Gauge Freedom and Tidal Interactions of Compact Binary Systems

by

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The study of compact binary tidal interactions has become a very active area of research in recent years, encouraged further by the detection of gravitational wave signals by laser interferometers. Tidal deformations of material bodies should measurably alter the gravitational wave signals detected, teaching us about the internal structure of these compact objects. To explore these effects, one can carry out a matching calculation stitching together two metrics, one calculated close to our central body and one in the far field. This work looks to simplify these lengthy calculations by taking advantage of the gauge freedom present in perturbation theory. We explore two different options: the EZ gauge, originally developed by Steven Detweiler, and the widely used Regge-Wheeler gauge. We discover the EZ gauge, unfortunately, does not aid in these calculations. It also includes an unavoidable singularity at the event horizon, further limiting its usefulness. The Regge-Wheeler gauge, however, does maintain the form of a post-Newtonian metric when transformed, a result which drastically simplifies the matching procedure.
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CHAPTER 1

INTRODUCTION

1.1 LIGO AND THE ERA OF GRAVITATIONAL WAVE ASTRONOMY

On September 14, 2015 the Laser Interferometer Gravitational-Wave Observatory (LIGO) made the first ever detection of gravitational waves [1]. Emitted from a binary black hole merger $410^{+160}_{-180}$ Mpc away, these ripples in the fabric of Space had been travelling towards the Earth for over a billion years. Ever so slightly altering the length of a 4 km tube, an effect smaller than the radius of a proton, it can’t be oversold just how incredible this discovery was. This completely changed our view of the physical world, beginning the era of gravitational wave astronomy. This provides us with an entirely new method to observe and understand the universe, and is strongest test of Einstein’s General Theory of Relativity ever conducted.

Gravitational waves were first proposed by Henri Poincaré in 1905 [2]. Noting that gravity propagating at speeds greater than $c$ would contradict the principles of relativity, he suggested the idea of an onde gravifique (gravitational wave), analogous with electromagnetic waves. They were subsequently theorized by Albert Einstein based on his Theory of General Relativity (GR) [3]. However, even as GR passed countless experimental tests [4–6], there was still a large debate: are gravitational waves (GWs) a physical phenomenon or simply a mathematical artifact [7]? Even Einstein had his doubts they were real [8], and even if they did exist, he was certain the effects

1
would be too small to measure [3]. Famously, in 1936 Einstein nearly published a paper with Nathan Rosen in *Physical Review* stating the firm conclusion that gravitational waves did not exist. Luckily, due to his stubbornness, Einstein pulled the paper after clashing with a reviewer (not because he had yet realized the flaw in their calculations) [8]. This manuscript was eventually published by *The Journal of the Franklin Institute*, with the correct conclusion that gravitational waves do exist [9].

This debate was not fully settled until after Einstein’s death in 1955. A fundamental property of GR is known as covariance, the idea that physical laws are completely independent of the choice of coordinates. However, since one chooses a coordinate system for calculations, it can be quite difficult to distinguish physical and mathematical effects. This was the basis for the gravitational wave debate - are they real or just a result of the mathematics? A significant step to overcome this hurdle was cleared by Felix Pirani in 1956. His work covariantly showed that particles would move back and forth as a gravitational wave passed by [10]. This was then extended to the seminal "Sticky Bead" thought experiment proposed by Richard Feynman:

"It is simply two beads sliding freely (but with a small amount of friction) on a rigid rod. As the wave passes over the rod, atomic forces hold the length of the rod fixed, but the proper distance between the two beads oscillates. Thus, the beads rub against the rod, dissipating heat."

This argued that not only are gravitational waves real, but they must also carry energy - an observable property. The presentation of this thought experiment at the Chapel Hill conference of 1957 [11], along with its subsequent publication by Hermann Bondi in *Nature* [12], is often considered the point when gravitational waves became widely accepted as a physical phenomenon.

After this debate had been largely settled, the next step was to begin work on experimentally detecting gravitational waves. Throughout the 1960’s and 70’s several groups began work on designing a gravitational wave detector [13]. The design used by LIGO, a laser interferometer with freely hanging mirrors, was first proposed in 1962 by Michael Gertsenshtein and Vladislav Pustovoit [14]. Independently, the same idea was proposed several years later by Joseph Weber and
Rainer Weiss [13]. Weiss continued to pursue this line of investigation and in 1984, along with Kip Thorne and Ronald Drever, they formed the leadership of the new Caltech and MIT collaboration: LIGO [13].

The general experimental design of LIGO consists of two identical, modified Michelson interferometers (for a more thorough discussion see the first detection paper [1] and references therein). One detector is located in Hanford, Washington and the other in Livingston, Louisiana, separated by a distance of 3002 km. This separation is important to distinguish noise from a true signal, as any real event should appear coincidentally at each observatory (within the \( \lesssim 10 \) ms travel time). Each observatory has two perpendicular 4 km arms with large, freely hanging mirrors at each end acting as test masses (see Fig. 1.1). A high-powered laser is then sent through a beam splitter and reflected repeatedly inside the optical cavity of each arm using a power-recycling mirror. This allows a build up of 100 kW circulating within each arm, greatly increasing the sensitivity of the detector [1]. A passing gravitational wave will then change the effective length of each optical cavity, which can be detected through precise measurements of the laser signal. In practice, the experiment works to measure the gravitational wave strain \( h(t) \), projected onto the plane of the detector. Defining the unperturbed arm lengths \( L_x = L_y = L = 4 \) km, the gravitational wave strain is

\[
h(t) := \frac{\Delta L(t)}{L} \tag{1.1.1}
\]

where \( \Delta L(t) = \delta L_x(t) - \delta L_y(t) \).

The strain measured is then related to the gravitational radiation of our physical system by the Quadrupole formula:

\[
h^{jk}_{TT} = \frac{2G}{c^4 R} I^{jk}_{TT} \tag{1.1.2}
\]

where the subscript TT indicates the transverse-tracefree projection, an overdot indicates differentiation with respect to proper time, \( h^{jk} \) describes the gravitational potential in the wave zone far from our mass distribution, and \( I^{jk} \) is the Newtonian mass quadrupole moment

\[
I^{jk}(t) := \int \rho(\tau, x) x^j x^k d^3 x
\]

where \( \rho \) is the bodies density dependent on proper time, \( \tau \), and spatial position, \( x \). Here, it's worth noticing the differences between gravitational and electromagnetic
radiation. The later is generated by changing dipole moments while former is only produced by a changing quadrupole.

Several decades later, to say the LIGO collaboration has been a success would be an understatement. Along with the first binary black hole detection, GW150914, they have also detected four other black hole mergers \[15\textsuperscript{-}18\] and one possible detection which has not reached statistical significance \[19\]. Of these black hole detections, it is worth highlighting the fourth one, GW170814, as it was the first event seen both by the LIGO and Virgo collaborations (a European collaboration similar to LIGO using a 3 km detector located near Pisa, Italy) \[17\]. The collaboration also received international recognition when the 2017 Nobel Prize in Physics was awarded to Rainer Weiss, Barry C. Barish, and Kip S. Thorne "for decisive contributions to the LIGO detector and the observation of gravitational waves" \[20\]. Finally, and most relevant to this work, on August 17, 2017 the LIGO/Virgo collaboration detected gravitational radiation emitted from the inspiral of a binary neutron star merger \[21\]. This detection was combined with an extensive observing campaign providing related measurements across the electromagnetic spectrum \[22\]. This included a short gamma ray burst 1.7s after merger, helping to explain the origin of these mysterious signals \[23\].

The LIGO/Virgo collaboration, along with many other scientists have used these observations to begin answering countless open questions in physics. To date gravitational wave observations have provided the most stringent strong field tests of General Relativity \[24\]; placed constraints on the speed of gravitational waves \[25\]; and independently measured the Hubble parameter \(H_0\) \[26\] among countless other tests. Currently, both the LIGO and Virgo observatories are undergoing upgrades to increase the detector sensitivity and should resume operations in early 2019 \[27\]. Working in conjunction with several other observatories in various stages of planning or operation e.g. GEO600 \[28\], Kagra \[29\], and LIGO-India \[30\], the rate of gravitational wave detections will continue at an accelerated pace. The era of gravitational wave astronomy is well under way.
Simplified diagram of the LIGO detectors (not to scale). An incident gravitational wave would lengthen one arm while shortening the other. Each arm will be lengthened and shortened once per gravitational wave cycle. *Inset (a):* Location of each LIGO observatory, shown with approximate light travel time and each detector’s orientation. *Inset (b):* Sensitivity of each detector near the first detection, GW150914, expressed in terms of equivalent gravitational wave strain. This is limited by various sources of noise e.g. thermal noise in the mirrors or quantum noise arising from the discrete nature of light. This image is taken from Fig.3 of Ref. [1].
1.2 TIDAL INTERACTIONS AS A PROBE OF NEUTRON STAR STRUCTURE

All of the binary black hole mergers detected by LIGO are hugely significant events, especially considering the statistical tests which can be performed with a growing set of mergers (see e.g. [31–33]). However, for many applications, the merger of a binary neutron star (BNS) system is a far more interesting detection. Neutron stars are the most compact material bodies known to exist in the universe. With a compactness of \( GM/c^2 R \lesssim 0.30 \), neutron stars approach the theoretical limit for the compactness of a uniform density star: \( 4/9 \sim 0.44 \) [34]. For reference, Schwarzschild black holes have a compactness of \( 1/2 \). The extreme conditions present in a material body provide an ideal laboratory to test not only gravitational theories but also nuclear models, an advantage over binary black hole observations.

Since the first pulsar was observed by Jocelyn Bell in 1967 [35], our understanding of neutron star structure has greatly increased. Recent observations have precisely measured stars with a wide range of masses spanning \( 1.17–2.01 \, M_\odot \) and radii of \( 9.9–11.2 \, \text{km} \) [36]. Some recent observations even claim to have observed neutron star masses around \( 2.3 \, M_\odot \) [37], and the BNS merger by LIGO suggests radii around \( 11.9 \, \text{km} \) [38]. These measurements already put very strong constraints on the possible equations of state (EoS) describing the interior of neutron stars. Low energy nuclear experiments, such as nucleon-nucleon scattering below 350MeV have also been used to constrain the EoS below nuclear saturation density [39, 40]. Saturation density, \( \rho_{\text{sat}} = 2.8 \times 10^{14} \, \text{g cm}^{-3} \), is the average density of nuclei found on Earth [36]. This allows physicists to describe the outer layers of a neutron star (see Fig. 1.2) with a high degree of accuracy. However, the interior which can greatly exceed \( \rho_{\text{sat}} \), is not well constrained and currently there are no terrestrial experiments which can explore such extreme pressures and densities.

To try and further constrain the high density EoS, astronomers have sought to independently measure neutron star masses and radii. However, this has proven extraordinarily difficult, especially radius measurements. In theory, if the distance to a neutron star is known, one can
Interior structure of a neutron star expressed in terms of the nuclear saturation density $\rho_{\text{sat}}$. The outer regions with $\rho \lesssim \rho_{\text{sat}}$ are well constrained by terrestrial experiments. The inner regions well above $\rho_{\text{sat}}$ are still poorly understood but gravitational wave detections from binary neutron star mergers could provide a new experimental technique to probe the interior structure. Figure adapted from Robert Schulze under Creative Commons License [41].

approximate the star as a blackbody, estimating its surface area with the Stefan-Boltzmann law. However, one must account for the star’s atmosphere, magnetosphere, variable emissivity, and a range of other factors leaving a lot of uncertainty in these measurements [36]. In contrast, mass can be fairly well constrained, at least in binary systems where the measurement of post-Keplerian orbital parameters such as the advance of periastron or the Einstein delay\(^1\) can be measured. The uncertainties present in radius measurements mean the constraints on the high density EoS are also not well defined by astronomical observations.

With the first detection of a binary neutron star merger, physicists now have a new tool for probing the high density EoS. Tidal interactions alter the orbital dynamics of binary systems with at least one material body, an effect that should be measurable in the gravitational radiation emitted.

\(^1\)Einstein delay is a combination gravitational redshift and special relativistic time delay.
There are two main methods proposed for experimentally measuring these tidal effects. The first involves detecting tidal disruptions that occur close to merger \([42, 44]\). As a neutron star inspirals closer to its companion body, the differential strength of the tidal field across the star may become large enough to cause a disruption. This would sharply dampen the gravitational wave emission at a specific cut off frequency, which is strongly dependent on the internal structure \([42, 44]\). Unfortunately, this approach is not without its challenges. Extracting detailed information about the internal structure involves match filtering with a known gravitational wave template. Thus the effects are degenerate with other parameters such as individual spins and masses. Producing these templates is also very computationally expensive \([45]\). Another issue comes from the frequency at which this damping occurs. Neutron stars are not expected to be tidally disrupted until very close to merger i.e. at higher frequencies approaching the limits of LIGO’s sensitivity. Notably, the actual merger of the binary neutron star system GW170817 was not seen by LIGO, as this occurred at higher frequencies outside of LIGO’s sensitivity band \([21]\).

The second approach was first proposed by Flannagan and Hinderer in Ref. \([45]\). Instead of considering tidal disruptions near merger, they choose to examine the early inspiral portion. Here the neutron star will be well separated from its companion with radiative frequencies \(f_{GW} \lesssim 400\) Hz, well within LIGO’s highest sensitivity band (see Fig. 1.1 inset (b)). At this stage, the primary physical effect driving orbital evolution are the neutron star’s fundamental \(f\)-modes. These can be treated as a forced, damped harmonic oscillator driven by the companion’s tidal field. Since the orbital frequency evolves on the radiation reaction timescale, significantly longer than the frequency of the modes, this system evolves adiabatically and the star’s quadrupole deformation will track the companion body’s tidal field. This mass quadrupole moment will affect the orbital dynamics during inspiral, leaving a distinct imprint on the emitted gravitational radiation. Flannagan and
Hinderer showed this will introduce a tidal phase shift to the gravitational wave signal emitted:

\[ \delta \Psi = -\frac{3\pi^{5/3}}{8} \left( \frac{f_{\text{gw}}}{c^3} \mathcal{M} (M + M') \right)^{5/3} \left[ (12q + 1) \left( \frac{2GM}{c^2} \right)^5 K_2 + (12q^{-1} + 1) \left( \frac{2GM'}{c^2} \right)^5 K_2' \right] , \]

where \( \mathcal{M} := (MM')^{3/5} / (M + M')^{1/5} \) is the binary’s chirp mass and \( q := M'/M \) is the mass ratio. Measuring this phase shift in emitted gravitational waves would allow us to probe the interior structure of neutron stars by determining \( K_2 \) and \( K_2' \), the tidal Love numbers of the neutron star and its companion.

Tidal Love numbers were first introduced by their namesake A.E.H. Love in the context of Newtonian tides on Earth \[47\]. If we consider a body embedded in an external quadrupolar tidal field \( \mathcal{E}_{ab}(t) := -\partial_{ab} U_{\text{ext}} \), it will become tidally deformed as described by the mass quadrupole moment

\[ I_{ab} = -\frac{2}{3G} R^5 k_2 \mathcal{E}_{ab} \]

where \( I_{ab} \) was previously defined in the context of Eq. (1.1.2). \( R \) is the body’s radius, \( k_2 \) is the dimensionless tidal Love number and it is understood that \( \mathcal{E}_{ab} \) is evaluated at our tidally deformed body’s centre of mass, after differentiation. The factors of \(-2/3\) are purely conventional and the Love number introduced here is related to the version introduced in Eq. (1.2.1) by

\[ K_2 := \left( \frac{2GM}{c^2 R} \right)^{-5} k_2. \]

To Newtonian order, the gravitational potential outside a tidally deformed body is given by

\[ U = \left( \frac{GM}{r} \right) - \left[ \frac{1}{2} \mathcal{E}_{ab} x^a x^b \right] - \left\{ 2k_2 \left( \frac{R}{r} \right)^5 \mathcal{E}_{ab} x^a x^b \right\} . \]

Here, \( x^a \) represents Cartesian coordinates, mass centred on the reference body, and the separation \( r \) is defined as \( r := \sqrt{\delta_{ab} x^a x^b} \). The term in round brackets represents the potential from a spherically symmetric mass monopole, the square brackets describes the external tidal field, and the curly brackets contain the body’s response to that tidal field - dependent on the Love number \( k_2 \).

\(^2\)The notation of Eq. (1.2.1) differs from that used originally in Ref. [43], we instead choose to adopt the notation of eq. (1.1.1) from the PhD Thesis of Landry [46]. A lot of the work of this thesis is built off the work of Landry and we regularly adopt his notation throughout.

\(^3\)The subscript 2 indicates this is related to quadrupolar \((l = 2)\) tidal deformations. In general there are infinite higher order multipole Love numbers, for a discussion see Chapter 2 of Ref. [4].
This Newtonian description of tides works well in the limit of weak field gravity, but is insufficient for the strong, relativistic fields found in compact binaries. The first complete description of relativistic Love numbers was introduced by Damour and Nagar [48]. Independently, a similar construction was introduced by Binnington and Poisson [49], who note their general results agree barring scale factors. These authors discovered that two Love numbers $K^\text{el}_2$ (equal to $K_2$ from Eq. (1.2.1)) along with $K^\text{mag}_2$ are required to describe relativistic tidal deformations. Referred to as the gravitoelectric and gravitomagnetic Love numbers respectively, they are named for their analogous behaviour when compared with electromagnetism. The gravitoelectric Love number relates to tides raised by the second body’s mass distribution (like electric charge distributions), characterized by the tidal field $E_{ab}$. The gravitomagnetic Love number corresponds to tides raised by mass currents (like magnetic charge currents), characterized by the tidal field $B_{ab}$. This gravitomagnetic tidal field has no analogue in Newtonian gravity. In their work, Binnington and Poisson also conclusively show that the tidal Love numbers vanish completely for black holes [49].

This was then extended by Landry and Poisson to the case of a rotating central body [50]. They discovered that the tidal fields $E_{ab}$ and $B_{ab}$ couple to the body’s spin angular momentum, an effect arising from the non-linearity of the field equations. To account for this coupling, they introduced four new tidal Love numbers. $\mathcal{E}^q$ and $\mathcal{B}^q$ describe the quadrupolar and octupolar deformations caused by spin coupling with $E_{ab}$; $\mathcal{B}^q$ and $\mathcal{K}^o$ describe the quadrupolar and octupolar deformation from spin coupling with $B_{ab}$. Like $K^\text{el}_2$ and $K^\text{mag}_2$, these Love numbers appear as integration constants when solving the field equations and vanish for black holes.

1.3 Overview

The work required to determine all relevant tidal moments and their associated Love numbers is quite laborious and the primary goal of this thesis was to simplify these calculations. Our starting place is the matching calculation developed by Poisson with various collaborators [49–55]. This combines black hole perturbation theory in the local neighbourhood of our central body and a post-Newtonian (PN) description of the tidal field far from the black hole. Although each
technique describes a different region of our tidally deformed spacetime, they can be matched in an overlapping region of validity (see Fig. 1.3). This final introductory section aims to outline this matching calculation and the structure of this thesis.

A post-Newtonian system consisting of a black hole (left, black) and a normal star (right, yellow). The post-Newtonian domain is pictured as a blue ellipse, and it excludes the fuzzy white region surrounding the black hole. The black-hole domain is pictured as the red fuzzy region, which extends all the way down to the black hole. The matching of the black-hole and post-Newtonian metrics is carried out in the overlap between the black-hole and post-Newtonian domains. Figure adapted from Paper 2 [56]

We begin with a central compact body of mass $M$ and radius $R$. It will be embedded in some exterior tidal field characterized by a secondary mass $M_2$, separation scale $b$, and velocity $v^2 \sim \frac{M+M_2}{b}$. Throughout this work we adopt natural units with $G = c = 1$ unless otherwise stated. We also work with a metric convention of $(-, +, +, +)$. Ch. 2 begins by formally defining the gravitoelectric and gravitomagnetic tidal moments $E_{ab}$ and $B_{ab}$ along with higher order multipoles. We use the framework developed by Poisson and Vlasov [54], which defines them from the Weyl
Tensor in a tidally perturbed Minkowski space. These tidal moments are then packaged into irreducible scalar, vector, and tensor potentials for insertion within the metric. Finally, for the sake of completing calculations, these potentials are then decomposed into the spherical harmonic basis developed by Martel and Poisson in Ref. [51].

In Ch. 3 we then proceed with a review of black hole perturbation theory, specifically the framework of Martel and Poisson which constructs a perturbed Schwarzschild metric using spherical harmonics [51]. This is used to develop a tidally perturbed metric ansatz, initially gauge independent. The metric ansatz is then specialized to the three different gauges which are used throughout this work: the Light Cone Gauge [57], the Regge-Wheeler Gauge [58], and the EZ Gauge [59]. This chapter concludes with a discussion of first order gauge transformations. These sections are especially important, as the primary goal of this thesis is to simplify the matching calculation by exploiting the gauge freedom of black hole perturbation theory. In past work (see e.g. Ref. [55]) the perturbed spacetime was first calculated in the light cone gauge and then transformed to a harmonic gauge to facilitate PN matching. This gauge transformation, discussed in Sec. 4.4, is quite laborious and finding a gauge in which this can be avoided would greatly simplify the problem.

Ch. 4 begins by solving the vacuum field equations for the EZ gauge. This allows for the determination of all radial functions which appear as coefficients to the spherical harmonic expansion. We find the EZ gauge contains an unavoidable singularity at the event horizon - a property intrinsic to the gauge and not specific to our problem. This is the first major result arising from our work and has been published as Ref. [60], hereafter Paper 1. At this point, we look to match our tidally perturbed metric with the exterior PN spacetime. This is first conducted in the light cone gauge (a reproduction of the calculations in Ref. [55]), and subsequently in the EZ and Regge-Wheeler gauges. We find that the Regge-Wheeler gauge allows for direct matching with a post-Newtonian metric, without an intermediate gauge transformation. This is the second major result arising from our work and has been published in Ref. [56], hereafter Paper 2.

Finally, we arrive at Ch. 5 which begins with a summary of the remaining work completed in Paper 2. Using the Regge-Wheeler gauge simplification, Poisson has calculated the higher
order tidal multipole moments $E_{abc}, B_{abcd}$ etc. as while as time derivatives of these moments. We conclude by summarizing the main results of this thesis and discussing some future research opportunities.
CHAPTER 2

CONSTRUCTING THE TIDAL ENVIRONMENT

2.1 TIDAL MULTIPOLe MOMENTS

The focus of this chapter will be describing the tidal field in which our reference compact object will be embedded. The formalism used was first introduced by Zhang [61] who showed the local tidally perturbed metric of a vacuum spacetime, can be described with two types of tidal multipole moments. This was then specialized to look at the tidal deformation of a nonrotating black hole by Poisson & Vlasov [54], who developed the irreducible tidal potentials used in this thesis.

We will consider a central compact body with mass $M$ and radius $R$ affected by an exterior mass distribution with mass $M_2$. The system is characterized by a separation scale $b$ and velocity scale $v^2 \sim \frac{M + M_2}{b}$. We will use lower case Latin indices to span 1-3 i.e. the Cartesian spatial dimensions which are raised and lowered with the Euclidean metric $\delta_{ab}$ or its inverse. Greek indices will span all dimensions 0-4 and these are raised and lowered with background metric $g_{\alpha\beta}$ or its inverse.

Consider the local neighbourhood, $\mathcal{N}$, around the timelike geodesic world line, $\gamma$, of our reference body. The multipole moments introduced by Zhang are defined in terms of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ defined within $\mathcal{N}$. Constructed from the trace-free part of the Riemann tensor, the Weyl tensor characterizes the tidal fields present in vacuum spacetimes. Here, the specifics of the tidal field and thus the Weyl tensor are taken to be arbitrary.

We start by parametrizing our geodesic as $z^\alpha(\tau)$, where $\tau$ is the proper time. We can construct a
basis starting with the central body’s velocity \( u^\alpha = \frac{dz^\alpha}{d\tau} \), which is tangent to \( \gamma \), and an orthonormal triad, \( e^\alpha_a(\tau) \), which are all orthogonal to \( u^\alpha \). Using this basis, our Weyl tensor can be decomposed as

\[
C_{a0b0}(\tau) = C_{\alpha\gamma\beta\delta} e^\alpha_a u^\gamma e^\beta_b u^\delta, \quad C_{abc0}(\tau) = C_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c u^\delta, \quad C_{abcd}(\tau) = C_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d.
\]

(2.1.1)

The tidal quadrupole moments are then defined as

\[
E_{ab} := (C_{a0b0})^{STF}, \quad B_{ab} := \frac{1}{2}(\epsilon_{acd} C_{cd}^{b0})^{STF},
\]

(2.1.2)

where \( \epsilon_{acd} \) is the permutation symbol and STF stands for symmetric trace-free. The higher order tidal moments are then defined as successive covariant derivatives of the quadrupole moments.

\[
E_{abc} := (C_{a0b0;c})^{STF}, \quad B_{abc} := \frac{1}{2}(\epsilon_{ajk} C_{jk}^{b0;c})^{STF},
\]

(2.1.3)

and

\[
E_{abcd} := (C_{a0b0;c;d})^{STF}, \quad B_{abcd} := \frac{1}{2}(\epsilon_{ajk} C_{jk}^{b0;c;d})^{STF}.
\]

(2.1.4)

The tidal moments \( E_{ab...} \) and \( B_{ab...} \) are known as the gravitoelectric and gravitomagnetic moments respectively. The gravitoelectric moments describe the tidal fields created by mass distributions, directly analogous to standard tides as described by Newtonian theory. The gravitomagnetic moments then describe the tides created by mass currents. Gravitomagnetic tides have no analogue in Newtonian gravity. These names derive from the analogous behaviour of Maxwell’s equations, where static charge distributions generate electric fields and charge currents generate magnetic fields.

These tensors are taken to be spatially constant within the local neighbourhood and are functions of coordinate time. They are also STF tensors by their Weyl tensor definitions. Under a parity
transformation $u^\alpha \rightarrow u^\alpha$ and $e^\alpha_a \rightarrow -e^\alpha_a$, the tidal moments will change according to

\begin{align*}
\mathcal{E}_{ab} &\rightarrow \mathcal{E}_{ab}; & \mathcal{B}_{ab} &\rightarrow -\mathcal{B}_{ab}; \quad (2.1.5a) \\
\mathcal{E}_{abc} &\rightarrow -\mathcal{E}_{abc}; & \mathcal{B}_{abc} &\rightarrow \mathcal{B}_{abc}; \quad (2.1.5b) \\
\mathcal{E}_{abcd} &\rightarrow \mathcal{E}_{abcd}; & \mathcal{B}_{abcd} &\rightarrow -\mathcal{B}_{abcd}. \quad (2.1.5c)
\end{align*}

Thus, the gravitoelectric tensors are even parity while the gravitomagnetic tensors are odd parity.

Recall our tidal environment is characterized by $M_2, b$, and $v$, thus our quadrupolar tidal moments will scale as

\[ \mathcal{E}_{ab} \sim \frac{M_2 b^3}{b^3}, \quad \mathcal{B}_{ab} \sim \frac{M_2 v b^3}{b^3}. \quad (2.1.6) \]

Higher order moments are each suppressed by another factor of $\frac{1}{b}$. Similarly, each time derivative of the moments would be suppressed by a factor of $\frac{v}{b}$, e.g.,

\[ \dot{\mathcal{E}}_{abc} \sim \frac{M_2}{b^4}, \quad \dot{\mathcal{B}}_{ab} \sim \frac{M_2 v}{b^4}. \quad (2.1.7) \]

For much of this work, we will neglect time derivatives and higher multipole moments, $l > 2$, because of this suppression.

These multipole moments are then to be packaged into irreducible potentials for insertion into the metric. First, one must introduce the radial unit vector

\[ \Omega^a = \frac{x^a}{r}. \quad (2.1.8) \]

As described in Sec II of Poisson & Vlasov [54], the multipole moments and radial unit vector are packaged into scalar, vector, and tensor potentials which obey the same parity rules as Eq. (2.1.5).

There are three kinds of gravitoelectric potentials

\[ \mathcal{E}^q := \mathcal{E}_{ab} \Omega^a \Omega^b, \quad \mathcal{E}^q_a := \gamma_a^b \mathcal{E}_{bc} \Omega^c, \quad \mathcal{E}^q_{ab} := 2\gamma_a^c \gamma_b^d \mathcal{E}_{cd} + \gamma_{ab} \mathcal{E}^q, \quad (2.1.9) \]
and two gravitomagnetic potentials

\[ B^q_a := \epsilon_{acd} \Omega^c B^d_b \gamma^b_a, \quad B^q_{ab} := \epsilon_{abcd} \Omega^e B^d_e \gamma^e_b + \epsilon_{bca} \Omega^e B^d_e \gamma^e_a. \] (2.1.10)

The superscript \( q \) is included as notation to indicate these are of quadrupolar order, and the tensor \( \gamma^b_a := \delta^b_a - \Omega_a \Omega^b \) projects the vector and tensor potentials transverse to our radial unit vector \( \Omega^a \).

### 2.2 SPHERICAL HARMONIC DECOMPOSITION

The tidal potentials will now be expressed in terms of scalar, vector, and tensor spherical harmonics to be integrated while solving the field equations. This takes advantage of the direct connection between STF tensors and the spherical harmonics (see e.g. Ch 1 of Ref. [4]). However, the first step before decomposition is the transformation from Cartesian to spherical coordinates. This is facilitated with the Jacobian matrix

\[ \Omega^a_A := \frac{\partial \Omega^a}{\partial \theta^A}. \] (2.2.1)

where \( \Omega^a_A \) is the radial unit vector and \( \theta^A \) represents the angular coordinates on the unit two sphere. In spherical coordinates, these vectors take the explicit form \( \Omega^a = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi] \) and \( \theta^A = [\theta, \phi] \). Note, all vector and tensor quantities with capital Latin indices span the angular coordinates of \( \theta^A \). Our transformed potentials are

\[ E^q_A := E^q_a \Omega^a_A, \quad E^q_{AB} := E^q_{ab} \Omega^a_A \Omega^b_B, \] (2.2.2)

and

\[ B^q_A := B^q_a \Omega^a_A, \quad B^q_{AB} := B^q_{ab} \Omega^a_A \Omega^b_B, \] (2.2.3)

while the scalar potential \( E^q \) remains unchanged.

Note that these potentials have no radial dependence as they are functions of only \( \theta^A \). The remaining, radial degree of spatial freedom present in our generic tidal field is contained within a set of radial functions. These functions, which are obtained by integrating the field equations, are
discussed further in Sec. 4.1.

With only angular dependence, our potentials are ready to be packaged into the spherical harmonics. Our decomposition is based on the framework developed by Martel & Poisson [51], presented in the context of black hole perturbation theory. The gravitoelectric potentials are decomposed using the real valued scalar harmonics $Y^\ell m(\theta^A)$ (listed in Table 2.1) and the even parity vector and tensor harmonics

$$Y^\ell m_A := D_A Y^\ell m, \quad Y^\ell m_{AB} := \left[ D_A D_B + \frac{1}{2} \ell (\ell + 1) \Omega_{AB} \right] Y^\ell m,$$

where $\Omega_{AB}$ is the metric of the unit two-sphere. The gravitomagnetic potentials require the odd-parity vector and tensor harmonics

$$X^\ell m_A := -\epsilon_A^B D_B Y^\ell m, \quad X^\ell m_{AB} := -\frac{1}{2} \left( \epsilon_A^C D_B + \epsilon_B^C D_A \right) D_C Y^\ell m.$$

Here, $D_A$ is the covariant derivative operator compatible the unit two-sphere metric, $D_A \Omega_{BC} = 0$ and $\epsilon_{AB}$ is the Levi-Civita tensor on the unit two-sphere with $\epsilon_{\theta\phi} = \sin \theta$.

As discussed in Ref. [4], a STF tensor of rank $l$ contracted with $l$ copies of the radial unit vector is equivalent to a spherical harmonic expansion. In the context of our tidal potentials we have

$$E^{ab}_{\Omega^a \Omega^b} = \sum_m E^q_m Y^{2m}, \quad E^{aq}_{ab \Omega^a \Omega^b} = \sum_m B^q_{ab} Y^{2m}.$$ 

Thus we have packaged the five independent components of each quadrupole moment into the spherical harmonic coefficients $E^q_m$ and $B^q_m$. This will depend on the normalization adopted for the harmonics, and for this work we will follow the convention laid out by Poisson & Vlasov [54]. Their definitions for the real valued harmonics are reproduced in Table 2.1 and their spherical harmonic coefficients are listed in Table A.1 in Appendix A.

The full decomposition into spherical harmonics is given by

$$E^q = \sum_m E^q_m Y^{2m}, \quad E^q_A = \frac{1}{2} \sum_m E^q_m Y^{2m}_A, \quad E^q_{AB} = \sum_m E^q_m Y^{2m}_{AB}$$

(2.2.7a)
The dipole ($l = 1$), quadrupole ($l = 2$), and octupole ($l = 3$) modes listed for the real valued spherical harmonics. The abstract index $m$ describes the $\phi$ dependence e.g. $Y^{3,3c}$ is proportional to $\cos (3\phi)$. To simplify the expressions we write $C := \cos \theta$ and $S := \sin \theta$.

\[ B^A_A = \frac{1}{2} \sum_m B^A m X_A^{2m}, \quad B^A_{AB} = \sum_m B^A m X_{AB}^{2m}. \quad (2.2.7b) \]

The coefficients here come from differentiation of Eq. (2.2.6) to obtain all vector and tensor harmonics. This is explained in Appendix A of Ref. [54].
CHAPTER 3

BLACK HOLE PERTURBATION THEORY

3.1 BACKGROUND SPACETIME

The objective of this section will be to fully develop the tidally perturbed metric describing the
near field around our central body. We will use the framework of black hole perturbation theory
developed by Martel & Poisson in Ref. [51]. This was then used in the first matching calculation
by Taylor & Poisson [53] who calculated the tidal moments $E_{ab}$ and $B_{ab}$ for a non-rotating black
hole to leading post-Newtonian order. The higher order tidal moments were then formally defined
by Poisson & Vlasov in Ref. [54] and utilized to explore the tides around a spinning black hole by
Poisson [55] and a spinning material body by Landry & Poisson [50]. Finally, the non spinning
case was then reexplored, calculating the higher multipole tidal moments and their derivatives (e.g.
$E_{abc}$ or $\dot{E}_{ab}$) by Poisson & Corrigan in Paper 2 [56].

Black hole perturbation theory, which has a rich history, was initially laid out in the seminal
work of Tulio Regge and John Archibald Wheeler [58]. This was extended by Vishveshwara [62]
and Zerilli [63] and then developed using gauge-invariant formalisms in Ref. [51, 64–72]. It has
been used to study several areas of interest such as the gravitational radiation from a point particle
orbiting a Schwarzschild black hole (see e.g. Ref. [73] or the living review [74]), the gravitational
self force in curved spacetimes [75–77], quasinormal modes of a black hole [78], the collision of
two black holes in a close limit approximation [79], and the application here - tidal deformations
of compact bodies [50, 52, 53, 55, 57, 80].
3.2 PERTURBED SPACETIME

We begin with the background metric of our isolated central body, described by the unperturbed Schwarzschild metric

$$ds^2 = -f dv^2 + 2dvdr + r^2d\Omega^2.$$  (3.2.1)

Here $f := 1 - \frac{2M}{r}$ and $d\Omega^2 := \Omega_{AB}d\theta^Ad\theta^B$. We have adopted ingoing Eddington-Finkelstein coordinates with the advanced time $v$ defined as $v := t + r + 2M \ln(\frac{r}{2M} - 1)$.

Starting with this background Schwarzschild metric, $g_{\alpha\beta}$, we can write our tidally deformed metric, to linear order, as

$$g_{\alpha\beta} = g_{\alpha\beta}^{\text{back}} + p_{\alpha\beta}$$  (3.2.2)

where $p_{\alpha\beta}$ describes our perturbation and $g_{\alpha\beta}$ describes the complete metric.

Following the work of Martel & Poisson [51] we will then decompose our perturbed metric into a basis of scalar, vector, and tensor spherical harmonics. The specific definitions and normalization of the harmonics were introduced in Sec. 2.2 specifically Eq. (2.2.4) and Eq. (2.2.5). Using this basis we can decompose $p_{\alpha\beta}$ as

$$p_{ab} = \sum_{lm} h^l_{ab} Y^{lm},$$  (3.2.3a)

$$p_{aB} = \sum_{lm} j^l_a Y^{lm}_B,$$  (3.2.3b)

$$p_{AB} = r^2 \sum_{lm} \left(K^{lm} \Omega_{AB} Y^{lm} + G^{lm} Y^{lm}_{AB}\right),$$  (3.2.3c)

in the even parity sector, and

$$p_{ab} = 0,$$  (3.2.3d)

$$p_{aB} = \sum_{lm} h^l_{a} X^{lm}_B,$$  (3.2.3e)

$$p_{AB} = \sum_{lm} h^l_{2} X^{lm}_{AB}$$  (3.2.3f)
in the odd parity sector. The functions \( \{ h_{ab}^{lm}, j_a^{lm}, K^{lm}, G^{lm}, h^l_{a2} \} \) depend only on \( x^a \). Note, the tensorial potentials are not defined for \( l = 1 \).

One can then match up this decomposition of our perturbed metric, \( p_{\alpha\beta} \), with the tidal potentials written in terms of the spherical harmonics (see Sec. 2.2). For example \( \mathcal{E}^q \) is proportional to \( Y^{2m} \) so it will appear in the even parity parts of \( p_{ab} \) and \( p_{AB} \). Following this technique we can determine how each of our potentials will fit into the perturbed metric. This is explained thoroughly by Landry in his PhD thesis [46], but that work was completed in the Regge-Wheeler gauge. Here we choose to work independent of any gauge; gauge selection will be left to Sec. 3.3.

Looking first at the quadrupole tidal potentials \( \mathcal{E}_{ab} \) and \( \mathcal{B}_{ab} \), we find a generic gravitoelectric perturbation is described by

\[
\begin{align*}
p_{ab} &= r^2 e_{ab}^q(r) \mathcal{E}^q, \\
p_{aB} &= r^2 e_{aB}^q(r) \mathcal{E}^q, \\
p_{AB} &= r^4 e_{AB}^q(r) \mathcal{E}^q + r^4 e^q_{\star}(r) \mathcal{E}_{AB}^q.
\end{align*}
\]

Here, the factors of \( r \) have been included to ensure the correct dimensions for a metric perturbation. The dimensionless functions \( \{ e_{ab}^q, e_{aB}^q, e_{AB}^q, e^q_{\star} \} \) contain the radial dependence of the tidal perturbations and have been separated from \( \{ h_{ab}, h_a, K, G \} \). These will be determined in Sec. 4.1 by integrating the vacuum field equations. For the gravitomagnetic perturbations we have

\[
\begin{align*}
p_{aB} &= r^3 b_{aB}^q(r) \mathcal{B}_{B}^q, \\
p_{AB} &= r^4 b_{AB}^q(r) \mathcal{B}_{AB}^q.
\end{align*}
\]

Combining the background metric of Eq. (3.2.1) with the perturbations of Eq. (3.2.4) and (3.2.5) one can obtain a general, and gauge independent, metric ansatz for the spacetime around our tidally deformed, non-rotating body.

\[
g_{vv} = -f + r^2 e_{vv}^q \mathcal{E}^q, 
\]
\[ g_{\nu \nu} = 1 + r^2 e_\nu^q \mathcal{E}^q, \quad (3.2.6b) \]
\[ g_{\tau \tau} = r^2 e_\tau^q \mathcal{E}^q, \quad (3.2.6c) \]
\[ g_{\nu A} = r^3 e_\nu^q \mathcal{E}_A^q + r^3 b_\nu^q B_A^q, \quad (3.2.6d) \]
\[ g_{\tau A} = r^2 e_\tau^q \mathcal{E}_A^q + r^3 b_\tau^q B_A^q, \quad (3.2.6e) \]
\[ g_{\nu B} = r^2 \Omega_{AB}^q + r^4 e_\nu^q \Omega_{AB} \mathcal{E}^q + r^4 e_\nu^q \mathcal{E}_{AB}^q + r^4 b_\nu^q B_{AB}^q. \quad (3.2.6f) \]

Note, the explicit functional dependence on \( r \) has been suppressed compared to above e.g. \( e^q(r) \rightarrow e^q \).

### 3.3 Gauge Freedom

The previous section constructed our tidally perturbed metric ansatz in a completely general way. However, the perturbation equations contain multiple redundancies known as gauge freedom. To account for this, one must choose a specific gauge, eliminating these residual degrees of freedom. Exploring the gauge freedom present within the perturbation equations was a large part of the novel work completed in this thesis. In particular, the main goal was the simplification of matching calculations required for the determination of the tidal potentials.

In this section we will introduce three different gauges, briefly discuss the advantages of each, and rewrite the metric of Eq. (3.2.6) in these three gauges. It should also be noted that, although it is helpful to complete calculations in a specific gauge, it is best to present results as gauge-invariant or ideally observable quantities. This ensures the results found are not mathematical artefacts.

Conceptually, one needs to fix a gauge because of the auxiliary degrees of freedom introduced by perturbing our coordinates. In this work, calculations are all completed using ingoing Eddington-Finkelstein coordinates, \((v, r)\), for the background Schwarzschild spacetime. When considering only the background metric, these coordinates have a very specific geometric meaning as discussed below in Subsec. 3.3.1. However, after perturbing our metric, the background coordinates become slightly shifted, both losing their geometric meaning and reintroducing 4 degrees of freedom - hence the need to fix a gauge.
In practice, fixing a gauge means one can set four components of Eq. (3.2.3) to zero. There are three redundant degrees of freedom in the even parity sector and one in the odd parity sector. In the language of Martel & Poisson this means that three of the even parity functions \( \{ h_{vv}, h_{vr}, h_{rr}, j_v, j_r, K, G \} \) and one of the odd parity functions \( \{ h_v, h_r, h_2 \} \) will be set to zero (from this point forward we will drop all \( lm \) labels for clarity). As mentioned previously, the sums of Eq. (3.2.3) are only defined for \( l \geq 2 \). We will have to select additional gauge restrictions to handle dipole \( (l = 1) \) perturbations.

### 3.3.1 Light-Cone Gauge

As mentioned above, ingoing \((v, r)\) coordinates for the Schwarzschild spacetime have a number of clear geometric properties. This enables a direct connection between calculated quantities and their physical significance. The light cone gauge, first developed by Preston & Poisson [57], is designed to preserve a number of these key geometric properties. In particular, the authors designed this gauge so: (i) the time coordinate \( v \) is constant on ingoing light cones, (ii) the angular coordinates \( \theta \) and \( \phi \) are constant on the null-generators of these light cones, and (iii) the radial coordinate \( r \) is an affine parameter for the generators. Preserving these geometric properties allows for an easier connection between calculations and the physical world, a challenge affecting most other choices of gauge.

Formally, the light cone gauge is created by setting

\[
\begin{align*}
    h_{vr} &= h_{rr} = j_r = 0, \\
    h_r &= 0,
\end{align*}
\]  

(3.3.1)

in the even parity sector, and

\[
    h_r = 0,
\]  

(3.3.2)

in the odd parity sector. Poisson [55] also discovered that perturbations proportional to \( r^2 K^{lm} \Omega_{AB} Y^{lm} \) produce differential equations dependent only on \( K \) and fully decoupled from the remaining field
equations. They choose to adopt the simplest solution, $K = 0$, which we also use here.\footnote{Even this doesn’t remove all residual gauge freedom. The radial functions of the light-cone gauge contain a number of integration constants $\{c, e^{d,q,o}, \gamma^{d,q,o}\}$. These can be fixed as done in Sec. VI of Ref. \cite{55} to preserve the geometrical meaning of our coordinates. They can also be set to zero, since they are integration constants with no physical meaning, as discussed in Sec. III of Ref. \cite{50}.} By doing this, our tidally perturbed metric in the light-cone gauge is

\begin{align}
 g_{vv}^{LCG} &= -f - r^2 q^q e^q, & (3.3.3a) \\
 g_{vr}^{LCG} &= 1, & (3.3.3b) \\
 g_{rr}^{LCG} &= 0, & (3.3.3c) \\
 g_{vA}^{LCG} &= -\frac{2}{3} r^2 (q^q q^q - b^q B^q_A), & (3.3.3d) \\
 g_{rA}^{LCG} &= 0, & (3.3.3e) \\
 g_{AB}^{LCG} &= r^2 \Omega_{AB} - \frac{1}{3} r^4 (q^q q^q e_{AB}^q - b^q B_{AB}^q). & (3.3.3f)
\end{align}

Here we have adopted the notation of Landry & Poisson for our radial functions e.g. $e_1, b_4$ etc. \cite{50}. Note than in Eq. (3.2.6) our ansatz was constructed as the sum of all possible perturbations. Here, we have factored out negative signs where applicable. This makes the behaviour of all radial functions consistent - each made up of a growing and decaying mode.

### 3.3.2 Regge-Wheeler Gauge

Named for Tulio Regge and John A. Wheeler, the first physicists to lay out the framework for black hole perturbation theory \cite{58}, this gauge was utilized in their foundational paper. Since then it has commonly been employed throughout the literature e.g. Ref. \cite{50,51,81–83}. It is defined by setting

\begin{align}
 j_v = j_r = G = 0, & (3.3.4) \\
 h_2 &= 0. & (3.3.5)
\end{align}
in the odd parity sector. For \( l = 1 \) we adopt the auxiliary gauge conditions of Landry & Poisson, \( K^{1,m} = j^{1,m} = h^{1,m}_r = 0 \). The tidally deformed metric ansatz in the Regge-Wheeler gauge is

\[
\begin{align*}
    g^{RW}_{rr} &= -f - r^2 e^q_{rr} \mathcal{E}^q, \\
    g^{RW}_{vr} &= 1 + r^2 e^q_{vr} \mathcal{E}^q, \\
    g^{RW}_{v\alpha} &= 2 \frac{3}{3} r^3 h^q_{\alpha} \mathcal{B}^q, \\
    g^{RW}_{\alpha r} &= -2 \frac{2}{3} r^3 h^q_{\alpha} \mathcal{B}^q + r^3, \\
    g^{RW}_{\alpha\beta} &= r^2 \Omega_{\alpha\beta} - r^4 e^q \mathcal{E}^q_{\alpha\beta}.
\end{align*}
\]

(3.3.6a) 
(3.3.6b) 
(3.3.6c) 
(3.3.6d) 
(3.3.6e) 
(3.3.6f)

Here, we again adopt the notation of Landry & Poisson but performed a coordinate transformation. They presented their metric ansatz with \( t \) as the time coordinate while we use \( v \) for consistency.

Looking again at the perturbation equations of Eq. (3.2.3), Martel & Poisson defined six gauge invariant quantities \([51]\). Four in the even parity sector:

\[
\tilde{h}_{ab} := h_{ab} - \nabla_a \epsilon_b - \nabla_b \epsilon_a,
\]

(3.3.7)

and

\[
\tilde{K} := K + \frac{1}{2} l(l + 1) G - \frac{2}{r} r^a \epsilon_a,
\]

(3.3.8)

where

\[
\epsilon := j_a - \frac{1}{2} r^2 \nabla_a G;
\]

(3.3.9)

and two in the odd parity sector:

\[
\tilde{h}_a := h_a - \frac{1}{2} \nabla_a h_2 + \frac{1}{2} r_a h_2,
\]

(3.3.10)

where \( r_a := \frac{\partial r}{\partial x^a} \). Since \( j_a = G = h_2 = 0 \) in the Regge-Wheeler gauge, the harmonic expansion

\footnote{Again, there is residual gauge freedom for \( l = 1 \). This appears as two integration constants, \( \{ c^d, \gamma^d \} \), in the radial functions of Eq. (3.3.6) \([50]\).}
coefficients become gauge invariant: \( h_{ab} = \tilde{h}_{ab} \) and \( K = \tilde{K} \).

### 3.3.3 EZ Gauge

The final gauge we will explore in this thesis is known as the EZ-Gauge. First developed by the late Steven Detweiller, it was published by Thompson, Chen, & Whiting [59] who had gone through Detweiller’s unpublished work, sharing much of it posthumously. The gauge is designed to remove all purely angular parts of the perturbation. Formally this is accomplished by setting

\[
j_v = K = G = 0, \quad (3.3.11)
\]

in the even parity sector and

\[
h_2 = 0, \quad (3.3.12)
\]

in the odd parity sector. Our perturbed metric then becomes

\[
ge^{EZ}_{vv} = -f + r^2 \epsilon^q_{vq} \mathcal{E}^q, \quad (3.3.13a)
\]

\[
ge^{EZ}_{vr} = 1 - r^2 \epsilon^q_{r} \mathcal{E}^q, \quad (3.3.13b)
\]

\[
ge^{EZ}_{rr} = r^2 \epsilon^q_{rr} \mathcal{E}^q, \quad (3.3.13c)
\]

\[
ge^{EZ}_{vA} = r^3 b_i^q \mathcal{B}_A^q, \quad (3.3.13d)
\]

\[
ge^{EZ}_{rA} = -r^2 \epsilon^q_v \mathcal{E}^q_A - r^3 b_i^q \mathcal{B}_A^q, \quad (3.3.13e)
\]

\[
ge^{EZ}_{AB} = r^2 \Omega_{AB}. \quad (3.3.13f)
\]

The EZ gauge has not been widely adopted in the literature, but Thompson, Chen, & Whiting [59] provide a simple example, examining a point particle in a circular orbit around a Schwarzschild black hole.
3.4 Gauge Transformations

As mentioned above, the choice of gauge is completely arbitrary, however, for many calculations it is often useful or necessary to switch between gauges. This is facilitated by a gauge transformation vector. Poisson & Martel have laid out a general framework for constructing a gauge vector, which we adopt here [51].

A gauge transformation is generated by the vector field \( \Xi_\alpha \). This can be decomposed using the spherical harmonic basis constructed in Sec. 2.2 as

\[
\Xi_a = \sum_{lm} \xi_{lm} Y_{lm}^a, \tag{3.4.0a}
\]

\[
\Xi_A = \sum_{lm} \xi_{\text{even}} Y_{lm}^A, \tag{3.4.0b}
\]

in the even parity sector and

\[
\Xi_A = \sum_{lm} \xi_{\text{odd}} X_{lm}^A, \tag{3.4.0c}
\]

for odd parity perturbations. To first order, the spherical harmonic coefficients listed in Eq. (3.2.6) will transform as

\[
h_{vv} \rightarrow h_{vv} - 2 \partial_v \xi_v + \frac{2M}{r^2}, \tag{3.4.1a}
\]

\[
h_{vr} \rightarrow h_{vr} - \partial_r \xi_v - \partial_v \xi_r - \frac{2M}{r^2}, \tag{3.4.1b}
\]

\[
h_{rr} \rightarrow h_{rr} - 2\partial_r \xi_r, \tag{3.4.1c}
\]

\[
j_v \rightarrow j_v - \partial_v \xi_{\text{even}} - \xi_v, \tag{3.4.1d}
\]

\[
j_r \rightarrow j_r - \partial_r \xi_{\text{even}} - \xi_r + \frac{2}{r} \xi_{\text{even}}, \tag{3.4.1e}
\]

\[
K \rightarrow K - \frac{2f}{r} \xi_r - \frac{2}{r^2} \xi_v + \frac{l(l+1)}{r^2} \xi_{\text{even}}, \tag{3.4.1f}
\]

\[
G \rightarrow G - \frac{2}{r^2} \xi_{\text{even}} \tag{3.4.1g}
\]
in the even parity sector, and

\[ h_v \rightarrow h_v - \partial_v \xi_{\text{odd}}, \]  
(3.4.2a)

\[ h_r \rightarrow h_r - \partial_r \xi_{\text{odd}} + \frac{2}{r} \xi_{\text{odd}}, \]  
(3.4.2b)

\[ h_2 \rightarrow h_2 - 2\xi_{\text{odd}}. \]  
(3.4.2c)

in the odd parity sector, again dropping the \( lm \) labels for clarity.
CHAPTER 4

DETERMINING THE TIDALLY DEFORMED SPACETIME

Selecting one of the gauges introduced Sec. 3.3 is largely a matter of personal preference. Each gauge has its pros and cons as briefly discussed in Sec. 3.3 and most have been widely used throughout the literature. For example, Poisson [55] calculated the radial functions for the perturbed metric of a slowly rotating black hole first in the light cone gauge (to take advantage of the geometric significance) and then in the harmonic gauge (for matching with a PN expansion). Subsequently, Landry & Poisson [50] calculated the equivalent radial functions in the Regge-Wheeler gauge. This chapter begins by solving the vacuum field equations using the EZ gauge in Sec. 4.1. This will then be compared to the Regge-Wheeler solutions found by Landry & Poisson, using a gauge transformation. We will then discuss the newly discovered event horizon singularity, intrinsic to the EZ gauge in Sec. 4.2 - the main result of Paper 1. In Sec. 4.3 we review the basics of post-Newtonian theory, presenting the PN metric of Taylor & Poisson [53] required for our matching calculation. Finally, in Sec. 4.4 we will complete the matching calculation in the light-cone, EZ, and Regge-Wheeler gauges. Here we will discuss one of the main results from Paper 2: the matching simplifications enabled by the Regge-Wheeler gauge.
Radial functions found in the metric ansatz of Eq. (3.3.13). These are found as solutions to the vacuum field equations $R_{ab} = 0$ and are made of two linearly independent modes, one growing with $r$ and the other decaying. The decaying mode can be identified as it is multiplied by the bodies Love numbers $K_{2e}^l$ and $K_{2m}^m$.

4.1 SOLVING THE FIELD EQUATIONS

4.1.1 EZ Gauge

To solve for the radial functions, the perturbed metric ansatz of Eq. (3.3.13) will be used to calculate the Ricci tensor, $R_{ab}$, expanded to linear order in powers of $E^q_m$ and $B^q_m$. This expansion is then plugged into the vacuum field equation

$$R_{ab} = 0,$$  \hspace{1cm} (4.1.1)

and solved using the spherical harmonic basis described in Sec. 2.2. The field equations reduce to a system of coupled differential equations where the gravitoelectric and gravitomagnetic sectors fully decouple from each other. Solving this system of differential equations, we determine the radial functions $\{e^q_{vv}, e^q_{vr}, e^q_{rr}, b^q_v, b^q_r\}$ which are presented in Table 4.1.
Table 4.2: Radial Functions in the Regge-Wheeler Gauge: Non-Spinning Body

\[ e^{tt}_t = f^2 + \frac{2}{x^5} \left[ -30x^3(x-1)^2 \ln f - \frac{5}{2}x(2x-1)(6x^2 - 6x - 1) \right] K^e_2 \]
\[ e^{rr}_r = f^{-2}e^{tt}_t \]
\[ e^q = 1 - \frac{1}{2x^2} + \frac{2}{x^7} \left[ -15x^3(2x^2 - 1) \ln f - 5x^2(6x^2 + 3x - 1) \right] K^e_2 \]
\[ b^t_q = -\frac{3}{x^2} \left[ 20x^4(x-1) \ln f + \frac{5}{3}x(12x^3 - 6x^2 - 2x - 1) \right] K^{mag}_2 \]

Radial functions for the metric ansatz Eq. (4.1.3) as calculated by Landry & Poisson [50]. These are expressed in terms of \( x := \frac{r}{2M}, f := 1 - \frac{1}{2x}, \) and a number of different integration constants. Functions within square brackets behave as \( 1 + O\left( \frac{1}{x} \right) \) when \( x \gg 1 \)

Schematically, this system of differential equations takes the form

\[ \mathcal{L}^j_k w^k_1 = 0, \quad (4.1.2) \]

where \( \mathcal{L}^j_k \) is a second order differential operator and \( w^k_1(r) \) is the collection of radial functions [50].

Each radial function can further be decomposed into a mode which grows as \( r \) increases and one which decays; these modes are linearly independent. Physically, the growing mode has been identified with the external tidal field, and the decaying mode is associated with the central body’s response to this tidal field [50]. This is why the decaying modes are multiplied by the bodies gravitational love numbers \( K^e_2 \) and \( K^{mag}_2 \). The Love numbers appear as integration constants while solving the system of differential equations.

### 4.1.2 Regge-Wheeler Gauge

Solving the vacuum field equations in the Regge-Wheeler gauge has been previously completed by Landry & Poisson in Ref. [50]. They completed their work in standard Boyer-Lindquist coordinates \( (t, r, \theta, \phi) \) and considered the case of a spinning material body. We simplify their metric ansatz to the non-spinning case, yielding

\[ g^{RW}_{tt} = -f - r^2 e^{tt}_t e^q, \quad (4.1.3a) \]
where the radial functions are listed in Table 4.2.

Transforming to \((v, r)\) coordinates these results can then be compared with the radial functions of Table 4.1. To facilitate this comparison, we calculate the gauge transformation vector described in Sec. 3.4 finding

\[
\begin{align*}
\xi_{s_r}^{\text{RW} \rightarrow \text{EZ}} &= 0, \\
\xi_{s_r}^{\text{RW} \rightarrow \text{EZ}} &= \frac{r}{2f} K^{\text{RW}}, \\
\xi_{\text{even}}^{\text{RW} \rightarrow \text{EZ}} &= 0, \\
\xi_{\text{odd}}^{\text{RW} \rightarrow \text{EZ}} &= 0.
\end{align*}
\]

Using this gauge vector, we can express the radial functions of the EZ gauge using the Regge-Wheeler radial functions. We find

\[
\begin{align*}
e^{q}_{vv} &= -e^{q}_{tt} - \frac{M}{r} e^{q}, \\
e^{q}_{rr} &= -f^{-1} e^{q}_{vv}, \\
e^{q}_{rr} &= -f^{-2} (e^{q}_{tt} + f(e^{q}_{rr} - 3e^{q} - r \partial_{r} e^{q}) + r e^{q} \partial_{r} f), \\
e^{q}_{r} &= \frac{1}{2} r^{3} f^{-1} e^{q}, \\
b^{q}_{v} &= \frac{1}{3} b^{q}_{t}, \\
b^{q}_{r} &= -f^{-1} b^{q}_{t},
\end{align*}
\]

where all radial functions to the left of an equality are in the EZ gauge and all function to the
right are in the Regge-Wheeler gauge (Table 4.2). Thankfully, this did indeed validate our results. Functions calculated with the vacuum field equations matched those calculated through a gauge transformation, as expected.

### 4.2 EZ GAUGE SINGULARITY

Examining Table [4.1], one quickly notices the inverse of the Schwarzschild mass function, \( f := 1 - \frac{2M}{r} \), present in a majority of the radial functions listed. This means that a majority of these radial functions contain a singularity at \( r = 2M \), the location of the unperturbed event horizon. While examining this property of the radial functions, it has been found that this singularity is an unavoidable (and unfavourable) property intrinsic to the EZ gauge. This is the main result presented in Paper 1, and the arguments supporting this are laid out below.

While exploring this singularity, we began by lifting the assumption of time independent perturbations. However, the divergence at \( r = 2M \) remained in the radial functions of the EZ gauge even after lifting this assumption. Furthermore, the same factor of \( f^{-1} \) is present in the \( \xi_r \) component of the gauge vector transforming from the RW gauge to the EZ gauge (Eq. (4.1.4)). This suggests the singularity may be a fundamental property of the gauge itself.

To look at the fully general, time dependent case, the gauge vector from any gauge to EZ was calculated

\[
\xi_v = j_v^{\text{old}} - \frac{1}{2} r^2 \partial_r G^{\text{old}},
\]

\[
\xi_r = \frac{r}{2f} \left[ K^{\text{old}} - \frac{2}{r} j_v^{\text{old}} + r \partial_v G^{\text{old}} + \frac{1}{2} l(l + 1) G^{\text{old}} \right],
\]

\[
\xi_{\even} = \frac{1}{2} r^2 G^{\text{old}},
\]

\[
\xi_{\odd} = \frac{1}{2} h_2^{\text{old}}.
\]

Once again, the singularity caused by \( f^{-1} \) remains in the \( r \) component of this vector. Thus, the EZ gauge is unavoidably singular at the event horizon. This is the main result presented in Paper 1.

It is important to note; this singularity doesn’t require that the EZ gauge should be completely
discarded. For example, standard Schwarzschild coordinates have been widely used in the literature, despite the well-known singularity at $r = 2M$. With a little care to separate coordinate (or gauge) singularities from actual physical effects, the EZ gauge could still be used. This issue is also relevant only when considering black hole physics. The metric we have calculated is only valid outside our central compact object, and the surface of any material body exists well beyond $r = 2M$. Applied to the largest motivation of the work, neutron star tides, this singularity would become irrelevant.

### 4.3 Post-Newtonian Metric

Far away from our central body, our spacetime will no longer be accurately described by our perturbed metric. Instead, we will need to use a post-Newtonian expansion to describe the weak field regime. Post-Newtonian theory has been well developed by countless authors and used extensively throughout the literature. For a summary of the fundamentals of Post-Newtonian Theory we turn to Ref. [4]. Note, for the remaining sections in this chapter we cease the use of natural units, restoring $G$ and $c$ which were previously set to unity. We will also drop the assumption of a non-spinning body for the remainder of this chapter.

A general Post-Newtonian metric can be expressed as

\[
\begin{align*}
g_{00} &= -\frac{1}{2} \frac{1}{c^2} U + \frac{2}{c^4} (\Psi - U^2) + O(c^{-6}), \\
g_{0a} &= -\frac{4}{c^3} U_a + O(c^{-5}), \\
g_{ab} &= (1 + \frac{2}{c^2} U) \delta_{ab} + O(c^{-4}),
\end{align*}
\]

where $U$ is a Newtonian potential, $U_a$ a vector potential, and $\Psi$ a Post-Newtonian potential. We can further decompose $\Psi$ as

\[
\Psi = \psi + \frac{1}{2} \partial_{tt} X,
\]

with $\psi$ another potential and $X$ the superpotential. The metric here is presented in harmonic
coordinates \( x^a = \{ct, x^a \} \), satisfying the harmonic condition

\[
\partial_t U + \partial_a U^a = 0.
\] (4.3.3)

These are centred on the barycentre of our Post-Newtonian system. We also require our metric to be a solution to the vacuum field equations

\[
\begin{align*}
\nabla^2 U &= 0, \quad \text{(4.3.4a)} \\
\nabla^2 U_a &= 0, \quad \text{(4.3.4b)} \\
\n\nabla^2 \psi &= 0, \quad \text{(4.3.4c)} \\
\n\nabla^2 X &= 2U. \quad \text{(4.3.4d)}
\end{align*}
\]

Here, \( \nabla^2 \) is the traditional Laplacian operator in flat space.

Specializing to our system, we follow the matching criteria of Taylor & Poisson \[53\]. They treat the central black hole as a post-Newtonian monopole of mass \( M \), which moves along the world line \( z(t) \). This was then extended to the spinning case in Ref. \[55\], adding a spin dipole moment \( S \) but no higher multipole moments. The post-Newtonian potentials are

\[
\begin{align*}
U(t, x) &= \frac{GM}{R} + U_{\text{ext}}(t, x), \quad \text{(4.3.5a)} \\
U^a(t, x) &= -\frac{G(R \times S)^a}{2R^3} + \frac{GMu^a}{R} + U_{\text{ext}}^a(t, x), \quad \text{(4.3.5b)} \\
\psi(t, x) &= \frac{GM\mu}{R} + \frac{Gp \cdot R}{R^3} + \psi_{\text{ext}}(t, x), \quad \text{(4.3.5c)} \\
X(t, x) &= GMR + X_{\text{ext}}(t, x), \quad \text{(4.3.5d)}
\end{align*}
\]

where the black-hole terms are clearly distinguished from the external terms \[55\]. With this we find that

\[
\Psi(t, x) = \frac{GM}{R} \left( \mu + \frac{1}{2} u^2 \right) - \frac{GM}{2R^3} (u \cdot R)^2 - \frac{GM}{2R^3} \alpha \cdot R + \frac{Gp \cdot R}{R^3} + \psi_{\text{ext}}(t, x), \quad \text{(4.3.6)}
\]
where $\Psi_{ext} = \psi_{ext} + \frac{1}{2} \partial_{tt} X_{ext}$. Here, we define a separation vector, $R := x - z(t)$ with $R := |R|$, and the velocity and acceleration of the body $u(t) := d\mathbf{z}/dt$, $u^2 := \mathbf{u} \cdot \mathbf{u}$, and $a(t) := d\mathbf{u}/dt$.

The term $\mu(t)$ is a Post-Newtonian correction to the monopole mass, $M$, and $p(t)$ the analogous correction to the spin dipole $S$. These are required for successful matching of the Post-Newtonian metric.

The final step required before matching can occur is to transform the Post-Newtonian metric from the barycentric coordinates $(t, x^a)$ to another harmonic frame affixed to our central body $(\bar{t}, \bar{x}^a)$. This was first done by Racine & Flanagan [84] building off the Post-Newtonian Theory of Reference Frames by Kopeikin [85], Damour, Soffel, & Xu [86], and Kopeiken & Vlasov [87]. For a summary of this transformation see e.g. Ref. [4] or Ref. [53]. In the central-bodies frame, the Post-Newtonian potentials become

$$\tilde{U} = \frac{GM}{\bar{r}} + 0U + 1U_a \bar{x}^a + 2U_{ab} \bar{x}^a \bar{x}^b + O(\bar{r}^3), \quad (4.3.7a)$$
$$\tilde{U}^a = -\frac{G(\bar{x} \times S)^a}{2\bar{r}^3} + 0U^a + 1U^a b \bar{x}^b + 2U_{bc} \bar{x}^b \bar{x}^c + O(\bar{r}^3), \quad (4.3.7b)$$
$$\bar{\Psi} = \frac{Gq \cdot \bar{x}}{\bar{r}^3} + \frac{GM}{\bar{r}} (\mu + \mathbf{A} - 2\nu^2) + 0\Psi + 1\Psi_a \bar{x}^a + 2\Psi_{ab} \bar{x}^a \bar{x}^b + O(\bar{r}^3), \quad (4.3.7c)$$

where $\bar{r}^2 := \delta_{ab} \bar{x}^a \bar{x}^b$, $q := p - 2uxS - M(H - Au)$, and all other symbols are presented in Table 4.3.

### 4.4 MATCHING WITH POST-NEWTONIAN METRIC

#### 4.4.1 Light-Cone Gauge Post-Newtonian Matching

To fully understand the Post-Newtonian matching calculation, this author reproduced the matching first completed by Poisson in Ref. [55]. We begin by considering a slowly rotating black hole, characterized by the dimensionless spin magnitude $\chi$. To describe the effects of spin, we introduce a dipole spin potential $\chi^d_A$, defined in Appendix A. Due to the non-linearity of the field equations, the bodies spin angular momentum also couples to the gravitoelectric and gravitomagnetic tidal
Table 4.3: Potentials of the Post-Newtonian Metric

\[ 0 U = \frac{1}{2} v^2 - \dot{A} + \dot{U}_{\text{ext}} \]
\[ 1 U_a = -a_a + \partial_a \dot{U}_{\text{ext}} \]
\[ 2 U_{a b} = \frac{1}{2} \partial_{a b} \dot{U}_{\text{ext}} \]
\[ 0 U^a = \dot{U}_{\text{ext}} - v^a \ddot{U}_{\text{ext}} + \frac{1}{4} (2 \dot{A} - v^2) - \frac{1}{4} \ddot{H}^a + \frac{1}{4} c_{b c} v^b R^c + \frac{1}{4} \gamma^a \]
\[ 1 U^a_b = \partial_b \ddot{U}_{\text{ext}} - v^a \partial_b \ddot{U}_{\text{ext}} + \frac{3}{8} v^a a_b + \frac{1}{4} a^a v_b + \frac{1}{4} \delta^a_b \left( \frac{4}{3} \dot{A} - 2 v^c a_c \right) - \frac{1}{4} c_{b c} \dot{R}^c + \frac{1}{4} \gamma^a_b \]
\[ 2 U^a_{b c} = \frac{1}{2} \partial_{b c} \ddot{U}_{\text{ext}} - \frac{1}{2} v^a \partial_{b c} \ddot{U}_{\text{ext}} + \frac{3}{20} (\delta^a_b \dot{a}_c + \delta^a_c \dot{a}_b) - \frac{1}{10} \dot{a}^a \delta_{b c} + \frac{1}{8} \gamma^a_{b c} \]

\[ 0 \Psi = \ddot{\Psi}_{\text{ext}} - 4 v_a \dot{U}^a_{\text{ext}} + 2 v^2 \ddot{U}_{\text{ext}} + A \partial_t \dot{U}_{\text{ext}} + (H^a - A v^a) \partial_a \dot{U}_{\text{ext}} + \frac{1}{2} \ddot{A}^2 - A \dot{v}^2 + \frac{1}{4} v^4 + \ddot{H}^a v_a - \dot{C} \]
\[ 1 \Psi_a = \partial_a \ddot{\Psi}_{\text{ext}} - 4 v_b \partial_a \dot{U}^b_{\text{ext}} + \left( \frac{5}{2} v^2 - \dddot{A} \right) \partial_a \dot{U}_{\text{ext}} - \frac{1}{2} v_a v^b \partial_b \dot{U}_{\text{ext}} + v_a \partial_t \ddot{U}_{\text{ext}} + A \partial_t a \dot{U}_{\text{ext}} \]
\[ + (H^b - A v^b) \partial_{a b} \ddot{U}_{\text{ext}} + \left( \dddot{A} - \frac{3}{2} v^2 a_c \right) v_a - \epsilon_{a b c} \left( \partial_b \dot{U}_{\text{ext}} R^c + v^b \dot{R}^c \right) \]
\[ 2 \Psi_{a b} = \frac{1}{2} \partial_{a b} \ddot{\Psi}_{\text{ext}} - 2 v_c \partial_{a b} \dot{U}^c_{\text{ext}} + \left( \frac{3}{2} v^2 - \dddot{A} \right) \partial_{a b} \dot{U}_{\text{ext}} - \frac{1}{2} v^c v_a \partial_b \dot{U}_{\text{ext}} + v_a \partial_t \ddot{U}_{\text{ext}} + \frac{1}{2} A \partial_{a b} \ddot{U}_{\text{ext}} \]
\[ - a_a \partial_a \dot{U}_{\text{ext}} + \frac{1}{2} \delta_{a b c} v^c \partial_a \dot{U}_{\text{ext}} - \epsilon_{p(a} R^b \partial_{b) c} \ddot{U}_{\text{ext}} + \frac{1}{2} (H^c - A v^c) \partial_{c a b} \ddot{U}_{\text{ext}} + \frac{1}{2} a_a a_b - v_a \dot{a}_b \]
\[ + \frac{1}{2} \delta_{a b} (v^c \dot{a}_c) - \frac{1}{6} \delta_{a b} A^{(3)} - \frac{1}{2} \gamma_{a b} \]

Decomposition of Post-Newtonian potentials appearing in Eq. (4.3.7). External potentials are evaluated at \( \bar{x} = 0 \) after differentiation. The quantities \{ A, z^a, H^a, R^a, C, \gamma_a, \gamma_{a b}, \gamma_{a b c} \} appear in the transformation from the barycentric to central body frame. They are determined by matching the Post-Newtonian and perturbed Schwarzschild metrics.
fields. The $\chi E$ type couplings are described by the potentials $\{F^d, \hat{E}^q, \hat{E}^q_A, \hat{E}^q_{AB}, F^o, F^o_A, F^o_{AB}\}$ while the $\chi B$ couplings are described by $\{K^d, K^d_A, \hat{B}^q, \hat{B}^q_A, K^q, K^q_A, K^q_{AB}\}$. All of these potentials are also defined in Appendix A and the superscripts $d, q, o$ label the potentials as dipolar, quadrupolar, or octupolar order. Using these potentials, the perturbed metric ansatz was initially calculated in the light-cone gauge and is presented in Eq. (4.6) of Ref. [55]. This allowed a thorough discussion of the event horizon geometry - taking advantage of the LCG’s geometric significance (see Sec. V-VII of that work).

To facilitate matching with the Post-Newtonian metric, this was then transformed to the harmonic gauge first developed by Taylor & Poisson [53] and adapted to the spinning case by Poisson [55]. The construction of this gauges follows. We first define a set of scalar fields

$$ X^{(0)} = T := v - r - 2M \log \left( \frac{r}{2M} - 1 \right), $$

(4.4.1b)

$$ X^{(1)} = X := (r - M) \sin \theta \cos \phi, $$

(4.4.1c)

$$ X^{(2)} = Y := (r - M) \sin \theta \sin \phi, $$

(4.4.1d)

$$ X^{(3)} = Z := (r - M) \cos \phi. $$

(4.4.1e)

These are each required to be solutions to the wave equation

$$ g^{\alpha\beta} \nabla_\alpha \nabla_\beta X^{(\mu)} = \frac{1}{\sqrt{-g}} \partial_\alpha (g^{\alpha\beta} \partial_\beta X^{(\mu)}) = 0, $$

(4.4.2)

in the Schwarzschild spacetime. Here, $\nabla_\alpha$ is the covariant derivative and $g$ the metric determinant. First, considering only the background metric, we select Eq. (4.4.1) as our coordinates. With this choice, Eq. (4.4.2) reduces to $\partial_\beta (\sqrt{-g} g^{\alpha\beta}) = 0$ (the usual statement of the Harmonic condition, see e.g. Ref. [4]).

We also require that $X^{(\mu)}$ are harmonic coordinates for both the perturbed spacetime and the unperturbed background. Thus, in addition to Eq. (4.4.2), the scalar fields must satisfy

$$ \nabla^\text{back}_\alpha \left( \psi^{\alpha\beta} \partial_\beta X^{(\mu)} \right) = 0, $$

(4.4.3)
with

\[ \psi^{\alpha\beta} := p^{\alpha\beta} - \frac{1}{2} p g^{\alpha\beta} - p^{\alpha\mu} p_\mu^{\beta} + \frac{1}{2} p p^{\alpha\beta} - \frac{1}{8} (p^2 - 2 p_{\mu\nu} p^{\mu\nu}) g^{\alpha\beta}. \] (4.4.4)

Here, \( p := g^{\alpha\beta} p_{\alpha\beta} \) is the trace of our perturbed metric and \( g^{\alpha\beta} = g^{\alpha\beta} - p^{\alpha\beta} + p^{\alpha\mu} p_\mu^{\beta} \) is the inverse metric, to second order in our perturbation (required to account for the spin couplings).

To transform from the metric ansatz first presented in the light cone gauge, one must complete a second order gauge transformation to a gauge which satisfies Eq. (4.4.3). In general, a perturbation \( p_{\alpha\beta}^{\text{old}} \) initially presented in an old gauge, can be represented in a new gauge, to second order, by

\[ p_{\alpha\beta}^{\text{new}} = p_{\alpha\beta}^{\text{old}} - \mathcal{L}_\xi g^{\alpha\beta}_{\text{back}} - \mathcal{L}_\xi p_{\alpha\beta}^{\text{old}} + \frac{1}{2} \mathcal{L}_\xi \mathcal{L}_\xi g^{\alpha\beta}_{\text{back}}. \] (4.4.5)

Here, \( \xi^\alpha \) is the generating vector field, and \( \mathcal{L}_\xi \) indicates Lie differentiation in the direction of \( \xi^\alpha \).

This transformation is completed in two steps. A first order gauge transformation is used to express the rotational and tidal perturbations of the LCG in a harmonic gauge. Then, a second order transformation is completed to account for the spin coupled tidal potentials [55]. To calculate these transformations, Eq. (4.4.5) is inserted into Eq. (4.4.3), generating a nonlinear, second order differential equation for the vector \( \xi^\alpha \). This gauge vector can be further decomposed as

\[ \xi^\alpha = \xi^\alpha [\chi] + \xi^\alpha [\mathcal{E}] + \xi^\alpha [\mathcal{B}] + \xi^\alpha [\chi \mathcal{E}] + \xi^\alpha [\chi \mathcal{B}], \] (4.4.6)

where square brackets indicate the component of our gauge vector related to each type of tidal potential. The solution to the \( \mathcal{E} \) and \( \mathcal{B} \) parts of our transformation were previously found by Taylor & Poisson [53],

\[ \xi_v[\mathcal{E}] = -\frac{1}{3} r^3 f \mathcal{E}^q, \quad \xi_r[\mathcal{E}] = \frac{1}{3} r^3 \mathcal{E}^q, \quad \xi_A[\mathcal{E}] = -\frac{r^5 f^2}{3(r - M)} \mathcal{E}^q_A, \] (4.4.7)

and

\[ \xi_v[\mathcal{B}] = 0, \quad \xi_r[\mathcal{B}] = 0, \quad \xi_A[\mathcal{B}] = \frac{1}{3} r^2 (r^2 - 6M^2) \mathcal{B}^q_A. \] (4.4.8)
Radial functions appearing in the gauge transformation vectors of Eq. \((4.4.10)\) and Eq. \((4.4.11)\). Again, \(f := 1 - 2M/r\).

Also to first order, Poisson [55] found the spin potential will transform as

\[
\xi_A[x] = -\frac{M^2 r}{r - M} \chi_A^d. \quad (4.4.9)
\]

Finally, the spin coupled potentials will transform with the gauge vectors

\[
\begin{align*}
\xi_v[\chi E] &= r^3 p^\beta \chi \partial_\phi \mathcal{E}^q, \quad (4.4.10a) \\
\xi_r[\chi E] &= r^3 p^\beta \chi \partial_\phi \mathcal{E}^q, \quad (4.4.10b) \\
\xi_A[\chi E] &= r^4 p^\beta \chi \partial_\phi \mathcal{E}^q + r^4 p^d \mathcal{F}^d_A + r^4 p^o \mathcal{F}^o_A, \quad (4.4.10c)
\end{align*}
\]

and

\[
\begin{align*}
\xi_v[\chi B] &= r^3 q^d v \mathcal{K}^d + r^3 q^o v \mathcal{K}^o, \quad (4.4.11a) \\
\xi_r[\chi B] &= r^3 q^d r \mathcal{K}^d + r^3 q^o r \mathcal{K}^o, \quad (4.4.11b)
\end{align*}
\]
\[ \xi_A[\mathcal{B}] = r^4 q^d K_A^d + r^4 q^o K_A^o + r^4 q^q \chi \partial_q \mathcal{B}^q. \] (4.4.11c)

Here, the functions \( \{p^d, p^o, p^q, p^r, p^o, q^d, q^o, q^q, q^r, q^o\} \) are found by integrating the differential equations for \( \xi_A[\mathcal{E}] \) and \( \xi_A[\mathcal{B}] \). They are listed in Table 4.4.

After this lengthy calculation, the result is the following metric ansatz, presented in the harmonic gauge

\[ g_{vv}^{HG} = -f - r^2 e^q \mathcal{E}^q - r^2 \hat{\mathcal{E}}^q + r^2 k_{vv}^d K^d + r^2 k_{vv}^o K^o, \] (4.4.12a)
\[ g_{vr}^{HG} = 1 + r^2 e^q \mathcal{E}^q - r^2 \hat{\mathcal{E}}^q + r^2 k_{vr}^d K^d + r^2 k_{vr}^o K^o, \] (4.4.12b)
\[ g_{rr}^{HG} = -2r^2 e^q \mathcal{E}^q - r^2 \hat{\mathcal{E}}^q + r^2 k_{rr}^d K^d + r^2 k_{rr}^o K^o, \] (4.4.12c)
\[ g_{rA}^{HG} = \frac{2M^2}{r} \chi^d + \frac{2}{3} r^3 b^q \mathcal{B}^q_A - r^3 \hat{\mathcal{E}}^q + r^3 f_{rv}^d \mathcal{F}^d_A + r^3 f_{rv}^o \mathcal{F}^o_A - r^3 b^q \hat{\mathcal{B}}^q_A, \] (4.4.12d)
\[ g_{rA}^{HG} = -r^2 \mathcal{E}^q + r^2 \hat{\mathcal{E}}^q + r^2 k_{rA}^d K^d + r^2 k_{rA}^o K^o, \] (4.4.12e)
\[ g_{AB}^{HG} = r^2 \Omega_{AB} - r^4 e^q \Omega_{AB} \mathcal{E}^q - r^4 \hat{\mathcal{E}}^q \Omega_{AB} \hat{\mathcal{E}}^q - r^4 \chi \partial_q \mathcal{B}^q_{AB} - r^4 \hat{\mathcal{B}}^q_{AB} + r^4 k_{AB}^d K^d + r^4 k_{AB}^o K^o. \] (4.4.12f)

All radial functions are listed in Table 4.5.

The next step in preparing the metric for matching is a transformation from \((v, r, \theta, \phi)\) coordinates to Cartesian Harmonic Coordinates (Eq. (4.4.1)). This is completed in two steps. First, we transform to another spherical coordinate system \((t, R, \theta, \phi)\). We have adopted the traditional time coordinate \(t\), and a harmonic radial coordinate, \(R := r - M\), which is related to Cartesian coordinates through the usual relations. Our metric becomes

\[ g_{tt} = g_{vv}, \] (4.4.13a)
\[ g_{tR} = f^{-1} g_{vv} + g_{vr}, \] (4.4.13b)

There are a number of integration constants appears within these radial solutions which can be freely set while still keeping our metric in the harmonic gauge. Poisson used this freedom to simplify the expressions for the gauge vectors and resulting metric perturbations as much as possible.
Table 4.5: Radial Functions in the Harmonic Gauge

<table>
<thead>
<tr>
<th>radial function</th>
<th>expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{uv}^q$</td>
<td>$f^2$</td>
</tr>
<tr>
<td>$e_{vr}^q$</td>
<td>$f$</td>
</tr>
<tr>
<td>$e_{tr}^q$</td>
<td>$1$</td>
</tr>
<tr>
<td>$e_0^q$</td>
<td>$-\frac{(3r^2 - 6Mr + 4M^2)M}{3r(r - M)^2}$</td>
</tr>
<tr>
<td>$c_0$</td>
<td>$\frac{r^2}{3r - M}$</td>
</tr>
<tr>
<td>$c_0^q$</td>
<td>$\frac{(3r^2 - 6Mr + 2M^2)M}{3r^2(r - M)}$</td>
</tr>
<tr>
<td>$b_0^q$</td>
<td>$f$</td>
</tr>
<tr>
<td>$b_0$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Radial functions appearing in the metric ansatz Eq. (4.4.12). Here, $f := 1 - 2M/r$. 

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\[
g_{RR} = f^{-2} g_{vA} + 2 f^{-1} g_{vrr} + g_{rr}, \quad (4.4.13c)
\]
\[
g_{tA} = g_{vA}, \quad (4.4.13d)
\]
\[
g_{RA} = f^{-1} g_{vA} + g_{rA}, \quad (4.4.13e)
\]
\[
g_{AB} = g_{AB}. \quad (4.4.13f)
\]

Second, we transform to Cartesian coordinates \( x^a = R \Omega^a(\theta^A) \). Our metric becomes

\[
g_{tt} = g_{tt}, \quad (4.4.14a)
\]
\[
g_{ta} = g_{tR} \Omega_a + \frac{1}{R} g_{tA} \Omega^A_a, \quad (4.4.14b)
\]
\[
g_{ab} = g_{RR} \Omega_a \Omega_b + \frac{1}{R} g_{RA} (\Omega_a \Omega^A_b + \Omega^A_a \Omega_b) + \frac{1}{R^2} g_{AB} \Omega_a \Omega^B_b. \quad (4.4.14c)
\]

Recall, our unit vector is defined as \( \Omega^a := [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta] \) and the transformation matrix \( \Omega^A_a := \delta_{ab} \Omega^{AB} \partial_B \Omega^a = R \partial \theta^A / \partial x^a \). These expressions here can become quite messy, but fortunately only the \( g_{tt} \) and \( g_{ta} \) components are required for a complete matching.

To complete the Post-Newtonian expansion, the radial functions of Table 4.5 are plugged into the Harmonic Cartesian metric, all factors of \( G \) and \( c \), which were previously set equal to 1, are restored, and we Taylor expand in powers of \( c^{-1} \). We then insert the definitions of our tidal potentials, replace the dimensionless spin with \( S^a = \frac{GM^2}{c} \chi^a \), and drop all higher order terms obtaining

\[
g_{tt} = -1 + \frac{2GM}{c^2 R} - 2 \left( \frac{GM}{c^2 R} \right)^2 - \frac{1}{c^2} \left( 1 - \frac{2GM}{c^2 R} \right) \mathcal{E}_{ab} x^a x^b + 2 \frac{1}{c^4} \mathcal{B}_{ab} \hat{S}^b x^a, \quad (4.4.15a)
\]
\[
g_{ta} = \frac{2G (x \times \hat{S})_a}{c^3} - \frac{2}{3c^3} \epsilon_{abc} B^c_d x^b x^c - \frac{\gamma^d}{c^3} \epsilon_{abc} \mathcal{E}^c_d \hat{S}^d x^b. \quad (4.4.15b)
\]

Here, \( \hat{S}^a := S^a / M \) and \( (x \times \hat{S})_a := \epsilon_{abc} x^b S^c \). Thus, we have verified the results of Ref. [55].
4.4.2 EZ Gauge Post-Newtonian Matching

The Post-Newtonian matching of the light-cone gauge metric was quite laborious, in particular the second order gauge transformation. It is possible, however, that a non-harmonic gauge could keep the Post-Newtonian form of the metric. This would greatly simplify our calculations, removing a difficult gauge transformation and leaving only the coordinate transformation described in Eq. (4.4.13) and Eq. (4.4.14). Removing all angular perturbations, it was thought that the EZ gauge could allow for a clean matching with the Post-Newtonian metric.

Unfortunately, after working through the calculations, it was found that the EZ gauge did not maintain the post-Newtonian form of the metric. Looking at Eq. (4.3.1c) we see that the spatial components of a Post-Newtonian metric, \( g_{ab} \), are conformally flat i.e. the spatial metric has the form

\[
\hat{h}_{ab} = \psi^4 \delta_{ab}, \tag{4.4.16}
\]

where \( \psi(x^a) \) is some scalar field \[89\]. Because the EZ gauge did not maintain this property of conformal flatness, it is not possible to directly match it with our post-Newtonian metric (without first transforming to the Harmonic gauge). Considering both the singularity discovered in Sec. 4.2 along with the mathematical difficulty of the PN matching, it appears that the EZ gauge does not provide any real benefits when compared with other choices. At this point, we choose to cease working with the EZ gauge.

4.4.3 Matching the Regge-Wheeler Gauge

Looking to match the Regge-Wheeler gauge we consider the full metric ansatz and the radial functions solved for by Landry & Poisson \[50\], including the spin terms, defined in Appendix A

\[
\begin{align*}
g_{tt}^{RW} &= -f - r^2 \epsilon_{tt}^q \mathcal{E}^q + r^2 \epsilon_{tt}^q \hat{\mathcal{E}}^q + r^2 k_{tt}^d \mathcal{K}^d - r^2 k_{tt}^o \mathcal{K}^o, \tag{4.4.17a} \\
g_{tr}^{RW} &= r^2 \epsilon_{tr}^q \hat{\mathcal{E}}^q, \tag{4.4.17b} \\
g_{rr}^{RW} &= f^{-1} - r^2 \epsilon_{rr}^q \mathcal{E}^q + r^2 \epsilon_{rr}^q \hat{\mathcal{E}}^q + r^2 k_{rr}^d \mathcal{K}^d - r^2 k_{rr}^o \mathcal{K}^o. \tag{4.4.17c}
\end{align*}
\]
Here, the radial functions for the non-spinning terms were previously given in Table 4.2 and the remaining radial functions are now presented in Table 4.6. The matching procedure is almost identical to that of the harmonic gauge, except our metric already uses $t$ as our time coordinate and so we skip that step.

We begin by changing radial coordinate, converting from $r$ to $R := r - M$. Then, the metric is transformed from spherical $(t, R, \theta, \phi)$ coordinates to Cartesian $(t, x, y, z)$ coordinates as described in Eq. (4.4.14). Finally, we restore all factors of $c$ and $G$, and Taylor expand our metric components in powers of $c^{-1}$.

Luckily, it was discovered that the metric presented in the Regge-Wheeler gauge does maintain the Post-Newtonian form. Following the procedure described above, the $g_{tt}$ and $g_{ta}$ components produce exactly the same expressions as Eq. (4.4.15) - at least to 1.5PN order. This discovery was quite astounding, as there was no prior reason to believe the Regge-Wheeler gauge should have this property. It is also quite beneficial, as this greatly reduces the amount of work required for the determination of all tidal moments. Using the Regge-Wheeler gauge, one can completely skip the second order gauge transformation described in Subsec. 4.4.1. This is one of the main results of Paper 2, which we will discuss in the following chapter.
Table 4.6: Radial Functions in the Regge-Wheeler Gauge: Spinning Body

\[
\begin{align*}
\hat{e}_{tt}^q &= \frac{2}{x^2} \left[ -30x^3(x^2 - 1)^2 \ln f + \frac{5}{2} x (2x - 1)(6x^2 - 6x - 1) \right] (\xi^q - \frac{1}{120}) + f^2 \hat{e}^q \\
\hat{e}_{tr}^q &= -\frac{1}{3x^3} \left[ 15x^4 (3x - 1) \ln f + \frac{5}{4} x^2 (36x^3 - 30x^2 + 1) \right] K_2^q + \frac{1}{4x^2} - \frac{1}{12x} \\
\hat{e}_{rr}^q &= f^{-2} \hat{e}^q \\
\hat{e}^q &= \frac{2}{x^2} \left[ -15x^3 (2x^2 - 1) \ln f - 5x^2 (6x^2 + 3x - 1) \right] (\xi^q - \frac{1}{120}) + \frac{1}{2x^2} (2x^2 - 1) \xi^q \\
\hat{b}_{t}^q &= -\frac{2}{x^2} \left[ 20x^4 (x - 1) \ln f + \frac{5}{3} x (12x^3 - 6x^2 - 2x - 1) \right] (\tilde{\xi}^q - \frac{1}{120}) + \frac{2}{3} f \tilde{\xi}^q \\
\hat{b}_{r}^q &= \frac{3}{4x^3} \left[ - \frac{20}{3} x^5 (2x - 1) \ln f - \frac{10}{9} x^2 (12x^3 + x + 1) \right] K_2^{mag} + \frac{1}{12} \frac{2x - 1}{x^2} \\
k_{tt}^d &= -\frac{2}{x^2} \left[ \frac{15}{2} (20x - 9)(x - 1)x^4 \ln f + \frac{5}{8} x \left( 240x^5 - 468x^4 + 242x^3 - 16x^2 - x + 2 \right) \right] K_2^{mag} \\
&\quad + \frac{1}{x^3} \frac{c^d}{2} + \frac{1}{20} \frac{20x^2 - 49x + 88}{x^2(x - 1)} \\
k_{tr}^d &= \frac{6}{x^2} \left[ \frac{15}{2} x^6 (x - 1) \ln f + \frac{5}{8} x^3 \left( 12x^4 - 18x^3 + 4x^2 + x + 2 \right) \right] K_2^{mag} + \frac{3}{20} \frac{2x^2}{x^2(x - 1)^3} \\
k_{rr}^d &= \frac{1}{x^2} \left[ -10x^3 (x - 1)(280x^3 - 420x^2 + 140x + 1) \ln f \right] K_2^{mag} \\
&\quad + \frac{1}{x^3} \frac{c^d}{2} + \frac{1}{20} \frac{20x^4 - 53x - 59}{x^2(x - 1)} \\
k_{tt}^o &= \frac{7}{x^2} \left[ \frac{10}{x^2} x^6 (280x^3 - 420x^2 + 140x + 1) \ln f \right] K_2^{mag} \\
&\quad + \frac{7}{x^3} \frac{c^o}{2} + \frac{1}{20} \frac{20x^4 - 53x - 59}{x^2(x - 1)} \\
k_{tr}^o &= \frac{2}{x^2} \left[ -420x^4 (2x^2 - 1)(x - 1) \ln f - 7x^2 (120x^4 - 240x^3 + 130x^2 - 10x - 1) \right] K_2^{mag} \\
&\quad + \frac{2}{x^3} \frac{c^o}{2} + \frac{1}{20} \frac{20x^4 - 53x - 59}{x^2(x - 1)} \\
k_{rr}^o &= \frac{7}{x^2} \left[ \frac{10}{x^2} x^6 (280x^3 - 420x^2 + 140x + 1) \ln f \right] K_2^{mag} \\
&\quad + \frac{7}{x^3} \frac{c^o}{2} + \frac{1}{20} \frac{20x^4 - 53x - 59}{x^2(x - 1)} \\
f_{t}^d &= -\frac{1}{28x^2} \left[ -84x^4 (10x^3 - 10x^2 + 1) \ln f - 14x^3 (60x^3 - 30x^2 - 10x + 1) \right] K_2^{el} + \frac{1}{2x^4} \tilde{\xi}^d + \frac{1}{2x^2} \gamma^d + \frac{1}{2x^2} \\
f_{r}^d &= -\frac{1}{10x^2} \left[ -\frac{3}{2} x^2 (5x - 2) \ln f - \frac{3}{4} x (10x + 1) \right] K_2^{el} \\
&\quad + \frac{3}{2x^2} \left[ 210x^3 (3x - 2)(x - 1) \ln f + \frac{7}{2} x^2 (180x^4 - 210x^3 + 30x^2 + 5x + 1) \right] \tilde{\xi}^o + \frac{1}{2} f (3x - 2) \tilde{\xi}^o - \frac{5}{12x^2} + \frac{1}{6x^2}
\end{align*}
\]

Radial functions for the spin coupled potentials appearing in Eq. (4.4.17) as previously calculated by Landry & Poisson [50]. These are expressed in terms of \( x := \frac{\rho}{2\pi} \), \( \tilde{f} := 1 - \frac{1}{x} \), and a number of different integration constants. Functions within square brackets behave as \( 1 + O(\frac{1}{x}) \) when \( x \gg 1 \).
CHAPTER 5

APPLICATIONS AND CONCLUSIONS

5.1 HIGHER ORDER TIDAL MULTipoLES - RESULTS OF PAPER 2

Discovering the Regge-Wheeler gauge maintains the form of a post-Newtonian metric has greatly simplified the matching calculations required to determine tidal moments. As a direct application of this result, in Paper 2, Poisson worked to calculate the higher order tidal multipole moments $E_{ab}, E_{abc}, E_{abcd}$ and $B_{ab}, B_{abc}, B_{abcd}$ of a non-rotating black hole - a direct follow-up to the work of Taylor and Poisson [53]. Here we shall try to provide a brief summary of this work to highlight the advantages provided by the Regge-Wheeler gauge. This is by no means a complete description and interested readers would benefit from a more thorough reading of Paper 2.

We begin by considering the familiar quadrupolar tidal potentials, but also include the higher order octupolar ($E_{abc}, B_{abc}$) and hexadecapolar ($E_{abcd}, B_{abcd}$) tidal moments as defined in Eq. (2.1.3) and Eq. (2.1.4). We also now consider various time derivatives of the tidal moments: $\dot{E}_{ab}, \ddot{E}_{ab}, \dot{E}_{abc}$ and the equivalent gravitomagnetic derivatives. Previously, these moments had been ignored as higher order moments are each suppressed by an extra factor of $\frac{1}{b}$ and time derivatives by a factor of $\frac{v}{b}$. Considering these additional functions greatly complicates the perturbed metric ansatz, making the field equations more difficult to solve. Luckily, the field equations decouple for each value of $l$, but there is still another layer of difficulty added to the already lengthy matching calculation. This
Irreducible tidal potentials constructed from the quadrupolar \((l = 2)\), octupolar \((l = 3)\), and hexadecapolar \((l = 4)\) tidal moments. All numerical factors are inserted to remain consistent with the normalization of Zhang [61]. Angular brackets indicates these tensors have been symmetrized and made tracefree i.e. \(A_{\langle abc \rangle} := A_{abc}^{STF}\).

is one of the big advantages of the Regge-Wheeler gauge. The effort saved while matching can motivate the tackling of more difficult problems.

The next steps of Paper 2 should now be familiar to the reader. We begin by combining these tidal moments into irreducible potentials, listed in Table 5.1. These potentials differ from those previously calculated in Sec. 2.1 in a number of ways. First, we don’t require the transverse projector \(\gamma_{ab}^b\). This is simply a result of using the Regge-Wheeler gauge which sets all perturbations proportional to the tensors \(Y_{AB}^{lm}\) and \(X_{AB}^{lm}\) to zero. Second, the only coupled potentials previously considered were of the form \(E \chi\) and \(B \chi\). Although we are considering the non-spinning case, it is no longer valid to ignore potentials of the form \(EE\), \(BB\), and \(EB\). The scaling of these quadratic and mixed potentials are comparable to the potentials from higher order moments, and so they can no longer be neglected.

These tidal potentials are then transformed from Cartesian to spherical spatial coordinates using the transformation matrix previously defined in Eq. (2.2.1),

\[
\Omega_A^a = \frac{\partial \Omega^a}{\partial \theta^A}. \tag{5.1.1}
\]
Our vector potentials will transform as before e.g.

$$B^0_A := B^0_a \Omega^a_A.$$  \hfill (5.1.2)

while the scalar potentials remain unchanged.

These transformed potentials can then be decomposed using the spherical harmonic basis of Martel and Poisson [51]. The decomposition follows the same procedure as that described in Ch. 2, but for brevity the full spherical harmonic decomposition will not be reproduced here. Finally, we can use these tidal potentials to construct the metric ansatz for a tidally perturbed, non-rotating black hole

$$g_{tt} = -f + r^2 \varepsilon_1^E \xi^E + r^3 \varepsilon_2^E \dot{\xi}^E + r^4 \varepsilon_3^E \ddot{\xi}^E + r^3 \varepsilon_4^E \xi^o + r^4 \varepsilon_5^E \xi^h + r^4 \left( p_m^m P^m + q_m^m Q^m \right)$$

$$+ r^4 \left( \mu_t \alpha q + q_{tq}^T Q^q \right) + r^4 \left( \mu_t h q^h h^h + g_{tt}^h Q^h \right) + r^4 \left( g_{tt}^d G^d + g_{tt}^o G^o \right) + O(r^5),$$

$$g_{rr} = r^3 \varepsilon_1^E \dot{\xi}^E + r^4 \varepsilon_2^E \ddot{\xi}^E + r^3 \varepsilon_3^E \xi^o + r^4 \left( \mu_r \alpha q + q_{rr}^T Q^q \right) + r^4 \left( g_{rr}^d G^d + g_{rr}^o G^o \right) + O(r^5),$$

$$g_{rA} = r^3 \varepsilon_1^E \dot{B}_A^E + r^4 \varepsilon_2^E \ddot{B}_A^E + r^5 \varepsilon_3^E \dot{B}_A^E + r^4 \varepsilon_5^E \xi^h + r^5 \varepsilon_6^E \xi^h + r^5 \left( h_q^h H_A^q + h_r^h H_A^r \right) + O(r^6),$$

$$g_{rA} = r^4 \varepsilon_1^E \dot{B}_A^E + r^5 \varepsilon_2^E \ddot{B}_A^E + r^5 \varepsilon_3^E \dot{B}_A^E + r^5 \left( h_q^h H_A^q + h_r^h H_A^r \right) + O(r^6),$$

$$g_{AB} = r^2 \left[ 1 + r^2 \varepsilon_1^E \xi^E + r^3 \varepsilon_2^E \dot{\xi}^E + r^4 \varepsilon_3^E \ddot{\xi}^E + r^3 \varepsilon_4^E \xi^o + r^4 \varepsilon_5^E \xi^h + r^4 \left( p_m^m P^m + q_m^m Q^m \right)$$

$$+ r^4 \left( \mu_r \alpha q + q_{rr}^T Q^q \right) + r^4 \left( \mu_r h q^h h^h + g_{rr}^h Q^h \right) + r^4 \left( g_{rr}^d G^d + g_{rr}^o G^o \right) + O(r^5) \right].$$  \hfill (5.1.3f)

The radial functions, as before, are found by solving the vacuum field equations \( R_{\alpha\beta} = 0 \) and are listed in Tables XIII-XVI of Paper 2. Appearing throughout these radial functions are a set of integration constants which can be arbitrarily chosen. This introduces the idea of choosing a specific calibration\(^1\) For example, in the "horizon calibration" one can fix these integration constants, so the perturbed event horizon remains situated at \( r = 2M \). This was the calibration used for the light cone gauge metric in Ref. [55]. To simplify the matching procedure, we introduce the idea of a "post-Newtonian calibration". This ensures that an \( M/r \) expansion of all radial functions

\(^1\)This is equivalent to the integration constants previously discussed as residual gauge freedom e.g. the constants \( \{ c, c^d, q^o, \gamma^d, q^o \} \) found in the light-cone gauge.
begins with the largest power possible. Take for example, the radial function

$$\dot{e}_{tt}^q = -\frac{1}{x}[e_1^q + 2 \ln f] f^2 - \frac{52}{15 x} + \frac{148}{15 x^2} - \frac{28}{15 x^3} - \frac{16}{3 x^4} - \frac{8}{3 x^5}, \quad (5.1.4)$$

where $x := r/M$ and $f := 1 - 2/x$. If, for example, we set our integration constant $e_1^q$ to unity, the $M/r$ expansion of our radial function begins at linear order,

$$\dot{e}_{tt}^q = -\frac{67}{15} \left( \frac{M}{r} \right) + \frac{268}{15} \left( \frac{M}{r} \right)^2 - \frac{286}{15} \left( \frac{M}{r} \right)^3 + \frac{32}{15} \left( \frac{M}{r} \right)^6 + O\left( \left( \frac{M}{r} \right)^7 \right). \quad (5.1.5)$$

However, if we set $e_1^q = -52/15$ this eliminates the three leading terms resulting in the expansion

$$\dot{e}_{tt}^q = \frac{32}{15} \left( \frac{M}{r} \right)^6 + O\left( \left( \frac{M}{r} \right)^7 \right) . \quad (5.1.6)$$

Therefore, this choice of integration constant defines the "post-Newtonian calibration". Following this procedure for all integration constants, the metric of Eq. (5.1.3) can then be matched with the post-Newtonian metric, following the general procedure laid out in Sec. [4.4]. Here, we leave the remaining results to be explain in Paper 2 if the reader is interested. It includes a more thorough discussion of calibration, showing that all radial functions can be written in terms of the "bare" radial functions i.e. all integration constants set to zero. The general case of switching between calibrations is discussed - somewhat analogous to gauge transformations. The perturbed metric is then matched with a PN expansion, and the tidal moments are found both in the black hole’s frame and the barycentric frame. These moments, found to leading post-Newtonian order, are independent of any internal structure as these affects appear at higher order. Therefore, these results are equally applicable to a neutron star as the considered case of a non-rotating black hole.

Next, the specific case of a two-body system with circular orbits is considered as an illustrative example. Finally, the paper concludes with a discussion of the geometry of the tidally deformed event horizon.
5.2 **CONCLUSION**

Throughout this thesis we have been interested in exploring the tidal effects present within compact binary systems. Motivated by the dawning era of gravitational wave astronomy, scientists hope to tightly constrain the high density EoS for neutron stars by examining the gravitational radiation emitted during inspiral and merger. It has been previously shown that the internal structure of a neutron star will noticeably alter the dynamics of an inspiralling binary though tidal interactions. This effect, which appears as a phase shift in the gravitational wave signal, is dependent only on the tidal Love numbers, $K^e_2$, of each body [45]. Tidal Love numbers, named after A.E.H. Love, characterizes a body's response to an external tidal field and is strongly dependent on the internal structure. However, to be able to extract information from any detected gravitational wave signals we must first understand the nature of relativistic tides present in compact binaries.

Consider a generic tidal field, described to leading order by the quadrupolar gravitoelectric and gravitomagnetic tidal moments $E_{ab}$ and $B_{ab}$. These have been previously calculated to leading Post-Newtonian order, for a non-rotating black hole by Taylor & Poisson in Ref. [53]. Their framework uses a matching calculation which combines a tidally perturbed Schwarzschild metric close to the central black hole with a post-Newtonian expansion far away. These two solutions are then matched in an overlapping region of validity, fully determining the spacetime and the associated tidal moments. This was then extended to the rotating case by Poisson [55], and a rotating material body by Landry & Poisson [50]. Although effective, this matching calculation is quite complex and tedious to perform. The overall goal of this project was to simplify these calculations by exploiting the gauge freedom present in the equations of black hole perturbation theory.

We begin in Ch. [2] by fully characterizing the tidal environment by considering a non-rotating compact body embedded in an external, quadrupolar tidal field. We use a framework which packages these tidal effects into a set of irreducible potentials, which are subsequently decomposed using a spherical harmonic basis. These potentials, and their decompositions, are then packaged into a tidally perturbed Schwarzschild metric in Ch. [3]. This is first done in a general, gauge independent way and then specialized to the three different gauges used throughout this work: the
light-cone gauge [57], Regge-Wheeler gauge [58], and the EZ gauge [59]. By choosing a gauge to work in, we remove the residual degrees of freedom present in the perturbation equations.

We then proceed in Ch. 4 by solving the vacuum field equations in the EZ gauge. This is compared to the work of Landry & Poisson [50], who had previously done this in the Regge-Wheeler gauge. We discover that the EZ gauge has a singularity at $r = 2M$ which is an unavoidable, intrinsic property of the gauge. This is the main result presented in Paper 1. From here we look to match with a Post-Newtonian metric. After a brief review of PN theory, the matching is completed first in the light-cone gauge, then successively the EZ and Regge-Wheeler gauges. It is discovered, that the Regge-Wheeler gauge maintains the Post-Newtonian form of the metric, allowing it to be matched directly without an intermediate gauge transformation. Thus, we have found a simplification arising from our gauge freedom, just as was intended. This is one of the main results presented in Paper 2. As a direct application of this work, we review Poisson’s calculations of the higher order tidal moments $E_{abc}, E_{abcd}$ etc. along with time derivatives. This was facilitated by the direct matching afforded by the Regge-Wheeler gauge.

Concluding this project, we look forward to possible future applications. An obvious next step would be to extend the matching calculations previously completed using the simplified Regge-Wheeler framework. For example, the work of Poisson [55] and subsequently Landry & Poisson [50], looked at a rotating black hole and material body respectively. However, they assumed their body was slowly rotating, allowing terms quadratic in spin to be neglected. Relaxing this assumption would be a valuable exploration for astrophysical settings, allowing us to understand the universality of previous results. One could also lift the assumption of slowly evolving tides, probing the previously unexamined dynamical regime. Finally, the Regge-Wheeler framework developed in this thesis has already proven to be quite fruitful, but is only known to match a post-Newtonian metric up to 1.5PN order (as was done in Ch. 4) and 1PN order for the higher order multipoles as in Paper 2. Determining if the Regge-Wheeler gauge maintains the form of a PN metric to higher orders would allow the computation of tidal potentials beyond leading PN order. Even if the Regge-Wheeler gauge doesn’t maintain its PN form, the calculation of our tidal moments beyond leading post-Newtonian order would still be a worthwhile pursuit. Examining
any of these future projects would allow us to develop a better analytical understanding of the tides of a compact binary system. This provides an essential complement to the numerical simulations and astrophysical observations required to fully understand the structure of neutron stars.
APPENDIX A

SPIN COUPLED TIDAL MOMENTS

The moments discussed in Ch. 2 are sufficient to fully describe a quadrupolar tidal environment provided our reference body is non spinning. Due to the non-linear nature of the field equations, if we consider a spinning system, we will have new tidal potentials generated from the body’s spin angular momentum coupling to the quadrupole moments $E_{ab}$ and $B_{ab}$. These $\chi E$ and $\chi B$ type potentials were first explored thoroughly by Poisson in Ref. [55] in the context of a spinning black hole. This was later extended by Landry & Poisson [50] for the case of a material body.

The spin angular momentum of our central body is described with the tensor $S^{\alpha\beta}$ on our world line $\gamma$. Using our parallel transported basis $\{u^\alpha, e^\alpha_a\}$, we can define the spin vector

$$S_a := \frac{1}{2} \epsilon_{abc} e^b_\alpha e^c_\beta S^{\alpha\beta},$$

which is odd under a parity transformation,

$$S_a \rightarrow -S_a.$$  

One can then define the dipole spin potential

$$\chi^d_a := \epsilon_{abc} \Omega^b \chi^c,$$

where $\chi_a = S_a/M^2$ is the body’s dimensionless spin vector, and the superscript $d$ indicates this is
of dipolar order. We assume our body is slowly rotating so $\chi \ll 1$, so terms quadratic in $\chi$ can be neglected. Expressed in spherical coordinates, our spin potential becomes

$$\chi^d_A := \chi^d_a \Omega^a_A, \quad (A.0.4)$$

There will also be a large set of irreducible potentials which describe the coupling between $\chi^a$, $\mathcal{E}_{ab}$, and $\mathcal{B}_{ab}$. We begin by looking at the gravitoelectric coupling which produces the odd parity tensors

$$\mathcal{F}_a := \mathcal{E}_{ab} \chi^b, \quad (A.0.5a)$$

$$\mathcal{F}_{abc} := (\mathcal{E}_{ab} \chi^c)^{STF}, \quad (A.0.5b)$$

and the even parity tensor

$$\hat{\mathcal{E}}_{ab} := 2\chi^c \epsilon_{cda} \mathcal{E}^d_b. \quad (A.0.5c)$$

Round brackets indicates the symmetrization of indices. Similarly, one can combine $\chi_a$ and $\mathcal{B}_{ab}$ to define the two even parity tensors

$$\mathcal{K}_a := \mathcal{B}_{ab} \chi^b, \quad (A.0.6a)$$

$$\mathcal{K}_{abc} := (\mathcal{B}_{ab} \chi^c)^{STF}, \quad (A.0.6b)$$

and the odd parity tensor

$$\hat{\mathcal{B}}_{ab} := 2\chi^c \epsilon_{cda} \mathcal{B}^d_b. \quad (A.0.6c)$$

These spin coupled tidal moments are then packaged into a set of potentials for insertion into the metric. This involves combining Eq. (A.0.5) and Eq. (A.0.6) with our radial unit vector $\Omega^a$, the permutation symbol $\epsilon_{abc}$, and transverse projector $\gamma^b_a$ in various irreducible ways. In the gravitoelectric sector, the $\mathcal{F}$ type tensors generate the odd parity dipolar vector potential

$$\mathcal{F}^d_a := \epsilon_{abc} \Omega^b \mathcal{F}^c, \quad (A.0.7)$$
and the odd parity octupolar vector and tensor potentials

\[
F^o_\alpha := \epsilon_{acd} \Omega^c F^d_{bc} \Omega^b \Omega^e, \quad F^o_{ab} := (\epsilon_{acd} \Omega^c F^d_{ef} \gamma^e_b + \epsilon_{bcd} \Omega^c F^d_{ef} \gamma^e_a) \Omega^f. \tag{A.0.8}
\]

Similarly, the hatted gravitoelectric tensor \( \hat{E}_{ab} \) leads to the even parity scalar, vector, and tensor potentials

\[
\hat{E}^d := \hat{E}_{ab} \Omega^a \Omega^b, \quad \hat{E}^q_a := \gamma^a_b \hat{E}_{bc} \Omega^c, \quad \hat{E}^q_{ab} := 2 \gamma^c_a \gamma^d_b \hat{E}_{cd} + \gamma_{ab} \hat{E}^q, \tag{A.0.9}
\]

all of quadrupolar order.

In the gravitomagnetic sector, our type \( K \) spin coupled tensors lead to the even parity dipolar scalar and vector potentials

\[
K^d := K_{\alpha} \Omega^\alpha, \quad K^d_{\alpha} := \gamma^a_b K_{\alpha b}, \tag{A.0.10}
\]

and the even parity octupolar scalar, vector, and tensor potentials

\[
K^o := K_{abc} \Omega^a \Omega^b \Omega^c, \quad K^o_a := \gamma^a_d K_{abc} \Omega^b \Omega^c, \quad K^o_{ab} := 2 \gamma^e_a \gamma^d_b K_{dec} \Omega^e + \gamma_{ab} K^o. \tag{A.0.11}
\]

Finally, the hatted gravitomagnetic tensor \( \hat{B}_{ab} \) can be used to define the odd parity quadrupole vector and tensor potentials

\[
\hat{B}^q_a := \epsilon_{acd} \Omega^e \hat{B}^d_{\gamma b} \Omega^b, \quad \hat{B}^q_{ab} := \epsilon_{acd} \Omega^e \hat{B}^d_{\gamma e} + \epsilon_{bcd} \Omega^e \hat{B}^d_{\gamma a}. \tag{A.0.12}
\]

As in Eq. \((2.2.2)\) and Eq. \((2.2.3)\) we need to transform these potentials from Cartesian to spherical coordinates, enabling the spherical harmonic decomposition. Again, this will be facilitated with the Jacobian matrix of Eq. \((2.2.1)\). The vector and tensor potentials will transform as

\[
F^d_A = F^d_{a} \Omega^a_A, \quad K^o_{AB} = K^o_{ab} \Omega^a_A \Omega^b_B. \tag{A.0.13}
\]
The relations between $B_m^\Omega$ and $B_{ab}$ are identical to those between $E_m^\Omega$ and $E_{ab}$. Similarly, the relations between $K^\Omega_m$, $K_a$ and $\chi E_m^\Omega$ are identical to those between $F_m^\Omega$, $F_a$ and $\chi E_m^a$; the relations between $B_m^\Omega$, $B_{ab}$ and $\chi B_m^\Omega$ are identical to those between $E_m^\Omega$, $E_{ab}$ and $\chi E_m^a$; and the relations between $K^\Omega_m$, $K_{abc}$ and $\chi B_m^\Omega$ are identical to those between $F_m^\Omega$, $F_{abc}$ and $\chi E_m^a$.

and so on, while the scalar potentials remain unchanged.

We then take advantage of the identities connecting our STF tensors and the spherical harmonics

$$\chi^a_s \Omega^a = \sum_m \chi^d_m Y^{1m}, \quad F^a_s \Omega^a = \sum_m F^d_m Y^{1m}, \quad K^\Omega_a = \sum_m K^d_m Y^{1m}, \quad (A.0.14a)$$

$$\hat{E}_{ab} \Omega^a \Omega^b = \sum_m \hat{E}^d_m Y^{2m}, \quad \hat{B}_{ab} \Omega^a \Omega^b = \sum_m \hat{B}^d_m Y^{2m}, \quad (A.0.14b)$$

$$F_{abc} \Omega^a \Omega^b \Omega^c = \sum_m F^0_m Y^{3m}, \quad K_{abc} \Omega^a \Omega^b \Omega^c = \sum_m K^0_m Y^{3m}. \quad (A.0.14c)$$

Again, differentiating these identities as described in Appendix A of Ref. [54], we obtain the complete collection of harmonic decompositions:

$$\chi^d_a = \sum_m \chi^d_m \chi^1_m, \quad (A.0.15a)$$

<table>
<thead>
<tr>
<th>$l = 1$</th>
<th>$l = 2$</th>
<th>$l = 3$</th>
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</thead>
<tbody>
<tr>
<td>$E_0^q = \frac{1}{2}(E_{11} + E_{22})$</td>
<td>$E_{1c}^q = E_{13}$</td>
<td>$F_0^q = \frac{1}{2}(F_{113} + F_{223}) = \frac{2}{3} \chi E_m^q$</td>
</tr>
<tr>
<td>$\chi_0^q = \chi$</td>
<td>$E_{1s}^q = E_{23}$</td>
<td>$F_{1c}^q = \frac{1}{3}(F_{111} + F_{122}) = -\frac{4}{15} \chi E_{1c}^q$</td>
</tr>
<tr>
<td>$\chi_{1c}^q = \chi_1 = 0$</td>
<td>$E_{2c}^q = \frac{1}{2}(E_{11} - E_{22})$</td>
<td>$F_{1s}^q = \frac{1}{3}(F_{112} + F_{222}) = -\frac{4}{15} \chi E_{1s}^q$</td>
</tr>
<tr>
<td>$\chi_{1s}^q = \chi_2 = 0$</td>
<td>$E_{3s}^q = \hat{E}_{12}$</td>
<td>$F_{2c}^q = \frac{1}{3}(F_{113} - F_{223}) = \frac{1}{3} \chi E_{2c}^q$</td>
</tr>
<tr>
<td>$F_0^d = F_3 = -2 \chi E_0^q$</td>
<td>$\hat{E}<em>0^d = \frac{1}{2}(\hat{E}</em>{11} + \hat{E}_{22}) = 0$</td>
<td>$F_{2s}^q = F_{123} = \frac{1}{3} \chi E_{2s}^q$</td>
</tr>
<tr>
<td>$F_{1c}^d = F_1 = \chi E_{1c}^q$</td>
<td>$\hat{E}<em>{1c}^d = \hat{E}</em>{13} = -\chi E_{1s}^q$</td>
<td>$F_{3c}^q = \frac{1}{3}(F_{111} - 3F_{122}) = 0$</td>
</tr>
<tr>
<td>$F_{1s}^d = F_2 = \chi E_{1s}^q$</td>
<td>$\hat{E}<em>{2c}^d = \hat{E}</em>{23} = \chi E_{1c}^q$</td>
<td>$F_{3s}^q = \frac{1}{3}(3F_{112} - F_{222}) = 0$</td>
</tr>
<tr>
<td>$E_{2c}^q = \frac{1}{2}(E_{11} - E_{22}) = -2 \chi E_{2s}^q$</td>
<td>$\hat{E}<em>{2s}^d = \hat{E}</em>{12} = 2 \chi E_{2c}^q$</td>
<td></td>
</tr>
</tbody>
</table>
\[ \mathcal{F}_A^d = \sum_m \mathcal{F}_m^d X_A^{1m}, \]  
\[ (A.0.15b) \]

\[ \mathcal{K}_A^d = \sum_m \mathcal{K}_m^d Y^{1m}, \quad \mathcal{K}_A^d = \sum_m \mathcal{K}_m^d Y^{1m}, \]  
\[ (A.0.15c) \]

\[ \hat{E}^q = \sum_m \hat{E}_m^q Y^{2m}, \quad \hat{E}_A^q = \frac{1}{2} \sum_m \hat{E}_m^q Y^{2m}, \quad \hat{E}_{AB}^q = \sum_m \hat{E}_m^q Y_{AB}^{2m}, \]  
\[ (A.0.15d) \]

\[ \hat{B}_A^q = \frac{1}{2} \sum_m \hat{B}_m^q Y_{AB}^{2m}, \quad \hat{B}_{AB}^q = \sum_m \hat{B}_m^q Y_{AB}^{2m}, \]  
\[ (A.0.15e) \]

\[ \mathcal{F}_A^o = \frac{1}{3} \sum_m \mathcal{F}_m^o X_A^{3m}, \quad \mathcal{F}_{AB}^o = \frac{1}{3} \sum_m \mathcal{F}_m^o X_{AB}^{3m}, \]  
\[ (A.0.15f) \]

\[ \mathcal{K}_A^o = \sum_m \mathcal{K}_m^o Y^{3m}, \quad \mathcal{K}_A^o = \frac{1}{3} \sum_m \mathcal{K}_m^o Y^{3m}, \quad \mathcal{K}_{AB}^o = \frac{1}{3} \sum_m \mathcal{K}_m^o Y_{AB}^{3m}, \]  
\[ (A.0.15g) \]

Combined with Eq. (2.2.7), these potentials provide completely describe the spacetime around a slowly rotating compact body embedded in a quadrupolar tidal field.

The spin coupled perturbations defined in above can also be packaged into our metric perturbation using the same technique as Sec. 3.2. For the \( \chi \mathcal{E} \) type potentials we find

\[ p_{ab} = r^2 \hat{e}_a^q(r) \hat{E}_r, \]  
\[ (A.0.16a) \]

\[ p_{AB} = r^2 \hat{e}_a^q(r) \hat{E}_{AB}, \]  
\[ (A.0.16b) \]

\[ p_{AB} = r^4 \hat{e}_a^q(r) \Omega_{AB} \hat{E}_r + r^4 \hat{e}_a^q(r) \hat{E}_{AB}, \]  
\[ (A.0.16c) \]

in the even parity sector and

\[ p_{AB} = r^3 f_a^d(r) \mathcal{F}_B^d + r^3 f_a^o(r) \mathcal{F}_B^o, \]  
\[ (A.0.16d) \]

\[ p_{AB} = r^4 f_a^o(r) \mathcal{F}_{AB}^o, \]  
\[ (A.0.16e) \]
in the odd parity sector. For the $\chi B$ coupled potentials we have

\begin{align}
    p_{ab} &= r^2 k_{ab}^d K^d + r^2 k_{ab}^o K^o, \\
    p_{aB} &= r^3 k_a^d K_B^d + r^3 k_a^o K_B^o, \\
    p_{AB} &= r^4 k^o \Omega_{AB} K^o + r^4 k^o(r) K_{AB}^o,
\end{align}

(A.0.17a)

(A.0.17b)

(A.0.17c)

in the even parity sector and

\begin{align}
    p_{aB} &= r^3 \hat{b}_a^q(r) \tilde{B}_B^q, \\
    p_{AB} &= r^4 \hat{b}_a^q(r) \tilde{B}_{AB}^q,
\end{align}

(A.0.17d)

(A.0.17e)

in the odd parity sector.
REFERENCES


[27] Update on the start of ligo’s 3rd observing run, 2018.


