by any instantaneous function that governs growth. The fish is a Lake Whitefish.](image)

This figure provides an image of a Lake Whitefish, which, conceptually exists as a resource that can grow internally. This growth can be represented in a basic manner by

\[ \frac{dx}{dt} = rx \left( 1 - \frac{x}{k} \right) \]  

(2.1)

which is Verhulst’s equation, otherwise known as the logistic equation. The parameters are then, \( x \) as a resource, \( r \) is a growth rate, and, \( k \) is a carrying capacity (Schaefer [1991]). The growth rate typically encompasses natural birth, and, mortality while carrying capacity abstractly represents the the total number of animals that might be supported by the amount of available items (e.g. food, oxygen, breeding grounds etc.) that a resource can use.

2.2 Harvest of a Resource

Natural resources, or more appropriately “the commons”, are the primary source of discussion surrounding the historical development of sustainability science, in regards to the fishery. An influential paper by H. S. Gordon (1954), entitled “The Economic Theory of a Common-Property Resource: The Fishery” gave researchers a culmination of past thoughts surrounding humanity’s view of fish as a resource. This idea encapsulated the harvest of fish, and, Figure 2.2 gives an abstract representation of this building block. Our reader can think of management as currently as representing a group fishers.
Including fishers as a mathematical harvesting concept was an idea that spawned from World Wars I and II. Prior to both of the World Wars prominent researchers believed that fish were inexhaustible as a resource (Gordon, 1954). An idea amongst the populace held that fishers had negligible influence on populations of fish around the world (Gordon, 1954). However, as (Gordon, 1954) highlighted, during the World Wars, fishing decreased severely, and as a result, people saw an incredible increase in the growth of their fisheries post-war. Clearly humans had an impact on their fisheries, and the fish that reside within. From this observation (Gordon, 1954) outlined the economics of a fishery, and with it created the precursor to what would be referred to as the “Gordon-Scheafer model”, (G-S). M.B. Schaefer (1991) linked the
basic economics that Gordon (1954) provided and coalesced it into a workable fishery model under his own assumptions. The G-S model is based on the Verhulst equation (Schaefer, 1991). Gordon (1954) and Schaefer (1991) would combine this model with a “harvest” term, derived from their ideas of a fisheries economics wherein a fish population grows logistically, and is harvested at a desired rate. A generalized view of this model is seen as:

\[
\frac{dx}{dt} = g(x) - h(x),
\]

where \( g(x) \) is the growth term, and \( h(x) \) is the harvest term and where \( x \) is the total fish population. Their analysis brought about the usage of a now commonly known term: maximum sustainable catch (Schaefer, 1991). Going forward, it will be referred to as maximum sustainable yield or MSY. MSY is effectively the maximum yield or catch that can be obtained by a fisher onto a fish stock for an infinite time period without depleting the stock. We can then see a quick depiction of Equation 2.1 for it’s dynamics vs growth in Figure 2.3 where, the red dot represents maximum sustainable yield (known as MSY).

Depending on how a model is framed, MSY can be thought of as either a group of fishers contributing to the maximum yield or a set of individuals whose contributions also contribute to this maximum through their summation. In this way, one can think of MSY as the maximum possible population level that is viable to harvest prior to
any decrease in the growth of a population. MSY occurs when $x = k/2$, i.e. MSY occurs when the total population of fish in the lake at any time $t$ is at exactly half the carrying capacity of the lake. A fishery can then harvest at a yield less than or equal to the estimated growth rate of the fish stock (Schaefer 1991). This proceeds as the basis for the next step towards a modern model. For our readers interest, additional information on the G-S model, as well as MSY is given in Appendix B Section B.1

### 2.3 Social System

The introduction of harvest, and the idea of the fisher brought about the inclusion of other concepts from other domains of research. Social/political science had been developing interesting theories parallel to fishery science. Treatment of the human
side of fishery management was not given the attention it deserved. Luckily, past research and current have only borrowed some basic tenets of social science. Inclusion of social norms, governance, cooperation, defection, property rights, utility, profit, loss, opportunity cost, imitation dynamics are just some of the social and economic components that have been borrowed and developed (Atzenhoffer, 2011; Bauch and Bhattacharyya, 2012; Brandt and Merico, 2013; McClanahan et al., 2016; Richter and Dakos, 2015; Sethi and Somanathan, 1996; Sugiarto et al., 2015; Sumaila, 2013a,b; Yodzis et al., 2016). However, it is important to understand what degree of each of these components are put to use. Some concepts have depth appropriate for their own field of research, and as such we must frame our research to the boundaries they have currently set. We will present the idea of cooperation and defection below coupled with their dynamics governed by temptation and peer pressure. Prior to describing these processes we direct attention to Figure 2.4 where a visual depiction of how these social concepts fit into the developing model.
2.3.1 Cooperators and Defectors

Abstracted, cooperators and defectors are fairly simple to understand. Cooperators are individuals who act together to accomplish a task. Defectors are those that oppose the cooperators’ stance. That is, they do not work together to accomplish a task. However, it does not mean that this task is not completed, it just requires that they do so in a way that is non-cooperative. The lack of cooperation can cause that it causes distress between the cooperators and defectors.
Cooperators

Within the context of fisheries management, cooperators and defectors are defined and with fisheries and their relationship to the fishery in mind (Richter and Dakos 2015). A cooperator is typically an individual who will fish sustainably with respect to other cooperators (Richter and Dakos 2015; Sethi and Somanathan 1996). That is, they fish to the maximum possible yield that the fish stock can sustain alongside other cooperators that are also fishing (Richter and Dakos 2015). This allows for the cooperators to fish for long term monetary gain (Richter and Dakos 2015). Not all papers have considered this idea; in some cases, cooperators are excluded and models have considered completely non-cooperative situations (Sumaila 2013a,b).

Defectors

Defectors, much like their abstract definition, in the fisheries context do not follow the same pathway as the cooperators. They instead fish selfishly, fishing to the maximum short term monetary benefit (Richter and Dakos 2015). That is, they fish as much as possible without considering the long term sustainability of the fish stock (Richter and Dakos 2015). In this thesis, we carry over this idea with the inclusion of social norms as the immediate governing body of the cooperators and defectors.
2.3.2 Temptation and Peer Pressure

These two groups of individuals are governed, and, enact social norms among each-other as well. Due to the breadth of those concepts (governance and social norms), we direct the reader to Appendix B Section B.3.2 for an in-depth discussion of each with respect to our research. From here we can introduce the concepts of temptation and peer pressure, both of which are direct results of social norms research (Chung and Rimal, 2016). These two concepts are briefly discussed below, to provide added context to this situation.

**Temptation**

In social settings it is possible that a person might switch to being against the common norm. This invokes an instance of defection. Defection can come about in different ways, however, in more recent fishery studies, it has been formulated around temptation to defect (Richter and Dakos, 2015; Richter and Grasman, 2013). Temptation to defect is when an individual, given their current circumstances, deems it beneficial, usually monetarily, to go against the norm of the common group (Richter and Dakos, 2015; Richter and Grasman, 2013).

**Social Pressure**

Conversely, individuals who are currently against the common social norm, can be brought back. In this sense, the ones who are no longer following the social
norm, can sustain peer or social pressure to return \cite{richter2015 richter2013 richter dako2015 richter2013}. To feel or become under the influence of social pressure, an individual must be exposed to those who are currently following the common social norm \cite{chung2016 richter dako2015 richter dako2015 richter dako2013}. 

2.4 Economics

Sections 2.1 and 2.2 covered resource dynamics, while Section 2.3 outlined the social dynamics. We can then move onto the final piece which ties these concepts together, economics. In reference to fish as a resource, it is often associated with the economics of their specific market, or, at least this was conceptualized by \cite{gordon1954}. Figure 2.5 demonstrates the expanded system with the inclusion of cooperators, defectors, (under a management), and their interaction with the resource through harvest, and, profit.
In addition to profit, we also explore another key element of the economics of a fishery, alternative wage sectors. A relevant concept to this section, The Tragedy of the Commons, is outlined in Appendix B section B.4. Its discussion is lengthy, but its philosophy is important to the inclusion of economics in fishery research.

2.4.1 Profit

The tragedy of the commons captures a few distinct ideas that are present in great detail within more current fisheries research (Hardin, 1968). Profit can be given generally as:

$$\pi(t, x(t)) = S(x, t) - C(x, t),$$
where $\pi(x,t)$ is profit, $S(x,t)$ is sales and $C(x,t)$ is cost. Both sales and cost can be dependent on the commodity, $x$, and on time, $t$, depending on how the author frames these. Note that, $S(x,t) \geq 0$ and $C(x,t) \geq 0$, and, in addition, $\pi \geq 0$ when $S(x,t) \geq C(x,t)$ or $\pi < 0$ when $S(x,t) < C(x,t)$. Seeing this equation, it is obvious that profit encapsulates the total net gain or loss associated with a commodity.

### 2.4.2 Alternative Revenue

A particular concept that is directly relevant to our research is the concept of alternative revenue streams. In general, this can be simply presented as:

$$\pi(t,x(t)) = S(x,t) - C(x,t) + S_{alt}(x,t),$$ \hspace{1cm} (2.2)

where, in Equation 2.2 $S_{alt}(x,t)$ is a revenue associated with some alternative “stream” where a stream is a market that is not your primary market. In the form of Equation 2.2 it might be concluded that it just be added to the sales. Because of this, alternative revenue streams are usually bounded in such a way as to provide an interesting form of complexity. In this way, one may think of this alternative revenue stream as a revenue that could be gained given that we invest some amount of time or effort into selling something else. We will see in Chapter 3 how this type of assumption alters how profits behave.
2.5 Socio-Ecological Models

Evolutionary Game Theory (EGT) evaluates competitions between populations with respect to their strategic approaches over a time period (Atzenhoffer 2011; Bauch and Bhattacharyya 2012; Sethi and Somanathan 1996; Sumaila 2013a,b). Typically, a population or set of populations are defined, this can be a group of fishers, for example. A set of strategies are given to these population groups. Time evolution occurs, and the populations compete using their strategies. Although EGT is an interesting subject to delve into, due to its incredibly similar nature to socio-ecological systems (otherwise known as SES, and, often referred to as SES models), we will spend time on SES models instead of focusing on any finite differences between them. Appendix B Section B.5 outlines an important EGT model, and discusses its importance as well as its differences as an immediate precursor to SES models.

With the development of EGT in regards to the fishery, and with the incredibly diverse development of resource dynamics there have been shifts in the way researchers perceive and model situations such as ours. Socio-ecological models are a more modern-day introduction to questions surrounding sustainability science. They typically seek to understand how a coupled system behaves. In the case of the fishery, social and economic components combine into the social segment of an SES, while the evolution of the resource are governed by the ecological segment.
Richter and Dakos (2015) have presented an SES model wherein they coupled fish, as an uncertain resource, with Allee effects. Alongside these resource dynamics, they have included a social component consisting of a differential equation attempting to cover the ideas of temptation and defection. Mathematically Richter and Dakos (2015) fish dynamics are given as

$$\frac{dx(t)}{dt} = rx(t)(x(t) - x_{\text{crit}}) \left(1 - \frac{x(t)}{k}\right) dt - H(t)dt + \sigma_x dw,$$

where $r$ is given as the intrinsic growth rate, $k$ is the carry capacity, $H(t)$ is harvest at time $t$, $x(t)$ is fish population at time $t$, and $x_{\text{crit}}$ is a critical threshold fish population level. We see present within this equation the usage of both the Allee effects, (see Appendix B Section B.3), as well as logistic growth. Finally $\sigma_x dw$ is a stochastic effect. We note that $k \geq 0$, and $x_{\text{crit}} \geq 0$. Much like the historical models, this SES depends on a scheme surrounding how fishers, as humans, interact. Presented is an interesting temptation and defection dynamic not previously used before (Richter and Dakos 2015). As pointed out by Richter and Dakos (2015), models that utilize items such as imitation dynamics or replicator dynamics, as was seen in Sethi and Somanathan (1996), are useful for determining the emergence of new or dominating strategies. That is to say, if one wanted to see which harvest strategy, for example, would emerge out of a set system, these dynamics would be applicable. However, in situations where one already knows which strategies are being used, something else
might be more appropriate (Richter and Dakos, 2015). Richter and Dakos (2015) cooperator-defector dynamics are given as:

$$\frac{dC(t)}{dt} = \frac{\gamma C(t)D(t)}{n} - \beta C(t) \left(1 - \frac{\pi_C(t)}{\pi_D(t)}\right),$$  \hspace{1cm} (2.4)$$

where $C(t)$ is the total cooperator population at time $t$, $D(t)$ is the total defector population at time $t$, $\gamma \geq 0$ is the strength of peer pressure, $n = C(t) + D(t) \geq 0$ is total population of fishers, $\beta \geq 0$ is the strength of temptation, $\pi_C(t) \geq 0$ is the cooperator profits at time $t$, and $\pi_D(t) \geq 0$ is the defector profits at time $t$.

We see that if $\frac{\gamma C(t)D(t)}{n} > \beta C(t)(1 - \frac{\pi_C(t)}{\pi_D(t)})$ then the cooperators see growth, and, if $\frac{\gamma C(t)D(t)}{n} < \beta C(t)(1 - \frac{\pi_C(t)}{\pi_D(t)})$ then the cooperator population will see decline.

Contained in Chapter 3 will be a full presentation of these equations, but for the sake of the reader, they are intially presented here as a contextual introduction. The case analyzed by Richter and Dakos (2015) deals with a single fishing entity consisting of one set of cooperators and one set of defectors. Instead, we have a set of fishing entities the SON and the OMNRF; also, each with their own management strategies in regards to their fishing zones. We must then consider a set of two cooperators and two defectors with a governance interaction equation. Chapter 1 introduced these managements in detail. Figure 2.6 finalizes the conceptualization of this non-cooperating dual management fishery system, encompassing all of the concepts previously mentioned in this chapter, where, management S is the SON,
and management O is the OMNRF.

Figure 2.6: Conceptual depiction of a non-cooperating dual management fishery, S represents the SON, and O represents the OMNRF.

In addition, Figure 2.6 depicts a red arrow between our two managements. The red arrow signifies that there is zero interaction between these two systems, then, the only way that these two groups can govern their populations of fishers is through direct knowledge of the resource alone. We present in Chapter 3 a socio-ecological approach to a system of two fishing managerial bodies operating with distinctly different management strategies on the same body of water. Our model will be defined in accordance with the base model, extended, and finally analyzed.
Chapter 3

Methods

3.1 Original Model

We introduced the original model in Section 2.5, that is, the two differential equation system governed by Equations 2.3 and 2.4 or, topically, an SES system. An SES system usually, as far as we know it, consists of at least a resource and a social component. In the case of this SES model, it has exactly these two components. It is assumed in this model that the animal, or stock, that is evolving follows a traditional growth model, like Equation 2.1. Chapter 2 talked heavily about the development of this assumption and the dynamics of this growth equation. It is important to understand, however, that this is the basis for the additions that Richter and Dakos (2015) include. We present the overarching concept in Figure 3.1, which displays the general flow of the systems involved in the original model. Also given with this diagram is the
Natural Growth with Fluctuations.

\[ \frac{dC}{dt} = \gamma\pi \frac{n}{C} - \beta C (1 - \pi_C \pi_D) \]

\[ n := \text{Total Fisher Population} \]
\[ x(t) := \text{Resource} \]
\[ C := \text{Cooperators} \]
\[ D := \text{Defectors} \]
\[ \beta := \text{Temptation Coefficient} \]
\[ \gamma := \text{Peer Pressure Coefficient} \]
\[ H := \text{Total Harvest} = \sum_{i=1}^{n} h_i(t) = h^C + h^D \]
\[ h^C := \text{Total Cooperator Harvest} = qx(t)^\alpha e^C(t) \]
\[ h^D := \text{Total Defector Harvest} = qx(t)^\alpha e^D(t) \]
\[ e^C := \text{Cooperator Effort} = \max(0, \min(\hat{e}, R)) \]
\[ e^D := \text{Defector Effort} = \begin{cases} \hat{e} & \text{if } x^\alpha(t) \geq \frac{(w + m)}{P(t)q} \\ 0 & \text{if } x^\alpha(t) < \frac{(w + m)}{P(t)q} \end{cases} \]
\[ \pi_C := \text{Cooperator Profit} = P(t)qx^\alpha e^C - we^C + m(\hat{e} - e^C) \]
\[ \pi_D := \text{Defector Profit} = P(t)qx^\alpha e^D - we^D + m(\hat{e} - e^D) \]
\[ P(t) := \text{Price of Resource} = \bar{P} + \eta(t) \]
\[ dx := \Delta \text{Resource} = rx(t)(x(t) - x_{\text{crit}})(1 - \frac{x(t)}{K})dt - Hdt + \sigma_x dw \]
\[ \frac{dC}{dt} := \Delta \text{Cooperators} = \frac{\gamma CD}{n} - \beta C (1 - \frac{\pi_C}{\pi_D}) \]

Figure 3.1: Diagram for basic flow of data between systems for original model.
entirety of the governing functions used to simulate their single cooperator-defector system. Overall, the cooperators and defectors harvest from a resource, contributing to a decrease in that resource. The resource itself grows instantaneously, using the positive portion of Equation 2.3 alongside the fluctuations which provide either a small positive or negative influence on the resource. This harvest directly influences the profit that the cooperators and defectors obtain alongside the price of the fish they are selling and a potential opportunity cost or gain associated with their alternative revenue stream. These profits influence the relative profit difference with the temptation coefficient, $\beta$, to determine how many cooperators in that year leave to the defection side, and subsequently the peer pressure term, $\gamma$, with the proportioned cooperator-defector interactions lead to a positive increase of cooperators going into the next year. The simulation model that this diagram represents then repeats yearly until a determined time.

Allee effects were also decidedly included as a form of altering the traditional parabolic growth curve, and perhaps inducing a collapse. The collapse would be caused by a lack of reproductive partners, captured by the $x(t) - x_{\text{crit}}$ term. It is claimed by the original authors that this had the additional use of introducing multiple steady states, this claim goes without analysis (Richter and Dakos 2015). Also discussed was the importance of including stochastic effects in regard to the growth of a stock. This behavior is mostly non-deterministic, although within reason, however, important to their assumptions, is the type of stochastic effect that was chosen. In their
case, they present an equation of \( \frac{dx}{dt} = (allee) \ast (growth) \ast -(harvest) \ast \pm ("noise"); \)
where, “noise” is given as \( \sigma_x \frac{dW}{dt} \). As it is mentioned in Table 3.1 \( \sigma_x \) is the standard deviation of \( x(t) \). We are given that \( dW = \epsilon \sqrt{dt} \) where \( \epsilon \) is an error term. Going forward however, this stochastic effect is limited to the simulation model, (and it is discussed here for completeness), but, for the sake of our analysis will not be appending this stochastic term to our extended model.

The last term to be discussed from the original growth stochastic differential equation model is the harvest term, namely, \( H(t) \).Knowing that we have a total population of \( n \) humans/fishers, we have \( H(t) \) given as:

\[
H(t) = \Sigma_{i=1}^{n} h_i(t),
\]

where this is the sum of all individual, \( h_i \), \( i = 1 \ldots n \) harvests, of fish \( x(t) \). Individual harvest is then given as:

\[
h_i(t) = qx^\alpha(t)e_i(t), \quad (3.1)
\]

which gives the catchability coefficient or technology coefficient \( q \), an elasticity constant \( \alpha \) and individual effort \( e_i(t) \). As was discussed in Chapter 2 this is loosely related to the pre-dynamic harvest equation used by the G-S model. The technology coefficient, \( q \), acts as a multiplicative constant, combined with the effort, on the resource. One often assumes that \( q \) acts as an indicator of how “advanced” a fisher is,
hence the technology aspect. A quick conceptual example: If one fisher has a strong net, their $q$ will be higher than another fisher whose net is filled with holes. We may visualize these concepts on Figure 3.1, namely the blue resource box along with the harvest arrows aimed at the resource from the two populations of fishers.

Since cooperators, $C$, and defectors, $D$, are governed by different managing strategies, we assume that their efforts, $e^C$ and $e^D$, respectively, follow different rules. Each of these are governed differently. It is assumed that cooperators all engage in the same effort; thus, rather than sum all of their individual efforts, the total population acts together within the confines of a sustainable harvest, together. That is:

\[ e^C(t) = \max(0, \min(\hat{e}, R)), \quad (3.2) \]

\[ R = a + x(t)b, \quad (3.3) \]

$\hat{e} \geq 0$ is effort associated with an alternative revenue stream and $R$ is an effort, associated only with cooperators, $R$ is talked about in greater detail below. This alternative revenue stream gives both the cooperators and defectors a choice of where to put their effort, the value is constant and may be chosen at the time of simulation. Equation 3.2 is an interesting approach to the choice of effort for cooperators. Earlier, it was mentioned that, unlike EGT, the effort is not summed. This is more of a semantic difference between a sum and a chosen value. Technically, one calculates
the effort that the entire population engages in total, and from there, if it is needed, figures out the individual effort level that a single individual engages in. However, since all cooperators will be engaging in such a way as to give the exact same effort each, the idea of determining an individuals’ effort is not needed. In reality, this is a simplifying assumption. People that cooperate, in reality, will try their best to do so within a margin of each other, but achieving perfect harmonious effort is a bit far-fetched. This could also have been alleviated with a simple noise term, but it has not been. Of more notable influence on the effort of the cooperators is the $R$ term given in both Equation 3.2 and extracted in Equation 3.3, which is considered a harvest rule. The harvest rule acts as a way to limit and approach an optimal effort level for the entire cooperator population. The simple assumption aims to achieve the maximum possible group effort plausible to fish a sustainable yield. In this way, they will fish with maximum sustainable yield in mind. This is given as:

$$x_{msy} = \sqrt{\frac{k^2 - kx_{crit} + x_{crit}^2 + k + x_{crit}}{3}},$$  \hspace{1cm} (3.4)$$

obtained by maximizing Equation 2.3 with respect to the non-stochastic terms. Once again, as a refresher, the MSY is the maximum possible resource that can be obtained indefinitely without causing the resource to decrease. Then associated biomass with the MSY is given in Equation 3.4. Also given is the maximum effort associated with
Equation 3.4
\[ e_{msy} = \frac{x_{msy}(x_{msy} - x_{crit}) \left(1 - \frac{x_{msy}}{k}\right)}{nq x_{msy}^\alpha} \] (3.5)

We arrive at Equation 3.5 by substitution of Equation 3.4 into Equation 3.1. Then the graph of \( R \) is a straight line, as one may have determined from its equation. Our cooperators, at the start of a year, choose a value off of this line up to the point where \( x(t) = x_{msy} \), in which case they have reached the maximum possible effort conceivable for their purposes. The values associated with Equation 3.3 [Richter and Dakos (2015)], claim that \( a \) is a “precautionary” term derived from the fishery itself. With this in mind, at \( x_{msy} \) we have:

\[ b = \frac{(e_{msy} - a)}{x_{msy}} \] (3.6)

This finally brings the description of \( R \) to an end. Moving on, before dissecting the defectors’ effort levels, it is more appropriate to introduce the economic aspect of this model. [Richter and Dakos (2015)] base quite a bit of the social dynamics and resource dynamics links on the profit that cooperators and defectors achieve. The basis for individual profit is given as:

\[ \pi_i(t, x(t)) = P(t)qx(t)^\alpha e_i - we_i + m(\dot{e} - e_i), \] (3.7)
where $m$ is revenue associated with an alternative market corresponding to $\hat{e}$, $w$ is a constant cost associated with fishing effort $e_i(t)$. As well, $P(t)$ models price, given as:

$$P(t) = \bar{P} + \eta(t)$$  \hspace{1cm} (3.8)

As such, $P(t)$ is a price term, governed by an average price $\bar{P}$ plus the error term $\eta(t)$. This error term is normally distributed, and, it has zero mean with standard deviation $\sigma_p$. One can see that for cooperators, given that their effort levels are no different from the alternative $\hat{e}$ the highest possible cooperator profit (for fishing) is:

$$\pi_i(t, x(t)) = P(t)qx(t)\alpha - w\hat{e},$$  \hspace{1cm} (3.9)

Equation 3.7 equals zero when $e_i(t) = \hat{e}$. For cooperators, this is more appropriate for when $e^C(t) = \hat{e}$ when $R \geq \hat{e}$. Clearly then, when there is zero effort investment in fishing, we get that Equation 3.7 tends to $\pi_i(t, x(t)) = m\hat{e}$. Unlike the cooperators, the defectors are not limited by MSY and subsequently are not limited in the effort they choose to engage with onto the fish stocks. Instead, Richter and Dakos (2015) decided to frame the defector effort around their view of profit. In light of this, they chose to maximize Equation 3.7 in accordance with $e_i(t)$. This leads to:

$$e^D(t) = \begin{cases} 
\hat{e} & \text{if } x^\alpha(t) \geq \frac{(w+m)}{P(t)q} \\
0 & \text{if } x^\alpha(t) < \frac{(w+m)}{P(t)q}
\end{cases}$$
then it is seen that defectors choose to place the highest possible effort at all points of fishing given that the amount of resource in the lake $x^\alpha(t)$ is higher than or equal to $\frac{(w+m)}{P(t)q}$. In contrast, they will never invest effort if the above statement is not true. The final piece of this model is given in Chapter 2 as Equation 2.4. This is the social aspect of this model that is not grounded in economics. Instead, Equation 2.4 governed purely the amount of individuals entering or leaving a state of cooperation. That is, it governs the cooperator population level. Within this equation is a term, $\gamma$ given as:

$$\gamma = \lambda(\nu - \phi), \quad (3.10)$$

where $\lambda$ is considered a Poisson parameter, $\nu$ is a utility loss, and $\phi$ is a cost associated with sanctioning. This $\gamma$ term, that is, Equation 3.10 represents a multitude of assumptions. Authors Richter and Dakos (2015) introduced an idea, partially novel when compared with Sethi and Somanathan (1996), in which they stray away from their idea of replicator dynamics and instead frame their problem around “temptation” and “peer pressure”. As we outlined in Chapter 2, there are many processes associated with social norms, in the case of Richter and Dakos (2015), they incorporated aspects from this, like cooperators and defectors, and with it also brought in assumptions that govern sanctioning which is governed by Equation 3.10 in this specific instance. In this way, the use of this equation starts to make more sense. A cooperator and a defector “encounter” each other, that is, they somehow “view” or “meet” each other in such a way as to allow for a cooperator to sanction a defector.
This sanction is captured by \( \nu \) as a utility loss for a defector. With this punishment comes an associated cost, \( \phi \). In order for social norms to hold, it is always more acceptable to be a cooperator than a defector. In this way, they frame this encounter as being weighed in favor of the cooperators, or \( \nu > \phi \).

In the original paper outlined by Richter and Dakos (2015), Equation 3.10 was introduced using the temptation and peer pressure concepts under the guise of a discrete process. We believe that this was attempted in such a way as to follow at least one method of deriving a differential equation. Instead of introducing this concept first it seemed a bit more appropriate to show where social norms fit in with the cooperator and defector populations. However, because of this backward introduction we have neglected the \( \lambda \) term and its purpose. Now, here, consider a differential equation wherein one considers a change in cooperator population over a time interval such that:

\[
\frac{dC}{dt} = Df(U_c, U_d) - Cg(\pi^C, \pi^D), \tag{3.11}
\]

where Equation 3.11 is a generalized, or, pre-derived version of Equation 2.4. Then, one considers that peer pressure, or the first term, is some process that incorporates the defector population and \( f(U_c, U_d) \) which is a function of both population’s utility; and, temptation, which incorporates the cooperator population, and \( g(\pi^C, \pi^D) \) which is a function of both populations profit. Noting that, despite using utility, it is done
so very loosely, as it is never fully utilized in the form of a utility equation. It is here that one can begin to envision the usage of Equation 3.10. For the first term, considered the peer pressure term, if this is considered a discrete process then one may assume that encounters occur across a time step of $t$ to $t + \delta t$ causing a change of cooperators:

$$C(t + \delta t) - C(t) = \frac{\lambda \nu C(t)D(t)\delta t}{n} - \frac{\lambda \phi C(t)D(t)\delta t}{n},$$

noting that, as previously stated, because of Equation 3.10, we arrive at:

$$C(t + \delta t) - C(t) = \frac{\gamma C(t)D(t)\delta t}{n},$$

(3.12)

currently then, $\frac{\gamma C(t)D(t)\delta t}{n}$ is always positive given the current constraints. One can finally see the purpose of $\lambda$, knowing that $\gamma$ is always positive frames this Poisson parameter as exactly that, it decides the amount of events, meetings, or encounters that occur. In this case, as it is captured best using $\gamma$, Equation 3.10 entails the number of defectors that encounter a cooperator at a certain proportion $\lambda$ and are subsequently sanctioned, incurring a cost to both sides that is always lopsided, and thus is non-negative in nature. That is, they are pressured back into the cooperator population.
Given that time is continuous, Richter and Dakos (2015) acknowledge this, and approximate Equation 3.12 into a continuous time variant. This variant is simply the first term present in our generalized Equation 3.11 and specifically in Equation 2.4. Before arriving at the full continuous time version of Equation 3.11 we must now consider the assumptions associated with the second term of Equation 3.11 namely the temptation term: $C(t)g(\pi^C, \pi^D)$. This temptation term was framed as a proportion of $C(t)$ viewing the profits of $D(t)$ relative to their profits. Introduced then is a proportion term $\beta$ where $\beta C(t)$ is a proportion of cooperators at some time $t$. The present derivation assumes a jump in development by making the assumption that $g(\pi^C, \pi^D)$ is considered time continuous (Richter and Dakos, 2015). That is to say, these authors give this component as:

$$g(\pi^C, \pi^D) = \beta \left(1 - \frac{\pi^D}{\pi^C}\right), \quad (3.13)$$

where this is non-negative for all values of $\frac{\pi^D}{\pi^C} \geq 0$. However, it is not-constrained by this inequality. That is to say, $\pi^C$ can be greater than $\pi^D$. In the larger picture, Equation 3.13 is utilized in the second term of Equation 3.11. The additional derivation of terms present can also be visually summarized on Figure 3.1, wherein the social components are highlighted on the orange box and the economic components exist as a flow labeled by the arrow pointing towards said box. The economical contributions also add to the social evaluation via the negative term between cooperators and
defectors in the orange box. With this in hand, we have fully presented the original model that will be utilized for extension in the sections following.

3.2 New Model

There was much deliberation concerning the direction that additions to a model would receive. Initially, our scenario started from a posed question: given that there are two mutually non-agreeing managements operating on a large scale resource, like a fish stock, how do they affect the fish? Here, we will present the aspects of our scenario that needed to be derived, and the approaches we took to arrive at their inception. Before moving on we direct our reader to Table 3.1 as a reference point for all of the symbols that have been used, and, will be used in subsequent chapters.

In order to frame our scenario properly, we first list a few key assumptions. A critical first assumption requires that we consider only two managerial bodies on Lake Huron. That is to say, we consider only the SON and the OMNRF to be the acting governing entities on their respective fishing grounds. Although there is no real spatial component to this model, we can naively assume that both bodies have their own unique populations. That is, there is a specific SON population, and a specific OMNRF population. If this were not the case, we would have a problem that resembles the work done by Sugiarto et al, where a population of individuals would be considered in a heterogeneous state of some variety, then, being able to mix and net-
### Table 3.1: Table of Symbols used throughout the text.

<table>
<thead>
<tr>
<th>Item</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(t)$</td>
<td>Total resource biomass at time $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$x_{\text{crit}}$</td>
<td>Critical resource biomass</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>Standard Deviation of normal distribution associated with $x(t)$ with zero mean and variance 1</td>
<td>0.075</td>
</tr>
<tr>
<td>$q$</td>
<td>Constant of Technology</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$r$</td>
<td>Intrinsic growth rate of resource biomass</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$k$</td>
<td>Carrying capacity for resource biomass</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$n$</td>
<td>Total human population</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$H(t)$</td>
<td>Total harvest at time $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$C(t)$</td>
<td>Total human cooperator population</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$D(t)$</td>
<td>Total human defector population</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\pi^c(t)$</td>
<td>Total cooperator profit at time $t$</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$\pi^d(t)$</td>
<td>Total defector profit at time $t$</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$e^c(t)$</td>
<td>Total cooperator effort at time $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$e^d(t)$</td>
<td>Total Defector effort at time $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\hat{e}$</td>
<td>Effort Endowment</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Constant of Peer Pressure</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Constant of Defection</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Elasticity constant</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$P(t)$</td>
<td>Total price of fish at time $t$</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$\eta_P(t)$</td>
<td>Normally Distributed Error Term With Zero Mean and Standard Deviation $\sigma_P$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\sigma_P$</td>
<td>Standard Deviation of Total Price of a Single Fish at a time $t$</td>
<td>20</td>
</tr>
<tr>
<td>$m$</td>
<td>Net Revenue Associated with Alternate Economic Activity</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$w$</td>
<td>Net Cost of Deploying Effort</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$n_S$</td>
<td>SON total fisher population</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$C_S$</td>
<td>SON total fisher cooperator population</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$D_S$</td>
<td>SON total fisher defector population</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$n_O$</td>
<td>OMNRF total fisher population</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$C_O$</td>
<td>OMNRF total fisher cooperator population</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$D_O$</td>
<td>OMNRF total fisher defector population</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\gamma_S$</td>
<td>SON peer pressure coefficient</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\gamma_O$</td>
<td>OMNRF peer pressure coefficient</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\beta_S$</td>
<td>SON temptation coefficient</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\beta_O$</td>
<td>OMNRF temptation coefficient</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\pi^c_S(t)$</td>
<td>SON cooperator profit at time $t$</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$P_S(t)$</td>
<td>SON market price of fish at time $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$P_O(t)$</td>
<td>OMNRF market price of fish at time $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$h^d_S(t)$</td>
<td>SON total defector harvest at time $t$</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$h^c_S(t)$</td>
<td>SON total cooperator harvest at time $t$</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$h^d_O(t)$</td>
<td>OMNRF total defector harvest at time $t$</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$h^c_O(t)$</td>
<td>OMNRF total cooperator harvest at time $t$</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$H_O(t)$</td>
<td>Total harvest</td>
<td>$\in \mathbb{R}$</td>
</tr>
<tr>
<td>$w_S$</td>
<td>SON wage cost</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$w_O$</td>
<td>OMNRF wage cost</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$e^d_S(t)$</td>
<td>SON total defector effort at time $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$e^c_S(t)$</td>
<td>SON total cooperator effort at time $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$e^d_O(t)$</td>
<td>OMNRF total defector effort at $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$e^c_O(t)$</td>
<td>SON total cooperator effort at time $t$</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\hat{e}_S$</td>
<td>SON alternative effort</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\hat{e}_O$</td>
<td>OMNRF alternative effort</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$R_S$</td>
<td>SON linear control rule</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$h_Q$</td>
<td>Harvest quota (associated with OMNRF)</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$e_Q$</td>
<td>Quota effort (associated with OMNRF)</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>OMNRF saturation coefficient</td>
<td>$\in \mathbb{R}_{\geq 0}$</td>
</tr>
<tr>
<td>$\Gamma_a$</td>
<td>OMNRF Quota Alteration (decrease)</td>
<td>0 $\leq \Gamma_a &lt; 1$</td>
</tr>
<tr>
<td>$\Gamma_b$</td>
<td>OMNRF Quota Alteration (increase)</td>
<td>$\Gamma_b &gt; 1$</td>
</tr>
</tbody>
</table>
work among both sets of human populations. A management body, as was inferred in Chapter 2 Section 2.3 is considered to be a combination of governances onto a population of people. In our case, we have taken our management bodies to be acting in different manners, however, both fit the idea of external and internal governance. Abstracted, the SON and OMNRF are intended to represent larger bodies of people that act in wholly uncooperative ways. Uncooperative in this case is not a reference to a ‘cooperator and defector’ system but instead a higher level interaction between the two management bodies. The reader must keep this in mind during the detailing of the extensions of this model. Expansion includes the introduction of these management bodies, and the additional interactions, and ownership of the original model that they indicate. Their respective sub-population’s will have associated cooperator dynamics, markets, profit, effort and harvest. Incidentally, our observational goal, the fish population, will also be affected by these changes, and as such an exploration of these modifications will be explored.
**Figure 3.2:** Diagram for basic flow of data between systems for new model.

\[
H_S := \text{Total SON Harvest} = \Sigma_{i=1}^{n} h_i^S(t) = h_C^S + h_D^S
\]

\[
h_C^S := \text{Total SON Cooperator Harvest} = qx(t)\alpha e_C^S(t)
\]

\[
h_D^S := \text{Total SON Defector Harvest} = qx(t)\alpha e_D^S(t)
\]

\[
e_C^S := \text{SON Cooperator Effort}
\]

\[
e_D^S := \text{SON Defector Effort}
\]

\[
\pi_C^S := \text{SON Cooperator Profit} = P(t)qx\alpha e_C^S - we_C^S + m(\hat{e} - e_C^S)
\]

\[
\pi_D^S := \text{SON Defector Profit} = P(t)qx\alpha e_D^S - we_D^S + m(\hat{e} - e_D^S)
\]

\[
P(t) := \text{Price of Resource for SON}
\]

\[
\frac{dC_S}{dt} := \Delta \text{Cooperators} = \frac{\gamma C_S D_S}{n_S} - \beta_S C_S(1 - \frac{C_S}{n_S})
\]

**Influences**

\[
\text{Resource}
\]

Natural Growth with Fluctuations.
3.2.1 A Dual Management System

Originally, the base model was tailored for a single system of fishers. Regardless, the first gap to cross is the expansion of this original model into a dual-fisher-population model. Conceptually, one may view the larger picture by referencing Figure 3.2 which outlines the total process of the new model. Luckily for this thesis there are no additional interactions that must be introduced to pair up from the original model. Prior to introducing these new dynamics a few constructs need to be defined.

Management Bodies

In the opening paragraphs of Section 3.2 management bodies were introduced and framed within the context of governance much like preceded it in Chapter 2. However, what objectively constitutes a management body has yet to be concretely defined within the walls of this thesis. A management body acts first as a label. In our case these labels will take place as subscripts on terms related to the said management body. Labels, in this case, are related to the bodies we must observe, that is, the SON and the OMNRF who will be represented by the \( S \) and \( O \) subscripts respectively. Then, any individual, any function, or any property may be said to be under the management of one of either the SON or the OMNRF if and only if they are labeled with either of their respective subscripts.
Fisher Populations

Now that labeling has been defined we may start to split this model into its constituent components. A requirement of a system with two different management bodies is populations of individuals. Now, it is possible that one may think of these bodies as managing one pool of individuals, but this is not the case. Instead, each management body, namely the SON and OMNRF manage their own sets of individuals. In this way, much like in Section 3.1 we saw that for the model presented by Richter and Dakos (2015) a ‘population’ in their sense requires a total population $n$, and two sub-populations that makeup the total population, that is the cooperators and defectors, $C(t)$ and $D(t)$. This was for a situation with one governing, or management body on a body of water. In this case, a first step to expanding this idea, is introducing the SON and OMNRF as containing their own set of total population as well as cooperators and defectors. So, let the SON label their total population of individuals as $n_S$. Furthermore, let the SON label their sets of cooperators and defectors as $C_S(t)$ and $D_S(t)$. In a similar manner, let the OMNRF label their total population of individuals as $n_O$ and, in addition, let their cooperator and defector sub-populations be labeled as $C_O(t)$ and $D_O(t)$. Then, explicitly:

$$n_S = C_S(t) + D_S(t),$$

$$n_O = C_O(t) + D_O(t),$$
where $n_S, n_O \in \mathbb{R}_{\geq 0}$ to be chosen at the time of simulation, and elaborated on subsequently, and, $C_O(t), C_S(t), D_O(t), C_S(t) \in \mathbb{R}_{\geq 0}$. That is, the combination of cooperator and defector populations equals the total population of that management body. The cooperator and defector sub-populations associated with their respective total populations are considered to be under the same social constraints as their single system counterparts which were explained abstractly in Chapter 2 Section 2.3 Subsection 2.3.1. However, unlike their counterparts, they do not immediately adopt similar constraints associated with their effort levels, profit, or any other items of their inclusion. Instead one may just think of these sub-populations currently as sets of selfish and non-selfish individuals. As an explicit clarification, an individual in our case always refers to a fisher; so, in the event that one is used over the other, both are completely equivalent in definition.

### 3.2.2 Cooperator and Defector Dynamics

Co-Evolving Cooperator Populations

In the previous section, subpopulations for the individuals that comprise the SON and OMNRF total populations were formulated. From a top down approach, almost in reverse of the derivation of the Richter and Dakos (2015) model, a logical next step is outlining the dynamics of each of these populations. We may borrow the derivation stemming from Equation 3.11 wherein the dynamics are outlined and defined. In fact for each separate population, that is, for the SON total population consisting of its
cooperators and defectors, the scheme fits perfectly. The same can also be said for the OMNRF total population and its substituents. An important fact to note is the assumption we defined earlier in Subsection 3.2.1 where we stated that each of these populations, or in this case the labels associated with these populations, excluded these individuals from each-others labels. This enables us to state from an abstract viewpoint that the SON and OMNRF individuals never interact with individuals across their sub-populations. In addition, with this other assumption outlined, there are no additional dynamics or cross overs required between the cooperator or defector groups among both populations. One may say that the SON and OMNRF do not mix. However, that is not to say that it was not thought of. It is highly likely that the individuals who fish on the lake under the control of either management body do in fact interact with each other; but, to consider these dynamics is outside the scope of this thesis. It is possible to then write a set of co-evolving, non-interacting, differential equations for both the SON and OMNRF cooperator dynamics:

\begin{align}
\frac{dC_S(t)}{dt} &= \gamma_S C_S(t) D_S(t) - \beta_S C_S(t) \left( 1 - \frac{\pi_C^S(t, x(t))}{\pi_D^S(t, x(t))} \right), \\
\frac{dC_O(t)}{dt} &= \gamma_O C_O(t) D_O(t) - \beta_O C_O(t) \left( 1 - \frac{\pi_C^O(t, x(t))}{\pi_D^O(t, x(t))} \right),
\end{align}

where once again \(\gamma_S, \beta_S, \pi_C^S(t, x), \pi_D^S(t, x)\) are the coefficient of peer pressure, coefficient of temptation, profit of cooperators and profit of defectors for the SON. Then, \(\gamma_O, \beta_O, \pi_C^O(t, x), \pi_D^O(t, x)\) are the constant coefficient of peer pressure, constant coef-
ticient of temptation, profit of cooperators and profit of defectors for the OMNRF. For completeness, $\gamma_S, \beta_S \gamma_O, \beta_O \in \mathbb{R}_{>0}$ are left as parameters to be properly quantified in a real setting. For now, they are left as a construct of their respective ideology: temptation and peer pressure.

### 3.2.3 A Note on Social Norms and Economics

For the new model we have presented most of the overarching items needed to conduct analysis. There is quite a fair amount of tailoring required to justify parameters and inclusions like temptation, peer pressure, wage, alternative markets, and profits associated with a dual management system. Although these concepts are important to the model, they only require contextual fitting with respect to the original model in a way that makes sense to our situation. For the readers sake, we present the justification of these terms and concepts in Appendix C. The final, and most important set of concepts remains the tailoring of effort for both the SON and OMNRF. The respective equations from these justifications will be presented at the end of this Chapter convenience.

### 3.2.4 Effort

The effort given by individuals in a lake determines how much fish will be harvested, taking into account other factors. Richter and Dakos (2015) decided to create a framework which utilized an alternative revenue stream, as we saw earlier. In addition
to this they introduced a harvest control rule. With it comes an idea known as “adaptive capacity” which describes how a group of cooperators achieve maximum sustainable yield (MSY). With that in mind, we can actually tailor these ideas to fit our new dual system. Note, when we talk about effort in this case, effort is considered to be *per person*, that as, as our reader will see with total harvest, effort is a small portion of the total cooperator effort, or defector effort.

**SON Effort**

In the original model, cooperators fished up to the population level that contributed to the MSY. The defectors of this total population would then fish up to a maximum level \( \hat{e} \). However, the SON in their fishing practices are quite reactionary, and have, in the past 20 years or so closed their waters entirely to the commercial fishers and public, citing that the MSY was reached ([OMNRF](#) 2005a). The closing of their waters also incorporates the amount of fish collected/harvested by the OMNRF in that same year. This reactionary style will allow us to frame this scenario in a similar fashion. Seeing as the SON will in fact react to the harvests of the OMNRF, we can assume that at *any* point in that year they will choose to close their waters or not to close their waters. This is very similar to the way the current model operates. The SON are also considered to use methods of prediction allowing them to consider how they approach harvesting of fish. In their case they actually do employ similar methods to that of “adaptive capacity” ([SON](#) 2016). The defectors of any management body
will seek to harvest at a non-controlled rate or amount, and both of these bodies will act the same way. So, the SON cooperators and defectors behave as such:

\[ e^C_S(t, x) = \max(0, \min(\dot{e}_S, R_S)) \]  

\[ e^D_S(t) = \begin{cases} 
\dot{e}_S & \text{if } x^\alpha(t) \geq \frac{(w_S + m_S)}{P(t)/q_S}, \\
0 & \text{if } x^\alpha(t) < \frac{(w_S + m_S)}{P(t)/q_S} 
\end{cases} \]  

\[ R_S = a_S + x(t)b_S, \]  

\[ b_S = \frac{(e_{msy} - a_S)}{x_{msy}}, \]  

where, Equations 3.16 and 3.17 govern cooperator, \( e^C_S(t, x) \), and defector, \( e^D_S(t) \), efforts at time \( t \). As well, as seen in the original model, Equations 3.18 and 3.19 are equivalent to Equations 3.3 and 3.6. Worth noting is the labels on the parameters in Equations 3.18 and 3.19 namely: \( a_S \), \( b_S \), and \( R_S \). These were added to give ownership to the parameters, although, it is more to differentiate them from the original parameters from the base model, noting that OMNRF never require these parameters, which is why there will not be an equivalent value for each for the OMNRF. Explanation for this fact is given later in this subsection. There is no definable difference behind these equations and the equations given in the original model. The cases present for \( e^D_S(t) \) (and as we will see for \( e^D_O(t) \)), are tailored in a way as to ensure that the defectors
are always making maximum profit. In fact

\[ x^\alpha(t) \geq \frac{(w_S + m_S)}{P_S(t)q_S} \implies \pi^C_S(t, x(t)) \geq m(\epsilon_S) \]

\[ x^\alpha(t) < \frac{(w_S + m_S)}{P_S(t)q_S} \implies \pi^C_S(t, x(t)) < m(\epsilon_S) \]

This does not address the asymptotic behaviour that can be exhibited by \( \frac{(w_S + m_S)}{P(t)q_S} \).

Although from a physical standpoint allowing \( P_S(t) \to 0 \) or \( q_S \to 0 \) makes little sense (i.e. having free fish or no ability to catch fish), from a mathematical perspective we must account for these possibilities. It should be apparent then, that for analysis, we note that \( \frac{(w_S + m_S)}{P_S(t)q_S} \to \infty \), if \( P_S(t)q_S \to 0 \), i.e. a vertical asymptote.

An additional item to point out is the lack of assignment to the value \( e_{msy} \). The effort at MSY is in fact calculated based on both contributions from individuals in both management bodies. Then, there is an additional large assumption that must be addressed. Statement about MSY effort implies that both the SON and OMNRF are fishing the same population of fish, indicating that the MSY for both managements are the exact same. In Appendix \( \text{[3]} \) the idea of metapopulation is discussed and explained. On lake Huron, and specifically in regards to Lake Whitefish there has been an attempt from our group to determine the makeup of Lake Whitefish population or populations. However, at this point there is insufficient data to support the idea of a patchy population of fish. Instead, assume that both SON and OMNRF harvest
their fish from the same population of Lake Whitefish during the seasons in which fishing is possible.

**OMNRF Effort**

The most major change to the base model is the effort given by the OMNRF when fishing. A large divergence in effort is apparent when compared to the management practices of the SON, however, especially in the real life circumstances surrounding the OMNRF’s practices in regards to their cooperator population.

In Subsubsection 3.2.4 Lake Whitefish were described as being one whole cohesive population, not subject to any metapopulation concepts, to the best of current knowledge. In this consideration, the SON view the lake with this in mind, and thus react according to the total population of fish in the lake. However, this is not the case for the OMNRF. When it comes to the OMNRF fisher population, there is an extension required, (when it comes to their effort levels), in contrast with the original model assumptions. They behave as if there is no other governing body fishing along side of them. That is, they will ignore, or, not consider the fishing levels of the SON when it comes to their calculation, and subsequently they will not close their waters or alter their MSY. The decision to assume this comes from talks with the SON wherein they cite that the OMNRF have in fact ignored their predictive models or used odd responses to MSY wherein they have kept fishing even when MSY is breached (SON, 2016). In this way, regardless of whether or not the SON contribute to the MSY of
the lake, it will only alter the cooperative effort prior to any massive social change over, or so this is the idea. We can then capture this by altering how we consider the OMNRF to give effort while fishing. From talks previous, it seems that the OMNRF fish with quotas in mind. These quotas are unknown, so, it is assumed here that the base quota is given as some portion of the MSY of the Lake Whitefish population in Lake Huron.

Then the OMNRF will follow a “quota” scheme for their cooperative effort levels in accordance with the assumptions above. Let the initial OMNRF cooperator effort be

\[ e_Q^C(t) = \max(0, e_Q), \]

when \( t = 1 \) in this model; and, where \( e_Q \in \mathbb{R} \) is the starting quota effort and \( e_{msy} \geq e_Q \geq 0 \). This allows the OMNRF to set a quota that is either less than or equal to the effective total effort needed by both groups to fish up to the MSY or less than this effort. In this way, the OMNRF can give initial consideration to the lake Whitefish and restricts them to an initial cooperative level of effort. If they are being cautious, an appropriate level might be to assume that both groups will invest exactly half of the effort required to achieve MSY, i.e. \( e_Q = \frac{e_{msy}}{2} \). The effect this has on the population of fish will be explored, as it is an important point to consider when simulating. The quota effort level speaks only to the initial year that the OMNRF individuals start fishing. Seeing as we are free to play with parameters to determine
aspects of this quota system, we may speak in general terms. The OMNRF typically continue to fish at the same effort level of the year prior so long as certain constraints on their harvest are met. Namely, if the total reported harvest of the OMNRF is within the range of the MSY (that is bounded by a set percentage of harvest from the year prior) then they will continue to use the same effort levels as the year prior for the year current. Only when the MSY is breached significantly, or when it is not met within a range will they alter their effort levels. We must then explicitly define the harvest of the OMNRF. Total harvest is the summation of both the sub-populations total net harvest namely:

$$H_O(t, x) = h_O^P(t, x) + h_O^C(t, x),$$

where $h_O^P(t, x) = q_O x^o(t) e_P^O(t) (n_O - C_O(t))$ and $h_O^C(t, x) = q_O x^o(t) e_C^O(t) C_O(t)$ are OMNRF defector and cooperator harvests (representing total harvest of both sub-populations) and $q_O$ is the technological coefficient associated with the OMNRF. In anticipation, we have brought forth harvest for OMNRF here, but, harvest will be further explained in its own section with respect to the growth dynamics. Now, in line with previous logic, at $t = t_0$ then we have,

$$H_O(t_0, x) = q_O x^o(t_0) e_P^O(t_0) (n_O - C_O(t_0)) + q_O x^o(t_0) e_Q C_O(t),$$
or,

\[ H_O(t_0, x) = h_O^P(t_0, x) + h_Q, \]

where, \( h_Q \) is the quota harvest associated with the OMNRF cooperator effort at \( t = t_0, e_Q \). For cooperator effort past the initial phase of harvest, we must consider the evolution of their effort over time. We stated above that if the cooperator effort falls within a set range of harvest quota, then, suppose we have \( H_O(t^*) \in [\delta h_Q, h_Q] \) for some \( t^* \). Then \( \forall t \) \( H_O(t) \in [\delta h_Q, h_Q] \) such that:

\[ e_C^O(t) = e_C^O(t^*), \]

where \( \delta \in [0, 1] \) is a reference point to either keep an explicit quota or to develop a loose or tight range around the harvest required to alter effort levels. By the same logic and in extension to cases where the OMNRF cooperators over or under fish we fully define \( e_C^O(t) \) to be:

\[
e_C^O(t) = \begin{cases} 
\text{max}(0,e_Q) & \text{, if } t = 1 \\
e_C^O(t^*) & \text{, if } H_O(t^*) \in [\delta h_Q, h_Q] \\
\Gamma_a e_C^O(t^*) & \text{, if } H_O(t^*) > h_Q \\
\Gamma_b e_C^O(t^*) & \text{, if } H_O(t^*) < \delta h_Q 
\end{cases}
\]
where, in Equation 3.20 $\Gamma_a \in \mathbb{R}_{\geq 0}$ and $\Gamma_b \in \mathbb{R}_{\geq 0}$ are scaling coefficients determined by the OMNRF to decrease or increase effort levels for the current year regardless of the previous years effort levels. However, since both sets of fishing groups are fishing on the lake, we know then that at some points, they might over-fish, or they might under-fish using the scheme above. $\delta \in [0, 1]$ will help to encompass the range that they consider “meeting quota”, if $H_O(t)$ falls within the quota range, then they won’t alter their effort levels for the year after. However, if they go over, or if they go under, they will alter their effort levels subsequently. This leads the way for $\Gamma_a$ and $\Gamma_b$, coefficients of decrease, and, increase respectively. One can think of these parameters as a percentage increase or decrease in estimated quota effort levels given that one of these cases occurs in any of the years during simulation. They are bounded by constraints, explicitly: $\Gamma_b e^C_O(t^*) < e^C_O(t^*)$ implying $\Gamma_b < 1$ and if $\Gamma_a \in \mathbb{R}_{\geq 0}$ then $0 \leq \Gamma_a < 1$. Conversely, $\Gamma_a e^C_O(t^*) > e^C_O(t^*)$ implying then that $\Gamma_a > 1$ with no further constraints.

Much like for the SON, the defectors for the OMNRF do not deviate in behavior from the original models assumptions. Then, the are purely selfish and operate as such:

$$e^D_O(t) = \begin{cases} \hat{e}_O & \text{, if } x^\alpha(t) \geq \frac{(w_O+m_O)}{P_O(t)q_O} \\ 0 & \text{, if } x^\alpha(t) < \frac{(w_O+m_O)}{P_O(t)q_O} \end{cases}$$
where, under the constraints of OMNRF variants on wage $w_O$, alternative revenue $m_O$, price $P_O(t)$ and technology $q_O$ determine the effort levels that govern their defectors. 

The last item to discuss is the resource that these two management bodies, with their individuals fishers harvest from.

### 3.2.5 Growth and Harvest

Referring back to both Chapter 2 Section 2.2 and Chapter 3 Section 3.1 we saw the introduction and then usage of the Gordon-Schaeffer differential equation for a dynamic resource

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) - H(x)$$

To reiterate, $x$ is our resource, which represents the Lake Whitefish of lake Huron. \cite{richter2015} expanded on this differential equation to bring about the inclusion of stochasticity, seen in Equation 2.3 and visually in Figure 3.1. For harvest we introduce a new term, to differentiate it from its original meaning:

$$H_T(t, x(t)) = H_S(t, x(t)) + H_O(t, x(t)),$$

where, $H_O(t, x(t))$ has been given in Subsection 3.2.4 and,

$$H_S(t, x(t)) = h^O_S(t, x(t)) + h^C_S(t, x(t)),$$
where \( h_D^S(t, x(t)) = q_S x^\alpha(t) e_D^S(t)(n_S - C_S(t)) \) and \( h_S^C(t, x(t)) = q_S x^\alpha(t) e_C^S(t) C_S(t) \) are SON defector and cooperator harvests (representing total harvest of both subpopulations). Expanding this, we get: then

\[
H_T(t, x(t)) = (q_S e_D^S(t)(n_S - C_S(t)) + q_S e_C^S(t, x) C_S(t) \\
+ q_O e_D^O(t)(n_O - C_O(t)) + q_O e_C^O(t) C_O(t)) x^\alpha(t) 
\]

(3.21)

\( H_T \) is the total harvest from all sub-populations from each management body. Given what was said earlier about the growth of the resource, we may substitute into Equation 2.3 for \( H \) our new harvest

\[
\frac{dx}{dt} = r x(t)(x(t) - x_{crit}) \left( 1 - \frac{x(t)}{k} \right) dt - H_T(t, x), 
\]

(3.22)

where, Equation 3.22 is the differential equation for the growth of our resource with a new harvest given by Equation 3.21. Note, there is no actual spatial component to this model, any additional considerations can be forgone with this in mind (looking back to sections dealing with metapopulation(s) and the assumptions we have made), in that way we have really just tacked on a different harvest. In actuality, one may dissolve this new model back into the original (ignoring the changes to effort for OMNRF) if one eliminates a management body from the problem and assumes parameter equivalence.
Discontinuities in Effort

Our reader might have noted that there are discontinuities present within efforts $e_S^D, e_D^D,$ and $e_O^C$. These are wild jumps that can occur for the defined situations they can occupy. From a biological, or phenomenon based standpoint, they make sense, but from an analytic perspective they pose a problem. An approach to handling situations like these involves approximating the jumps using an appropriate continuous function. A function that can fit our jumps is

$$z(t) = \frac{1}{2} + \frac{\text{arctan}(Lt)}{\pi},$$

where $L > 0$, and, we note the derivative of $z(t)$ evaluated at $t = 0$ demonstrates

$$\left. \frac{dz}{dt} \right|_{t=0} = \frac{L}{\pi},$$

which implies that as $L \to \infty$ then the slope of $z(t)$ at $t = 0$ approaches infinity, which can be used to approximate the jump. We then assume that our efforts $e_S^D, e_D^D,$ and, $e_O^C$ can be approximated by variants of $z(t)$ such that the discontinuities are removed for the process of analysis.
3.3 Summary of Equations

For the sake of clarity, the entire model with its new additions is provided below. It is split into the differential system followed up with each supplementary intermediate equation. Then, we have:

\[ \frac{dx(t)}{dt} = rx(t)(x(t) - x_{crit}) \left( 1 - \frac{x(t)}{k} \right) - H_T(t, x, C_S, C_O), \]

\[ \frac{dC_S(t)}{dt} = \frac{\gamma_s C_S(t) D_S(t)}{n_S} - \beta_s C_S \left( 1 - \frac{\pi^C_C(t, x)}{\pi^D_S(t, x)} \right), \]

\[ \frac{dC_O(t)}{dt} = \frac{\gamma_o C_O(t) D_O(t)}{n_O} - \beta_o C_O(t) \left( 1 - \frac{\pi^C_O(t, x)}{\pi^D_O(t, x)} \right), \]

which represents our set of differential equations given by a resource growth equation and two cooperator equations. Within each of these equations, as we have seen, is quite a bit of intermediate functionality. Zeroing in on the resource dynamics we have:

\[ H_T(t, x(t)) = (q_s e^D_S(t)(n_S - C_S(t)) + q_s e^C_S(t, x)C_S(t) + q_o e^D_O(t)(n_O - C_O(t)) + q_o e^C_O(t)C_O(t)) x^a(t) \]

furthermore, we focus on each sub-population’s effort levels:

\[ e^C_S(t) = \max(0, \min(\hat{e}_S, R_S)), \]
\[ e^D_S(t) = \begin{cases} \hat{e}_S, & \text{if } x^\alpha(t) \geq \frac{(w_S + m_S)}{P(t)sq_S}, \\ 0, & \text{if } x^\alpha(t) < \frac{(w_S + m_S)}{P(t)sq_S} \end{cases}, \]

\[ R_S = a_S + x(t)b_S, \]

\[ b_S = \frac{(e_{msy} - a_S)}{x_{msy}}, \]

for the SON, and for the OMNRF we get:

\[ e^C_O(t) = \begin{cases} \max(0, e_Q), & \text{if } t = 1 \\ e^C_O(t^*), & \text{if } H_O(t^*) \in [\delta h_Q, h_Q] \\ \Gamma_a e^C_O(t^*), & \text{if } H_O(t^*) > h_Q \\ \Gamma_b e^C_O(t^*), & \text{if } H_O(t^*) < \delta h_Q \end{cases}, \]

\[ e^B_O(t) = \begin{cases} \hat{e}_O, & \text{if } x^\alpha(t) \geq \frac{(w_O + m_O)}{P(t)q_O}, \\ 0, & \text{if } x^\alpha(t) < \frac{(w_O + m_O)}{P(t)q_O} \end{cases}, \]

each of the efforts rely on a market price as well as sub-population profits given as:

\[ P_S(t) = \bar{P}_S(t) + \eta(t), \]

\[ P_O(t) = \bar{P}_O(t) + \eta(t), \]
and,

\[ \pi^C_S(t, x(t)) = P_S(t)h^C_S(t, x) - w_S e^C_S(t, x) + m_S(\dot{e}_S - e^C_S(t, x)), \]

\[ \pi^D_S(t, x(t)) = P_S(t)h^D_S(t, x) - w_S e^D_S(t) + m_S(\dot{e}_S - e^D_S(t)), \]

\[ \pi^C_O(t, x(t)) = P_O(t)h^C_O(t, x) - w_O e^C_O(t) + m_O(\dot{e}_O - e^C_O(t)), \]

\[ \pi^D_O(t, x(t)) = P_O(t)h^D_O(t, x) - w_O e^D_O(t) + m_O(\dot{e}_O - e^D_O(t)), \]

which, as we now know, feeds into the cooperator dynamics for their respective sub-populations defined as:

\[ n_S = C_S(t) + D_S(t), \]

\[ n_O = C_O(t) + D_O(t) \]

Now we have presented the entirety of the new additions to the original model developed by Richter and Dakos (2015). The reader can refer to Table 3.1 for symbols contained throughout the text.
Chapter 4

Analysis

Our analysis will be presented from a deterministic standpoint for a selection of parameters that are in line with the original authors’ ideas for their model. With that in mind, this section will classify the extended model utilizing its fixed points, providing a derivation of a Jacobian, and finally giving a summary of eigenvalues for both a general case and for a specific set of parameters. We will provide the reader with an idea of how this model behaves with respect to perfect conditions. In addition to this analysis, we demonstrate that solutions to this model are indeed restricted to the positive orthant of the solution space. Proceeding from this chapter will be a re-presentation of the parameters we have extracted from literature from both the OMNRF and SON on their Lake Whitefish reports from 2016. From there, the focus will be on both providing an restricted analysis from the point of view of a cooperating system.
4.1 Invariant Orthant

Our model encompasses a physical situation, as such we would like to expect physically plausible results from this model. Physical being realistic in any real-world setting. A prime example of a non-physical solution would be either an imaginary or negative value for $x(t)$, as these would represent a population of fish that is impossible in our actual world. A nice check for the avoidance of situations like this comes from application of the following theorem to a differential model such that we can prove invariance of our solutions in the positive orthant space.

**Theorem 4.1.1** If

$H1. \bar{f}(t, \bar{x}(t)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous in $\bar{x}(t)$ for $t \geq 0 \ \forall \ \bar{x}(t)$, and, it is is continuous for $t$.

$H2. \bar{f}(t, \bar{x}(t))$ satisfies $f_i(t, \bar{x}(t)) \geq 0$ if, for $t > 0$, $x_i = 0$, and, $x_j \geq 0$, $i \neq j$.

Then any solution to our differential equations $\frac{d\bar{x}(t)}{dt} = \bar{f}(t, \bar{x}(t))$, with $\bar{x}(0) = x_0$ in the positive orthant, stays inside of the positive orthant (Kunze, 2014)

Direct application of this theorem is possible with our non-autonomous model. Since we initially allowed for the arctan approximation of our efforts, we can state that our model is in fact continuously differentiable, $C^1$, for local regions, and, local $C^1$ implies Lipschitz continuity. This satisfies the first hypothesis of the invariant
orthant theorem. The second hypothesis can be satisfied if we let:

\[
\frac{dx(t)}{dt} = f_1(x(t), C_S(t), C_O(t)) \\
= r x(t)(x(t) - x_{crit}) \left(1 - \frac{x(t)}{k}\right) - H_T(t, x(t), C_S(t), C_O(t))
\]

(4.1)

\[
\frac{dC_S(t)}{dt} = f_2(x(t), C_S(t), C_O(t)) \\
= \frac{\gamma_S C_S(t) D_S(t)}{n_S} - \beta_S C_S(t) \left(1 - \frac{\pi_S^C(t, x(t))}{\pi_S^D(t, x(t))}\right)
\]

(4.2)

\[
\frac{dC_O(t)}{dt} = f_3(x(t), C_S(t), C_O(t)) \\
= \frac{\gamma_O C_O(t) D_O(t)}{n_O} - \beta_O C_O(t) \left(1 - \frac{\pi_O^C(t, x(t))}{\pi_O^D(t, x(t))}\right)
\]

(4.3)

then solving each of these we see:

\[
f_1(0, C_S(t), C_O(t)) = r(0)(0 - x_{crit}) \left(1 - \frac{0}{k}\right) - H_T(0, C_S(t), C_O(t)) = 0 - 0,
\]

assuming \( C_S(t), C_O(t) \geq 0 \),

\[
\Rightarrow f_1(0, C_S(t), C_O(t)) = 0
\]

\[
f_2(x(t), 0, C_O(t)) = \frac{\gamma_S(0) D_S(t)}{n_S} - \beta_S(0) \left(1 - \frac{\pi_S^C(x(t))}{\pi_S^D(x(t))}\right) = 0 - 0,
\]

assuming \( x(t), C_O(t) \geq 0 \),

\[
\Rightarrow f_2(x(t), 0, C_O(t)) = 0
\]
and finally,

\[ f_3(x(t), C_S(t), 0) = \frac{\gamma_O(0)D_O(t)}{n_O} - \beta_0 0 \left( 1 - \frac{\pi_C^O(x(t))}{\pi_D^O(x(t))} \right) = 0 - 0, \]

assuming \( x(t), C_O(t) \geq 0, \)

\[ \implies f_3(x(t), C_S(t), 0) = 0 \]

Thus we can say that for our system of differential equations, and solution that starts in the positive orthant, will stay in the positive orthant. Ensuring that our solutions that are physical, never become nonphysical.

An interesting result of the application of the invariant orthant theorem yields an additional result from our system. Normally, as it is stated by the conclusion of the invariant orthant theorem, solutions stay in the invariant orthant when starting inside of it. However, for our system, we have a stronger result. Our system restricts any solution that starts or enters the face of the invariant orthant to the face of said orthant. Figure 4.1 demonstrates an example sketch for the solution space of this result. For that example, it implies that if we have a solution to our system that allows the OMNRF cooperators to cease existence (that is, full defection), then at that point, it is up to the SON subsystem to counter act this. This further implies that the other results could be possible, that is, the opposite situation, and, finally, the less physical solution wherein the resource dies but there is still a solution that exchanges between the other two social systems.
4.2 Autonomous Model

It has been possible to apply our logic and theorems to the current standing model. Currently, our system is non-autonomous that is, it depends explicitly on the independent variable, $t$,

$$\frac{d\bar{x}(t)}{dt} = \bar{f}(t, \bar{x}(t)).$$
such non-autonomous models typically require more sophisticated analytic solution
techniques (Miino et al., 2015). We instead complete an analysis on the related
autonomous model by assuming

\[ e_C^C(t, x(t)), e_D^C(t), e_O^C(t), e_O^D(t) = \text{constant}. \]

Then

\[ H(t, x(t), C_S(t), C_O(t)) = H(x(t), C_S(t), C_O(t)), \]

\[ \pi_C^C(t, x) = \pi_S^C(x), \]

\[ \pi_S^D(t, x) = \pi_S^D(x), \]

\[ \pi_O^C(t, x) = \pi_O^C(x), \]

and finally,

\[ \pi_O^D(t, x) = \pi_O^D(x) \]

Then this leads to an autonomous version our System 4.1-4.3 given by

\[ \frac{dx(t)}{dt} = rx(t)(x(t) - x_{crit}) \left( 1 - \frac{x(t)}{k} \right) - H_T(x(t), C_S(t), C_O(t)), \quad (4.4) \]

\[ \frac{dC_S(t)}{dt} = \gamma_S C_S(t) D_S(t) \left( 1 - \frac{\pi_C^C(x(t))}{\pi_S^D(x(t))} \right), \quad (4.5) \]
\[
\frac{dC_O(t)}{dt} = \gamma O C_O(t) D_O(t) - \beta O C_O(t) \left( 1 - \frac{\pi O(x(t))}{\pi O(x(t))} \right), \quad (4.6)
\]

where Equations 4.4, 4.5, and 4.6 now have reliance on state variables only.

### 4.2.1 Fixed Points

A fixed or critical point of the differential equation

\[
\frac{dx}{dt} = f(x(t))
\]

is any point, \(x_0\), such that

\[
f(x_0) = 0.
\]

Extending to a system of equations, denote by \(\bar{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)\) the fixed point to the system

\[
\bar{x}' = \bar{f}(x),
\]

where \(\bar{x}^*\) satisfies

\[
\bar{f}(\bar{x}^*) = \begin{pmatrix}
f_1(x_1^*) \\ f_2(x_2^*) \\ f_3(x_3^*)
\end{pmatrix} = \begin{pmatrix}
0 \\ 0 \\ 0
\end{pmatrix} = \bar{0}
\]
From the form given in Equations 4.1, 4.2, and 4.3 we may write that \((x^*, C^*_S, C^*_O)\) are a plane of fixed point solutions to our system of differential equations that satisfy:

\[
\begin{align*}
  f_1(x^*, C^*_S, C^*_O) &= 0 \\
  f_2(x^*, C^*_S, C^*_O) &= 0 \\
  f_3(x^*, C^*_S, C^*_O) &= 0
\end{align*}
\]

which when applied to our model provides some trivial and non-trivial fixed points for our system. The trivial fixed points consist of: **Set 1**

\[
\begin{align*}
  x^* &= 0 \\
  C^*_S &= 0 \\
  C^*_O &= 0
\end{align*}
\]

**Set 2**

\[
\begin{align*}
  x^* &= 0 \\
  C^*_S &= 0
\end{align*}
\]

\[
C^*_O = \frac{n_0(\beta_O(e^O_{mO} + e^O_{wO} - e^P_{wO}) + \gamma_O(e^P_{mO} + e^P_{wO} - \hat{e}_omO))}{\gamma_O(e^P_{mO} + e^P_{wO} - \hat{e}_omO)}
\]
Set 3

\[ x^* = 0 \]

\[ C^*_O = 0 \]

\[ C^*_S = \frac{n_S(\beta_S(e^S_{C}m_S + e^S_{C}w_S - e^S_{D}w_S) + \gamma_S(e^D_{S}m_S + e^D_{S}w_S - \hat{e}sw_S))}{\gamma_S(e^D_{S}m_S + e^D_{S}w_S - \hat{e}sm_S)} \]

and finally in combination, Set 4

\[ x^* = 0 \]

\[ C^*_S = \frac{n_S(\beta_S(e^S_{C}m_S + e^S_{C}w_S - e^S_{D}w_S) + \gamma_S(e^D_{S}m_S + e^D_{S}w_S - \hat{e}sw_S))}{\gamma_S(e^D_{S}m_S + e^D_{S}w_S - \hat{e}sm_S)} \]

\[ C^*_O = \frac{n_O(\beta_O(e^O_{C}m_O + e^O_{C}w_O - e^O_{D}w_O) + \gamma_O(e^D_{O}m_O + e^D_{O}w_O - \hat{e}ow_O))}{\gamma_O(e^D_{O}m_O + e^D_{O}w_O - \hat{e}om_O)} \]

These solutions are not particularly interesting in terms of their potential description of our system. When solving for the remaining, non-trivial fixed points we required computer assistance. Our code is provided at [https://github.com/rkett/dual-fishery-management-model](https://github.com/rkett/dual-fishery-management-model) wherein our reader can find simulation models, and, the current analytical model. The analytical model was solved using a simple set of procedures in Maple [https://www.maplesoft.com/](https://www.maplesoft.com/) We solved for a total of 8 sets of fixed points each pushing further and further into the realm of complexity and length. The final fixed point (set 8) represents the coexistence case. Due to its
complexity and length we do not display it here. Of interest now is the stability of these equilibria, that is, how solutions to our system behave in a neighborhood of these equilibria. Moving forward, we direct the reader back to this section if there is any mention of “sets” here on out, as, we will refer to these as sets 1, ..., 8 whenever they are required.

4.2.2 Linearization of Non-linear Autonomous Differential System

Since stability analysis for linear systems is well known and relatively simple to carry out, when it comes to classification of equilibria to nonlinear systems we often "linearize" in an attempt to utilize these methods. Linearization of a non-linear system utilizes a Taylor Expansion of the vector-valued function $\bar{f}$, to the first degree centered around a given fixed point $\bar{x}^*$. That is, building our approximation around $\bar{x}^*$ we see

$$\bar{f}(\bar{x}) = \bar{f}(\bar{x}^*) + \frac{D\bar{f}(x)(\bar{x} - \bar{x}^*)}{1!} + \frac{D^2\bar{f}(x)(\bar{x} - \bar{x}^*)^2}{2!} + \ldots$$

(4.7)

and,

$$J = D\bar{f}(\bar{x}^*) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \ldots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$
is the Jacobian, of the system. It is reasonable then to assume that a good approximation around the local neighborhood of \( \bar{x}^* \) can be approximated by the linear terms of the Taylor Expansion, and, that higher order terms will approach 0, under these assumptions 4.7 leads to

\[
\bar{f}(\bar{x}) \approx D\bar{f}(\bar{x}^*)(\bar{x} - \bar{x}^*)
\]

as \( \bar{f}(\bar{x}^*) = 0 \) when \( \bar{x}^* \) is an equilibrium point. Thus in a neighborhood of \( \bar{x}^* \), our nonlinear system can be approximated by the linear system

\[
\frac{d\bar{x}}{dt} = D\bar{f}(\bar{x}^*)(\bar{x} - \bar{x}^*)
\]

Applying these ideas to our model 4.4, 4.5, and 4.6 with \( x(t) = x_1, C_S(t) = x_2 \) and \( C_O(t) = x_3 \) and \( f_1 \ldots f_3 \) are as denoted

\[
\begin{align*}
\frac{dx(t)}{dt} &= f_1(x(t), C_S(t), C_O(t)) \\
&= rx(t)(x(t) - x_{crit}) \left( 1 - \frac{x(t)}{k} \right) - H_T(x(t), C_S(t), C_O(t)) \\
&= f_2 \\
\frac{dC_S(t)}{dt} &= f_2(x(t), C_S(t), C_O(t)) \\
&= \frac{\gamma_S C_S(t) D_S(t)}{n_S} - \beta_S C_S(t) \left( 1 - \frac{\pi_S^G(x(t))}{\pi_S^H(x(t))} \right) \\
&= f_2
\end{align*}
\]
\[
\frac{dC_O(t)}{dt} = f_3(x(t), C_S(t), C_O(t))
\]
\[
= \gamma_o C_O(t) DO(t) - \beta_o C_O(t) \left(1 - \frac{\pi_0^O(x(t))}{\pi_0^O(x(t))}\right)
\]
\[
= f_3
\]
then our Jacobian is given by
\[
J(f_1, f_2, f_3) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\
\frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3}
\end{bmatrix}
\]

It can be shown that solutions to linear systems \( \tilde{x}' = A\tilde{x} \) involve terms of the form \( e^{Re(\lambda)t} \) where \( \lambda \) is an eigenvalue of \( A \in \mathbb{R}^{n\times n} \). Then, the evaluation of the determinant of the Jacobian at a fixed point
\[
\det J(f_1(x_1^*, x_2^*, x_3^*), f_2(x_1^*, x_2^*, x_3^*), f_3(x_1^*, x_2^*, x_3^*))
\]
will yield three eigenvalues, \( \lambda \) (possibly repeated). The limiting behavior of these solutions (which relies on the sign of \( Re(\lambda) \)) allows one to analyze the behaviour of the system of equations (Chicone, 2006). In the case of a nonlinear system that has been linearized, global stability of an equilibrium point can be determined only in some circumstances as the following theorem states.
Theorem 4.2.1 Let $\bar{x}^*$ be a fixed point of the non-linear autonomous system $\frac{dx}{dt} = \bar{f}(\bar{x})$. Let the linear approximation of this system then be represented by $\frac{dx}{dt} = D\bar{f}(\bar{x}^*)(\bar{x} - \bar{x}^*)$. If $D\bar{f}(\bar{x}^*)$ has eigenvalues with non-zero real parts, then the stability of $x^*$ in the non-linear system coincides with the linearized system.

Applying these ideas to our system, we must solve for a total of seven partial derivatives as certain cross terms which the reader can verify by going back through our system are not present, implying that certain partial derivatives are 0. Let us solve these one at a time:

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_1}{\partial x},$$

$$\frac{\partial f_1}{\partial x(t)} = \frac{\partial}{\partial x} \left[ rx(t)(x(t) - x_{\text{crit}})(1 - \frac{x(t)}{k}) - H_T(x, C_S, C_O) \right]$$

$$= \frac{\partial}{\partial x(t)} \left[ rx(t)(x(t) - x_{\text{crit}})(1 - \frac{x(t)}{k}) \right] - \frac{\partial}{\partial x} [H_T(x(t), C_S(t), C_O(t))],$$

and letting $\frac{\partial}{\partial x(t)} H_T(x(t)) = H_{T_x}(x(t), C_S(t), C_O(t))$ for simplicity, we then have

$$\frac{\partial f_1}{\partial x(t)} = \frac{\partial}{\partial x(t)} \left[ rx(t)(x(t) - x_{\text{crit}})(1 - \frac{x(t)}{k}) \right] - H_{T_x}(x(t), C_S(t), C_O(t)) \quad (4.8)$$

$$= \frac{\partial}{\partial x(t)} \left[ (rx^2(t) - rx(t)x_{\text{crit}})(1 - \frac{x(t)}{k}) \right] - H_{T_x}(x(t), C_S(t), C_O(t))$$

$$= \frac{\partial}{\partial x(t)} \left[ (rx^2(t) - \frac{rx^3(t)}{k} - rx_{\text{crit}}x(t) + \frac{rxx_{\text{crit}}^2(t)}{k}) \right] - H_{T_x}(x(t), C_S(t), C_O(t))$$

$$= 2rx(t) - \frac{3rx^2}{k} - rx_{\text{crit}} + \frac{2rxx_{\text{crit}}x(t)}{k} - H_{T_x}(x(t), C_S(t), C_O(t)).$$
= 2rx(t) - \frac{3rx^2}{k} - rx_{crit} + \frac{2rx_{crit}x(t)}{k} - E_T(C_S(t), C_O(t))

as \( H_T(x(t), C_S(t), C_O(t)) = x(t)E_T(C_S(t), C_O(t)) \). Now, due to the dependence that total harvest has on both \( C_S(t) \) and \( C_O(t) \) we must generate derivatives of \( f_1 \) with respect to those two variables. Starting with:

\[
\frac{\partial f_1}{\partial x_2} = \frac{\partial f_1}{\partial C_S(t)},
\]

noting from the form of \( f_1 \) we know that the first set of terms are never changing with respect to \( C_S(t) \), then

\[
\frac{\partial f_1}{\partial C_S(t)} = \frac{\partial}{\partial C_S(t)} [-H_T(x(t), C_S(t), C_O(t))]
\]

\[
= \frac{\partial}{\partial C_S(t)} [-x(t)E_T(C_S(t), C_O(t))]
\]

\[
= -x(t) \frac{\partial}{\partial C_S(t)} \left[ q_S (e^D_S(n_s - C_S(t)) + e^C_S C_S(t)) + q_O (e^D_O(n_o - C_O(t)) + e^C_O C_O(t)) \right];
\]

which reduces to:

\[
\frac{\partial f_1}{\partial C_S} = q_S x(t) (e^D_S - e^C_S),
\]
and, once again, due to the form that total harvest takes on, and the form of \( f_1 \), we have, by similar processes that:

\[
\frac{\partial f_1}{\partial C_O(t)} = q_O x(t) \left( e^D_O - e^C_O \right)
\]

Moving on from here, the next calculations focus upon finding the other two differential equations with respect to both \( x(t) \) as well as \( C_S(t) \) and \( C_O(t) \); then, we start with:

\[
\frac{\partial f_2}{\partial x} = \frac{\partial f_2}{\partial x(t)}
\]

\[
\frac{\partial f_2}{\partial x(t)} = \frac{\partial}{\partial x(t)} \left[ \frac{\gamma_S C_S(t) D_S(t)}{n_S} \right] - \frac{\partial}{\partial x(t)} \left[ \beta_S C_S(t) \left( 1 - \frac{\pi^C_S (x(t))}{\pi^D_S (x(t))} \right) \right],
\]

now, noting that the first term has zero \( x(t) \) dependence, we may write:

\[
= - \frac{\partial}{\partial x(t)} \left[ \beta_S C_S(t) \left( 1 - \frac{\pi^C_S (x(t))}{\pi^D_S (x(t))} \right) \right]
\]

\[
= - \frac{\partial}{\partial x(t)} \left[ \beta_S C_S(t) \right] + \frac{\partial}{\partial x(t)} \left[ \beta_S C_S(t) \frac{\pi^C_S (x(t))}{\pi^D_S (x(t))} \right], \tag{4.9}
\]

once again, the first term in Equation \( 4.9 \) is independent of \( x(t) \) so Equation \( 4.9 \) reduces to:

\[
= \frac{\partial}{\partial x(t)} \left[ \beta_S C_S(t) \frac{\pi^C_S (x(t))}{\pi^D_S (x(t))} \right],
\]
and, expanding the derivative we have:

$$\frac{\partial}{\partial x(t)} \left[ \beta_S C_S(t) \frac{\pi^C_S(x(t))}{\pi^D_S(x(t))} \right] = \beta_S C_S(t) \left( \frac{\pi^C_S(x(t)) \pi^D_S(x(t)) - \pi^C_S(x(t)) \pi^D_S(x(t))}{(\pi^D_S(x(t)))^2} \right),$$

(4.10)

where $\pi^C_S(x(t))$ and $\pi^D_S(x(t))$ are partial derivatives of $\pi^C_S(x(t))$ and $\pi^D_S(x(t))$ with respect to $x(t)$. Because of the piece-wise nature of each of the intermediate forms of these profits, we will proceed to write them into substituent components. These components are assumed to be substituted back into Equation 4.10 at the end of the derivation. Now,

$$\pi^C_{S_x}(x(t)) = \frac{\partial}{\partial x(t)} \left[ P_S h^C_S(x(t)) - w_S e^C_S + m_S (\hat{e}_S - e^C_S) \right]$$

$$= \frac{\partial}{\partial x(t)} \left[ P_S h^C_S(x(t)) - w_S e^C_S + m_S \hat{e}_S - m_S e^C_S \right]$$

$$= \frac{\partial}{\partial x(t)} \left[ P_S h^C_S(x(t)) \right] - \frac{\partial}{\partial x(t)} \left[ w_S e^C_S \right] + \frac{\partial}{\partial x(t)} \left[ m_S \hat{e}_S \right] - \frac{\partial}{\partial x(t)} \left[ m_S e^C_S \right]$$

$$= \frac{\partial}{\partial x(t)} \left[ P_S h^C_S(x(t)) \right] - \frac{\partial}{\partial x(t)} \left[ w_S e^C_S \right] - \frac{\partial}{\partial x(t)} \left[ m_S e^C_S \right]$$

$$= \frac{\partial}{\partial x(t)} \left[ P_S h^C_S(x(t)) \right]$$

where $h^C_{S_x}(x(t))$ is partial derivatives of SON cooperator harvest respect to $x(t)$, then

$$h^C_{S_x}(x(t)) = q_S e^C_S$$
returning to look at Equation 4.10 we see that we must calculate $\pi^D_S(x(t))$, then

$$
\pi^D_S(x(t)) = \frac{\partial}{\partial x(t)} \left[ P_S h^D_S(x(t)) - w_S e^D_S + m_S e^D_S - m_S e^D_S \right],
$$

noting that once again, $e^D_S$ is constant we may write:

$$
\pi^D_S(x(t)) = P_S(t) h^D_S(x(t)),
$$

and much like its cooperative counterpart, $h^D_S(x(t))$ is the partial derivative of SON defector harvest with respect to $x(t)$ leading to:

$$
h^D_S(x(t)) = q_S e^D_S,
$$

then finally, we are left with the last governing partial derivative to fulfill Equation 4.10:

$$
\pi^D_S(x(t)) = P_S q_S e^D_S,
$$

Moving on, the next step is to calculate $\frac{\partial f_2}{\partial x_2}$ which is:

$$
\frac{\partial f_2}{\partial x_2} = \frac{\partial f_2}{\partial C_S(t)},
$$
where,

\[
\frac{\partial f_2}{\partial C_S(t)} = \frac{\partial}{\partial C_S(t)} \left[ \frac{\gamma_S C_S(t) D_S(t)}{n_S} \right] - \frac{\partial}{\partial C_S(t)} \left[ \beta_S C_S(t) \left( 1 - \frac{\pi_S^C(x(t))}{\pi_S^D(x(t))} \right) \right]
\]

\[
= \frac{\partial}{\partial C_S(t)} \left[ \frac{\gamma_S C_S(t)(n_S - C_S(t))}{n_S} \right] - \beta_S \left( 1 - \frac{\pi_S^C(x(t))}{\pi_S^D(x(t))} \right)
\]

\[
= \frac{\partial}{\partial C_S(t)} \left[ \frac{\gamma_S C_S(t) n_S - \gamma_S(C_S(t))^2}{n_S} \right] - \beta_S \left( 1 - \frac{\pi_S^C(x(t))}{\pi_S^D(x(t))} \right)
\]

\[
= \gamma_S - \frac{2\gamma_S C_S(t)}{n_S} - \beta_S \left( 1 - \frac{\pi_S^C(x(t))}{\pi_S^D(x(t))} \right),
\]

(4.11)

noting that \( n_S \) is constant with respect to the change in \( C_S(t) \) and \( D_S(t) \). Much like the approach used to calculate \( \frac{\partial f_2}{\partial x(t)} \) we now want:

\[
\frac{\partial f_3}{\partial C_S(t)} = \frac{\partial}{\partial C_S(t)} \left[ \frac{\gamma_O C_O(t) D_O(t)}{n_O} - \beta_O C_O(t) \left( 1 - \frac{\pi_O^C(x(t))}{\pi_O^D(x(t))} \right) \right]
\]

\[
= - \frac{\partial}{\partial x(t)} \beta_O C_O(t) \left( 1 - \frac{\pi_O^C(x(t))}{\pi_O^D(x(t))} \right)
\]

\[
= \frac{\partial}{\partial x(t)} \beta_O C_O(t) \frac{\pi_O^C(x(t))}{\pi_O^D(x(t))}
\]

\[
= \beta_O C_O(t) \left( \frac{\pi_O^C(x(t)) \pi_O^D(x(t)) - \pi_O^C(x(t)) \pi_O^D(x(t))}{\pi_O^D(x(t))^2} \right),
\]

where:

\[
\pi_O^C(x(t)) = \frac{\partial}{\partial x(t)} \left[ P_O(t) h_O^C(x(t)) - w_O e_O^C + m_O (e_O - e_O^C) \right],
\]
recalling from Equation 4.11 we have

\[ \pi^{C}_{O}(x(t)) = P_{O}e_{O}^{C}, \quad (4.13) \]

and,

\[ \pi^{D}_{O}(x(t)) = \frac{\partial}{\partial x(t)} \left[ P_{O}h_{O}^{D}(x(t)) - w_{O}e_{O}^{D} + m_{O}(\dot{e}_{O} - e_{O}^{D}) \right], \]

once again, we arrive at:

\[ \pi^{D}_{O}(x(t)) = P_{O}e_{O}^{D}, \quad (4.14) \]

substituting Equations 4.13 and 4.14 into Equation 4.12 we obtain:

\[ \frac{\partial f_{3}}{\partial x(t)} = P_{O} \left( e_{O}^{C} - \frac{e_{O}^{D} \pi^{C}_{O}(x(t))}{\pi^{D}_{O}(x(t))} \right) \]

The final derivative is of \( \frac{\partial f_{3}}{\partial x_{3}(t)} \), or,

\[ \frac{\partial f_{3}}{\partial x_{3}(t)} = \frac{\partial f_{3}}{\partial C_{O}(t)}, \]

then, following almost the exact same procedure as was done for \( \frac{\partial f_{2}}{\partial x_{2}} \) we have that

\[ \frac{\partial f_{3}}{\partial C_{O}(t)} = \gamma_{O} - \frac{2\gamma_{O}C_{O}(t)}{n_{O}} - \beta_{O} \left( 1 - \frac{\pi^{C}_{O}(x(t))}{\pi^{D}_{O}(x(t))} \right) \]
4.2.3 Stability

The size and complexity of the fixed points leads to some incredibly non-terse eigenvalues for each of the sets of fixed points. As a result demonstration of certain limited eigenvalues are presented. Since we have so many functional forms provided, and, non-retrieved parameters, we did require some assistance from the original authors of the base model that our extensions are based on. Our analysis begins with a brief overview of the base model parameters and their values, followed by some exploration into a few different cases using these values. From there we demonstrate a set of values, found through human ingenuity, that propose a coexistence solution between all three populations that govern our system. The parameters that were found will be motivated by a few observations from both our numerical solutions to the model as well as from simple trial-and-error.

Vanilla Stability Analysis

Provided in Table 4.1 is the set of original parameters used by Richter and Dakos (2015). These parameters are then mirrored across their respective SON and OMNRF counterparts. The distinct reason we decided on using these parameters was due to the fact that for the base model, they provided a fixed point in $x^*$ and $C^*$ that was coexistent for a time period of 100 years. Then a logical jump to our model was to mirror these values, as the two managements do not interact, it provides a good basis to assume that they would be “exactly the same”. For reading simplicity,
when we refer to the base model parameters, in their form provided in Table 4.1, we will denote them: vanilla. A numeric search can then conducted first using these vanilla parameters, we will present a few cases and note observations of the equilibria. Specifically we can inquire about the trivial case, the dual existence (of cooperators), and finally talk a bit about potential 3-way coexistence before moving on. Then, up first is:

**Situation \( A_{\text{vanilla}} \):** \((x^*, C_S^*, C_O^*) = \text{Set 1, (Total Annihilation)}\)

\[
\det(J_{\text{set 1}}) = \begin{bmatrix}
-0.2 - 0.7e_D^S - 0.7e_D^O & 0 & 0 \\
0 & -0.1 + \frac{0.2(0.6 - 2e_D^S)}{0.6 - 2e_D^S} & 0 \\
0 & 0 & -0.1 + \frac{0.2(0.6 - 2e_D^O)}{0.6 - 2e_D^O}
\end{bmatrix}
\]

where we note, that for the trivial case we have a nice diagonal matrix, as such the eigenvalues, denoted \( \lambda_i \), where \( i = 1 \ldots 3 \), can be immediately taken from the diago-
nal. One of the first interesting observations comes from manipulating our constant variants of effort. In fact, one can produce an asymptotically stable trivial equilibrium \((\lambda_1, \lambda_2, \lambda_3) < 0\) so long as \(e^C_S \geq 0.3, 0 < e^D_S \leq 0.3, e^C_O \geq 0.3,\) and, \(0 < e^D_O \leq 0.3\). Then, we can note a few items of physicality about this situation: cooperators are able to fish with greater effort than their defector counter parts. Figure 4.2 demonstrates asymptotic behaviour that falls upon the value of the defector effort in either management. This asymptotic behaviour befalls the value of 0.3 for either management’s defector effort. However, interesting to note, in the way that the biological model is framed we have a switching mechanism which implies that the only allowable values for \(e^D\) are 0 or 0.6 if we want to match the actual model. The last item to address is the fact that it is actually possible when \(e_C = 0.05\) to have a scenario where in either \(\lambda_1 = 0\) or \(\lambda_2 = 0\). Since the linearization is not reliable globally when a zero eigenvalue occurs, other methods such as the construction of a Lyapunov function, or the investigation of stability using non-linear methods.
Figure 4.2: Asymptotic Behaviour of $\lambda_2$ and $\lambda_3$ for Set 1 Using Vanilla Parameters. for constant efforts ranging from 0 to 0.6 for both $C$ and $D$. This result applies to either management.

**Situation $B_{vanilla}$**: $(x^*, C^*_S, C^*_O) = \text{Set 4, (Coexistence of Cooperators, Resource Annihilation)}$

$$\text{det}(J_{set\ 2}) = \begin{bmatrix} -0.2 - 0.7 e^C_S - 0.7 e^C_O & 0 & 0 \\ 70000 \frac{e^C_S}{0.6-2e^C_S} & -0.3 + \frac{0.2(0.6-2e^C_O)}{0.6-2e^D_O} & 0 \\ 70000 \frac{e^C_O}{0.6-2e^C_S} & 0 & -0.3 + \frac{0.2(0.6-2e^C_O)}{0.6-2e^D_O} \end{bmatrix}$$

In the case where we can produce the cooperation of both SON and OMNRF cooperators in the absence of any fish, we note that the form of the eigenvalues are *effectively* the same as the case wherein total annihilation occurs. The most distinct difference being that $\lambda_1$ requires a choice of cooperator efforts instead of defector efforts like in
**Situation A.** The eigenvalues are real and negative when we have a choice of efforts such that \( e^C_S = e^C_O = e^D_S = e^D_O = 0.6 \). In observation of these parameters, we also notice, through comparison with **Situation A** that when this is true, we actually see that the eigenvalues produced for Set 1 are now unstable. While this observation is minimal in its scope, it is helpful in determining the potential fine line that these two scenarios operate on. This does indicate barring any odd behaviour for differences in constants, that when we lack a resource there is a possibility that everyone cooperates, or, everyone defects. This plays into an instance of greed versus altruism as when we are near a point of no return, we can say with some certainty that it is possible to have everyone working together or nobody at all, and, it relies on how they have been distributing their effort.

**Situation \( C_{vanilla} \):** \((x^*, C^*_S, C^*_O) = \text{Set 8, (Non-physical Unstable Coexistence of Cooperators and Resource)}\)

Once again, due to the size, complicated nature, and lack of hardware, we must impose this last portion on the reader in good faith. For this last portion, we had a fixed point:

\[
x^* = 0.0514467632136880,
\]

\[
C^*_S = 98.7704052460465,
\]

\[
C^*_O = 102.52153063940,
\]

which is non-physical, as \( C^*_O > n_O \). The eigenvalues consist of \( \lambda_1 = +\text{Real}, \lambda_2 = \cdots \)
\[ \lambda_3 = -\text{Real} + \text{Imaginary} \]. Although their meaning is not required due to the physicality of the situation. However, as our reader can note from this, selectively playing with parameters in an educated manner allowed us to demonstrate that it is possible to achieve 3-way coexistence. From here, it should be obvious that we should explore situations that might gleam physical results.

**Non-Vanilla Stability Analysis**

In order to produce results that demonstrate more of the power and behaviour of the model it was required that more observations be made about other parameters. Up until this point we have kept everything constant at their vanilla values and altered only our constant efforts. Playing with the other parameters yields some nice behaviour. We first observe that the system has a large dependence on \( \gamma_S, \gamma_O, \beta_S, \) and, \( \beta_O \). This is likely because we have to hold effort constant, the typical biological framework allows for our social system to switch depending on the status of the resource, however when this is removed there is likely a larger dependence on peer pressure and temptation that drives the system. Secondly, the alternative market drives away cooperation. This will be talked about in a moment, but, it was noted when experimenting with the system that the alternative market has a sway on cooperation, and specifically the existence of the resource. Lastly \( e_C^O \) also sways the system towards total annihilation, it was noted that when the cooperators for OMNRF are given a small effort, 3-way coexistence was more likely. With these observations we
can finally talk about the parameters that were actually changed. Table 4.2 provides a table of values used to guide ourselves to a realistic 3-way coexistence. Values in blue are with respect to the SON, and red, to the OMNRF. We increased the influence of peer pressure, and decreased the influence of temptation across both systems. Then, inline with our observations, turned off the alternative market \((m_O = 0)\) for the OMNRF, and made the revenue rate \(m_S\) tiny in magnitude \((m_S = 0.0001)\) for the SON (this was a simple trial-and-error, other values could be used, but our results fit for this choice). Now that the parameters have been outlined, we have built ourselves up

\[
\begin{array}{|c|c|c|c|}
\hline
\text{SON} & \text{OMNRF} & \text{Global} \\
\hline
n_S & 100 & n_O & 100 & k & 10 \\
P_S & 5000 & P_O & 5000 & x_{\text{crit}} & 0.1 \\
w_S & 1 & w_O & 1 & r & 0.2 \\
\gamma_S & 0.7 & \gamma_O & 0.7 \\
\beta_S & 0.2 & \beta_O & 0.2 \\
\hat{e}_S & 0.6 & \hat{e}_O & 0.6 \\
q_S & 0.007 & q_O & 0.007 \\
m_S & 0.0001 & m_O & 0 \\
\hline
\end{array}
\]

Table 4.2: Parameters used by [Richter and Dakos 2015] mirrored across to the extended model with global parameters utilized for analysis, including the new, non-vanilla values given in blue and red.

to demonstrate coexistence we will split these up into two distinct results: unstable \((A_{\text{non-vanilla}})\), and, stable \((B_{\text{non-vanilla}})\). From there we will talk briefly about both and the physical nature they describe for our selective parameters.

Situation \(A_{\text{non-vanilla}}\): \((x^*, C^*_S, C^*_O) = \text{Set 8}, \ (\text{Unstable Coexistence of Cooperators and Resource})\)
The fixed point values obtained are:

\[ x^* = 9.99987920218154, \]
\[ C^*_S = 0.00285546312190942, \]
\[ C^*_O = 99.9971608183767, \]

which is unstable, as we have that all eigenvalues being real for, \( \lambda_1, \lambda_2 < 0 \), and, \( \lambda_3 > 0 \). This was achieved for effort values being \( e^C_S = 0.6, e^D_S = 9.4663e^{-31}, e^C_O = 9.4663e^{-31}, \) and, \( e^D_O = 0.6 \). The values for \( e^C_O \) as well as \( e^D_S \) were determined by simulating the biological model through numerical solving and feeding this to the analytic solver. One can think of these values as being sufficiently small to obtain a result such as this. These values actually directly feed into the next following scenario.

The physical meaning of this unstable fixed point solution indicates that we do not approach a realistic solution so long as we reduce the effort for select populations. In this case, we have instability likely because there is still presence of defector effort in the SON population. The next scenario drives the physical meaning better, so we will leave it to the next segment for the final discussion.

**Situation** \( B_{non\text{-}vanilla} \): \( (x^*, C^*_S, C^*_O) = \text{Set 8}, (\text{Stable Coexistence of Cooperators and Resource}) \)
The fixed point values obtained are:

\[ x^* = 9.99987920218154, \]

\[ C^*_S = 0.00285546312190942, \]

\[ C^*_O = 99.9971608183767, \]

and we have that all eigenvalues are real and \( \lambda_1, \lambda_2, \lambda_3 < 0 \), which is asymptotically stable. This was achieved for effort values being \( e^C_S = 0.6, e^D_S = 0, e^C_O = 9.4663e - 31 \), and, \( e^D_O = 0.6 \). The difference between these two results indicate that the difference between a stable and unstable result relies on there being no SON defector effort. This does not mean that this is the only pathway to this result, it is just the pathway we found for these parameters. In that case, we note that the fixed points are relatively large for the resource as well as the OMNRF cooperator population. However, the SON cooperator population is incredibly tiny, one can frame this as *most of the SON population has in fact defected*. But, we removed the effort from the SON defector population, then the only surviving SON cooperators can fish at the maximum rate. The OMNRF cooperators were also given a very tiny effort, and their defectors maximum effort. In finality, the OMNRF cooperators carried the system stable coexistence, as they do not give much effort. Important to note on this subject, this is only for *chosen* values, implying then that technically these equilibria is not generally stable as we have shown there is potential for instability through choice parameters as well.
4.3 Simulations

The analysis guides the reader through a few select sub-sets of equilibrium values. We were also able to numerically solve our system using the MATLAB programming language. Once again, the code developed is provided at https://github.com/rkett/dual-fishery-management-model. The results provided for this model indicate that stability is possible for long (100 years) periods of time. This stability was provided using the same values used in Subsubsection 4.2.3 for a market being on, \((m = 1)\), for both managements. In addition to those parameters, the model solved was that of the biological model with stochastic effects included. For additional context, the vanilla parameters were simulated as well.

The first step to generating a simulation for this model is to produce results from a numerical standpoint, specifically because of the tiny noise addition that was brought into the model itself. Although the analysis had a large focus on the deterministic results of this model, at the heart of a “real world” scenario there is the potential for noisy data. The introduction of noise does effectively nullify the finite aspects of the analysis, however, due to the restriction of the system to the positive orthant we can still expect results that are close enough to the deterministic outcome, within reason, hence the predominant focus on the term “simulation”. For our purposes, we can run single runs of the system with noise, however, as those reading might anticipate, this can lead to solutions of the system that have the potential for wildly different results.
Then, each of our scenarios will be run for a certain number of simulations such that the sample size is relatively reasonable. While it would be interesting to estimate how many simulations are actually required, this thesis will arbitrarily assign a large number, 100, to be exact, as the number of distinctly different simulations that will be run for each set of parameters there are to test. Now that this is understood, let us move onto the scenarios that are descriptive of the system’s behavior.

### 4.3.1 Vanilla Results

Results are given in Tables 4.3, 4.4, and 4.5. Each of these tables represents a summary of information pertaining to one of the three solutions \((x(t), C_S(t), \text{ and } C_O(t))\) with respect to the vanilla parameters given a different initial value of \(x(t_0)\). The idea of total: mean, max and min is as follows: each simulation is attributed with a mean of its respective solutions as explained above, and finally we take the mean of each of those combined to obtain a “total mean”. So each of these tables is then the culmination of many means across 100 simulations each at a time span of 100 years. In addition to these tables, there are a few select figures to observe as well.
Table 4.3: Table of total: mean, max and min values for $x(t)$ using vanilla parameters.

<table>
<thead>
<tr>
<th>Initial $x(t_0)$ Value</th>
<th>Total Mean $x(t)$</th>
<th>Total Max $x(t)$</th>
<th>Total Min $x(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0.2032</td>
<td>10.00</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{k}{2}$</td>
<td>0.1035</td>
<td>5.0000</td>
<td>0</td>
</tr>
<tr>
<td>$x_{crit}$</td>
<td>0.0015</td>
<td>0.2019</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.4: Table of total: mean, max and min values for $C_S(t)$ using vanilla parameters.

<table>
<thead>
<tr>
<th>Initial $x(t_0)$ Value</th>
<th>Total Mean $C_S(t)$</th>
<th>Total Max $C_S(t)$</th>
<th>Total Min $C_S(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>76.2131</td>
<td>99.9690</td>
<td>24.6070</td>
</tr>
<tr>
<td>$\frac{k}{2}$</td>
<td>77.0696</td>
<td>99.9740</td>
<td>24.7290</td>
</tr>
<tr>
<td>$x_{crit}$</td>
<td>89.9004</td>
<td>99.9950</td>
<td>48.9020</td>
</tr>
<tr>
<td>0</td>
<td>93.0689</td>
<td>99.9950</td>
<td>50.0000</td>
</tr>
</tbody>
</table>

Table 4.5: Table of total: mean, max and min values for $C_O(t)$ using vanilla parameters.

<table>
<thead>
<tr>
<th>Initial $x(t_0)$ Value</th>
<th>Total Mean $C_O(t)$</th>
<th>Total Max $C_O(t)$</th>
<th>Total Min $C_O(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>12.5697</td>
<td>69.8890</td>
<td>0.0005</td>
</tr>
<tr>
<td>$\frac{k}{2}$</td>
<td>11.8743</td>
<td>67.7800</td>
<td>0.0002</td>
</tr>
<tr>
<td>$x_{crit}$</td>
<td>5.5431</td>
<td>57.2140</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>7.3520</td>
<td>50.0000</td>
<td>0.2517</td>
</tr>
</tbody>
</table>
Figure 4.3: Graph of population biomass over 100 years for 100 different simulations using vanilla parameters.
Figure 4.4: Graph of \( C_S(t) \) over 100 years for 100 different simulations using vanilla parameters.
Figure 4.5: Graph of $C_O(t)$ over 100 years for 100 different simulations using vanilla parameters.
Figures 4.3, 4.4, and 4.5 give a visual indication of how $x(t)$, $C_S(t)$, and $C_O(t)$ behave under the new system when $x(t_0)$ is set at the carrying capacity, $k$, with $C_S(t_0) = \frac{n_S}{2}$ and $C_O(t_0) = \frac{n_O}{2}$.

### 4.3.2 Full Cooperation Results

Results following in Tables 4.6, 4.7, and 4.8 are similar to their vanilla counterparts, with the exception that we now have: $\beta_S = 0.1$, $\gamma_S = 0.5$, $\beta_O = 0.1$, and $\gamma_O = 0.4$. These changes constitute what is known as “full cooperation” parameters, and is actually a direct indication of a population that survives 100 years with full cooperation on both management bodies. This is reflected once again in our sets of figures for these full cooperation values. This time, for these parameters we have provided two sets of figures: Figures 4.6, 4.7, and 4.8 give a visual indication of how $x(t)$, $C_S(t)$, and $C_O(t)$ behave under the new system when $x(t_0)$ is set at the carrying capacity, $k$, with $C_S(t_0) = \frac{n_S}{2}$ and $C_O(t_0) = \frac{n_O}{2}$ under full cooperation parameters as indicated earlier. We also have Figures 4.9, 4.10, and 4.11 give a visual indication of how $x(t)$, $C_S(t)$, and $C_O(t)$ behave under the new system when $x(t_0)$ is set at the carrying capacity, $\frac{k}{2}$, with $C_S(t_0) = \frac{n_S}{2}$ and $C_O(t_0) = \frac{n_O}{2}$ once again under full cooperation parameters.
Figure 4.6: Graph of population biomass over 100 years for 100 different simulations using full cooperation parameters: $\beta_s = 0.1$, $\gamma_s = 0.5$, $\beta_o = 0.1$, and $\gamma_o = 0.4$, when $x(t_0) = k$
Figure 4.7: Graph of $C_S(t)$ over 100 years for 100 different simulations using full cooperation parameters: $\beta_S = 0.1$, $\gamma_S = 0.5$, $\beta_O = 0.1$, and $\gamma_O = 0.4$, when $x(t_0) = k$. 
<table>
<thead>
<tr>
<th>Initial $x(t_0)$ Value</th>
<th>Total Mean $x(t)$</th>
<th>Total Max $x(t)$</th>
<th>Total Min $x(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>5.4158</td>
<td>10</td>
<td>4.0504</td>
</tr>
<tr>
<td>$\frac{k}{2}$</td>
<td>4.8264</td>
<td>5.7450</td>
<td>2.6901</td>
</tr>
<tr>
<td>$x_{crit}$</td>
<td>0.0026</td>
<td>0.3254</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.6: Table of total: mean, max and min values for $x(t)$ using full cooperation parameters: $\beta_S = 0.1$, $\gamma_S = 0.5$, $\beta_O = 0.1$, and $\gamma_O = 0.4$.

<table>
<thead>
<tr>
<th>Initial $x(t_0)$ Value</th>
<th>Total Mean $C_S(t)$</th>
<th>Total Max $C_S(t)$</th>
<th>Total Min $C_S(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>88.6627</td>
<td>90.8580</td>
<td>50</td>
</tr>
<tr>
<td>$\frac{k}{2}$</td>
<td>88.0423</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>$x_{crit}$</td>
<td>97.9313</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>0</td>
<td>98.6135</td>
<td>100</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 4.7: Table of total: mean, max and min values for $C_S(t)$ using full cooperation parameters: $\beta_S = 0.1$, $\gamma_S = 0.5$, $\beta_O = 0.1$, and $\gamma_O = 0.4$.

<table>
<thead>
<tr>
<th>Initial $x(t_0)$ Value</th>
<th>Total Mean $C_O(t)$</th>
<th>Total Max $C_O(t)$</th>
<th>Total Min $C_O(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>74.3971</td>
<td>75.0930</td>
<td>50</td>
</tr>
<tr>
<td>$\frac{k}{2}$</td>
<td>75.0353</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>$x_{crit}$</td>
<td>61.2068</td>
<td>99.9890</td>
<td>50</td>
</tr>
<tr>
<td>0</td>
<td>80.5740</td>
<td>81.8050</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 4.8: Table of total: mean, max and min values for $C_O(t)$ using full cooperation parameters: $\beta_S = 0.1$, $\gamma_S = 0.5$, $\beta_O = 0.1$, and $\gamma_O = 0.4$. 
Figure 4.8: Graph of $C_O(t)$ over 100 years for 100 different simulations using full cooperation parameters: $\beta_S = 0.1$, $\gamma_S = 0.5$, $\beta_O = 0.1$, and $\gamma_O = 0.4$, when $x(t_0) = k$. 
Figure 4.9: Graph of population biomass over 100 years for 100 different simulations using full cooperation parameters: $\beta_S = 0.1$, $\gamma_S = 0.5$, $\beta_O = 0.1$, and $\gamma_O = 0.4$, when $x(t_0) = \frac{1}{2}$.
Figure 4.10: Graph of $C_S(t)$ over 100 years for 100 different simulations using full cooperation parameters: $\beta_S = 0.1$, $\gamma_S = 0.5$, $\beta_O = 0.1$, and $\gamma_O = 0.4$, when $x(t_0) = \frac{k}{2}$.
Figure 4.11: Graph of $C_O(t)$ over 100 years for 100 different simulations using full cooperation parameters: $\beta_S = 0.1$, $\gamma_S = 0.5$, $\beta_O = 0.1$, and $\gamma_O = 0.4$, when $x(t_0) = \frac{k}{2}$.
Chapter 5

Conclusions and Future Work

5.1 Conclusions

We can draw some limited conclusions from the work we have presented in this thesis. From our analysis we have shown that it is possible, for select values, to achieve stability among a set of three differential equations. We sought to enact a situation on Lake Huron with regards to the Lake Whitefish that resides within. This situation inquired about what would happen to the Lake Whitefish given that two managements whom operate their respective fisheries were non-cooperative in their dealings. These differential equations then represented the aforementioned fish, and, the dual management on Lake Huron. In our analysis we determined that there was a possible sensitivity for our model in regards to fisher effort, and, the social norms that governed their effort.
Future work will likely consist of approaching our analysis from additional viewpoints. We provided an analysis of our non-linear system in a reduced fashion. It could be possible to then explore our model using other methods as well. An interesting approach would be a bifurcation analysis of the variables associated with the concepts mentioned in our conclusion. Next steps would entail a parameter extraction, through the application of an inverse problem, for the real life situation on Lake Huron between the SON and the OMNRF. A case study of this situation could then be applied using our model. For other authors, it would be interesting to introduce complexity to the fish system, we have implemented functional forms for the social systems; but, we have neglected items like, predator prey interactions, stages, and multispecies considerations for the fish themselves.
Appendix A

Introductory Concepts

A.1 History of SON and OMNRF Fisheries

The tensions between the OMNRF and the SON have a long running history (OMNRF, 2005a). We note that the SON consists of two communities, the Chippewas of Nawash Unceded First Nation (Nawash), and Saugeen First Nation (Saugeen). In order to best appreciate the situation on Lake Huron a short history of these aboriginal communities comprising versus the Ontario provincial Crown and Canadian Federal Crown as well as the OMNRF (previously the Ministry of Natural Resources (MNR)), is required. Issues between these parties started in 1761 with assertion of land ownership by British troops (OMNRF, 2005a). From this point onward, the aboriginal communities in the area of interest, (Bruce, Peninsula), faced many hard times, including the forced removal from land, as well as signings of treaties with the
Ontario Provincial Crown (OMNRF 2005a). These treaties as well as discussions spanned from 1761 until 1857 wherein the Nawash and Saugeen were forced into their current reserves consisting of remote locations along the shores of Georgian Bay and Lake Huron (OMNRF 2005a). It should be noted that in 1847, the crown of Ontario clearly stated that the waters at or near Bruce Peninsula were in fact owned by the SON (OMNRF 2005a). By 1857, the SON has also lost most of its ability to fish in the waters that were rightfully theirs, due the increased intrusions by non-First Nation fishers at the time (OMNRF 2005a). Up until 1880, in various Ontario lakes, the waters and lands that first nations groups owned were continuously infringed upon by non-First Nation fishers. In addition, during this period one of the stocks native to Ontario, the sturgeon, became endangered due to poor fishing practices and due to its increased demand as a caviar supply (OMNRF 2005a). In the early 1900's Lake Trout had also been fished to the point of a stock collapse in lakes across Ontario (OMNRF 2005a). In a jump to 1981, we see that the Indigenous Peoples in the area had still been barred from fishing in Lake Huron (OMNRF 2005a). In 1984, the Nawash had been given limited licenses to fish, however, the Saugeen had not (OMNRF 2005a). In 1992, a ban was inevitably placed, on the Nawash after the collapse of a tentative fishing agreement between them and the Ontario Crown, partly due to MNR influence (OMNRF 2005a). The SON went to federal court to appeal the ban that same year (OMNRF 2005a). This ban had a one day removal and subsequently a re-instancing on June 12th 2005 (OMNRF 2005a). Immediately prior to this, the
famous trial of Jones-Nadjiwon began wherein the First Nations would battle for their treaty rights over land and fish (OMNRF 2005a). In 1993 the judge would rule that the treatment that the MNR had been giving their Nawash neighbours was “discriminatory, unconstitutional and unenforceable.” (OMNRF 2005a). Subsequently, the Ontario crown and the Nawash would re-establish talks for co-management in 1994. In 1995, an MPP, (who was also at the time in direct relation with the MNR), would lead a group to protest a First Nations vendor at a market in Owen Sound. Incidents of violence proceeded but none were harmed in the incident (OMNRF 2005a). This would indirectly lead to a string of petty incidents over the year of 1995, including the theft of at least 10,000 meters of First Nation fish netting, the releasing of a First Nations’ boat to open water, the defacing of a First Nations’ man’s fishing vessel, the burning of a fishing tug, and even the stabbing of a group of four First Nations’ men by a group of 35 non-First Nations’ individuals in the Owen Sound region (OMNRF 2005a). Finally, in 2005, a decision was reached allowing the SON to fish on their native waters (OMNRF 2005a). This, of course, is still met with contention, as

...Saugeen Ojibway Nations consider the Agreement to be a recognition and realization of their aboriginal and treaty rights to fish commercially. The MNR still calls it a license and the parties agree to disagree. (OMNRF 2005a)

This obviously continues to pressure both sides, as this disagreement, on a fundamental level, is quite important to consider. The SON will always consider this, and subsequent agreements, an entitlement to the lands they once owned historically.
As it was concluded in the Jones-Najiwon case, the SON were told that the licensing schemes used by the Ontario Crown were unjust (OMNRF 2005a). In addition, the judge of this case deemed that the lands they were fighting over were indeed the SON’s to claim (OMNRF 2005a). However, the precedent set by this case was ignored. Instead, as it is apparent by the quote above, the OMNRF, under certain pressures of their own, will not budge on their position (OMNRF 2005a).

While a co-management agreement was in place, this agreement has come to an end, specifically in March of 2018 (SON and MNR 2013). This co-management scheme has also experience substantial conflict. Personal correspondence with SON council members has led to the understanding that arguments over assessment techniques and reactions to said techniques have gotten quite “heated”. Although the agreement was meant to bring both sides together, it seems that this was never the case.

### A.2 Extended Discussion on Strategies

#### A.2.1 SON Strategies: Closing Waters

Participation in meetings with council members of the SON has given way to their fishing practices. Through this personal correspondence, we have learned that the SON are almost instantaneously reactionary to alterations in the lake whitefish populations within lake Huron. For example, given that they have reached total allowable
catch for the season, they will close the fishery for the rest of the season. In fact, this has been demonstrated at least twice by their managers on Lake Huron (OMNRF 2005a). Reactions like this are due to their concern for the sustainability of the Lake Whitefish, as it is both a source of income for their people, and an important First Nations’ fish due to its historical relationship to their ancestors (OMNRF 2005a).

A.2.2 OMNRF Strategies: Recreational Fishing

The OMNRF have a history of developing extensive policy to manage the resources under its control. Lake Huron consists of Ontario based and controlled commercial and recreational fisheries. According to the province of Ontario, recreational fisheries are quite important to the economic benefit of Ontario (OMNRF 2015a). Recreational catch of Lake Whitefish on Lake Huron, in our designated area of interest, that is, zone 13 or the main basin is limited to a possession limit of 25 units (OMNRF 2014, 2015b, 2016, 2017). Possession is defined as the number of fish that an individual may have on them at any place and time (Fis 2016, OMNRF 2014, 2015b, 2016, 2017). A person’s current possession number may decrease if they give away a fish, or eat their fish (Fis 2016). In 2017, a person may catch up to and hold, on any day, a maximum of 6-12 Lake Whitefish depending on their license (OMNRF 2017). This number has remained unchanged during the range of 2014 to 2017 in accordance to their fishing regulations summaries (OMNRF 2014, 2015b, 2016, 2017). While they have documented hardset policies on recreational fishing, it is harder to gleam their
policy when it comes to their commercial fishing.

A.3 Lake Whitefish Life History

Lake Whitefish can be found in the Great Lakes (ida, 2017; mic, 2017; Ihssen et al., 1981). There is potential variation among species of Lake Whitefish. This variation is considered, for this thesis sake, to be geographical. In that sense, we limit the life history parameters to the lake of study. Typically, Lake Whitefish spawn eggs during the winter seasons (mic, 2017), generally around October-November (ida, 2017; mic, 2017; Ihssen et al., 1981). Spawning is restricted to shallow waters (ida, 2017; mic, 2017). Female fish typically lay anywhere from 16401 eggs/kg to 26000 eggs/kg (ida, 2017; Ihssen et al., 1981). Lake Whitefish has been found to take approximately 167 days (natural incubation) to hatch in Lake Huron’s southern bay (Ihssen et al., 1981). Hatching occurs in March-April, where they enter the larvae stage (ida, 2017; mic, 2017; Ihssen et al., 1981). Time passes and the fish grow; at around the age of 4 years, they have grown into an adult fish (ida, 2017; mic, 2017; Ihssen et al., 1981). This age is marked by the time at which they are considered to be sexually mature. This is not a distinct value, some fish may mature at the age of 2 years or even the age of 5 years; some, depending on the lake’s history at an even higher age (ida, 2017).

A sampling of two highly cited studies of Lake Huron’s Lake Whitefish populations has produced some estimates of their mean length and weight. Casselman et al.
found that Lake Whitefish have a range of 41.9 cm to 50.7 cm mean length. Data was gathered for a mean age range of 3.6 to 9.4 years old and this age data was broken down into their minimums and maximums which were 2 years of age and 16 years of age, respectively (Casselman et al., 1981). The second study was performed within the South Bay of Lake Huron. Mean length and weight of fish in that area was approximately (41.1 ± 0.7) cm and (942 ± 69) g with a mean age of 9.4 years with a range of 4 to 14 years old (Ihssen et al., 1981).

Spatial distribution of Lake Whitefish particularly within the Great Lakes during spawning and off-spawning seasons is a secondary item for this study’s purposes. However, it will be discussed for completeness. Research has been conducted outlining the nature of movement among Lake Whitefish within Lake Huron; specifically from “detour” stocks and local stocks to the Northern regions of Lake Huron (Ebener et al., 2010). As it stands, for the Northern sections of Lake Huron, it seems that these fish typically spawn on the North Western side of Lake Huron (Ebener et al., 2010). Their ventures into the lake are limited at least from the perspective of tagging experiments (Ebener et al., 2010). Previous thesis work by Kathleen Ryan (2014) showed the movement patterns for the more relevant regions of Lake Huron. The area studied was Stokes Bay, located along the eastern coast of Lake Huron (Ryan and Crawford, 2014). Findings indicated that Lake Whitefish use this area as a spawning ground, much like the South Bay (Ryan and Crawford, 2014). After spawning season the fish leave coastal region and head along the coast, typically south to north (Ryan and
Table A.1: Summary of Known $r$ Values.

<table>
<thead>
<tr>
<th>Location</th>
<th>Value (rkg)</th>
<th>Standard Deviation (rkg)</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMB</td>
<td>1777364</td>
<td>N/A</td>
<td>(Jensen, 1976)</td>
</tr>
<tr>
<td>GB</td>
<td>7929000</td>
<td>3753000</td>
<td>(Hartford et al., 2007)</td>
</tr>
<tr>
<td>NGB</td>
<td>3210000</td>
<td>1700000</td>
<td>(Hartford et al., 2007)</td>
</tr>
<tr>
<td>WGB</td>
<td>799000</td>
<td>644000</td>
<td>(Hartford et al., 2007)</td>
</tr>
</tbody>
</table>

Table A.2: Summary of Known $k$ Values.

<table>
<thead>
<tr>
<th>Location</th>
<th>Value (rkg)</th>
<th>Standard Deviation (rkg)</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMB</td>
<td>0.22/0.50</td>
<td>N/A</td>
<td>(Jensen, 1976)</td>
</tr>
<tr>
<td>GB</td>
<td>0.25</td>
<td>0.09</td>
<td>(Hartford et al., 2007)</td>
</tr>
<tr>
<td>NGB</td>
<td>0.31</td>
<td>0.14</td>
<td>(Hartford et al., 2007)</td>
</tr>
<tr>
<td>WGB</td>
<td>0.36</td>
<td>0.13</td>
<td>(Hartford et al., 2007)</td>
</tr>
</tbody>
</table>

Crawford, 2014). This effect could be due to the water patterns observed within Lake Huron. Beletsky et al. demonstrated that circulatory patterns of water within Lake Huron follow the same circular pattern exhibited by both studies of Lake Whitefish. Water along the eastern coast of Lake Huron is pushed from south to north (Beletsky et al., 1999).

If our reader references Tables A.1 and A.2 they will see a few key parameters found by others in literature. Each of these parameter values pertain to a limited region of Lake Huron, namely the Northern Main Basin (NMB), or Georgian Bay (GB) with W indicating west and N, North. One will note then, the altered $r$ and $k$ values based on Hartford et al. (2007) and their work. In their case, they provided extrapolated results for each region based upon catch statistics and seeding with the
$r$, (known as natural growth rate), and, $k$ (known as the carrying capacity), values found by Jensen (1976).
Appendix B

Relevent Concepts Extended

B.1 Extended History of Fisheries Dynamics

The Gordon-Schaefer (G-S) Model and MSY became a widely utilized concept. Biological researchers attempted many different applications and modifications to the G-S model (Allen and McGlade 1986; Gordon 1954; Guo and Zou 2015; Jensen 1976; Jensen et al. 1982; Jensen 1978 1984; Pella and Tomlinson 1969; Schaefer 1991). Modifications to the G-S model allowed for skewedness of the original logistic curve, allowing for fish to behave in a way where their growth rate does not follow a perfect parabola (Pella and Tomlinson 1969). While their model does fit the historical data they analyzed, they conclude that their model may only act as a good benchmark for further research (Pella and Tomlinson 1969). However, it does provide an example that skewedness can occur in nature, and that it can be modeled.
In regards to the development of a new model, non-perfect growth rates have been considered much more in recent years, through ideas such as Allee effects (Myers, 2001; Richter and Dakos, 2015). One instance of this effect can be its effect on fish growth rates, wherein they can exhibit a dependence on a certain level of population density (or magnitude) in order to continue growing; else, they face the collapse of their population (Myers, 2001; Richter and Dakos, 2015). Models without skewedness were also applied to fisheries with similar results (Jensen, 1976; Jensen et al., 1982; Jensen, 1978, 1984). Introduced alongside the typical G-S model, was a dynamic yield term (Jensen, 1976). If yield is given as $y(t)$, then, much like typical yield or harvest, dynamic yield is:

$$\frac{dh(x)}{dt} = h(x)$$

where Equation B.1 is the change in harvest, $h(x)$ over time with respect to the resource $x$. In addition, predation, environmental effects, and, alterations to constants were all explored (Jensen, 1976; Jensen et al., 1982; Jensen, 1978, 1984). A common conclusion among all of these papers was that the model they provided was both a good fit to the data, and that it was ready to use as a management tool (Jensen, 1976; Jensen et al., 1982; Jensen, 1978, 1984). So long as fishery managers fished at or below the MSY predicted by any of their modified models, the fishery would continue to stay at a reasonable population level (Jensen, 1976; Jensen et al., 1982; Jensen, 1978, 1984). This conclusion is not exactly correct. In fact, there is quite a bit of controversy surrounding the usage of MSY (Ludwig et al., 1993; Pauly et al.,...
Pre-war fisheries were clearly being affected by humans and their activities. This was evident from the mass increase in fish after the world wars (Gordon, 1954). Still, very little was done to prevent those from overexploitation in these fisheries post-war (Pauly et al., 2002). MSY, although useful, was never readily implemented in certain fisheries (Pauly et al., 2002). Optimal effort levels were rarely achieved (Pauly et al., 2002). Consensus had now shifted to the view, as Ludwig et al. (1993) put it, that the concept and implementation of MSY was “unfortunate”. Some modified G-S models reflected this in their design and analysis through the inclusion of uncertainty or stochastic effects (Allen and McGlade, 1986; Beddington and May, 1977). Beddington and May (1977) altered the standard GS model to include an error term. Two scenarios were designed, one where they kept yield constant, and one where effort was kept constant (Beddington and May, 1977). In either case, the variance caused by the error term propagated through their harvests (Beddington and May, 1977). In the case of constant yield, the effort is variable, and their model found that with the increase in effort there was less predictability of the fish stock over time (Beddington and May, 1977). Overall, they concluded that with the introduction of uncertainty to their model the effect they witnessed would still be present with added complexity (Beddington and May, 1977). Additionally, because of the increase in the difficulty of predictability they conclude that MSY would be a poor measure, as it does not capture uncertainty (Beddington and May, 1977). Similar conclusions with respect to
MSY were brought about by Ludwig et al. (1993) as well. Discussion about including variance in these dynamic models was also voiced by the dissenters of MSY (Ludwig et al., 1993; Pauly et al., 2002).

Without speaking for a community of researchers, it reasonable to think that MSY is a less valid approach to sustainable fishing models and analysis. This does not mean that an industry, reliant on previous research can just throw away data or approaches using this concept. In fact, recent SON reports have used MSY as a statistical tool to aid in predicting total allowable catches for their fish stocks (Gillis, 2015). One can see the motivation here to provide a tool that is both modern in perspective and immediately implementable for those actually working in the fishing industry.

Many models still use the altered logistic equation, and to some degree with interesting approaches, such as discontinuous harvesting and SES models (Brandt and Merico, 2013; Guo and Zou, 2015; Richter and Dakos, 2015). While these have neglected the usage of MSY in analysis, they have instead developed much more comprehensive methods for predictability and management. Before continuing onto a discussion of SES models, a proper treatment of the economic and social sectors is appropriate. Authors turned to understanding resource management, not just as control of the fish, but also as control over the people (Ludwig et al., 1993).
B.2 Metapopulations

An interesting development in fisheries research coincided with the development of the game theoretical frameworks: metapopulation. Our reader can think of metapopulation roughly being described as a set of sub-populations of the same animal, or species within a pre-defined space \cite{Bisci2009}. One can consider, for example, that there exists a set of humans that live near a mountain. This set consists of two populations, one which lives on the southern base of the mountain and the other at the northern base. These two populations can interact with each other but at a distance. In some cases north will trade with the south and the opposite can occur. It is even possible that people move permanently to either the north or the south side from their side. In this way, technically, these are two sub-populations; also, they form a larger full human population. This type of population consideration has been covered by models that include ideas of two fleets operating on two distinct populations of the same fish \cite{Mchich2000}. These have also been extended further to include: non-constant migration rates, non-constant harvesting based on non-constant price, as well as extension to 3 fleet populations and finally N fleet populations \cite{Auger2010, Mchich2000, Mchich2002, Mchich2006}. Despite this, we do not consider a metapopulation of lake whitefish for this research, however, the historical involvement of this subject is applicable to the development of our model.
B.3 Allee Effects

Warder Clyde Allee’s (1931) book entitled “Animal Aggregations: a study in general sociology” brought about the idea of density dependence. In his review of literature at the time, quite a bit of experimentation had been conducted on the effects of population density in a space limited region (Allee, 1931). Researchers had shown that among bug populations, at maximum and minimum densities, the decline of population birth rates (Allee, 1931). A follow-up paper demonstrated one bound of this effect on goldfish populations placed in colloidal silver suspensions (Allee, 1932). That is, the fish proved to survive longer at higher population densities (Allee, 1932). The question remains as to whether or not fish exhibit the same effect at low population densities. Across the domain of fishery science, as it turns out, the Allee effect has had somewhat of a notation change. In some research, the effect has been brought up as the ‘50/500 rule’ or depensatory mechanisms (Freckleton, 1999). Regardless of this, the effect in question has been demonstrated. At critically low population densities, certain species of fish do exhibit a decrease in their per capita birth rates (Freckleton, 1999).

In fact, the basis of this thesis incorporates Allee effects as a lower bound on the population (Richter and Dakos, 2015). This is accomplished with a critical resource
term or $x_{\text{crit}}$. Isolated, this could be given as:

$$\frac{dx}{dt} = x - x_{\text{crit}}$$

If we imagine a fish population $x$ that is growing in time $t$; the change in this population relies on the current population level and some critical level already given as $x_{\text{crit}}$. The $x_{\text{crit}}$ term is meant to represent some lower bound that the fish experience within their population (Richter and Dakos 2015). The lower bound itself, is meant to encompass some density dependent effect, in their case, it is left as some point at which the stock collapses, which is easily captured by the differential equation. Both $x \geq 0$ and $x_{\text{crit}} \geq 0$. This implies that when $x = x_{\text{crit}}$ there is no change in the population level and that when $x < x_{\text{crit}}$ population growth is in decline. In addition, we can formulate an equation, in which we will see how the Allee effects affect the typical logistic growth curve:

$$\frac{dx}{dt} = (x - x_{\text{crit}}) \left( r x \left( 1 - \frac{x}{k} \right) \right)$$  \hspace{1cm} (B.2)

Figure B.1 depicts a graph of Equation B.2. It uses the same parameters as Figure 2.3 with the addition of the $x_{\text{crit}} = 200$ term. This is to give a visual demonstration of the Allee effect on a traditional growth model. It can be seen that the graph is no longer symmetric. In addition, the $x_{\text{crit}}$ term contributes to not just a declining growth rate, but a non-positive growth rate as well. This non-positive growth rate
occurs for \( \{x \in \mathbb{R} | 0 < x < x_{\text{crit}} \} \).

Figure B.1: Depicted is a graph of the Verhulst growth equation with additional multiplicative Allee effect over the total population.

B.3.1 Governance

If a group of people, or population have access to a common pool resource [Hardin (1968)] suggests that we will usually choose to exploit this resource without sustainability as a consideration [Hardin 1968]. Governance acts as the means to control this behaviour. We may define it in an abstract sense, according to the United Nations Educational, Scientific and Cultural Organization [UNE (2017)] governance is:

"...defined to refer to structures and processes that are designed to ensure accountability, transparency, responsiveness, rule of law, stability, equity
and inclusiveness, empowerment, and broad-based participation...represents the norms, values and rules of the game through which public affairs are managed in a manner that is transparent, participatory, inclusive and responsive.”

Governance is quite broad in scope, with its basis is within the realm of political science. For completeness, a relatively lengthy discussion of all aspects of governance would be ideal. But, for the sake of our reader we shall limit it to a short discussion on internal and external governance. External governance is purported by one major theory: centralized governance (Ostrom 1999). Centralized governance has been carried out by 3 main rules or principles. The first rule is that all of those who have access to a common pool resource will undoubtedly seek to maximize their output from that resource (Hardin 1968 Ostrom 1999). Secondly, centralized governance seeks generalized solutions to resource problems, and is not interested in the specifics of a situation (Ostrom 1999). Third, centralized governance assumes that, in the situation they are dealing with, there is in fact only one direction to govern(Ostrom 1999). This seems self-defining; however, it should be noted that centralized governance does not spawn out of nothing. Instead it is applied to existing communities, some of which might have multiple sub-communities, where, each might have their own agendas and rules (Ostrom 1999). Internal governance, is also difficult to define. For simplicity, we assume that internal governance is governance that has spawned out of the people of a system, referred to as self-governance (Ostrom et al. 1992). This type of governance can be thought of as an emergent behavior or at least an
assumption that the people in a system regulate themselves in some manner (Ostrom et al., 1992; Sethi and Somanathan, 1996). It is perhaps more appropriate that we link together internal and external governance. Polycentric governance borrows ideas from both centralized governance and self-governance (Ostrom, 1999). Focus is on a non-central governance, wherein many layers exist (Ostrom, 1999). Polycentric governance then relies on both centralized governance and self-governance existing in some manner together where the takeaway is that it is appropriate to have both types acting in unison for different situations. Where polycentric governance is simply external and internal governance working together in parallel to provide more appropriate situation based responses to the total governance of a system or resource (Ostrom, 1999).

B.3.2 Social Norms Extended Discussion

Social norms are effectively how people behave in relation to each other (Chung and Rimal, 2016). Much like governance, we seek to concentrate this concept down to the current thoughts surrounding fishery management science. Without getting into too much meta-discussion, it is noted that our focus for social norms will be on the ideas of “temptation” and “peer pressure” in regards to their dynamics with respect to the fisher (Richter and Dakos, 2015). Social norms, and their application are apparent in a large magnitude of fields, including focus in the social and economic aspects of fisheries research alongside sustainability science (Atzenhofer, 2011; Baumgartner and Quaas, 2011).
Individuals can be expected to behave in ways that appear to be or are directly a result of their peers’ behavior (Chung and Rimal, 2016).

Social Pressure Extended Discussion

Conversely, individuals who are currently against the common social norm, can be brought back. In this sense, the ones who are no longer following the social norm, can sustain peer or social pressure to return (Richter and Dakos, 2015; Richter and Grasman, 2013). To feel or become under the influence of social pressure, an individual must be exposed to those who are currently following the common social norm (Chung and Rimal, 2016; Richter and Dakos, 2015; Richter and Grasman, 2013). With this in mind, historically, social pressure was framed as observing ones’ strategy versus that of anothers’ strategy (Richter and Dakos, 2015; Sethi and Somanathan, 1996). This leads to imitation of strategies; however, much like Richter and Dakos (2015) emphasize, this leads to the emergence of only “successful” strategies. Of more importance to this fact is the potential avoidance of previous strategies, or those that just never appear (Richter and Dakos, 2015). With that in mind, the development of social pressure has arrived at the idea of persuasion or social punishment (Richter and Dakos, 2015; Richter and Grasman, 2013). Individuals are punished based on the idea that near-neighbours or some proportion of the norm conformists cause one to become socially sanctioned (Richter and Dakos, 2015). That is, they are considered either
“persuaded” or “punished” in some form. This can be either loosely or strictly framed. An example is the loose social pressure defined by Richter and Dakos (2015), where, social pressure is given as a simple constant of some Poisson process (Richter and Dakos 2015). Encounters of both conformists and non-conformists happen at some rate. As such, this rate determines who might go back to being a conformist given some constant of peer pressure (Richter and Dakos 2015). The higher the pressure constant, the more non-conformists will be pressured back to being conformists (Sethi and Somanathan 1996). An example of a strict framework has been given by Sugiarto et al. (2015), wherein they provided a discrete model which included social pressure within the context of a physical network. Their social pressure was formulated as a form of “ostracism” or exclusion, where, if surrounded by those conforming to the common social norm, the social pressure was increased (Sugiarto et al. 2015).

B.4 Tragedy of the Commons

While fisheries research borrowed the idea of the commons for the fish alone, economics branched off and continued to pursue the idea in regards to humans (Hardin 1968). Inexhaustible fishing resources were supplanted by Gordon (1954) and his ideas behind the fishing commons. Akin to these discussions, the tragedy that falls upon the fishing commons was and still is the central locus of most sustainability research. Our tragedy is encapsulated by the unrestricted usage of the commons and the utility that can be lost or gained from this open access use (Hardin 1968).
an example imagine a group of \( n \) families, each with the same amount of members \((\text{Hardin, 1968})\). They are all driven to increase their utility and have open access to a semi-limited resource, such as fish \((\text{Hardin, 1968})\). Each family member obtains the same amount of money and happiness from fishing \((\text{Hardin, 1968})\). A rational family would consider having another child (or more) creating a net benefit or increase in utility for that single family of +1 per new member \((\text{Hardin, 1968})\). However, in consequence this creates a net decrease in utility shared across all families, given as some form of decrease that is dependent on the utility lost from losing a fishing opportunity \((\text{Hardin, 1968})\). An imbalance in utility emerges. In this case the benefits always outweigh the costs associated with having more children \((\text{Hardin, 1968})\). Thus, the only rational way to continue, for all families, is to have as many children as possible \((\text{Hardin, 1968})\). Inherently, through this example, the consequences of open access can be quite damaging to the long term usage of a limited, or semi-limited, resource.

### B.5 Evolutionary Game Theory Continued

EGT is comprised of many components previously mentioned in Sections ?? through to and including 2.3.

A seminal paper by Sethi and Somanathan (1996) adapts EGT to the management of fish as a common pool resource. Fish dynamics were simply given as it was in the past, fish grow logistically, and are harvested governed by a dynamic harvest equation
dependent on the resource, technology, and effort levels. Additionally outlined, is the utilization of cooperators, defectors, enforcers, social norms and imitation dynamics. According to this specific paper, and in regards to a common pool resource, cooperators and defectors are populations of individuals who adopt low and high levels of effort respectively. It can be taken further where these population groups are given the ability to view the other strategies. Viewing of these other strategies might cause a certain subset of one population group to choose a different strategy of another population group. Strategy switching can be brought on by a few items. On top of these, there is the social aspect that arises from norm guided behaviour. These norms can have a frame of reference, such as external or internal governance. The norms guide the internal governance and the governance acts as a framework for the population to act within. In this case, and like many others after, this population is split into cooperators and defectors. Enforcers act as a special class of cooperator, wherein they act to cause a loss for the defectors through some form of sanction. Sanctions are typically seen as a loss of profit. Low effort is framed as those who use effort levels under MSY as a collective group. While high effort is considered as maximum effort to maximize profit or utility. They give this profit as:

\[ \pi_i = S_i(x_i) - K_i - l_i \]
where much like in the discussion of the original profit equation we see \( \pi_i \) as profit and \( S_i(x_i) \) as sales as a function of \( x_i \), fish. In alignment with the general profit equation, \( \pi_i \geq 0, S_i \geq 0, K_i \geq 0, \) and \( l_i \geq 0 \). In their case, they frame costs as sanction costs. These are \( K_i \) and \( l_i \) or punisher cost and punishee cost. That is, the cost of punishing and the cost of being punished. In their case, depending on whether or not an individual, \( i \), was a cooperator, sanctioner or defector determined which sales and cost was associated with said individual (Sethi and Somanathan, 1996).

The framework presented here is a culmination of all of the previous concepts defined previously. In fact, one might go as far as to say that the usage of EGT here acts as the first precursor to Socio-Ecological modeling. The analysis provided within Sethi and Somanathans paper goes into detail explaining both the emergence and internalization of social norms into a population (Sethi and Somanathan, 1996). Extracting this convoluted statement is the conclusion most of these papers face when everything is simulated and analyzed. That is, in most cases defection is the route most follow and cooperation always ceases to exist (Sethi and Somanathan, 1996). Mathematically, given the right parameters, cooperation or selflessness can be persistent, however, and slight perturbation to this usually leads to defection. The attractiveness of defection is always stable. While this might be the case for a lot of field studies of the subject, it is not always the case. In fact, as they state, it is possible that in small groups cooperation continues to exist (Sethi and Somanathan, 1996). In fact, this has been demonstrated in experiment as well. Given that a group
of individuals have enough communication, and given that they have no external governance the persistence of cooperation exceeds the selfish behaviour predicted by EGT (Ostrom et al. 1992). Notably, most game theoretical examples and EGT models usually find the alternative conclusion Sethi and Somanathan (1996); Sumaila (2013a,b).

At first glance, one might have trouble differentiating between EGT models and SES models, this is rightly justified. Both types of models are actually very similar. It is the approach that an author has to problems that changes how one might interpret their results, with respect to each modeling approach. SES models have been shown to model agents or individuals interacting over a network while harvesting a resource (Sugiarto et al. 2015). As well as modeling the perception that individuals have of a resource versus the harm that comes from continuing to harvest said resource (Yodzis et al. 2016). In addition, other SES models have looked at comparisons of long term versus short term production and the effect that these have on a resource and people’s satisfaction or the effect that certain levels of governance has on the production of a resource (Brandt and Merico 2013; McClanahan et al. 2016).

B.6 The Parameter $\alpha$

The $\alpha$ presented here is typically set to one, however, one can set it to some value less than one. It is often used, or represented for that matter, in the form of the
cobb-douglas function (Cobb and Douglas, 1928). This function uses $\alpha$ as means to determine the elasticity of an output.
Appendix C

Modelling Discussion Extended

C.1 Temptation and Peer Pressure in New Model

Something must be said for the parameters associated with the cooperator dynamics. In this case, the clarification lies with the two social coefficients: peer pressure, and temptation. These will not be parameterized explicitly here, but described in a general manner to be set in the simulation study. Although there is zero interaction between the SON and OMNRF individual populations we must assume a few key differences between the two which govern how Equations 3.14 and 3.15 will operate.

Richter and Dakos (2015) derived their social dynamics, but neglected to provide an adequate analysis of the parameters they created for said dynamics. The parameters with the least amount of explanation, but of incredible importance are the “temptation” and “peer pressure” terms. Their derivation are provided in Chapter 3 Section
3.1 The concept of these two terms was determined post-derivation, that is by our accord, they were considered to be aspects of social norms and internal governance. Economic benefit is then the main driving factor behind temptation. As Chung and Rimal (2016) pointed out, individuals can behave against a norm, so long as the utility they derive from doing so is net positive. Richter and Dakos (2015) derived Equation 2.4 as a way to capture these two concepts. Their dynamics are non-spatial however; then, as they pointed out it is only possible to consider proportions of people that cross between being a cooperator or a defector. This complements models that consider individuals meeting each other spatially, and determining at contact time a probability of switching. Instead, to realize a similar idea without spatial necessity they allow themselves to choose two parameters that may be tailored to influence this idea of group encounters. We saw this as the $\gamma$ and $\beta$ terms, or the social coefficients of “peer pressure” and “temptation”.

With the introduction of two management bodies we then must consider additional constraints and assumptions attributed to the temptation and peer pressure parameters. A key assumption is distinguishing between the two management bodies. In our case, as it can be seen in Equations 3.14 and 3.15 we have labeled these two terms according to the body they belong to. Then, we have in our case four total social coefficients: $\gamma_S$, $\gamma_O$, $\beta_S$, and $\beta_O$. Outlined in Chapter 1 was a significant portion of the SON’s current viewpoints. In their view, the Lake Whitefish are considered a sacred fish and a staple of their current society (OMNRF, 2005a; SON, 2016).
view of Lake Whitefish would lead one to think that perhaps even their defectors would be less inclined to over-fish because of this. This adds to the thought that their social norms connected to this resource might be tighter, or more restricted in their communities. A naive approach would be to remove the defectors from the equation entirely, but that would not be rational. It would be reasonable to assume that within any population, there would be those who could undermine the currently set social norm of their society. SON’s population should not be exempt from this assumption. In contrast, in the eyes of the OMNRF, Lake Whitefish are both part of a larger scheme of commercial profit fish and a draw for casual fishing participants (OMNRF, 2014, 2015b, 2016, 2017). In this way, it is assumed that the OMNRF put less exclusive emphasis on fishing Lake Whitefish, and, by extension have less of an inclination to protect the sustainability of lake whitefish attributed once again to their historical view of the fish, and their current political situation within Ontario. We then assume then that the OMNRF are profit driven, and as such will have higher temptation than that of the SON, who will have proportionally much fewer defectors than the OMNRF. Explicitly, this can be written as $\beta_O > \beta_S > 0$. In this way, internal governance is seen to restrict the defection of SON individuals more so than the internal governance of the OMNRF. Additionally, the first term of both Equations 3.14 and 3.15 cover meetings between both of their own exclusive internal groups of cooperators and defectors. Historically, the OMNRF took quite a few opportunities to limit the fishing of the SON; so, when the SON willingly introduce a self-imposed
limit, an additional viewpoint is to additionally assume that the OMNRF would help to enforce this as well. This assumption brings into consideration how much pressure their is to stay cooperative. If the SON follow the same logic as was mentioned above and the OMNRF as well then we assume that the proportion of cooperators converting from defectors will always be higher for the SON than the OMNRF: $\gamma_S > \gamma_O > 0$. Then, we have covered a large aspect of the social construction of this model, next is the economic portion which entails less about the governance of these groups, but instead the local emphasis on the cost and price of the lake whitefish in their respective regions.

C.2 Markets and Profit in New Model

Opting for a simple approach to start, there are some common assumptions that need to be addressed. These assumptions are common to the original model in this case and cover the concepts of profit, markets, alternative markets, and price of fish. Here we will define and state the respective differences in these concepts as well as the equations that will govern their operation.

In line with previous statements and assumptions, we have two bodies each with their own sub-populations consisting of cooperators and defectors. These cooperators and defectors are then considered to make their own profits. Individually this could be considered; however, we consider here the whole profit of the entire sub-population,
noting that a single agent’s profit would be a portion of the whole, as profit is uniformly distributed across both sub-populations for their respective methods of profit acquiring. Then, the SON body will have profit assigned to the cooperator sub-population of $\pi^C_S(t)$ and defector sub-population of $\pi^D_S(t)$. Contrasted with these are the OMNRF’s cooperator and defector sub-population profits of $\pi^C_O(t)$ and $\pi^D_O(t)$, respectively. Our profit fits similar facets of previously examined models namely general equations discussed in Chapter 2 Section 2.4 Subsections 2.4.1 and 2.4.2. More explicitly let $\pi^\bullet(t)$, where $\bullet$ may adopt either the SON ($S$) or OMNRF ($O$) label and $^*$ adopt either the cooperator ($C$) or defector ($D$) sub-population, at time $t$. Then,

$$\pi_t^\bullet(t, x) = \pi^\bullet_C(t, x) - \pi^\bullet_D(t, x) + A^\bullet(t, x),$$

where $\pi^\bullet_C(t, x)$, $\pi^\bullet_D(t, x)$, and $A^\bullet(t, x)$ represent, for their labels and respective sub-populations, the revenue, cost and alternative revenue at time $t$ with respect to resource $x$. The original model then defined each of these in subsequent order. Here we will do the same, noting that some items will be expanded on in further sections where it is more appropriate. In our case, both the SON and OMNRF obtain revenue from harvesting fish. Simply put, the amount of fish they sell, will be the revenue they obtain. This then requires one to know a price of fish $P(t)$ to conclude:

$$S^\bullet(t, x) = h^\bullet(t, x)P(t),$$  \hspace{1cm} (C.1)
where $h^*(t)$ is harvest associated with the body and sub-population that is designated. However, Equation C.1 brings into consideration a major question: which markets do the SON and OMNRF sell to? The original model had the cooperators and defectors selling to the same market, and utilizing the same alternative market (Richter and Dakos, 2015). This then also required, or assumed at the very least, that they would also be selling their fish at the exact same market price. This price was predetermined each year by Equation 3.8 wherein they added a “fluctuation” was added to an existing average price of fish. In our case, we have expanded this idea to include two bodies of cooperators and defectors as mentioned earlier. It is possible to say then that each of these bodies of individuals will sell to their own respective markets, with their own respective prices and fluctuations. This is the assumption that will be taken forward.

Both the SON and OMNRF will have different markets they will sell to, as it is in real life. These are explicitly different markets as well. In fact, the SON typically is restricted to their own local markets, and the OMNRF are restricted from the SON’s local markets (OMNRF, 2005a). This then leads us to an assumption about the price per fish. Richter and Dakos (2015) gave a simple function for price per fish, namely $P(t) = \bar{P}(t) + \eta(t)$. Since the SON and OMNRF will be selling to distinctly different fish markets we must reformulate $P(t)$ to account for this fact. Let the price of fish then be:

$$P^\bullet(t) = \bar{P}^\bullet(t) + \eta(t),$$  \hspace{1cm} (C.2)
noting that $\eta(t)$ is neither body nor sub-population dependent. This is because, both the SON and OMNRF, (and their subsequent sub-populations), would have similar or exactly the same fluctuations, $\eta(t)$, in their local markets due to outside global shifts. This is the assumption for our extension at the very least. One could argue that different local markets will feel the effects of global shifts differently, but that is left untouched in our case. Moving forward, with the alteration to $P(t)$ seen in Equation C.2 we must re-state Equation C.1 becomes:

$$S^\bullet(t, x) = h^\bullet(t, x)P^\bullet(t). \quad (C.3)$$

Equation C.3 now invites the usage of Equation C.2. The associated cost function for this equation entails the same logic as the previous model, with associated labels. In this way, cost, $C^\bullet(t)$ is given as:

$$C^\bullet(t, x) = w^\bullet e^\bullet(t), \quad (C.4)$$

where $w^\bullet \in \mathbb{R}_{\geq 0}$ is considered expenses given to items used for fishing, such as equipment cost, repair costs, fuel costs among any other items. This is left to be parameterized. Then, in Equation C.4 there is the $e^\bullet(t)$ term, which is considered the effort given at time $t$ for any sub-population (^\bullet) under a given body ^\bullet. Effort entails a complicated mechanism which will be touched upon in its own section. The reader should then accept that cost is attributed to a will-be-parameterized cost, $w$, multiplied by
the effort required to harvest fish thus accruing a net cost of some value.

Last to be given is $A^\star(t, x)$ which represents the alternative revenue stream. As it was originally formulated, it can either be a cost or a revenue depending on the effort given. In fact, in the case of Richter and Dakos (2015) the effort given acted as a maximum for the cooperator and defector population efforts. Namely, the last term of Equation 3.7 (extended to the whole sub-population): $m(\hat{e} - e^\star)$. Then, $A^\star(t, x)$ is given as:

$$A^\star(t, x) = m\cdot (\hat{e} - e^\star(t)),$$

where $m\cdot$ and $\hat{e}\cdot$ are potential revenue and maximum (or alternative opportunity effort) given for a designated body $\cdot$. This follows similar logic to both revenue and cost, wherein each of the bodies given will have their own associated alternative markets as well as an effort associated with said market. These alternative markets are assumed then to be separate from each other and thus have no influence or likeness to each other (even if this is not the case). We define an alternative revenue as a stream of money gained from not selling harvested fish. The total amount of revenue is controlled by the maximum effort minus the alternative effort allocated, and, when when a sub-population invests exactly alternative effort they achieve:

$$A^\star(t) = 0, \text{ when } \hat{e}\cdot = e^\star(t),$$
\[ A^*_\bullet(t, x) < 0, \text{ when, } \hat{e}_\bullet < e^*_\bullet(t), \]

\[ A^*_\bullet(t) > 0, \text{ when, } \hat{e}_\bullet > e^*_\bullet(t). \]

Now that these ideas are summarized, in a pedantic fashion, presented are the full set of profit equations which govern the SON cooperator and defector profits as well as the same for the OMNRF:

\[ \pi^C_S(t, x) = P_S(t)h^C_S(t, x) - w_S e^C_S(t, x) + m_S(\hat{e}_S - e^C_S(t, x)), \]  

\[ \pi^D_S(t, x) = P_S(t)h^D_S(t, x) - w_S e^D_S(t) + m_S(\hat{e}_S - e^D_S(t)), \]  

\[ \pi^C_O(t, x) = P_O(t)h^C_O(t, x) - w_O e^C_O(t) + m_O(\hat{e}_O - e^C_O(t)), \]  

\[ \pi^D_O(t, x) = P_O(t)h^D_O(t, x) - w_O e^D_O(t) + m_O(\hat{e}_O - e^D_O(t)), \]

where Equations \[C.5\] and \[C.6\] are restricted to SON cooperators and defectors; and, Equations \[C.7\] and \[C.8\] are restricted to OMNRF cooperators and defectors. In the same direction the entirety of this expansion has been going, the next step is to define and flesh out effort levels for both management bodies and their respective sub-populations.
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