Electrostatic Self-force as a Probe of Global Structure

by

Karl Davidson

A Thesis

presented to

The University of Guelph

In partial fulfilment of requirements

for the degree of

Master of Science

in

Physics

Guelph, Ontario, Canada

ABSTRACT

ELECTROSTATIC SELF-FORCE AS A PROBE OF GLOBAL STRUCTURE

Karl Davidson **Advisor:**
University of Guelph, 2018 **Professor Eric Poisson**

The self-force is an effect which occurs when the force-field produced by an object interacts with the curvature of spacetime. The object then reacts to the modification of its own field and effectively exerts a force on itself – the self-force. In this work, I investigate the electrostatic self-force on an electric charge, as well as an electric dipole in a locally flat spacetime, to show that these self-forces probe the global curvature of the spacetime. This spacetime is produced by joining two versions of Minkowski space at a hypersurface of constant radial distance from either of its two centers. This connection produces a singular distribution of curvature at this hypersurface and thus also a thin shell of matter described by a perfect fluid.

I find that the self-force on a charge is always directed away from this surface layer, regardless of the sign of the charge. For the dipole, the direction of the force is dependent upon both the position and orientation of the dipole moment. Both self-forces are found to approach infinity as they are made to approach the surface layer.

ACKNOWLEDGMENTS

I would like to thank my advisory committee members – Liliana Cabellero, Luis Lehner, and Eric Poisson – for taking the time to read my thesis, asking questions, and pushing me to a better understanding of my research. I also thank my research colleagues – Eamonn Corrigan, Phillipe Landry and Dylan Podkowka – for their support, constructive criticism and listening ear to all my questions. I thank my fiancée Rebecca Adams for her unwavering support of my entire academic career. Finally, I wish to specifically thank my advisor Eric Poisson for his guidance, insight and the occasional nudge in the right direction throughout my graduate work. The research presented below would not have been possible without his help and encouragement.

TABLE OF CONTENTS

List of Figures

CHAPTER 1 INTRODUCTION

1.1 THE SELF-FORCE

The self-force has been an interesting problem in physics for over 100 years, evolving from the aptly named radiation reaction force of electromagnetism to the gravitational self-force present in Extreme Mass-Ratio Inspirals (EMRIs) [1]. Consider a positive, stationary electric charge in flat spacetime (where the shortest path between two points is a straight line). This electric charge produces an electric field, isotropic about its position. Should this charge remain stationary, this electric field has no effect on the charge, but if it were to experience an acceleration it would radiate in a direction antiparallel to its jerk vector causing a change in its path. This is the so called radiation reaction force, hereafter referred to as the self-force.

Consider next the same electric charge in a curved spacetime where the electric field lines it produces can be imagined to bend with the curvature. This altered field produces a force on the charge, which is the self-force, matching the effect of an accelerating particle as per the equivalence principle. Finally, this concept can be extended further into the context of gravitation. For a small mass orbiting or inspiraling towards a much more massive object, introductory physics teaches that this mass travels along a path determined purely by the gravitational field of the larger mass. The smaller mass however produces its own, albeit much smaller, gravitational field which perturbs the background field of the larger mass. This perturbation gives rise to a change in path of the smaller mass, which is attributed to the gravitational self-force. The gravitational self-force will be briefly expanded upon in Sec. 1.3, but the following thesis focuses on the electrostatic self-force on stationary, electric objects in a curved spacetime.

1.2 A SHORT HISTORY

Beginning in 1903-1904, two scientists Abraham [2] and Lorentz [3] independently looked at how an electrically charged particle's path changes when it emits radiation. They found this so-called "radiation reaction" force to depend on the change in acceleration of the particle, as well as the square of the electric charge. A few years later, Poincaré wrote a paper [4] in response to Lorentz's work and confirmed his results. The problem wasn't looked at again until 1938 when Dirac generalized this result to special relativity [5]. In his work, he wrote the electromagnetic field in terms of the retarded and advanced field which left the important, "radiative" part of the electromagnetic field to be determined solely by the particle's world line, or curve that tracks its position through its existence. Over two decades later was the first attempt at an understanding of the radiation reaction force in a gravitational field. DeWitt and Brehme [6] worked to further generalize Dirac's result to curved spacetime. Their result was mostly correct, but was missing a critical piece involving the Riemann tensor in the advanced and retarded gravitational fields and thus the final equations of motion. This fact was pointed out by Hobbs in 1968 [7], who restated the curved spacetime generalization. The first well known application of Dewitt and Breheme's work (as corrected by Hobbs) was by Smith and Will in 1980 [8]. They worked out the exact electrostatic self-force on an electric charge held in place in Schwarzschild spacetime. Although this paper came to be the most famous, there were a few other works in between that did similar calculations, namely with weak field approximations [9] or first order quantum field theory calculations [10].

At the end of nearly another 20 year gap in 1997, came the foundational work by Mino, Sasaki and Tanaka [11] who derived the equations of motion for a point mass moving under the influence of gravitational self-force in a curved spacetime. The results of [11] were replicated in the same year by Quinn and Wald [12] with an axiomatic approach. These equations of motion became known as the MiSaTaQuWa equations, named after the five authors listed above. The first group arrived at their result in two different ways. First, they followed the formalism of Dewitt and Brehme, as corrected by Hobbs. Second, they used the method of matched asymptotic expansion developed by Burke in the early 1970s [13]. Quinn and Wald approached this problem axiomatically and achieved the same result as Mino, Sasaki and Tanaka. The same result was derived in other ways, namely by Galley and Hu using quantum effective field theory [14] and Harte using generalized Killing fields [15]. Much of this work has been done in the context of point particles, in Ref. [14] explicitly and as a limit for Ref. [11,12]. Point particles present problems in general relativity when solving the field equations due to their distributional nature, although this has been studied by authors such as Wald [16] and more extensively in a Living Review by Poisson, Pound and Vega [17]. Extension to small bodies has been the more recent endeavour. Previously mentioned Ref. [15] used the small body's center of mass. Other scientists such as Pound [18] and Gralla and Wald [19] have worked to generalize the MiSaTaQuWa equations to small bodies.

The MiSaTaQuWa equations have proven to be very useful in many realistic scenarios, even to this day. The main setback of these equations is that the Green's functions involved in their integration require knowledge of the particle's entire past history [20]. Since the establishment of the MiSaTaQuWa equations, there have been two developments to work around this fact, which are widely used in self-force calculations today, including this work. The first was the so-called "mode-sum" method developed by Barack and Ori [21], and independently by Mino, Nakano and Sasaki [22]. This method prescribes a functional decomposition of the relevant potential into modes. In modular form, the singular part of the potential can be removed mode-by-mode. The now regularized modes are then summed (or approximately so) for the regularized result. The mode sum method is the most commonly used in self-force calculations today. Some notable applications are such works as Ref. [23–25], although there are certainly many more. The second development was a new derivation of the self-force by Detweiler and Whiting [26] which implemented a decomposition of the Green's function into an inhomogenous field distorted by the curvature of the spacetime and a homogenous field which is entirely responsible for the self-force. Later in this work, these two parts are referred to as the *singular* and *regular* parts respectively.

This drew inspiration from the field splitting of Dirac's early work [5], and worked with a similar axiomatic procedure as Quinn and Wald in [12]. The following thesis makes use of a combination of the two methods, using a spherical harmonic decomposition and mode-sum, but using the *singular* and *regular* language of [26].

The work presented in this thesis follows more closely the work of Smith and Will [8] as it discusses the electrostatic self-force of a point charge in some globally curved spacetime. Smith and Will's result for the self-force on a point charge outside a Schwarzschild black hole was shown to be nearly universal such that there is actually very little dependence on the internal structure of the larger gravitating body. This was shown by Drivas and Gralla [27], who calculated the small correction which depends on the internal structure of the gravitating body. This aspect was investigated in full by Poisson and Isoyama [28], who concluded that self-force could indeed be used as probe of a body's internal structure.

Another few calculations by Unruh [29] and Burko, Liu and Soen [30] showed that a charge placed inside a spherical shell where spacetime was flat (curved outside the shell) experienced a self-force. This calculation nicely showed that in addition to probing the internal structure of a gravitating body, the self-force can also be used as a probe of global structure. The work of this thesis looks at a more extreme case where both sides of some spherical shell are flat, and the only occurrence of curvature is at the shell itself. This is certainly interesting in the case of the point charge, but becomes more so in the context of an electric dipole. Beyond the foundational work of Ref. [31], the case of self-force on an electric dipole has not been investigated in detail.

1.3 APPLICATIONS AND RELEVANCE

The self-force is an important quantity to account for in many applications. In the context of radiation reaction, it would certainly be of importance in particle accelerators sometimes taking the form of cyclotron radiation or Brehmsstrahlung radiation which are well known radiation mechanisms in particle physics applications. Today, physicists are most concerned with the gravitational self-force as it has many applications in massive body interactions, and gravitational wave (GW) physics. For example, as discussed in Ref. [28], the self-force is a way to look at the internal structure of objects that might be in a bound system. Due to the dependence of self-force on certain structural quantities of the system given by Drivas and Gralla in [27], the self-force can act as a probe of internal structure.

As another example, consider the currently in-progress planning/construction of the LISA (Laser Interferometer Space Antenna). In the past few years, the LIGO (Laser Interferometer Gravitational wave Observatory) has detected quite a few gravitational wave signals from merging black holes and most notably a binary neutron star merger [32]. Because of the relatively similar masses of these merging objects, the self-force (and perturbation theory) is not very applicable. However, should scientists be able to measure GWs produced by an EMRI, self-force effects would be much more relevant and measurable. This will hopefully be one of the achievements of the LISA detector once it is built [33]. Because of noise limitations, ground based detectors such as LIGO and Virgo are unable to detect the gravitational waves that originate from EMRIs. LISA will consist of three spacecrafts, separated by many millions of kilometers and will be placed in an Earth-like orbit around the sun [34]. The enormous size of this interferometer, as well its absence from Earthly noise, will allow LISA to detect much weaker and lower frequency GWs, which includes those produced by EMRIs under the effect of the self-force.

Unfortunately, the measurement and calculation of the self-force is not without its complications. There are a number of objectives of the self-force community that still require many calculations prior to measuring the self-force effects with LISA. These objectives are given in detail in Ref. [33]. These projects are of course in progress (some further along than others), with the most recent work coming from Ref. [35] on self-forces in arbitrary dimensions, Ref. [36] on the second-order self-force, and Ref. [37] on self-force in Kerr spacetime. Certainly more advancement is needed prior to the expected launch date of LISA ($\sim$ 2030s), but many are working towards this goal.

1.4 THESIS OVERVIEW

This section proceeds with outlining the problems of interest along with some of the context and ends with a brief plan of the structure of the thesis. From this point forward, this work uses Gaussian geometrized units such that $c = G = 4\pi\varepsilon_0 = 1$. The main portion of this thesis consists of two calculations of the electrostatic self-force. One on a point electric charge and the other on a point electric dipole. The self-force on each of these objects is produced by the curvature of the spacetime, as described in Sec. 1.2.

The curvature in these problems is produced by joining two Minkowski metrics together at a $2 + 1$ dimensional hypersurface discussed in depth in Sec. 2.1. The union of the two Minkowski metrics also results in closed spatial sections, or a closed universe. This hypersurface is home to a singular (described by a Dirac-delta function) mass distribution with specific properties determined by the two metrics. This singular mass distribution produces a singular distribution of curvature, which is enough to result in the self-force effect for both the charge and the dipole.

To calculate the self-force on a single charge, a second, oppositely charged object is required to satisfy Gauss' law in a closed universe. Both of the charges were placed along the z-axis (where $\theta = 0$ in spherical polar coordinates) for the use of axisymmetry. Doing so, however, introduced a Coulomb/interaction force which is easily distinguished from the self-force. In the case of the dipole, no such additional objects are required for obvious reasons. The dipole moment is given an arbitrary direction, and the resultant force on this dipole is produced purely by the self-force.

In the next chapter, I discuss the mathematical background used in this work, which includes a section on a description of the joining of the two metrics, the method of mode expansion, solving Maxwell's equations and the regularization technique used. The remaining chapters will then focus on each of the main problems discussed above, and end with a discussion of the results. An appendix is also included for the full derivations of some equations.

CHAPTER 2 BACKGROUND AND FRAMEWORK

2.1 THE THIN SHELL FORMALISM

In this section, a discussion of the thin-shell formalism as derived by Israel in Ref. [38] and modernized by Poisson in Sec. 3.7 of Ref. [39] is presented, along with its direct application to this work.

Consider a hypersurface Σ which partitions a spacetime into two regions $\mathcal{R}^-$ and $\mathcal{R}^+$, where the metric is given by

$$ds^2 = -dt^2 + dr^2 + R^2(r)d\Omega^2, \tag{2.1.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and

$$R(r) = \begin{cases} r & 0 \leq r < a \\ 2a - r & a \leq r < 2a \end{cases} . \tag{2.1.2}$$

Here, $R(r) = r$ is used in the region $\mathcal{R}^-$ and $R(r) = 2a - r$ is used in $\mathcal{R}^+$. Additionally, a is defined to be the radial position of Σ. The metric on either side of Σ describe a version of Minkowski space, and thus contain no curvature. By looking at the metric, it can be seen that as $r \to 0$ and $r \to 2a$ the surface area (characterized by $R(r)$) of the space goes to 0, implying a closed universe. The objective then becomes to find the conditions required for a smooth transition from $\mathcal{R}^-$ to $\mathcal{R}^+$ such that this metric union is a valid solution to the Einstein field equations. In the case of this work, the transition is not smooth, which will be shown to have a physical explanation.

The solution to this problem follows the prescription described in Sec. 3.7 of Ref. [39]. Briefly, the procedure follows the usual derivation of the Riemann tensor using the definition of the metric

$$g_{\alpha\beta} = g^{+}_{\alpha\beta}\Theta(\tau) + g^{-}_{\alpha\beta}\Theta(-\tau), \tag{2.1.3}$$

where $\Theta(\tau)$ is the Heaviside distribution, and τ is the proper length along some congruence of geodesics which cross Σ. A lengthy calculation finds that the only non-vanishing part of the Riemann tensor of the system is

$$R^{\alpha}{}_{\beta\gamma\delta} = \delta(\tau)\left(\left[\Gamma^{\alpha}{}_{\beta\delta}\right] n_{\gamma} - \left[\Gamma^{\alpha}{}_{\beta\gamma}\right] n_{\delta}\right), \tag{2.1.4}$$

where the square bracket notation is defined as $[A] \equiv A(\mathcal{R}^{+})\big|_{\Sigma} - A(\mathcal{R}^{-})\big|_{\Sigma}$. Additionally, the normal vector is defined to be directed perpendicular to the hypersurface Σ, $n_{\alpha} \equiv \partial_{\alpha} r$. By following the formalism of Ref. [39], it is easily shown that there also exists a non-vanishing Ricci tensor on the hypersurface. Therefore, by Einstein's field equations, we know that a Ricci tensor must be accompanied by a source of mass energy, thus we shall define a thin shell of matter, or surface layer, to exist on the hypersurface Σ. The stress-energy tensor of this thin shell is proportional to the extrinsic curvature of Σ as measured from either side, as shown below

$$S_{ab} = -\frac{1}{8\pi}\left(\left[K_{ab}\right] - \left[K\right] h_{ab}\right). \tag{2.1.5}$$

The result of Eq. (2.1.5) implies that if the extrinsic curvature on either side of the hypersurface is equal, the curvature singularity at the metric union vanishes. In the case of the current work, this is not true. However as previously stated, this implies the existence of a surface layer of matter at Σ with a stress-energy tensor S_{ab}. Solving for the extrinsic curvature on either side of the surface layer yields:

$$\mathbf{K}^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} \tag{2.1.6}$$

and

$$\mathbf{K}^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{a} & 0 \\ 0 & 0 & -\frac{1}{a}. \end{pmatrix} \tag{2.1.7}$$

Substituting these tensors into Eq. (2.1.5) gives the final result

$$\mathbf{S} = \begin{pmatrix} \frac{1}{2\pi a} & 0 & 0 \\ 0 & -\frac{1}{4\pi a} & 0 \\ 0 & 0 & \frac{1}{4\pi a}. \end{pmatrix} \tag{2.1.8}$$

Eq. (2.1.8) can be interpreted as a description of a perfect fluid. Reading the values straight from the tensor, the surface density σ, the pressure p and the mass M of the thin shell of matter are defined as

$$\sigma = S_{tt} = \frac{1}{2\pi a}, \tag{2.1.9a}$$

$$p = S_{BB} = -\frac{1}{4\pi a}, \tag{2.1.9b}$$

$$M = \sigma A = \frac{1}{2\pi a} 4\pi a^2 = 2a, \tag{2.1.9c}$$

where capital Latin subscripts represent the angular components of the tensor.

This thin shell of matter is what produces the singular distribution of curvature in the spacetime. Everywhere else, the spacetime is flat, but as will soon be shown, the singular hypersurface of curvature will be enough to produce a self-force on an electric charge and electric dipole in the system. This property alone makes this problem interesting and worth investigating, but it should be noted that the surface stress-energy tensor partially violates the weak energy condition given by

$$\sigma > 0, \qquad \sigma + p_1 + p_2 > 0, \tag{2.1.10}$$

as given in Chapter 2 of Ref. [39]. From the properties of the surface stress-energy tensor shown above, it is easy to see that the density is positive as required but the sum of the density and the pressures is identically zero, which is the source of this partial violation.

2.2 Maxwell's equations

2.2.1 Electric charge

Consider an electrically charged particle, stationary in the spatially closed universe. The Maxwell equations for such an object are

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} = 0, \qquad \nabla_\beta F^{\alpha\beta} = 4\pi j^\alpha, \tag{2.2.1}$$

where j^α is the current density and $F_{\alpha\beta}$ is the electromagnetic field tensor. For a system of two distinct point charges, the current density is given by

$$j^\alpha = q \int u_0^\alpha \delta(x, z_0) d\tau_0 - q \int u_1^\alpha \delta(x, z_1) d\tau_1, \tag{2.2.2}$$

where u_n^α and z_n^α are the 4-velocities and world lines of the relevant charges and τ_n are their respective proper times. Since the system is static, the substitution from proper to coordinate time is straightforward. Additionally, the spacetime comes with a timelike Killing vector t^α such that $t^t = 1$ is the only non-vanishing component, and $\nabla_\alpha t_\beta = 0$. This Killing vector simplifies some of the notation, particularly that of the 4-velocities of the charges, the current density and the electromagnetic vector potential. The 4-velocities simplify to $u_n^\alpha = t^\alpha$, and the current density to $j^\alpha = \rho t^\alpha$, where ρ is the charge density. Finally, the vector potential, as defined by the field tensor equation,

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha, \tag{2.2.3}$$

is simplified to $A_\alpha = \Phi t_\alpha$. The quantity Φ is defined to be the electrostatic potential. For the remainder of this calculation, I use spherical polar coordinates. Expanding the delta-functions and applying the Killing vector relation, the current density becomes the following charge density:

$$\rho = \frac{q}{R_0^2}\delta(r - r_0)\delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0) - \frac{q}{R_1^2}\delta(r - r_1)\delta(\cos\theta - \cos\theta_1)\delta(\phi - \phi_1), \tag{2.2.4}$$

where capital R is a generalized radial coordinate which is specified depending on which side of the surface layer the charge is on. The subscripts 0 and 1 are used for the remainder of the calculation as representations of the positive charge and negative charges respectively. The quantities r_0 and r_1 are left as the general location of the particle along the z-axis, which obviously means $\theta_0 = \theta_1 = 0$. The values of ϕ_0 and ϕ_1 are undefined, but are kept general for later use in the spherical harmonic decomposition.

Next, Maxwell's equations are simplified to:

$$\begin{aligned}
4\pi j_\alpha &= \nabla^\beta F_{\alpha\beta} \\
4\pi j_\alpha &= \nabla^\beta(\nabla_\alpha A_\beta - \nabla_\beta A_\alpha) \\
4\pi j_t &= \nabla^\beta(\nabla_t A_\beta - \nabla_\beta A_t) \\
-4\pi\rho &= \nabla^\beta(\nabla_t \Phi t_\beta + \nabla_\beta \Phi) \\
-4\pi\rho &= \nabla^\beta \nabla_\beta \Phi \\
-4\pi\rho &= \Box\Phi,
\end{aligned} \tag{2.2.5}$$

where the box operator is given by $\Box\Phi \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta\Phi$ and the sign changes come from lowering indices. Plugging Eq. (2.2.4) into Eq. (2.2.5), the field equation becomes

$$-\frac{4\pi q}{R_0^2}\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0)+\frac{4\pi q}{R_1^2}\delta(r-r_1)\delta(\cos\theta-\cos\theta_1)\delta(\phi-\phi_1)= \\ \frac{1}{R^2}\partial_r(R^2\partial_r\Phi)+\frac{1}{R^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Phi)+\frac{1}{R^2\sin^2\theta}\partial_{\phi\phi}\Phi. \tag{2.2.6}$$

2.2.2 Electric dipole

This time, consider a static dipole in flat spacetime. The current density again reduces to a simple charge density. The charge density ρ of a pure dipole is given by

$$\rho(\vec{r})\equiv j^t=-\vec{p}\cdot\nabla\delta(\vec{r}-\vec{r}_0), \tag{2.2.7}$$

where $\vec{p}$ is the dipole moment, and the delta function is scalarized as before. For this calculation however, it is difficult to represent the dipole in spherical coordinates because of its placement along the z-axis. To work around this, a momentary conversion to Cartesian coordinates is required. To give the dipole moment an arbitrary direction, each term in the dot product $\vec{p}\cdot\nabla$ is calculated separately. For the dipole purely in the z direction, the dipole moment aligns with $\hat{r}_0$ and the first term in Eq. (2.2.7) becomes

$$\begin{aligned}
\rho(\vec{r}) &= -p\hat{r}_0\cdot\nabla\delta(\vec{r}-\vec{r}_0)\\
\rho(\vec{r}) &= p\hat{r}_0\cdot\nabla_0\delta(\vec{r}_0-\vec{r})\\
\rho(\vec{r}) &= p\left(-\frac{1}{R^3(r_0)}\frac{\partial R(r_0)}{\partial r_0}+\frac{1}{R^2(r_0)}\partial_{r_0}\right)\delta(r_0-r)\delta(\cos\theta_0-\cos\theta)\delta(\phi_0-\phi).
\end{aligned} \tag{2.2.8}$$

Now, when the dipole moment points in either the x or y direction, it aligns with the $\hat{\theta}_0$ direction. Additionally, since ϕ_0 is generally undefined, the value is set to $\phi_0=0$ for the x direction and to

$\phi_0 = \pi/2$ for the y direction. In either case, the charge density of Eq. (2.2.7) reduces to

$$\rho(\vec{r}) = -\frac{p}{R_0}\hat{\theta}_0 \cdot \nabla\delta(\vec{r} - \vec{r}_0)$$
$$\rho(\vec{r}) = \frac{p}{R_0}\hat{\theta}_0 \cdot \nabla_0\delta(\vec{r}_0 - \vec{r})$$
$$\rho(\vec{r}) = \frac{p}{R_0^3}\partial_{\theta_0}\delta(r_0 - r)\delta(\cos\theta_0 - \cos\theta)\delta(\phi_0 - \phi). \tag{2.2.9}$$

In all cases, there is no ϕ derivative because the direction is undefined along the z-axis. The field equation then, is given simply by Eq. (2.2.5), where now ρ represents the sum of each of the terms described above.

2.3 SPHERICAL HARMONIC EXPANSION AND REGULARIZATION

As mentioned in Ch. 1 the regularization technique to be used was first presented by Barack and Ori in 2000 [21]. Regularization is a mathematical technique used to remove the singular portion of a physical relation without changing the physics. For example, in the calculation of the self-force, one calculates the total force acting on the charge. For a charged particle that produces a force-field of its own however, the electric field at its location is necessarily singular (infinite). Regularization is the method used to separate and remove the singular part of this field so that the self-force on the particle can be calculated. In the cases presented in this thesis, the electric potentials due to two static point charges or a point dipole are found with this method. To calculate the self-force, I begin by determining the net force acting on the charge (excluding the force required to hold it in place so that the mass can be ignored, i.e. the actual net force is necessarily zero). From the potential, the singular part is extracted as described above, and the electric field and self-force are calculated.

As presented in Ref. [21], this regularization can be done by first decomposing the potential into modes, based on a particular functional expansion. The chosen expansion for this work is a spherical harmonic expansion. In both cases, the electric potential is initially posed as

$$\Phi(r, \theta, \phi) = \sum_{\ell,m} \Phi_{\ell m}(r) Y_{\ell m}(\theta, \phi). \tag{2.3.1}$$

This expansion is simplified in both the case of the point charge and dipole, which will be expanded upon in the coming subsections. A similar expansion is done with the angular part of the three dimensional delta function, given below:

$$\delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0) = \sum_{\ell m} Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta_0, \phi_0). \tag{2.3.2}$$

With these expansions, regularization will be done in the following simple manner,

$$\Phi^R_{\ell m} = \Phi_{\ell m} - \Phi^S_{\ell m} \tag{2.3.3}$$

where Φ^S ans Φ^R represent the Detweiler-Whiting *singular potental* and the *regular potential* respectively [26]. This subtraction can be done mode-by-mode because each mode is regular at the position of the charge, where the full function would not be [21]. This avoids unnecessary confusion, and preserves rigour. Below, each case is further specified in spherical harmonic expansions, and simplified thereafter.

2.3.1 Electric charge

For two charges, the expansion is quite simple. Starting from the RHS of Eq. (2.2.5):

$$\begin{aligned}
4\pi\rho &= \frac{4\pi q}{R_0^2}\delta(r - r_0)\delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0) - \frac{4\pi q}{R_1^2}\delta(r - r_1)\delta(\cos\theta - \cos\theta_1)\delta(\phi - \phi_1) \\
&= \frac{4\pi q}{R_0^2}\delta(r - r_0)\sum_{\ell m} Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta_0, \phi_0) - \frac{4\pi q}{R_1^2}\delta(r - r_1)\sum_{\ell m} Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta_1, \phi_1) \\
&= \frac{4\pi q}{R_0^2}\delta(r - r_0)\sum_{\ell m} Y_{\ell m}(\theta, \phi)\sqrt{\frac{2\ell + 1}{4\pi}}\delta_{m0} - \frac{4\pi q}{R_1^2}\delta(r - r_1)\sum_{\ell m} Y_{\ell m}(\theta, \phi)\sqrt{\frac{2\ell + 1}{4\pi}}\delta_{m0}
\end{aligned}$$

$$= \left(\frac{4\pi q}{R^2(r)}\delta(r-r_0) - \frac{4\pi q}{R_1^2}\delta(r-r_1)\right)\sum_\ell \frac{2\ell+1}{4\pi}P_\ell(\cos\theta)$$

$$4\pi\rho_\ell = q(2\ell+1)\left(\frac{\delta(r-r_0)}{R_0^2} - \frac{\delta(r-r_1)}{R_1^2}\right)P_\ell(\cos\theta). \tag{2.3.4}$$

From the second to the third line above, both θ_0 and θ_1 where taken to zero. In the final step, the charge density is written as ρ_ℓ to simplify the final equation (i.e. remove the sum). Because the charge density simplifies to an expansion in Legendre polynomials, so must the potential:

$$\Phi(r,\theta,\phi) = \sum_{\ell m}\Phi_{\ell m}(r)Y_{\ell m}(\theta,\phi)\delta_{m0}$$

$$= \sum_\ell \sqrt{\frac{2\ell+1}{4\pi}}\Phi_{\ell 0}(r)P_\ell(\cos\theta).$$

Because of this, the potential admits the decomposition

$$\Phi(r,\theta) = \sum_\ell \Phi_\ell(r)P_\ell(\cos\theta), \tag{2.3.5}$$

where Φ_ℓ is given by

$$\Phi_\ell(r) \equiv \left(\frac{2\ell+1}{4\pi}\right)^{1/2}\Phi_{\ell 0}(r).$$

The potential is therefore axisymmetric, and the final form of the field equation becomes

$$-q\frac{(2\ell+1)}{R_0^2}\delta(r-r_0) + q\frac{(2\ell+1)}{R_1^2}\delta(r-r_1) = \partial_r^2\Phi_\ell + \frac{2}{R(r)}\partial_r R(r)\partial_r\Phi_\ell - \frac{\ell(\ell+1)}{R^2(r)}\Phi_\ell \tag{2.3.6}$$

where the third term on the RHS comes from the relation

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta Y_{\ell m}(\theta,\phi) + \frac{1}{\sin^2\theta}\partial_{\phi\phi}Y_{\ell m}(\theta,\phi) = -\ell(\ell+1)Y_{\ell m}(\theta,\phi),$$

and all angular factors of the potential have been removed. This equation will be solved later in Ch. 3.

2.3.2 Electric dipole

The potential decomposition for the dipole is less straightforward, given the angular derivatives of the spherical harmonics. Referring to Eq. (2.2.8), Eq. (2.2.9) and Eq. (2.3.2), the spherical harmonic expanded field equation is derived in parts as above. From this point forward, the dipole is assumed to be on the inside of the surface layer such that $R(r_0) = r_0$ and $\partial_{r_0} R_0 = 1$. For the dipole in the z direction, the field equation becomes

$$\begin{aligned}
\partial_\beta \partial^\beta \Phi &= -4\pi p \left(-\frac{1}{R^3(r_0)} \frac{\partial R(r_0)}{\partial r_0} + \frac{1}{R^2(r_0)} \partial_{r_0} \right) \delta(r_0 - r) \delta(\cos\theta_0 - \cos\theta) \delta(\phi_0 - \phi) \\
&= 4\pi p \left(\frac{1}{R^3(r_0)} \frac{\partial R(r_0)}{\partial r_0} - \frac{1}{R^2(r_0)} \partial_{r_0} \right) \delta(r_0 - r) \sum_{\ell m} Y_{\ell m}(\theta_0, \phi_0) Y^*_{\ell m}(\theta, \phi) \\
&= \frac{4\pi p}{R_0^2} \left(\frac{\partial_{r_0} R_0}{R(r_0)} \delta(r_0 - r) - \partial_{r_0} \delta(r_0 - r) \right) \sum_\ell \left(\frac{2\ell + 1}{4\pi} \right) P_\ell(\cos\theta) \\
\partial_\beta \partial^\beta \Phi_\ell &= \frac{4\pi p}{R_0^2} \left(\frac{\partial_{r_0} R_0}{R(r_0)} \delta(r_0 - r) - \partial_{r_0} \delta(r_0 - r) \right) \sum_\ell (2\ell + 1) P_\ell(\cos\theta), \qquad (2.3.7)
\end{aligned}$$

where the spherical harmonic sum reduced to the Legendre polynomials identically to the dual charge case. For the case of the x or y direction, the derivative of the spherical harmonics (with respect to θ) is taken. To avoid complication, this is done first. Starting from the angular momentum ladder operators (which are proportional to the θ derivative):

$$\frac{1}{2\hbar} \left(e^{-i\phi} L_+ - e^{i\phi} L_- \right) = \frac{1}{2\hbar} \left[\hbar e^{-i\phi} e^{i\phi} \left(\partial_\theta + i \cot\theta \partial_\phi \right) - \hbar e^{-i\phi} e^{i\phi} \left(-\partial_\theta + i \cot\theta \partial_\phi \right) \right] = \partial_\theta.$$

This allows us to write down the identity

$$\partial_\theta Y_{\ell,m} = \frac{1}{2} \left(e^{-i\phi} \sqrt{(\ell - m)(\ell + m + 1)} Y_{\ell,m+1} - e^{i\phi} \sqrt{(\ell + m)(\ell - m + 1)} Y_{\ell,m-1} \right), \qquad (2.3.8)$$

where the relation $L_\pm = \sqrt{(\ell \mp m)(\ell \pm m + 1)} Y_{\ell,m\pm1}$ is used to complete the calculation. This identity is used to simplify the derivative $\partial_{\theta_0} Y_{\ell,m}(\theta_0, \phi_0)$ where $\theta_0 \to 0$. With this, Eq. (2.3.8) reduces to

$$\partial_\theta Y_{\ell,m}(\theta_0,\phi_0) = \frac{1}{2}\left(e^{-i\phi_0}\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\ell(\ell+1)}\delta_{m,-1} - e^{i\phi_0}\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\ell(\ell+1)}\delta_{m,1}\right) \tag{2.3.9}$$

Now, starting from the field equation Eq. (2.2.5), and using the x direction (such that $\phi_0 = 0$), the previous result is implemented in the above derivation starting now from Eq. (2.3.7):

$$\begin{aligned}
\partial_\beta\partial^\beta\Phi =& -4\pi\frac{p}{R_0^3}\partial_{\theta_0}\delta(r_0-r)\delta(\cos\theta_0-\cos\theta)\delta(\phi_0-\phi) \\
=& -4\pi\frac{p}{R_0^3}\delta(r_0-r)\partial_{\theta_0}\sum_{\ell m}Y_{\ell m}(\theta_0,\phi_0)Y^*_{\ell m}(\theta,\phi) \\
=& -\frac{p}{R_0^3}\delta(r_0-r)\sum_{\ell m}Y^*_{\ell m}(\theta,\phi)\frac{\sqrt{\ell(\ell+1)}}{2}\left(e^{-i\phi_0}\sqrt{\frac{2\ell+1}{4\pi}}\delta_{m,-1} - e^{i\phi_0}\sqrt{\frac{2\ell+1}{4\pi}}\delta_{m,1}\right) \\
=& -4\pi\frac{p}{R_0^3}\delta(r_0-r)\sum_{\ell}\sqrt{\frac{2\ell+1}{4\pi}}\frac{\sqrt{\ell(\ell+1)}}{2}\left(Y^*_{\ell,-1}(\theta,\phi)-Y^*_{\ell,1}(\theta,\phi)\right) \\
\partial_\beta\partial^\beta\Phi =& -\frac{p}{R_0^3}\delta(r_0-r)\sum_{\ell}(2\ell+1)P^1_\ell(\cos\theta)\cos\phi
\end{aligned} \tag{2.3.10}$$

where the $P^1_\ell(\cos\theta)$ are the associated Legendre polynomials for $m = 1$. This process is identical for the dipole in the y direction, except $\phi_0 = \pi/2$, which changes the $\cos\phi$ to $\sin\phi$ because of the identity $\cos(\phi-\phi_0) = \cos(\phi-\pi/2) = \sin\phi$.

Finally, the potential decomposition must be specified further given the simplification of the charge density. In terms of Eq. (2.3.8), the potential can be decomposed as:

$$\begin{aligned}
\Phi(r,\theta,\phi) &= \sum_{\ell m}\Phi_{\ell m}(r)Y_{\ell m}(\theta,\phi)\left(\delta_{m,0} - \frac{2}{\sqrt{\ell(\ell+1)}}(\delta_{m,1}-\delta_{m,-1})\right) \\
&= \sum_{\ell}\left(\Phi^a_\ell(r)P_\ell(\cos\theta) + \Phi^b_\ell(r)P^1_\ell\cos(\phi-\phi_0)\right),
\end{aligned} \tag{2.3.11}$$

where

$$\Phi^a_\ell(r) = \sqrt{\frac{2\ell+1}{4\pi}}\Phi_{\ell,0},$$

and

$$\Phi^b_\ell(r) = -2\sqrt{\frac{2\ell+1}{4\pi}}\frac{1}{\sqrt{\ell(\ell+1)}}\Phi_{\ell,1}.$$

Furthermore, ϕ_0 is specified depending on which direction the dipole moment points. Putting everything together, there are three field equations to be solved, given by:

$$\partial_\beta\partial^\beta\Phi^z_\ell = -(2\ell+1)\frac{p^z}{r_0^2}\left(\frac{1}{r_0}\delta(r_0-r) + \partial_{r_0}\delta(r_0-r)\right) \tag{2.3.12a}$$

$$\partial_\beta\partial^\beta\Phi^x_\ell = -(2\ell+1)\frac{p^x}{r_0^3}\delta(r_0-r), \tag{2.3.12b}$$

$$\partial_\beta\partial^\beta\Phi^y_\ell = -(2\ell+1)\frac{p^y}{r_0^3}\delta(r_0-r), \tag{2.3.12c}$$

where superscripts x, y and z on the potential are simply labels, and do not imply vector components. Each of the above equations will be solved in Ch. 4.

CHAPTER 3 SELF-FORCE ON A POINT ELECTRIC CHARGE

3.1 THE BOUNDARY CONDITIONS

In this chapter, I work out the self-force acting on a point electric charge in the spacetime given by the metric

$$ds^2 = -dt^2 + dr^2 + R^2(r)d\Omega^2, \tag{3.1.1}$$

where

$$R(r) = \begin{cases} r, & r < a \\ 2a - r, & r > a, \end{cases} \tag{3.1.2}$$

as given originally in Eq. (2.1.1) and Eq. (2.1.2). For this calculation, I begin with Eq. (2.3.6) and split the potential into three regions. The potential and its derivative are assumed to be continuous across the electrically inert surface layer, but possibly discontinuous across each of the charges. Omitting the subscript ℓ, the potential takes the form

$$\Phi(r) \equiv \Phi_1(r)\Theta(r_0 - r) + \Phi_2(r)\Theta(r - r_0)\Theta(r_1 - r) + \Phi_3(r)\Theta(r - r_1), \tag{3.1.3}$$

where the positions of the charges are given here as $r_1 > r_0$ and the domain is given by $0 < r < 2a$. Ultimately however, the positions of the two charges with respect to one another are arbitrary when on the same side of the surface layer, as will be shown soon. For the following calculations, the positive charge is assumed to be inside the surface layer (i.e. $r_0 < a$); the calculation for the $r_0 > a$ is identical (due to symmetry) and only the final result is given later.

If the potential of Eq. (3.1.3) is plugged into Eq. (2.3.6), it can be greatly simplified by the fact that in each region, the potential satisfies the homogeneous equation. Furthermore, the definition of the potential provides terms dependent on compounded distributions such as $\delta(r-r_i)\Theta(r-r_j)$ and $\delta(r-r_i)\delta(r-r_j)$. For the first, $\delta(r-r_i)\Theta(r-r_j) \equiv \delta(r-r_i)$ if and only if $r_i > r_j$. Second, $\delta(r-r_i)\delta(r-r_j) \equiv 0$ unless $r_i = r_j$, which is not possible for this problem. The results of this straightforward calculation are the following boundary conditions:

$$[\Phi_\ell]_{r_0} = 0 \tag{3.1.4a}$$

$$[\Phi'_\ell]_{r_0} = -(2\ell+1)\frac{q}{r_0^2} \tag{3.1.4b}$$

$$[\Phi_\ell]_{r_1} = 0 \tag{3.1.4c}$$

$$[\Phi'_\ell]_{r_1} = (2\ell+1)\frac{q}{r_1^2} \tag{3.1.4d}$$

$$[\Phi_\ell]_{a} = 0 \tag{3.1.4e}$$

$$[\Phi'_\ell]_{a} = 0, \tag{3.1.4f}$$

where $[\Phi]_{r_0} = \Phi(r = r_0^+) - \Phi(r = r_0^-)$ denotes the jump across a charge. When the negative charge is located outside the surface layer (i.e. $r_1 > a$), only the fourth boundary conditions changes to $[\Phi'_\ell]_{r_1} = (2\ell+1)\frac{q}{(2a-r_1)^2}$.

3.2 SELF-FORCE CALCULATIONS

The calculations below will show the process of finding the electric potential in each region (defined differently for each case), and using the potentials in the regions about the positive charge at $r = r_0$ to calculate the self-force. The self-force on the negative charge would be identical because as will be shown, the self-force is dependent on q^2, leaving no dependence on the sign.

3.2.1 Case 1: $r_1 = 2a$ or $r_1 = 0$

In a first calculation, prior to including the second charge, it is found that the $\ell = 0$ mode is unsatisfied unless the potential and its derivative blow up at one end of the spacetime (i.e. $r = 0$ or $r = 2a$). The physical explanation for this of course is that another oppositely charged object is required in the system. This became obvious after realizing that Gauss' law requires a net zero charge in a closed universe, as it is here. Because of this, the preliminary calculation led to a simple monopole field created by a negative charge, $-q$ at $r = 0$ or $r = 2a$.

Starting with the general solution to Eq. (2.3.6), $\Phi_\ell = \{R^\ell, R^{-\ell-1}\}$, the boundary conditions for $r = r_0$ and $r = a$ listed in Eq. (3.1.4) are implemented along with the application of the potential singularity at each of the ends of the spacetime. Therefore, the solution to the homogeneous version of Eq. (2.3.6) for $\ell \neq 0$ is

$$\Phi_\ell(0 < r < r_0) = \frac{q}{r_0}\left(\frac{r}{r_0}\right)^\ell + \frac{q}{2\ell a}\left(\frac{rr_0}{a^2}\right)^\ell, \tag{3.2.1a}$$

$$\Phi_\ell(r_0 < r < a) = \frac{q}{r}\left(\frac{r_0}{r}\right)^\ell + \frac{q}{2\ell a}\left(\frac{rr_0}{a^2}\right)^\ell, \tag{3.2.1b}$$

$$\Phi_\ell(a < r < 2a) = \frac{q}{2\ell a}\left(\frac{Rr_0}{a^2}\right)^\ell + \frac{q}{2a}\left(\frac{Rr_0}{a^2}\right)^\ell, \tag{3.2.1c}$$

where $R = 2a - r$. The $\ell = 0$ solutions are defined up to an additive constant. For both $r_1 = 0$ and $r_1 = 2a$ this constant is simply q/r_0. When $r_1 = 0$ then, the potential is

$$\Phi_0(0 < r < r_0) = -\frac{q}{r} + \frac{q}{r_0}, \tag{3.2.2a}$$

$$\Phi_0(r_0 < r < a) = 0, \tag{3.2.2b}$$

$$\Phi_0(a < r < 2a) = 0. \tag{3.2.2c}$$

As previously stated, the solution is necessarily singular at $r = 0$, but the boundary conditions across $r = r_0$ and $r = a$ remain satisfied. Next, when $r_1 = 2a$ the potentials read

$$\Phi_0(0 < r < r_0) = \frac{q}{r_0}, \tag{3.2.3a}$$

$$\Phi_0(r_0 < r < a) = \frac{q}{r}, \tag{3.2.3b}$$

$$\Phi_0(a < r < 2a) = \frac{2q}{a} - \frac{q}{R}. \tag{3.2.3c}$$

This time the solution is singular at $r = 2a$, as expected, and the remaining boundary conditions are again satisfied.

The complete potential in each region is represented by three terms, described by

$$\Phi = \Phi_S + \Phi_R + \Phi_{\text{int}}, \tag{3.2.4}$$

where subscripts S, R and int stand for singular, regular and interaction respectively. In this case, the potential outside the shell has no singular term. In the other two potentials, the singular term is easy to spot as the one where the ℓ dependent part of each of the modes approaches unity as $r \to r_0$. Therefore, the singular, regular and interaction terms are given by

$$\Phi_S = \frac{q}{r_>} \sum_{\ell=0}^{\infty} \left(\frac{r_<}{r_>}\right)^{\ell} P_\ell(\cos\theta), \tag{3.2.5a}$$

$$\Phi_R = \frac{q}{2a} \sum_{\ell=1}^{\infty} \frac{1}{\ell} \left(\frac{r r_0}{a^2}\right)^{\ell} P_\ell(\cos\theta), \tag{3.2.5b}$$

$$\Phi_{\text{int}} = \begin{cases} -q/r & -q \text{ at } r_1 = 0 \\ 0 & -q \text{ at } r_1 = 2a \end{cases}, \tag{3.2.5c}$$

where $r_< \equiv \min(r, r_0)$ and $r_> \equiv \max(r, r_0)$. When $r_1 = 0$, the interaction between the two charges is very simple to describe. With the potential given above, there is a simple Coulomb interaction where the force is of course $F_{int} = -q^2/r_0^2$. However, when the charges are on opposite sides of the surface layer, there is no external force acting on the positive charge. Physically, this means the positive charge sees the negative charge as being distributed around it in a spherical shell, producing no force on the charge. Both of these statements are equivalent if the positive charge is located outside the surface layer. In that case, when $r_1 = 2a$, the Coulomb interaction reads $F_{\text{int}} = q^2/R_0^2$ and when $r_1 = 0$, the positive charge again sees a negatively charged spherical shell, producing no external force. This fact is brought about because the spacetime is symmetric about $r = a$, and either end of it (i.e. $r = 0$ and $r = 2a$) can be thought of as the center of this universe.

The singular potential described above is the potential that produces the spherically symmetric electric field about the charge. It is identified with the Detweiler-Whiting *singular potential* [26] and does not contribute to the self-force. The singular potential is also identical to the potential one would find in globally flat spacetime. The final potential Φ_R is identified as the Detweiler-Whiting *regular potential* and is entirely responsible for the self-force. After calculating the infinite sums above (see Appendix A for details), the analytic expressions for the potentials are:

$$\Phi_S(r,\theta) = \frac{q}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\theta}} \tag{3.2.6}$$

and

$$\Phi_R(r,\theta) = -\frac{q}{2a}\ln\left(\frac{a^2 - rr_0\cos\theta + \sqrt{a^4 + r^2r_0^2 - 2rr_0a^2\cos\theta}}{2a^2}\right). \tag{3.2.7}$$

From here, it is straightforward to calculate the force. First the electric field is given by $E_a^R = -\partial_a\Phi^R$ and the self-force by $F_r^{\text{self}} = qE_r^R(r = r_0, \theta = 0)$. In full form then,

$$F_r^{\text{self}} = -\frac{q^2 r_0}{2a^3}\frac{1}{1 - (r_0/a)^2}. \tag{3.2.8}$$

The r-component of the self-force is the only non-vanishing component. See Appendix A for further details on these derivations.

3.2.2 Case 2: $r_1 < a$

The following case presents a similar to calculation to above, now with $r_1 < a$ – a more general location than previously investigated. Because of this, there are now four regions in which to solve the potential. Additionally, the positive and negative charge can be placed at arbitrary locations with respect to one another for $r < a$. Doing so only changes the sign of one of interaction terms of the force depending on whether $r_1 < r_0$ or vice versa. Therefore without loss of generality, the case $r_1 < r_0$ is presented below and the difference is explained later.

The potentials for $\ell \neq 0$ in each of the regions are then given by

$$\Phi_\ell(0 < r < r_1) = \frac{q}{r_0}\left(\frac{r}{r_0}\right)^\ell - \frac{q}{r_1}\left(\frac{r}{r_1}\right)^\ell + \frac{q}{2\ell a}\left(\frac{rr_0}{a^2}\right)^\ell - \frac{q}{2\ell a}\left(\frac{rr_1}{a^2}\right)^\ell, \tag{3.2.9a}$$

$$\Phi_\ell(r_1 < r < r_0) = \frac{q}{r_0}\left(\frac{r}{r_0}\right)^\ell - \frac{q}{r}\left(\frac{r_1}{r}\right)^\ell + \frac{q}{2\ell a}\left(\frac{rr_0}{a^2}\right)^\ell - \frac{q}{2\ell a}\left(\frac{rr_1}{a^2}\right)^\ell, \tag{3.2.9b}$$

$$\Phi_\ell(r_0 < r < a) = \frac{q}{r}\left(\frac{r_0}{r}\right)^\ell - \frac{q}{r}\left(\frac{r_1}{r}\right)^\ell + \frac{q}{2\ell a}\left(\frac{rr_0}{a^2}\right)^\ell - \frac{q}{2\ell a}\left(\frac{rr_1}{a^2}\right)^\ell, \tag{3.2.9c}$$

$$\Phi_\ell(a < r < 2a) = \frac{q}{a}\left(\frac{Rr_0}{a^2}\right)^\ell - \frac{q}{a}\left(\frac{Rr_1}{a^2}\right)^\ell + \frac{q}{2\ell a}\left(\frac{Rr_0}{a^2}\right)^\ell - \frac{q}{2\ell a}\left(\frac{Rr_1}{a^2}\right)^\ell, \tag{3.2.9d}$$

where $R = 2a - r$. Now for $\ell = 0$, the potentials are given up to additive constant of $q/r_0 - q/r_1$:

$$\Phi_\ell(0 < r < r_1) = \frac{q}{r_0} - \frac{q}{r_1}, \tag{3.2.10a}$$

$$\Phi_\ell(r_1 < r < r_0) = -\frac{q}{r} + \frac{q}{r_0}, \tag{3.2.10b}$$

$$\Phi_\ell(r_0 < r < a) = 0, \tag{3.2.10c}$$

$$\Phi_\ell(a < r < 2a) = 0. \tag{3.2.10d}$$

Once again, the potential can be broken up into three parts: the regular, singular and interaction parts. The regular and singular parts are identical to Eq. (3.2.7) and Eq. (3.2.6) respectively, while the interaction part is more complicated than previously. The interaction potential is given in mode-sum form as

$$\Phi_{\text{int}} = \frac{q}{r}\sum_{\ell=0}^{\infty}\left(\frac{r_1}{r}\right)^{\ell} P_\ell(\cos\theta) - \frac{q}{2a}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{rr_1}{a^2}\right)^{\ell} P_\ell(\cos\theta); \tag{3.2.11}$$

see Appendix A details. In this case, the usual expression of $r_<^{\ell}/r_>^{\ell+1}$ is written more specifically in the case of $r_1 < r_0$ as above. The analytic form of the interaction potential is therefore

$$\Phi_{\text{int}} = -\frac{q}{\sqrt{r^2 + r_1^2 - 2rr_1\cos\theta}} + \frac{q}{2a}\ln\left(\frac{a^2 - rr_1\cos\theta + \sqrt{a^4 + r^2r_1^2 - 2rr_1a^2\cos\theta}}{2a^2}\right). \tag{3.2.12}$$

To find the interaction force acting on the positive charge, I again use the equations $E_a^{\text{int}} = -\partial_a\Phi^{\text{int}}$ and $F_r^{\text{int}} = qE_r^{\text{int}}(r = r_0, \theta = 0)$:

$$F_r^{\text{int}} = \mp\frac{q^2}{(r_1 - r_0)^2} + \frac{q^2 r_1}{2a^3}\frac{1}{1 - r_0r_1/a^2}. \tag{3.2.13}$$

As stated above, the only difference in replacing r_1 with r_0 is a sign change in a single term of the interaction potential. The minus sign corresponds to the case presented above, and the positive sign for the other. This equation nicely reduces to the interaction term of the previous case: $F_r^{\text{int}} = -q^2/r_0^2$ for $r_1 = 0$ as expected. This interaction shows the usual Coulomb force along with the second term which is due to the global curvature of the spacetime. This second term is directed towards the surface layer. However, the entire interaction force is net directed towards the other charge. Finally, I note that this force applies equally well to $r_1 = a$. Making this substitution produces the same expression as one calculated explicitly for $r_1 = a$.

3.2.3 Case 3: $r_1 > a$

This next case places the negative charge outside the surface layer (i.e. $r_1 > a$). For $\ell > 0$ then, the potentials are given by

$$\Phi_\ell(0 < r < r_0) = \frac{q}{r_0}\left(\frac{r}{r_0}\right)^{\ell} - \frac{q}{a}\left(\frac{rR_1}{a^2}\right)^{\ell} + \frac{q}{2\ell a}\left(\frac{rr_0}{a^2}\right)^{\ell} - \frac{q}{2\ell a}\left(\frac{rR_1}{a^2}\right)^{\ell}, \tag{3.2.14a}$$

$$\Phi_\ell(r_0 < r < a) = \frac{q}{r}\left(\frac{r_0}{r}\right)^\ell - \frac{q}{a}\left(\frac{rR_1}{a^2}\right)^\ell + \frac{q}{2\ell a}\left(\frac{rr_0}{a^2}\right)^\ell - \frac{q}{2\ell a}\left(\frac{rR_1}{a^2}\right)^\ell, \tag{3.2.14b}$$

$$\Phi_\ell(a < r < r_1) = -\frac{q}{R}\left(\frac{R_1}{R}\right)^\ell + \frac{q}{a}\left(\frac{Rr_0}{a^2}\right)^\ell + \frac{q}{2\ell a}\left(\frac{Rr_0}{a^2}\right)^\ell - \frac{q}{2\ell a}\left(\frac{RR_1}{a^2}\right)^\ell, \tag{3.2.14c}$$

$$\Phi_\ell(r_1 < r < 2a) = -\frac{q}{R_1}\left(\frac{R}{R_1}\right)^\ell + \frac{q}{a}\left(\frac{Rr_0}{a^2}\right)^\ell + \frac{q}{2\ell a}\left(\frac{Rr_0}{a^2}\right)^\ell - \frac{q}{2\ell a}\left(\frac{RR_1}{a^2}\right)^\ell, \tag{3.2.14d}$$

where $R = 2a - r$ and $R_1 = 2a - r_1$. For the $\ell = 0$ modes, there is no additive constant required, the modes are simply

$$\Phi_\ell(0 < r < r_0) = \frac{q}{r_0} - \frac{q}{a} \tag{3.2.15a}$$

$$\Phi_\ell(r_0 < r < a) = \frac{q}{r} - \frac{q}{a} \tag{3.2.15b}$$

$$\Phi_\ell(a < r < r_1) = -\frac{q}{R} + \frac{q}{a} \tag{3.2.15c}$$

$$\Phi_\ell(r_1 < r < 2a) = -\frac{q}{R_1} + \frac{q}{a}. \tag{3.2.15d}$$

Once more, the potential can be broken up into the Coulomb interaction, singular and regular potentials. The regular (responsible for the self-force) and singular terms are identical to the previous cases. The interaction term is given to be much different from before, this time not containing a simple Coulomb term as before. The potential is then given as

$$\begin{aligned}\Phi_{\text{int}} &= -\frac{q}{a}\sum_{\ell=0}^{\infty}\left(\frac{rR_1}{a^2}\right)^\ell P_\ell(\cos\theta) - \frac{q}{2a}\sum_{\ell=1}^{\infty}\left(\frac{rR_1}{a^2}\right)^\ell P_\ell(\cos\theta)\\ &= -\frac{qa}{\sqrt{a^4 + r^2R_1^2 - 2rR_1a^2\cos\theta}}\\ &\quad + \frac{q}{2a}\ln\left(\frac{a^2 - rR_1\cos\theta + \sqrt{a^4 + r^2R_1^2 - 2rR_1a^2\cos\theta}}{2a^2}\right).\end{aligned} \tag{3.2.16}$$

Refer to Appendix A for details. The electric field and the force are again calculated in the same way as before. Setting $r = r_0$ and $\theta = 0$, the interaction force is given as

$$F_r^{\text{int}} = \frac{q^2R_1}{2a^3}\frac{3 - r_0R_1/a^2}{(1 - r_0R_1/a^2)^2}. \tag{3.2.17}$$

This term vanishes if $R_1 = 0$ ($r_1 = 2a$), as expected from Sec. 3.2.1. As previously stated, this interaction force compares very little to the usual Coulomb force, and it is due to the global curvature of the spacetime. It is always directed towards the other charge, and thus the surface layer.

3.2.4 Results

Each of the potentials listed above are symmetric about the centre of the spacetime (i.e. $r = a$). Below I list each of the potentials for the case in which the charges are on opposite sides of the surface layer.

For $r_0 > a$ and $r_1 > a$, the potential and forces are given by:

$$\Phi_S = \frac{q}{\sqrt{R^2 + R_0^2 - 2RR_0 \cos\theta}}, \tag{3.2.18a}$$

$$\Phi_R = -\frac{q}{2a} \ln\left(\frac{a^2 - RR_0\cos\theta + \sqrt{a^4 + R^2R_0^2 - 2RR_0a^2\cos\theta}}{2a^2}\right), \tag{3.2.18b}$$

$$\begin{aligned}\Phi_{\text{int}} = &- \frac{q}{\sqrt{R^2 + R_1^2 - 2RR_1\cos\theta}} \\ &+ \frac{q}{2a} \ln\left(\frac{a^2 - RR_1\cos\theta + \sqrt{a^4 + R^2R_1^2 - 2RR_1a^2\cos\theta}}{2a^2}\right),\end{aligned} \tag{3.2.18c}$$

and

$$F_r^{\text{self}} = \frac{qR_0^2}{2a^3}\frac{1}{1 - (R_0/a)^2}, \tag{3.2.19a}$$

$$F_r^{\text{int}} = \pm\frac{q^2}{(R_0 - R_1)^2} - \frac{q^2R_1}{2a^3}\frac{1}{1 - R_0R_1/a^2}. \tag{3.2.19b}$$

Notice that there are sign differences for each of the terms. The $\pm$ is present in the usual Coulomb term for the same reason as was stated previously. For the remaining case, the singular and regular potentials and the self-force are identical to those listed directly above. The interaction terms are the only ones to change.

For $r_0 > a$ and $r_1 < a$, the interaction potential and force are given by

$$\Phi_{\text{int}} = -\frac{qa}{\sqrt{a^4 + R_0^2 r_1^2 - 2R_0 r_1 a^2 \cos\theta}} + \frac{q}{2a} \ln\left(\frac{a^2 - r_1 R\cos\theta + \sqrt{a^4 + R^2 r_1 - 2Rr_1 a^2 \cos\theta}}{2a^2}\right) \tag{3.2.20}$$

and

$$F_r^{\text{int}} = -\frac{q^2 r_1}{2a^3}\frac{3 - R_0 r_1/a^2}{(1 - R_0 r_1/a^2)^2}. \tag{3.2.21}$$

Again, the interaction force comes with a change in sign with respect to the previous interaction force where $r_0 < a$ and $r_1 > a$.

Finally, I plot the self and interaction forces as a function of the position r_0. Fig. 3.1 shows the self-force acting on a charge at the position $r = r_0$ where the value r_0 is varied over the domain $[0, 2a]$. Furthermore, the value of a is set to unity for simplicity. Fig. 3.1 plots the self-force as a

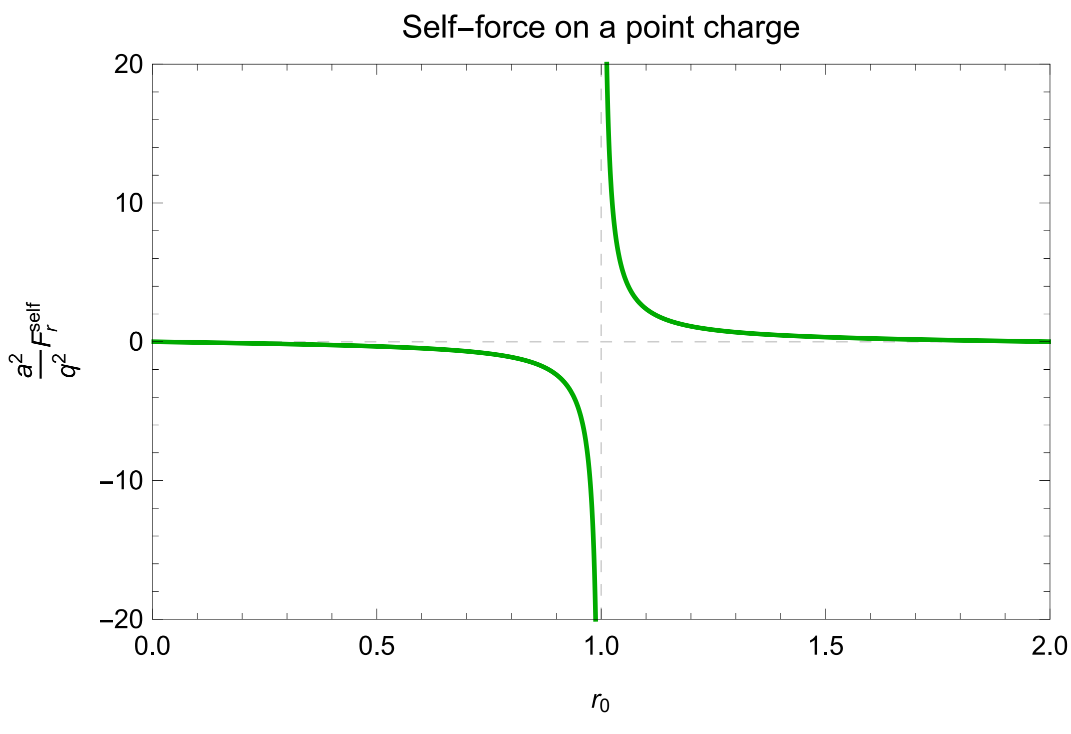

Figure 3.1: Self-force on a point charge at position $r = r_0$.

unitless quantity, i.e. $a^2/q^2 F_{\text{self}}$. As for the interaction force, the same scale is used where $a = 1$. Fig. 3.2 shows the interaction force acting on the positive charge at $r = r_0$ when the negative charge is located at $r_1 = 0.5$. Fig. 3.3 shows the interaction force again but where $r_1 = 1.5$ instead (where $a = 1$). Again, each force is plotted as a unitless quantity. The orange line in each plot signifies the location of the negative charge.

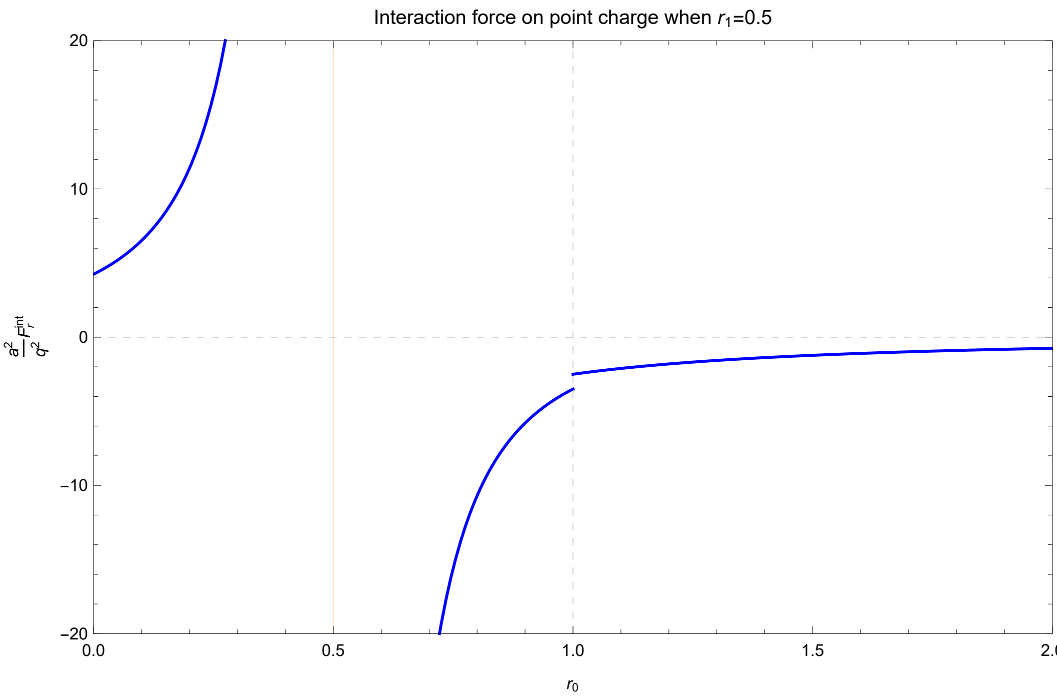

Figure 3.2: Interaction force on a point charge at position $r = r_0$ where a negative charge is located at $r_1 = 0.5$.

In both cases, the interaction force is seen to diverge near the position of the other charge, which is expected. Note the measurement of these forces is done by sliding the charge along the z-axis, and measuring the force required to hold it stationary. Furthermore, it is clear from the plots that the interaction force is discontinuous across the surface layer. Fortunately, this does not interfere with the original statement that the electric potential and field are continuous across the surface layer. To elaborate, for each configuration of the charges, the electric potential and field are necessarily continuous at $r = a$, but when we calculate the interaction force, we assume that the positive charge is moved along the z-axis through the universe. Therefore, because the curvature is what produces the altered interaction force, it is reasonable to assume that the singular nature of the curvature doesn't allow the force to remain continuous despite the fields themselves being continuous.

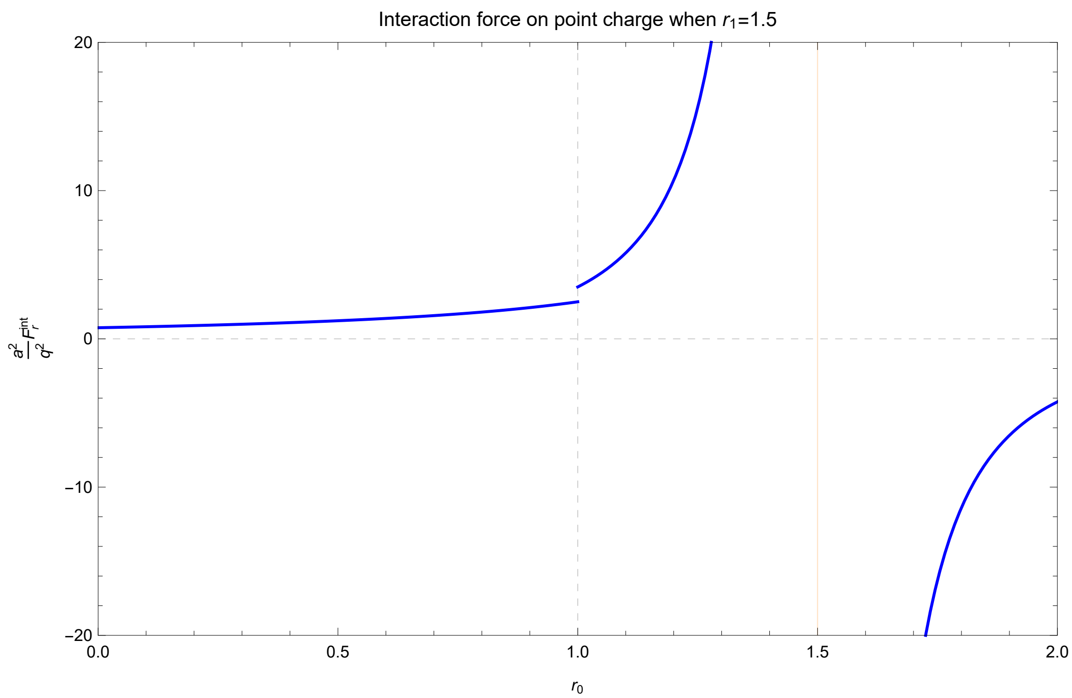

Figure 3.3: Interaction force on a point charge at position $r = r_0$ where a negative charge is located at $r_1 = 1.5$.

In these next few plots, I look at the comparison between the interaction force and the self-force acting on the positive charge. Given the strength of either force, we should consider the addition of the two forces and look at where a balance might occur, as well as where either force is necessarily stronger than the other. The first plot looks at the addition of the self-force and the interaction force when the negative charge is placed at $r_1 = 0.33$, where again a is set to unity for simplicity. In the second plot, the negative charge has been shifted to $r_1 = 0.66$. Because of the symmetry that we see in both the self-force and the interaction force, there is no need to consider the negative charge on the opposite side of the surface layer as no additional information will be found.

From Fig. 3.4 and Fig. 3.5 we can note a few distinct features. First, we see that when the charges are on the same side of the surface layer, the Coulomb force is always stronger. However, when the charges are on opposite sides of the surface layer, we see that there is a point at which the self-force becomes stronger as the $r_0 \to a$ from $2a$. For the plot where $r_1 = 0.33a$, the interaction and self-forces balance at the location $r_0 = 1.27a$ and when $r_1 = 0.66a$, the location of this balance shifts to $r_0 = 1.04a$. Following the trend seen here, it is easy to see that as $r_1 \to a$, the interaction

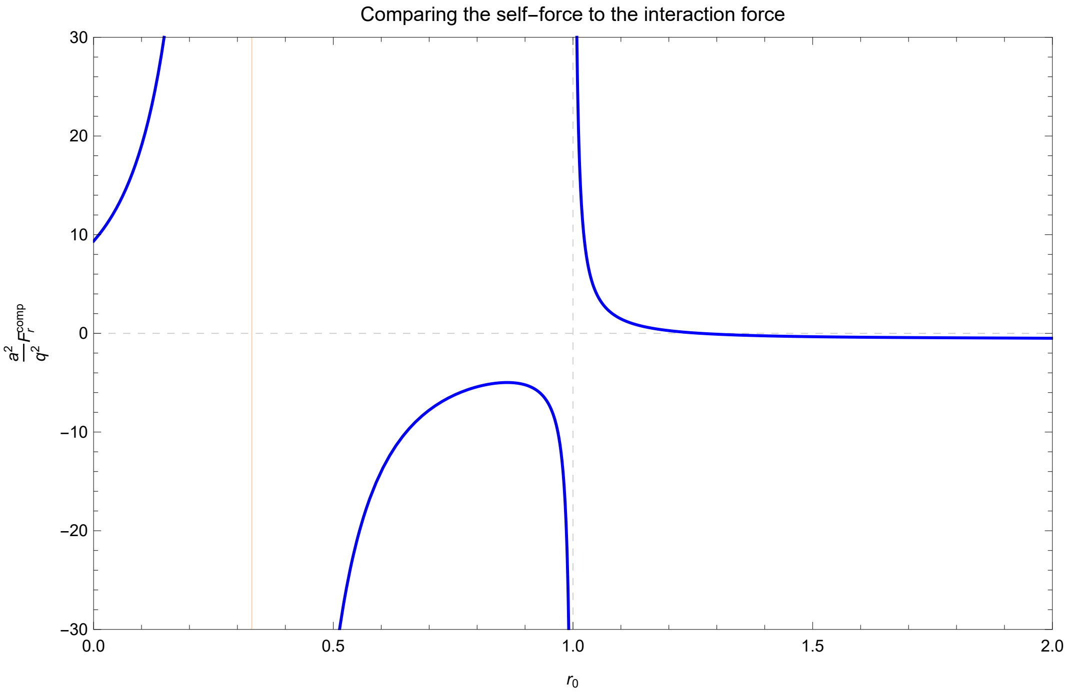

Figure 3.4: Addition of the self-force and interaction force on a point charge at position $r = r_0$ where a negative charge is located at $r_1 = 0.33$.

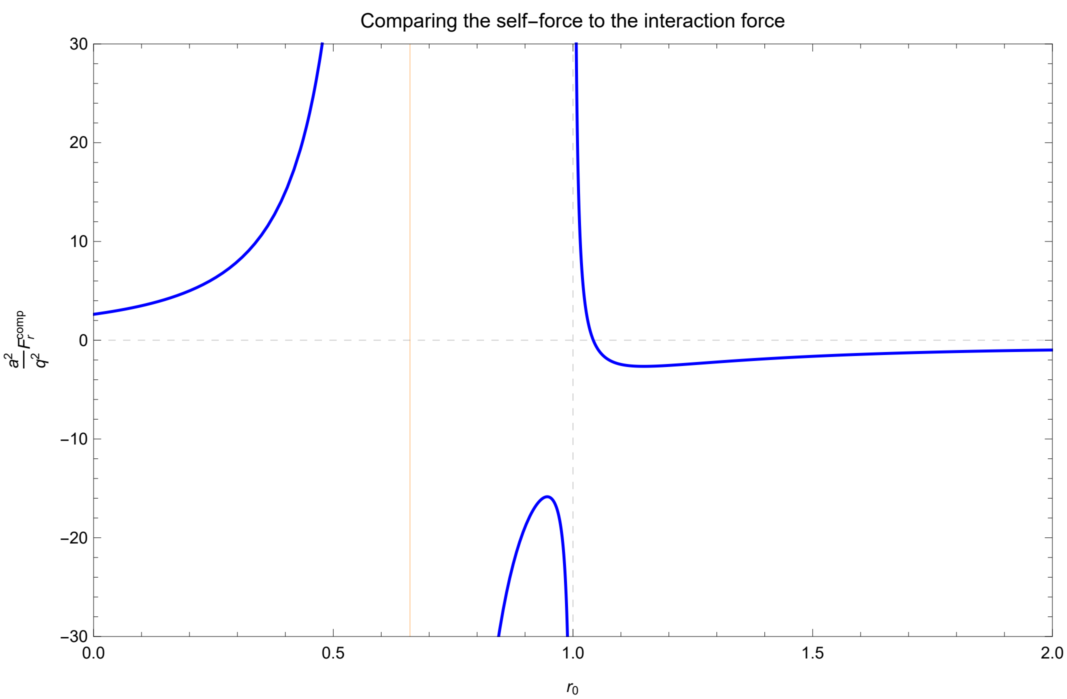

Figure 3.5: Addition of the self-force and interaction force on a point charge at position $r = r_0$ where a negative charge is located at $r_1 = 0.66$.

force becomes stronger for a larger domain, i.e. the self-force will almost always lose in this limit. However, if we place the negative charge on the surface layer, the charges can never touch because the self-force is stronger at this location. This can also be seen by taking the limit as $r_0 \to a$ of the ratio of the self-force to the interaction force. Because of this, when the negative charge is on the surface layer, the forces are balanced extremely close to the surface layer, but the self-force will always be strongest on the surface layer.

CHAPTER 4 SELF-FORCE ON A POINT ELECTRIC DIPOLE

4.1 THE BOUNDARY CONDITIONS

The boundary conditions for the dipole are calculated in much the same way as was in the case of the two point charges. This time however, the charge density is calculated differently as there is no longer a need for a second object in the spacetime. Gauss' law is perfectly satisfied for a single electric dipole. Once again, the electric potential and field are assumed to be continuous across the surface layer. Furthermore, given the arbitrary direction of the dipole, there will be a separate boundary condition associated with each direction. Starting with the $\hat{z}$ projection of the dipole moment, the field equation reads (as in Eq. (2.3.12a))

$$\partial_\beta \partial^\beta \Phi_\ell^z = -(2\ell+1)\frac{p^z}{r_0^2}\left(\frac{1}{r_0}\delta(r_0 - r) + \partial_{r_0}\delta(r_0 - r)\right). \tag{4.1.1}$$

It should be noted that this equation can be obtained equally well by using the charge density for Sec. 3. From the case of ρ of Eq. (2.2.4), a simple substitution of $r_1 \to r_0 - \delta r$ and a Taylor expansion to first order in δr yields

$$\begin{aligned}\rho &= \left(\frac{q}{R_0^2}\delta(r_0 - r) - \frac{q}{(R_0 - \delta r)^2}\delta(r_0 - r - \delta r)\right)\delta(\cos\theta_0 - \cos\theta)\delta(\phi_0 - \phi)\\ &= \left(\frac{q}{R_0^2}\delta(r_0 - r) - \left(\frac{q}{R_0^2} + 2\frac{q\delta r}{R_0^3}\partial_{r_0}R_0\right)\left(\delta(r_0 - r) - \partial_{r_0}\delta(r_0 - r)\delta r\right)\right)\delta(\cos\theta_0 - \cos\theta)\delta(\phi_0 - \phi)\end{aligned}$$

$$= \left(-2\frac{q\delta r}{R_0^3}\partial_{r_0}R_0\delta(r_0-r)+\frac{q\delta r}{R_0^2}\partial_{r_0}\delta(r_0-r)\right)\delta(\cos\theta_0-\cos\theta)\delta(\phi_0-\phi)$$
$$= \left(-2\frac{p^z}{R_0^3}\partial_{r_0}R_0\delta(r_0-r)+\frac{p^z}{R_0^2}\partial_{r_0}\delta(r_0-r)\right)\delta(\cos\theta_0-\cos\theta)\delta(\phi_0-\phi).$$

The boundary conditions are solved by splitting the potential into two regions, one on either side of the dipole along the z-axis, i.e. $\Phi \equiv \Phi_<(r)\Theta(r_0-r)+\Phi_>(r)\Theta(r-r_0)$. Plugging this potential into Eq. (4.1.1), the result is

$$\begin{aligned} 2(\Phi'_> - \Phi'_<)\delta(r-r_0)+&\frac{2R'}{R}(\Phi_>-\Phi_<)\delta(r-r_0)+(\Phi_>-\Phi_<)\delta'(r-r_0) \\ &= (2\ell+1)\frac{p^z}{R_0^2}\left(2\frac{\partial_{r_0}R_0}{R_0}\delta(r-r_0)+\delta'(r-r_0)\right), \end{aligned} \tag{4.1.2}$$

where a prime indicates differentiation with respect to r. The sign on the second term on the RHS was flipped by changing the derivative variable. The above expression is rearranged to collect delta functions and their derivatives, and then integrated to solve for the boundary conditions. The boundary conditions are derived as follows

$$\begin{aligned} \left(2(\Phi'_> - \Phi'_<)+\frac{2R'}{R}(\Phi_>-\Phi_<)-2(2\ell+1)\frac{p^z}{R_0^2}\frac{\partial_{r_0}R_0}{R_0}\right)\delta(r-r_0) \\ = \left((2\ell+1)\frac{p^z}{R_0^2}-(\Phi_>-\Phi_<)\right)\delta'(r-r_0) \\ 2[\Phi']_{r_0}+\frac{2R'_0}{R_0}[\Phi]_{r_0}-2(2\ell+1)\frac{p^z}{R_0^2}\frac{\partial_{r_0}R_0}{R_0} = [\Phi']_{r_0} \\ [\Phi']_{r_0}+\frac{2R'_0}{R_0}[\Phi]_{r_0}-2(2\ell+1)\frac{p^z}{R_0^2}\frac{R'_0}{R_0} = 0, \end{aligned} \tag{4.1.3}$$

where the square bracket notation is identical to the previous case and a prime now indicates differentiation with respect to r_0. From the above steps, it is clear that the boundary conditions are given by

$$[\Phi_\ell]_{r_0} = (2\ell+1)\frac{p^z}{R_0^2}, \tag{4.1.4a}$$

$$[\Phi'_\ell]_{r_0} = 0. \tag{4.1.4b}$$

The next step is to calculate the boundary conditions for the x and y terms of the potential (i.e. the terms which depend on p^x and p^y respectively). These two field equations are then

$$\partial_\beta \partial^\beta \Phi_\ell = -(2\ell+1)\frac{p^x}{R_0^3}\delta(r-r_0) \tag{4.1.5a}$$

$$\partial_\beta \partial^\beta \Phi_\ell = -(2\ell+1)\frac{p^y}{R_0^3}\delta(r-r_0) \tag{4.1.5b}$$

Just as before, the potential is broken into two regions about the dipole, and plugged into the field equation. The eventual result is

$$[\Phi']_{r_0} + \frac{2R_0'}{R_0}[\Phi]_{r_0} + (2\ell+1)\frac{p^i}{R_0^3} = 0, \tag{4.1.6}$$

which gives the appropriate boundary conditions,

$$[\Phi_\ell]_{r_0} = 0, \tag{4.1.7a}$$

$$[\Phi_\ell']_{r_0} = -(2\ell+1)\frac{p^i}{R_0^3}, \tag{4.1.7b}$$

where i is either x or y. Once again, this result can be replicated from the dual charge case by letting $r_1 \to r_0$, $\theta_1 \to \theta_0 - \delta\theta$, $\phi_1 \to \phi_0$ and letting $\delta\theta \to 0$ while keeping $p^i \equiv q r_0 \delta\theta$ fixed. A Taylor expansion in $\delta\theta$ from Eq. (2.2.4) yields

$$\begin{aligned}
\rho =& \frac{q}{r_0^2}\delta(r-r_0)\delta(\cos\theta - \cos\theta_0)\delta(\phi-\phi_0) - \frac{q}{r_1^2}\delta(r-r_0)\delta(\cos\theta - \cos(\theta_0+\delta\theta))\delta(\phi-\phi_0) \\
=& \frac{q}{r_0^2}\delta(r-r_0)\left(\delta(\cos\theta-\cos\theta_0) - \delta(\cos\theta-\cos\theta_0) + \partial_{\theta_0}\delta(\cos\theta_0-\cos\theta)\delta\theta\right)\delta(\phi-\phi_0) \\
=& \frac{q\delta\theta}{r_0^2}\delta(r-r_0)\partial_{\theta_0}\delta(\cos\theta_0-\cos\theta)\delta(\phi-\phi_0) \\
=& \frac{p^i}{r_0^3}\delta(r-r_0)\partial_{\theta_0}\delta(\cos\theta_0-\cos\theta)\delta(\phi-\phi_0),
\end{aligned}$$

which matches the charge density of the dipole.

4.2 SELF-FORCE CALCULATION

Below, the self-force is calculated for a dipole both inside and outside the surface layer. To do so, there are two necessary calculations, one including p^z and the other with p^x (p^y is identical up to the exchange of $\cos\phi$ with $\sin\phi$). This section will conclude (as with Ch. 3) with plots showing the magnitude of each component of the self-force.

4.2.1 Case 1: Dipole in the z direction

The potential modes are found by calculating the homogeneous versions of Eq. (4.1.1) and Eq. (4.1.5a) (and Eq. (4.1.5b)). The general solutions match the point charge case where $\Phi_\ell(r) = A_\ell R^\ell + B_\ell R^{-\ell-1}$, such that $R(r)$ is given by Eq. (2.1.2). Applying each of the boundary conditions fully specifies the potential in each of the the three regions, which are

$$\Phi_\ell(0 < r < r_0) = -(\ell+1)\frac{p^z}{r_0^2}\left(\frac{r}{r_0}\right)^\ell + \frac{p^z}{2ar_0}\left(\frac{rr_0}{a^2}\right)^\ell, \tag{4.2.1a}$$

$$\Phi_\ell(r_0 < r < a) = \ell\frac{p^z}{r_0 r}\left(\frac{r_0}{r}\right)^\ell + \frac{p^z}{2ar_0}\left(\frac{rr_0}{a^2}\right)^\ell, \tag{4.2.1b}$$

$$\Phi_\ell(a < r < 2a) = (2\ell+1)\frac{p^z}{2ar_0}\left(\frac{Rr_0}{a^2}\right)^\ell. \tag{4.2.1c}$$

It is clear which terms in Eq. (4.2.1) are singular at $r = r_0$. The single and regular potentials are then summed with the Legendre polynomials to give

$$\begin{aligned}\Phi_S &= -\frac{p^z}{r_0^2}\sum_{\ell=0}^{\infty}(\ell+1)\left(\frac{r}{r_0}\right)^\ell P_\ell(\cos\theta)\\ &= \frac{p^z(r\cos\theta - r_0)}{(r^2+r_0^2-2rr_0\cos\theta)^{3/2}}\end{aligned} \tag{4.2.2}$$

and

$$\Phi_R = \frac{p^z}{2ar_0}\sum_{\ell=0}^{\infty}\left(\frac{rr_0}{a^2}\right)^\ell P_\ell(\cos\theta)$$

$$= \frac{p^z a}{2r_0\sqrt{a^4 + r_0^2 r^2 - 2a^2 r r_0 \cos\theta}} \tag{4.2.3}$$

$$= \frac{p^z a}{2r_0 s^2}, \tag{4.2.4}$$

respectively. The value s is defined explicitly as

$$s^2 \equiv (a^4 + r_0^2 r^2 - 2a^2 r r_0 \cos\theta)^{1/2} \tag{4.2.5}$$

such that s has units of length. Notice that the singular term is equivalent to the potential of a dipole on the z-axis in flat spacetime. A more in depth derivation of each sum is given in Appendix A. Finally, as with the point charges, the singular potential here is related to the singular Detweiler-Whiting potential [26] and the regular potential to the regular Detweiler-Whiting potential. The self-force will again be entirely determined by the regular potential.

4.2.2 Case 2: Dipole in the x or y direction

Starting with the boundary conditions in Eq. (4.1.7) and the general solution to the homogeneous differential equation, the potential in the three different regions are given as

$$\Phi_\ell(0 < r < r_0) = \frac{p^i}{r_0^2}\left(\frac{r}{r_0}\right)^\ell + \frac{p^i}{2\ell a r_0}\left(\frac{r r_0}{a^2}\right)^\ell \tag{4.2.6a}$$

$$\Phi_\ell(r_0 < r < a) = \frac{p^i}{r r_0}\left(\frac{r_0}{r}\right)^\ell + \frac{p^i}{2\ell a r_0}\left(\frac{r r_0}{a^2}\right)^\ell \tag{4.2.6b}$$

$$\Phi_\ell(a < r < 2a) = \frac{p^i}{a r_0}\left(\frac{R r_0}{a^2}\right)^\ell + \frac{p^i}{2\ell a r_0}\left(\frac{R r_0}{a^2}\right)^\ell, \tag{4.2.6c}$$

where p^i represents both p^x and p^y, since the above equation doesn't change. Re-introducing the angular dependence, and calculating the sum (done explicitly in Appendix A) yields the potentials

$$\Phi_S^x = \frac{p^x}{r_0^2}\sum_{\ell=1}^{\infty}\left(\frac{r}{r_0}\right)^\ell \cos\phi P_\ell^1(\cos\theta)$$

$$= p^x \frac{r\sin\theta\cos\phi}{(r^2 + r_0^2 - 2rr_0\cos\theta)^{3/2}}, \tag{4.2.7a}$$

$$\Phi_S^y = \frac{p^y}{r_0^2}\sum_{\ell=1}^{\infty}\left(\frac{r}{r_0}\right)^{\ell}\sin\phi P_\ell^1(\cos\theta)$$

$$= p^y \frac{r\sin\theta\sin\phi}{(r^2 + r_0^2 - 2rr_0\cos\theta)^{3/2}}, \tag{4.2.7b}$$

and

$$\Phi_R^x = \frac{p^x}{2ar_0}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{rr_0}{a^2}\right)^{\ell}\cos\phi P_\ell^1(\cos\theta)$$

$$= p^x \frac{(a^2 + s^2)r_0 r\sin\theta\cos\phi}{2as^2(a^2 - rr_0\cos\theta + s^2)}, \tag{4.2.8a}$$

$$\Phi_R^x = \frac{p^y}{2ar_0}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{rr_0}{a^2}\right)^{\ell}\sin\phi P_\ell^1(\cos\theta)$$

$$= p^y \frac{(a^2 + s^2)r_0 r\sin\theta\sin\phi}{2as^2(a^2 - rr_0\cos\theta + s^2)}. \tag{4.2.8b}$$

These calculations were completed using the relation $P_\ell^1(\cos\theta) = -(d/d\theta)P_\ell(\cos\theta)$ which is derived in Appendix A.

4.2.3 Self-force on a dipole

As derived above (explicitly in the Appendix A) there are three parts to the full potential, the sum of which give the full potential:

$$\Phi_R(r,\theta,\phi) = \frac{(a^2 + s^2)(p^x r\sin\theta\cos\phi + p^y r\sin\theta\sin\phi)}{2as^2(a^2 - rr_0 + s^2)} + \frac{p^z a}{2r_0 s^2}. \tag{4.2.9}$$

Switching to Cartesian components, the full regular potential reads

$$\Phi_R(x,y,z) = \frac{(a^2 + s^2)(p^x x + p^y y)}{2as^2(a^2 - rr_0 + s^2)} + \frac{p^z a}{2r_0 s^2}, \tag{4.2.10}$$

where s is now given by

$$s^2 = (a^4 + r_0^2(x^2 + y^2 + z^2) - 2a^2 r_0 z)^{1/2} \tag{4.2.11}$$

Given this potential, it is straightforward to calculated the self-force acting on the dipole. It is written as

$$F_a^{\text{self}} = -p^b \partial_{ab} \Phi_R \tag{4.2.12}$$

or alternatively as

$$F_a^{\text{self}} = T_{abc} p^b p^c \tag{4.2.13}$$

where $p^a \equiv (p^x, p^y, p^z)$ and $\boldsymbol{T}$ is a rank 0,3 tensor which describes the self-force interaction.

After the derivatives, the limits $x \to 0$, $y \to 0$ and $z \to r_0$ are taken. The tensor $\boldsymbol{T}$ can then be written as

$$T_x = \begin{pmatrix} 0 & 0 & -\frac{r_0}{8a^5(1-(r_0/a)^2)^2} \\ 0 & 0 & 0 \\ -\frac{r_0}{8a^5(1-(a/r_0)^2)^2} & 0 & 0 \end{pmatrix}, \tag{4.2.14a}$$

$$T_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{r_0}{8a^5(1-(r_0/a)^2)^2} \\ 0 & -\frac{r_0}{8a^5(1-(a/r_0)^2)^2} & 0 \end{pmatrix}, \tag{4.2.14b}$$

$$T_z = \begin{pmatrix} -\frac{r_0}{4a^5}\frac{(3-(r_0/a)^2)}{(1-(r_0/a)^2)^3} & 0 & 0 \\ 0 & -\frac{r_0}{4a^5}\frac{(3-(r_0/a)^2}{(1-(r_0/a)^2)^3} & 0 \\ 0 & 0 & \frac{r_0}{a^5(1-(r_0/a)^2)^3}, \end{pmatrix} \tag{4.2.14c}$$

When contracted with the dipole moment twice, the self-force is found to have three components, which are written as

$$F_x^{\text{self}} = -\frac{r_0 p^x p^z}{4a^5(1-(r_0/a)^2)^2} \tag{4.2.15a}$$

$$F_y^{\text{self}} = -\frac{r_0 p^y p^z}{4a^5(1-(r_0/a)^2)^2} \tag{4.2.15b}$$

$$F_z^{\text{self}} = -\frac{r_0[(3-(r_0/a)^2)(p_x^2+p_y^2)+4p_z^2]}{4a^5(1-(r_0/a)^2)^3} \tag{4.2.15c}$$

Each of these components are symmetric about the surface layer, in that the self-force is always directed away from it, as shown in Fig. 4.1. Therefore, when the dipole is placed outside the shell, the self-force components are

$$F_x^{\text{self}} = \frac{R_0 p^x p^z}{4a^5(1-(R_0/a)^2)^2} \tag{4.2.16a}$$

$$F_y^{\text{self}} = \frac{R_0 p^y p^z}{4a^5(1-(R_0/a)^2)^2} \tag{4.2.16b}$$

$$F_z^{\text{self}} = \frac{R_0[(3-(R_0/a)^2)(p_x^2+p_y^2)+4p_z^2]}{4a^5(1-(R_0/a)^2)^3}, \tag{4.2.16c}$$

where $R_0 = 2a - r_0$.

As stated previously, the self-force acting on the dipole is not the same as the sum of the self-forces acting on the point charge. However, if one also includes the interaction force in this sum, we can replicate a specific case of the self-force on a dipole. Specifically, we first calculate the sum of the interaction forces and self-forces on both the positive and negative charge. The self and interaction forces on the negative charge are found by simply interchanging all instances of r_0 with r_1 and vice versa. Then we take the limit as $\delta r \to 0$ for $r_1 = r_0 - \delta r$, while keeping $q\delta r$ constant, and we find the self-force on a dipole for the dipole moment pointing solely in the z direction. Because the other self-forces are not easily calculated in this way, we see that there was a loss of generality in calculating the self-force along the z-axis only, since this same method should in theory also work in the other directions.

Finally, Fig. 4.1, Fig. 4.2, and Fig. 4.3 show the self-force on a dipole at location $r = r_0$ along the z-axis. Each plot shows a different orientation for the dipole stated in their respective captions. Additionally each of the dipole moments are normalized such that the magnitude of the dipole moment is always $|\vec{p}| = p$.

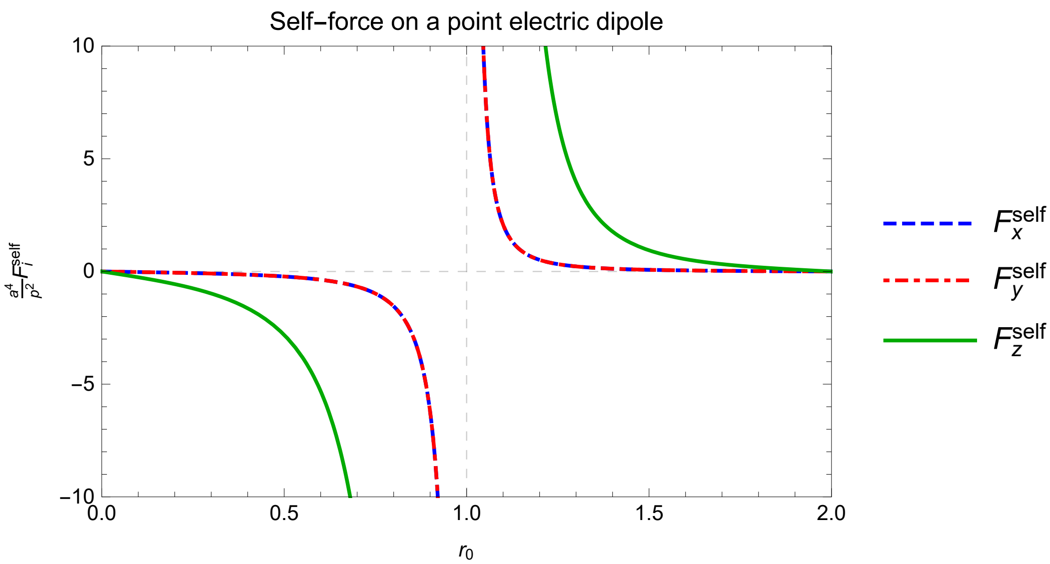

Figure 4.1: Self-force on a dipole located at position $r = r_0$, with components $p^z = p^y = p^x = p/\sqrt{3}$.

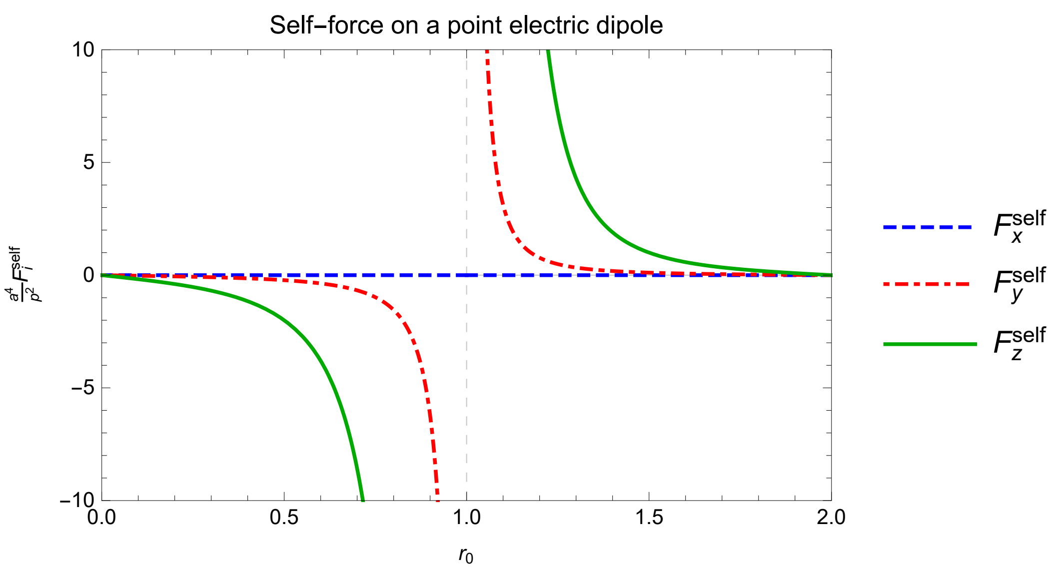

Figure 4.2: Self-force on a dipole located at position $r = r_0$, with components $p^z = p^y = p/\sqrt{2}$ and $p^x = 0$. The plot is identical for $p^y = 0$ instead of p^x, except the non-zero component becomes F_x.

For each plot, the self-force is plotted as a unitless quantity, $(a^4/p^2)F_{\text{self}}$, and for simplicity $a = 1$.

As a final note, Fig. 4.4 shows the graphical relation between the self-force on a dipole and self-force on a point charge. In Fig. 4.4, only F_z is plotted, and the dipole moment is given solely

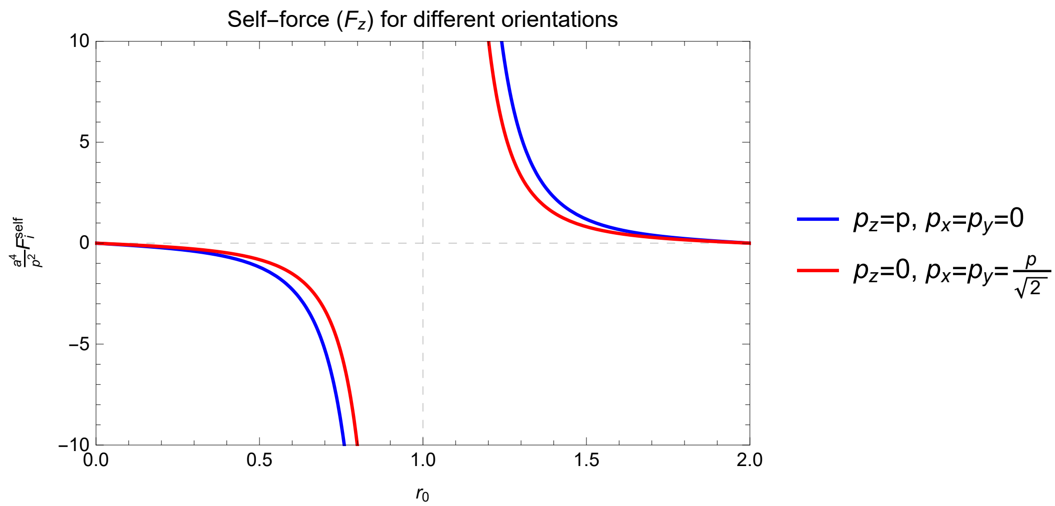

Figure 4.3: Self-force on a dipole located at position $r = r_0$, for two orientations of the dipole. In either case, the x and y components of the self-force vanish, leaving only the z component, plotted here.

by $p^z = p$. The legend of this plot states how each force is plotted as a unitless quantity. From this plot, it is clear that the dipole is repelled more strongly from the surface layer than the point charge.

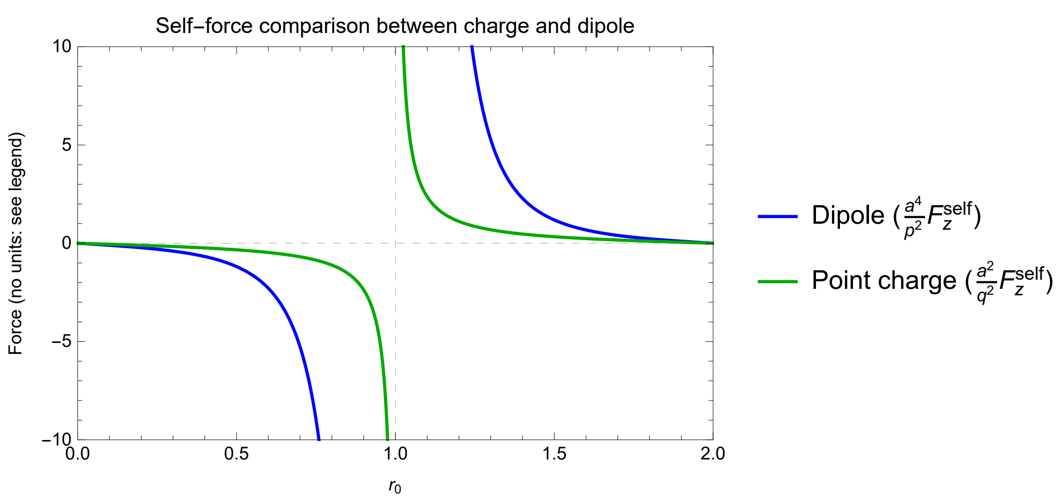

Figure 4.4: Comparison between the z-component of the self-force on a dipole and the self-force on a point charge.

CHAPTER 5 CONCLUSION

The work in this thesis has investigated the self-force acting on a point electric charge and a point electric dipole. Both objects are held in place by some external force (so that the net force is zero), and the self-force on the particle is measured. Due to Gauss' law, the point charge must be accompanied by a second oppositely charged object also held in place at some other position. In addition to the self-force, the two charges interact with one another via the usual Coulomb force as well as an interaction force that results from the global curvature. The self-force on a point charge (which is independent of the sign of the charge) is given in general by

$$\boldsymbol{F} = -\frac{q^2 r_0}{2a^3}\frac{1}{1-(r_0/a)^2}\boldsymbol{n}, \tag{5.0.1}$$

where $\boldsymbol{n} \equiv \boldsymbol{r_0}/r_0$. The self-force is symmetric about the surface layer meaning that we can treat either side as being "inside" the surface layer. Equivalently, both $r = 0$ and $r = 2a$ can be considered the centre of the universe. Unlike the self-force, the interaction force is dependent on the relative positions of the two charges as well as the surface layer. When the charges are both located at positions $r_0, r_1 < a$, the interaction force is given by

$$\boldsymbol{F}^{\text{int}} = \frac{q^2}{(r_0 - r_1)^2}\boldsymbol{r}_{01} + \frac{q^2 r_1}{2a^3}\frac{1}{1 - r_0 r_1/a^2}\boldsymbol{n}, \tag{5.0.2}$$

where $\boldsymbol{r}_{01} \equiv (\boldsymbol{r}_0 - \boldsymbol{r}_1)/|\boldsymbol{r}_0 - \boldsymbol{r}_1|$. Then, when the charges are on opposite sides of the surface layer (i.e. $r_1 > a, r_0 < a$) the interaction force changes to

$$\boldsymbol{F}^{\text{int}} = \frac{q^2 R_1}{2a^3} \frac{3 - r_0 R_1/a^2}{(1 - r_0 R_1/a^2)^2} \boldsymbol{r}_{01}, \tag{5.0.3}$$

where $R_1 = 2a - r_1$. Referring back to Fig. 3.2 and Fig. 3.3, it is easy to see that the interaction force is symmetric about the surface layer just as with the self-force as described above. We also recall the discontinuity of the interaction force across the surface layer. This discontinuity does not conflict with the requirement that the electric fields and the potentials are necessarily continuous across the surface layer, and is simply due to the singular nature of the curvature on the surface layer.

The self-force on the dipole obviously has no interaction force as Gauss' law remains satisfied. From Eq. (4.2.15), the self-force can be generalized to a coordinate independent form which is written as

$$\boldsymbol{F}^{\text{self}} = -\frac{p^2 r_0}{4a^5} \frac{1}{(1 - r_0^2/a^2)^3} \boldsymbol{f} \tag{5.0.4}$$

where

$$\boldsymbol{f} \equiv (1 - r_0^2/a^2)(\hat{\boldsymbol{p}} \cdot \boldsymbol{n})\hat{\boldsymbol{p}} + \left[(3 - r_0^2/a^2) + 2(r_0^2/a^2)(\hat{\boldsymbol{p}} \cdot \boldsymbol{n})^2\right] \boldsymbol{n}, \tag{5.0.5}$$

and $\hat{\boldsymbol{p}} = \boldsymbol{p}/p$. This generalized force holds much information. For example when the dipole is aligned with $\boldsymbol{n}$, the force is directed along $-\boldsymbol{n}$. This is also the case when the dipole moment is perpendicular to $\boldsymbol{n}$. For an arbitrary direction however, the force has components along both $\boldsymbol{n}$ and $\hat{\boldsymbol{p}}$. Due to the calculation being done along the z-axis, there was some loss of generality. We see this after finding the sum of the interaction and self-forces on each of the charges in the dual charge case and letting $r_1 \to r_0$. The result of this is the self-force on a dipole for the dipole oriented in the z direction only. This same method should in theory work for the other directions as well, assuming the calculation was done more generally. Because of this, we may also be able to do something very similar by adding the self-force on two dipoles to perhaps find the self-force on a quadrupole. This calculation is not done in this thesis, but would be an interesting endeavour.

The self-force on a point charge in curved spacetime has been an ongoing research endeavour for quite some time. This thesis works to extend the self-force to assess global structure. Discussed as well in Ref. [40], the self-force acts as probe of both internal structure (Ref. [28]) and global structure. From the plots at the end of Ch. 3 and Ch. 4, it is clear that the objects are repelled from the surface layer, which is the sole location of curvature in the spacetime. Therefore, calculating the self-force provides a direct way to view the structure of curvature of a spacetime.

As seen from the self-force plots in Ch. 4 and Ch. 3, the dipole and the charge respectively are repelled from the surface layer. This repulsion matches the self-force result first calculated by Smith and Will [8] who found the self-force to be directed away from the black hole which it is held outside of. Both these results and certainly others have found that the direction and strength of the self-force at any particular location can give information about where curvature might be strongest. In the case of this thesis, it is straightforward to see that one could identify the precise location of the curvature given the plotted information. This of course only being the case because the curvature is described by a delta function. To date there is still no intuitive or physical sense for why the self-force might act this way (i.e. point away from the curvature), but the consistency throughout many different calculations is at least comforting.

A natural extension of this work would be to consider the self-force on higher order multipoles, where I hypothesize the same effect - repulsion from the curvature - would appear. This hypothesis is based on the trend of the self-force on the charge as compared to the dipole although this has not been thoroughly explored. Another interesting avenue might be to take the inside of the shell to some semblance of Rindler space (as it can be made to look similar to a black hole) to further examine information the self-force finds about global structure. A calculation like this may help bridge the gap between the case of flat space on either side of the shell and Schwarzschild spacetime. Furthermore, this calculation again begs the question of why the self-force behaves like this, as stated above, which could certainly be explored by further complicating the spacetime until perhaps the opposite effect was seen, or something else.

APPENDIX A DERIVATION OF EQS. (3.2.7), (4.2.3), (4.2.6) AND (4.2.7)

Each of the following derivation make use of the generating function for Legendre polynomials given by

$$(1 - 2tx + t^2)^{-1/2} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x). \tag{A.0.1}$$

For each case, the derivation consists of reworking the sum to find this form, and then backtracking through each of the previous operations.

Eq. (3.2.6) uses the above form explicitly where $t = r_</r_>$ and $x = \cos\theta$. Eq. (3.2.7) is found with the following method,

$$\begin{aligned}
\Phi_R &= \frac{q}{2a}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{rr_0}{a^2}\right)^\ell P_\ell(\cos\theta) \\
&= \frac{q}{2a}\sum_{\ell=1}^{\infty}\frac{1}{\ell}t^\ell P_\ell(\cos\theta) \\
&= \frac{q}{2a}\sum_{\ell=1}^{\infty}\int \frac{d}{dt}\frac{1}{\ell}t^\ell P_\ell(\cos\theta)dt \\
&= \frac{q}{2a}\int\sum_{\ell=1}^{\infty} t^{\ell-1} P_\ell(\cos\theta)dt \\
&= \frac{q}{2a}\int\frac{1}{t}\sum_{\ell=1}^{\infty} t^{\ell} P_\ell(\cos\theta)dt \\
&= \frac{q}{2a}\int\frac{1}{t}\sum_{\ell=0}^{\infty} t^{\ell} P_\ell(\cos\theta)dt - \frac{q}{2a}\int\frac{dt}{t}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q}{2a}\int \frac{1}{t}\frac{1}{\sqrt{1+t^2-2t\cos\theta}}dt - \frac{q}{2a}\ln(t) \\
&= -\frac{q}{2a}\ln\left(1-t\cos\theta+\sqrt{1+t^2-2t\cos\theta}\right)+C \\
&= -\frac{q}{2a}\ln\left(\frac{a^2-rr_0\cos\theta+\sqrt{a^4+r^2r_0^2-2rr_0a^2\cos\theta}}{2a^2}\right),
\end{aligned} \tag{A.0.2}
$$

where the variable t was substituted for rr_0/a in the first step and reintroduced in the last. Furthermore, the integration constant is necessarily non zero in order to make the potential go to zero at $r=0$. The constant is therefore given a value of $-\ln(2)$. The rest of the sums in Ch. 3 follow either this prescription, the simple case described previously or some combination of the two.

The derivation for Eq. (4.2.2) follows a few of the same steps as above, making the t substitution and using the integral and derivative in the opposite order. The steps go as follows,

$$
\begin{aligned}
\Phi_S &= -\frac{p^r}{r_0^2}\sum_{\ell=0}^{\infty}(\ell+1)\left(\frac{r}{r_0}\right)^{\ell}P_\ell(\cos\theta) \\
&= -\frac{p^r}{r_0^2}\sum_{\ell=0}^{\infty}\frac{d}{dt}\int(\ell+1)t^\ell P_\ell(\cos\theta)dt \\
&= -\frac{p^r}{r_0^2}\int t\sum_{\ell=0}^{\infty}t^\ell P_\ell(\cos\theta)dt \\
&= \frac{1-t\cos\theta}{(1+t^2-2t\cos\theta)^{3/2}} \\
&= \frac{p^r(r\cos\theta-r_0)}{(r^2+r_0^2-2rr_0\cos\theta)^{3/2}}.
\end{aligned} \tag{A.0.3}
$$

There is no integration constant required here as the singular potential does not necessarily need to go to zero at $r=0$.

Eq. (4.2.7a) and Eq. (4.2.8a) both require slightly more care due to the associated Legendre polynomial. To start, it is necessary to derive the relation between $P_\ell^1(\cos\theta)$ and $P_\ell(\cos\theta)$. Start with the derivative of associated Legendre polynomials (specifically $m = 0$),

$$\frac{d}{d\theta}P_\ell(\cos\theta) = -\sin\theta\frac{d}{d\cos\theta}P_\ell(\cos\theta) = -(1-x^2)^{1/2}\frac{d}{dx}P_\ell(x)$$
$$-(1-x^2)^{1/2}\frac{d}{dx}P_\ell(x) \equiv P_\ell^{m+1}(\cos\theta)$$
$$\therefore \frac{d}{d\theta}P_\ell^m(\cos\theta) = -P_\ell^{m+1}(\cos\theta),$$

which is the desired result. This identity is implemented in the derivations of Eq. (4.2.7a), Eq. (4.2.7b), Eq. (4.2.8a) and Eq. (4.2.8b).

The derivation for Eq. (4.2.7a) is given by

$$\begin{aligned}
\Phi_S &= \frac{p^x}{r_0^2}\sum_{\ell=1}^{\infty}\left(\frac{r}{r_0}\right)^\ell \cos\phi P_\ell^1(\cos\theta) \\
&= -\frac{p^x}{r_0^2}\cos\phi\frac{d}{d\theta}\sum_{\ell=1}^{\infty}t^\ell P_\ell(\cos\theta) \\
&= -\frac{p^x}{r_0^2}\cos\phi\frac{t\sin\theta}{(1+t^2-2t\cos\theta)^{3/2}} \\
&= p^x\frac{r\sin\theta\cos\phi}{(r^2+r_0^2-2rr_0\cos\theta)^{3/2}}.
\end{aligned} \tag{A.0.4}$$

Now for the regular potential Eq. (4.2.8a),

$$\begin{aligned}
\Phi_R &= \frac{p^x}{2ar_0}\cos\phi\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{rr_0}{a^2}\right)^\ell P_\ell^1(\cos\theta) \\
&= -\frac{p^x}{2ar_0}\cos\phi\frac{d}{d\theta}\int\sum_{\ell=1}^{\infty}\frac{d}{dt}\frac{1}{\ell}t^\ell P_\ell(\cos\theta)dt \\
&= \frac{p^x}{2ar_0}\cos\phi\frac{t(1+s^{-2})\sin\theta}{1-t\cos\theta+s^2} \\
&= p^x\frac{(a^2+s^2)r\sin\theta\cos\phi}{2as^2(a^2-rr_0\cos\theta+s^2)}.
\end{aligned} \tag{A.0.5}$$

REFERENCES

[1] L. Barack, Classical and Quantum Gravity **26**, 213001 (2009).

[2] M. Abraham, Annalen der Physik **315**, 105.

[3] H. Lorentz, Proc. Acad. Science Amsterdam **6**, 809831 (1904).

[4] H. Poincaré, Rendiconti del Circolo matematico di Palermo **21**, 129 (1906).

[5] P. A. M. Dirac, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **167**, 148 (1938).

[6] B. S. DeWitt and R. W. Brehme, Annals of Physics **9**, 220 (1960).

[7] J. Hobbs, Annals of Physics **47**, 141 (1968).

[8] A. G. Smith and C. M. Will, Phys. Rev. D **22** (1980).

[9] C. M. DeWitt and B. S. DeWitt, Physics Physique Fizika **1**, 3 (1964).

[10] F. A. Berends and R. Gastmans, Annals of Physics **98**, 225 (1976).

[11] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).

[12] T. C. Quinn and R. M. Wald, Phys. Rev. D **56**, 3381 (1997).

[13] W. L. Burke, Journal of Mathematical Physics **12**, 401 (1971).

[14] C. R. Galley and B. L. Hu, Phys. Rev. D **72**, 084023 (2005).

[15] A. I. Harte, Classical and Quantum Gravity **25**, 235020 (2008), 0807.1150.

[16] R. M. Wald, ArXiv e-prints (2009), 0907.0412.

[17] E. Poisson, A. Pound, and I. Vega, Living Reviews in Relativity **14**, 7 (2011).

[18] A. Pound, Phys. Rev. D **81**, 024023 (2010).

[19] S. E. Gralla and R. M. Wald, Classical and Quantum Gravity **28**, 159501 (2011).

[20] P. Zimmerman, Self-force on a point particle falling into a Reissner-Nordstrom black hole, Master's thesis, The University of Guelph, 2011.

[21] L. Barack and A. Ori, Phys. Rev. D **67**, 024029 (2003).

[22] Y. Mino, H. Nakano, and M. Sasaki, Progress of Theoretical Physics **108**, 1039 (2002).

[23] P. Zimmerman and E. Poisson, Phys. Rev. D **90**, 084030 (2014).

[24] J. Kuchar, E. Poisson, and I. Vega, Classical and Quantum Gravity **30**, 235033 (2013).

[25] L. Barack and N. Sago, Phys. Rev. D **75**, 064021 (2007).

[26] S. Detweiler and B. F. Whiting, Phys. Rev. D **67**, 024025 (2003).

[27] T. D. Drivas and S. E. Gralla, Classical and Quantum Gravity **28**, 145025 (2011).

[28] S. Isoyama and E. Poisson, Classical and Quantum Gravity **29**, 155012 (2012).

[29] W. Unruh, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **348**, 447 (1976).

[30] L. M. Burko, Y. T. Liu, and Y. Soen, Phys. Rev. D **63**, 024015 (2000).

[31] E. Messaritaki, Phys. Rev. D **75**, 104011 (2007).

[32] LIGO Scientific Collaboration and Virgo Collaboration, B. P. e. a. Abbott, Phys. Rev. Lett. **119**, 161101 (2017).

[33] E. Poisson, The gravitational self-force, in *General relativity and gravitation. Proceedings, 17th International Conference, GR17, Dublin, Ireland, July 18-23, 2004*, pp. 119–141, 2004, gr-qc/0410127.

[34] Lisa: Laser interferometer space antenna, `https://lisa.nasa.gov/`, Accessed: 2018-03-20.

[35] A. I. Harte, E. E. Flanagan, and P. Taylor, Phys. Rev. D **93**, 124054 (2016).

[36] J. Moxon and E. Flanagan, Radiation-reaction force on a small charged body to second order, in *APS April Meeting Abstracts*, 2015.

[37] M. van de Meent, ArXiv e-prints (2017), 1711.09607.

[38] W. Israel, Nuovo cimento **44B**, 1 (1967).

[39] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, 2004).

[40] K. Davidson and E. Poisson, ArXiv e-prints (2018), 1803.09817.