

Game Theoretic Scenarios of Cybersecurity
Investment Models in a Sellers-Buyers Market

by

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ABSTRACT

GAME THEORETIC SCENARIOS OF CYBERSECURITY INVESTMENT MODELS IN A SELLERS-BUYERS MARKET

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We develop three economic models that look at sellers in a marketplace. Formulated within this paper is a Nash game, a generalized Nash game and a cooperative game. The sellers compete with one another to sell a higher volume of product to the buyers in the marketplace by investing in cybersecurity. The buyers are influenced by the average cybersecurity level of all sellers. In order to solve the games we calculate the Karush-Kuhn-Tucker (KKT) conditions for each game given that each seller is trying to maximize its own expected utility function. We find that both the Nash and generalized Nash games show similar characteristics when varying parameters of the games. It is also shown that when sellers agree to cooperate in these games their expected utilities are better off than when sellers compete.

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Chapter 1

Introduction

In this thesis we will study a cybersecurity investment game in a marketplace containing sellers and buyers of a single product. Nagurney and Nagurney [27] developed a model where sellers compete with one another in order to maximise each firm's expected profit by investing in cybersecurity. In this model the sellers are unique from one another (size, costs, etc.) and are trying to sell a single product to a marketplace of buyers each with their own preferences. We model this market competition problem using three related frameworks in game theory: a Nash game formulation of the firms' competition, a generalized Nash game and a cooperative game, where sellers agree to cooperate with one another. The method by which we are able to assert existence of Nash, generalized Nash and cooperative equilibria of the three games stated is that of using each games' associated Karush-Kuhn-Tucker (KKT) systems of conditions. We evaluate the three games using a Matlab solver in order to find any equilibrium solutions when implemented in the marketplace under several scenarios. We then compare these equilibria in order to see which game (if any) would produce higher expected profits with a higher average security level using diverse sensitivity analyses performed on

parameters of sellers and buyers.

1.1 Literature Review

According to Internet Live Stats [17], there are over 3.8 billion internet users around the world, where the first billion was reached in 2005. With the vast increase in users around the world as well as improving technology, the internet is more accessible now than ever before. Associated with the growth of internet usage is an increase in cybercrime. A recent example of a cyber attack started on May 12th, 2017, was a ransomware known as WannaCry. WannaCry spread around the world hitting thousands of targets, including public utilities and large corporations. In the end, WannaCry only netted 52 bitcoins (around \$130,000) but it also temporarily crippled National Health Service hospitals and facilities in the United Kingdom [33]. The Ponemon Institute [34] determined that cyber crime has increased 23 percent more than last year and is costing organizations, on average, \$11.7 million US. Another example occurred to a credit reporting company, Equifax Inc. According to Dobrygowski and Bohmayr [5] the privacy of 143 million customers was violated, which opened the doors for potential cascading breaches to occur.

Due to the rapid evolution of technology and the internet, information is a valuable asset that needs to be protected. This has generated recognition of the importance of cybersecurity investments and the methods of modelling these scenarios. Gordon and Loeb [11] provide a framework for considering information security as a response to competitor analysis systems, as well as a competitor analysis system defense plan. Although in our model the firms are trying to protect their information from hackers instead of their competitors, they may use a security investment strategy based on what they think their competitors will do.

The authors in [12] developed an economic model which determines the optimal amount to invest to protect given information using a security breach probability function. This model is based on a single decision-maker. Hausken [15] furthers the work on the security breach probability function work by creating four different breach functions based on the convexity and concavity of the model, and observes how each function type affects the model. Huang et al. [16] also create a model of information security investment based on the risks and probability of being attacked.

Nagurney and Li [26] construct a supply chain network model with information asymmetry in product quality. In this model consumers are only aware of the average quality of the product being sold, whereas the competing firms are aware of the quality of the product they produce. In our model we use these ideas such that sellers are aware of their own network security level, while the buyers are only aware of the average security level of the market. Another interesting cybersecurity game is proposed in Nagurney [25] in which she incorporates the average time that the stolen product/information takes to be delivered to the black market, with the demand price function of this product being a decreasing function with respect to time. Since cybercrime is a global problem, where the attackers can be located anywhere around the world, the formulation of the model in [25] can be used to explore potential policy interventions in order to reduce cybercrime. Shetty et al. [38] model a user's probability to incur damage from a cyberattack depending on both the user's security and the network security. It is first observed in [38] that when there is information asymmetry for the cyber-insurers there are moral hazards resulting in no existence of equilibrium. This is reflected in our model by the fact that sellers have perfect information on network security, while buyers only know the average security level of the marketplace. Rothschild and Stiglitz [36] create an economic model to analyse competitive markets in which characteristics of the commodities exchanged are not fully known to at least one of the parties to the transaction.

The paper focuses mainly on competitive insurance markets, but concludes that their findings could be applied to other markets such as competitive financial and labour markets. An optimization model can be found in [22] where the objective is to maximise total net benefit of players with cost and benefit functions being taken into account. It was found that when users cooperate, the largest amount of total net revenue is achieved. Another take on an economic model can be found in [2] where an economic model for non-profit firms is formulated by deriving the mathematical relationships between public value, services delivered, revenues, costs, service prices, resource prices and subsidies. The paper talks about cost and revenue functions and how they are used in standard economic models and how they are interpreted in optimization for non-profit organizations.

A good introduction to economic games and how they developed into where they are today can be found in the introductory book of Schmidt [37]. In the book [19], Kreps shows how game theory can be used to help understand and predict economic phenomena. By using methods formulated in the book it is clear that game theory is a viable resource for analysing non-cooperative and cooperative economic situations.

Since cybersecurity and cybercrime are a relatively new area of research, information on the subject is sparse. A common approach to this economic model is to use game theory since in a market firms are run by decision-makers who are competing with one another in order to maximise their expected profits. Economic models generally consist of many variables whose effects on a given model are often discussed in terms of a trade-off relation: changes in one variable lead to a positive outcome in a decision-maker's output, and a negative outcome in another's. Newhouse [32] studies an economic model based on a decision maker in a hospital setting trying to maximise their utility with a trade-off between quality and quantity of healthcare for its consumers. It is proven that strategic decision makers are boundedly rational, that power wins the battle over choice, and that chance matters (see

[7]). That is, a decision maker in a profit oriented firm will choose their strategy based on the best calculated payoff. The fact that chance matters, as reflected in [7], is incorporated in our model through the assumption that the sellers decide how much security to invest in based on a damage parameter and the probability of being attacked. Lu et al. [21] examine the attackers behaviours and model the strategic interaction between firms attempting to protect their information, and hackers trying to damage the information illegally. Cavusoglu et al. [4] compare game-theoretic and decision-theoretic approaches to modelling IT security investment levels and their effects on the seller and the hacker. This paper shows that cyber-attacks do not occur randomly from firm to firm but are usually directed at specific sellers for various reasons (money, weak network security, etc.). Yet another economic model which uses a game theory approach is [28] who model fossil fuel energy planning for government and private sectors. The model considers economic production rates with respect to government and private sector's prices, total demand, and government subsidies. Nasab et al. [28] solve for both non-cooperative and cooperative equilibrium, finding a similar outcome to [22] namely that when players cooperate, total profit is greater than in a non-cooperative scenario.

The idea of game theory was brought into consideration by Von Neumann and Morgenstern in [41]. In [41] they discuss the fundamental questions of economic theory in a different framework from past history. This is the start of using game theory or “games of strategy” as an application to solve economic problems consisting of players who are trying to maximise their expected profits. Solutions for an N -person game were developed by Nash [29] who proved the existence of the game's equilibrium points under assumptions for each player's payoff function and constraints. In [30] Nash shows that in a non-cooperative game where each player has their own strategy and restrictions, an equilibrium point exists given the payoff functions are bilinear. Another good introduction into Nash games is written by

Thomas [39]; the book introduces zero-sum games and then proceeds to extend the model to non-zero-sum games with further applications. Basar and Olsder [1] is often cited in the literature as a rich introduction to several game classes. They start with two-person nonzero-sum finite games, a Nash equilibrium solution is introduced and its properties and features are thoroughly investigated. Later, they extend the analysis to that of N-person games in normal form, presenting a proof for a theorem which states that every N-person finite game in normal form admits Nash equilibria in mixed strategies. This is useful for our results here, as we introduce a two-person game and extend the model to that of an N-person game.

An introduction to generalized Nash equilibrium games (GNE) is given by Rosen[35] in which he develops an environmental game where the strategy of each player not only depends on their own constraints, but may also depend on the strategy of every other player. Generalized Nash games are similar to Nash games in the concept that players are still in competition with one another, except, in a generalized game, all players may share a common binding constraint which affects differently the strategies used when compared to a Nash game.¹ Generalized Nash games have received an increasing amount of attention in the last two decades, though they were introduced much earlier, due to their ability to model complex competition situations under budget or resource constraints, especially in competitive markets. Harker [13] builds on Rosen’s proposed model to obtain ‘sharper’ results by creating a more easily understood and verifiable existence and uniqueness theorem for generalized equilibrium points. He also proposes efficient algorithms for finding equilibria of such problems, and sensitivity/stability results. The results found in the paper are used to simplify and analyze GNE games with shared constraints. Facchinei et al. [8] contains a good summary of combining the previous literature in generalized Nash games in order

¹These form a subclass of generalized Nash games, called “with shared constraints”. In general, not all players have shared constraints involving other players’ decision variables.

to develop an easier solution algorithm that requires all constraints to be smooth, convex functions.

In the two games described above (Nash and generalized Nash), all players compete with one another in order to maximise each individual's expected utility function. The last game we introduce in this paper is one where all players cooperate. In contrast, the idea behind a cooperative game is that all individual players are supposed to be able to jointly agree on a select joint plan of action in order to arrive at a solution, where the agreement is reasonable in the game. The idea of formulating cooperative games arose when the competitive model was formed and Nash [31] started by formulating a two-person cooperative game. Nash begins by formulating a way to model a two-person cooperative game by setting up the game into four steps: both players choose their mixed strategy (competitive), players inform one another of their threats, each player decides on their demand on the utility function, and lastly the payoffs are determined. Harsanyi [14] also presents his idea on a bargaining model while extending the idea to N -players, where each player in the N -player game can again be reduced first into two-player competitive sub games. These sub games are defined by each player interacting individually with the other $N - 1$ players to bargain on a cooperative solution. Again, Thomas [39] writes a very good introduction to the ideas of a cooperative game and the bargaining or negotiation set. Thomas [39] states that if players cooperate, then the payoff region will be larger than that of the corresponding non-cooperative game. In [23] McCain dives deeper into the understanding of when coalitions would form between players and states that when all players cooperate we have a grand coalition.

The closest inspirations for the methodology employed in this thesis are presented in the papers [3] and [40]. Zaccour et al. show that starting with a welfare function, which each player is trying to maximise, they can set up Nash, generalized Nash, and cooperative game scenarios. They analyze the resulting equilibria and payoff values. The authors refer to a

GNE game as a “joint implementation game” and an “umbrella game” respectively. These papers help us with the foundation framework for our games and we follow their approach when setting up our economic games. In order to solve the welfare games both [3] and [40] write the Lagrangian and Karush-Kuhn-Tucker conditions of each individual player and solve these, with the assumption that the solutions to the KKT systems lie in the interior of the strategy set of the game. Once all games are solved, they are then able to compare the games solutions to determine if and in what sense one game outperforms the others.

The method we use to solve our games in the thesis is also based on solving the systems of equations and inequalities given by writing out the Karush-Kuhn-Tucker (KKT) conditions. In the paper written by Karush [18], he acknowledges the necessary and sufficient conditions for determining a relative minimum of a function where all constraints are equal to zero. The paper proposes a method to solve the system when the constraints are inequalities. The idea of the KKT conditions started with Kuhn and Tucker in [20], where they formulated problems in linear programming for maximizing a linear function. It was shown that these problems can be transformed into an equivalent saddle value (minimax) problem for which they form the Lagrangian in order to solve the system. Since their introductory work, the KKT systems are widely known and used in optimization, game theory, multiobjective decision-making, etc. Wu [43] states conditions that are required in order to solve the KKT systems for optimization problems with inequality and equality constraints. It is also stated that this method can be applied to a zero-sum two person game in order to provide a solution. In [9] Facchinei et al. go through the GNE model and the conditions that must be satisfied in order for the KKT conditions attached to a generalized Nash game to be formulated. Then, in [9] and [6] they show how to formulate the KKT conditions corresponding to a GNE game. We followed their model closely when writing the KKT systems in chapter 2. For further information on the analysis and formation of the KKT conditions refer to Wu

[42]. We detail our presentation in the next subsection.

1.2 Mathematical background

In this section we will state definitions and theorems already existing in the literature that will be used in this thesis. The space we work in is the Euclidean space \mathbb{R}^n , unless stated otherwise. The following five definitions are basic definitions in regards to sets and set notation.

Definition 1 (Closure) *The closure \overline{S} of a set $S \in \mathbb{R}^n$ is defined to be the smallest closed set containing S .*

Definition 2 (Bounded) *A set $S \in \mathbb{R}^n$ is called bounded from above if there is a real number k such that $\|s\| \leq k, \forall s \in S$. The number k is called an upper bound of S . The terms bounded from below and lower bound are similarly defined.*

Therefore, a set S is bounded if it has both upper and lower bounds.

Definition 3 (Compactness) *A subset of \mathbb{R}^n is compact if the set is closed and bounded.*

Definition 4 (Convexity) *A set $S \in \mathbb{R}^n$ is called convex if for any two points $x, y \in S$, the line segment*

$$(1 - \lambda)x + \lambda y, \quad \lambda \in [0, 1],$$

lies in S .

Definition 5 (Interior) A point x is an interior point of a set $S \subset \mathbb{R}^n$ if there is an open n -dimensional ball $B \subset \mathbb{R}^n$ centred at x such that $B \subset S$. The interior of a set S is the set of all its interior points.

We now recall some basic definitions of function properties which are typical assumptions for mappings in optimization and game theory.

Definition 6 (Bounded) A function $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded if there exists a real number M such that:

$$\|f(x)\| \leq M \quad \forall x \in X.$$

Definition 7 (Convexity) A function $f : X \subseteq \mathbb{R}^n \rightarrow Y \subseteq \mathbb{R}$ is convex iff $\forall x, y \in X$, $\forall \lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Strict convexity is obtained when inequality is strict. A function that is concave satisfies:

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y).$$

Strict concavity is obtained when inequality is strict.

Theorem 1 If f_1, \dots, f_n are convex functions and $w_1, \dots, w_n \geq 0$, then,

$$f(x) = w_1 f_1(x) + \dots + w_n f_n(x)$$

is convex also. Similarly for concave functions.

Let us now consider N players, each player $i \in \{1, \dots, N\}$ having a finite-dimensional strategy set denoted by $S_i \subset \mathbb{R}^{n_i}$, which we consider to be closed and convex. Let now $S \subset \mathbb{R}^{n_1+n_2+\dots+n_N}$ be the overall set of strategies of the game, i.e., $S := S_1 \times S_2 \times \dots \times S_N$ and $\theta_i : S \rightarrow \mathbb{R}$ be player i 's payoff function.

Let us denote by $x_i \in S_i$ a strategy vector of player i , by $x \in S$ a strategy vector of all players, and $x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ a strategy vector of all players except player i .

Definition 8 (Nash Game) *A Nash Game is where each player $i \in \{1, \dots, N\}$ has to solve an optimization problem as follows:*

$$\begin{aligned} \max_{x_i} \theta_i(x_i, x_{-i}) \\ \text{s.t. } x_i \in S_i, \end{aligned}$$

thus the game can be thought of as the collection of each individual players' optimization problems.

Definition 9 (Nash Equilibrium) *A strategy vector $x^* \in S$ is a Nash equilibrium if no unilateral deviation in strategy by any single player is profitable for that player, that is:*

$$\forall i, x_i \in S_i : \theta_i(x_i^*, x_{-i}^*) \geq \theta_i(x_i, x_{-i}^*).$$

Since we see above that a Nash game is in fact a collection of optimization problems (one problem for each player), let us introduce now, in general, a typical nonlinear optimization problem and its associated KKT system. A nonlinear optimization problem (NLP) is a

problem of the form:

$$\begin{aligned} \min \text{ or } \max f(\underline{x}) \mathbb{R}^n \rightarrow \mathbb{R}, \\ \text{s. t. } \begin{cases} g_1(\underline{x}) \leq b_1, \\ g_2(\underline{x}) \leq b_2, \\ \vdots \\ g_m(\underline{x}) \leq b_m, \end{cases} \end{aligned} \tag{1.1}$$

where $g_1, \dots, g_m : \mathbb{R}^n \rightarrow \mathbb{R}$ and $b_1, \dots, b_m \in \mathbb{R}$, $\forall \underline{x} \in \mathbb{R}^n$.

To be able to assert existence of solutions to an optimization problem, as well as to be able to write the KKT conditions, we need to make the following assumptions:

Assumption 1 *Function $f(x)$ is convex (for a min NLP) or concave (for a max NLP) and of class \mathcal{C}^2 , and functions $g_1(x), \dots, g_m(x)$ are convex for every x and of class \mathcal{C}^2 . We also assume a so-called constraint qualification condition, for instance that the matrix $(\nabla g_1; \dots; \nabla g_m)$ is of full rank.*

Then we can give necessary and sufficient conditions for solving the general Lagrangian problem associated to the NLP above as follows:

Theorem 2 *Assume $x^* = (x_1^*, \dots, x_n^*)$ is a solution to our nonlinear problem (1.1)*

Then $\exists \lambda^ = (\lambda_1^*, \dots, \lambda_m^*)$ s.t. (x^*, λ^*) solves the system*

$$\begin{cases} \frac{\partial f}{\partial x_k}(x^*) - \sum_{j=1}^m \lambda_j^* \frac{\partial g_j}{\partial x_k}(x^*) = 0, & \forall k = \overline{1, n}, \\ \lambda_j^* (g_j(x^*) - b_j) = 0, & \forall j = \overline{1, m}, \\ \lambda_j^* \geq 0, & \forall j = \overline{1, m}. \end{cases} \tag{1.2}$$

Under the convexity assumptions, a point (x^, λ^*) satisfying (1.2) is a global optimal point*

of the NLP.

We show next that similar formulations hold for a generalized Nash game (see [6]). We keep the context of N players and their strategy sets and payoff functions of Definition 8.

Definition 10 (Generalized Nash Game) *A generalized Nash game (GNE) is an N -player game where player i ($i=1, \dots, N$) controls $x_i \in S_i \subset \mathbb{R}^{n_i}$ and solves the optimization problem*

$$\begin{aligned} & \max_{x_i} \theta_i(x_i, x_{-i}^*), \\ & \text{s.t. } x_i \in S_i \text{ and } \left\{ g^j(x_i, x_{-i}^*) \leq 0, j \in \{1, \dots, m_i\} \right\}, \end{aligned}$$

with $\theta_i : S \rightarrow \mathbb{R}$ being the player's payoff function, $g^j : S \rightarrow \mathbb{R}^{m_i}$ being the inequality constraints, and m_i the number of constraints of player $i \in \{1, \dots, N\}$ dependent on other players' optimal strategies.

Note that the strategy set of player i is a function of the vector x_{-i} , via the imposed constraints g^1, \dots, g^{m_i} . Thus, to better reflect the complexity of this game, we denote the strategy set of each player i to be:

$$K(x_{-i}) := \{x_i \in S_i \mid g^j(x_i, x_{-i}) \leq 0, j \in \{1, \dots, m_i\}\}.$$

Definition 11 *A feasible point x^* is an equilibrium point of the generalized Nash game if, for all players $i = \{1, \dots, N\}$, we have*

$$\theta_i(x_i^*, x_{-i}^*) \geq \theta_i(x_i, x_{-i}^*) \quad \forall x_i \in K(x_{-i}^*).$$

The following result is then known (see [6]):

Theorem 3 (KKT conditions GNE) *Let x^* be a solution of the generalized Nash game. Assuming a standard constraint qualification holds², it is well known that the following KKT conditions will be satisfied at x_i^* , for every player $i = \{1, \dots, N\}$:*

$$\nabla_{x_i} \theta_i(x_i, x_{-i}^*) + \sum_{j=1}^{m_i} \lambda_j^i \nabla_{x_i} g^j(x_i, x_{-i}^*) = 0,$$

$$\lambda_j^i \geq 0, \quad g^j(x_i, x_{-i}^*) \leq 0, \quad \lambda_j^i g^j(x_i, x_{-i}^*) = 0 \quad \forall j = \{1, \dots, m_i\},$$

where λ^i is the vector of Lagrange multipliers for player i .

If the players problems are convex, we have that if a point x^* , together with a suitable vector of multipliers $\lambda := (\lambda^1, \lambda^2, \dots, \lambda^N)$, satisfy the KKT conditions for every player i , then x^* is a solution of the GNE.

Remark 1 *In general we expect to have more than one generalized Nash equilibrium point for a GNE. This is usually because the strategy sets of all players are dependent on the others' choices.*

Last but not least, we formulate a cooperative scenario as a single maximization NLP, where players $\{1, \dots, N\}$ form a coalition. To ease the presentation, we assume for this stage that the strategy sets S_i of a player i consist of box constraints, i.e., they are of the form:

$$S_i := \{x_i \in \mathbb{R}^{n_i} \mid \underline{M}_i \leq x_i \leq \overline{M}_i\},$$

where \underline{M}_i is the lower bound of x_i and \overline{M}_i is the upper bound of x_i .

²One of the main used constraint qualifications is the Mangasarian-Fromowitz condition, which requires that, for $i \in \{1, \dots, N\}$, the following holds:

$$\nabla_{x_i} g^1(x) \lambda_1^i + \dots + \nabla_{x_i} g^{m_i}(x) \lambda_{m_i}^i = 0, \quad 0 \leq \lambda^i, \quad g^j(x) \leq 0, \quad \forall j \implies \lambda^i = 0$$

Definition 12 (Cooperative game) *The players then can agree to solve the optimization problem:*

$$\begin{aligned} & \max \sum_{i=1}^N \theta_i(x), \\ \text{s.t. } & \sum_{i=1}^N \underline{M}_i \leq \sum_{i=1}^N x_i \leq \sum_{i=1}^N \overline{M}_i. \end{aligned}$$

To write the KKT conditions for this problem we use the above formulation (1.2).

Chapter 2

The Model

In this chapter we give a brief overview of the marketplace cybersecurity investment game defined in [27]. After a Nash game has been formulated, we then proceed to formulate a Generalized Nash game and a cooperative game starting from a simple 2-seller, 1-buyer market. We then write all three games in a general market of m -sellers and n -buyers. While doing so we also calculate the KKT systems of each game and we give theoretical conditions for a comparison between various equilibrium points of the three models.

2.1 Cybersecurity Game Overview

Following [27] we consider the situation where we have m competitive firms who are selling identical products to n buyers shown in Fig 2.1. Each seller $i \forall i \in \{1, 2, \dots, m\}$ sells its product to buyer $j, \forall j \in \{1, 2, \dots, n\}$ via an electronic transaction which can occur online or in the seller's firm itself.

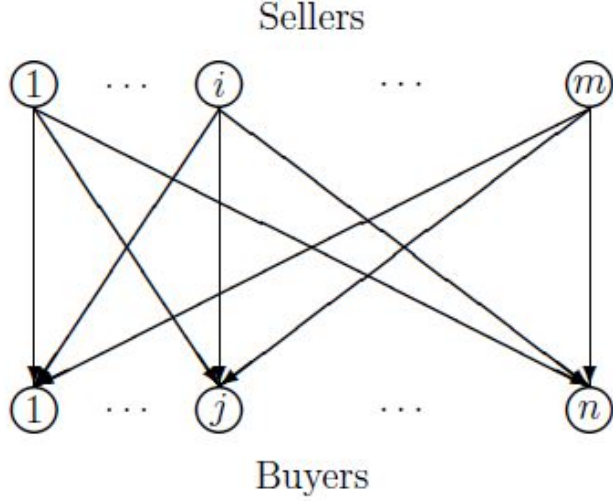


Figure 2.1: Network structure of the game theory model

For each seller denoted by i we let Q_{ij} denote the non-negative volume of product sold from seller i to buyer j . All product transactions Q_{ij} are combined into the vector $Q \in R_+^{mn}$. We denote the network security level for each seller i by s_i . Again we combine all the network security levels of all sellers in the game into the vector $s \in R_+^m$. Both vectors are assumed to be column vectors unless stated otherwise.

A seller's firm is assigned a network security level between 0 and 1, with 0 meaning no security and 1 meaning perfect security. Therefore,

$$0 \leq s_i \leq 1, \quad i = 1, \dots, m. \quad (2.1)$$

The volume of product sold by seller i to all buyers of the game is the strategic variable labelled as: $Q_i = (Q_{i1}, \dots, Q_{in}), \forall i = 1, \dots, m$. The average network security of the market game is denoted by \bar{s} , where:

$$\bar{s} = \frac{1}{m} \sum_{i=1}^m s_i. \quad (2.2)$$

The less network security that a firm has, the higher chance that a successful cyber attack will occur on that seller. Therefore, we define the probability p_i of a successful cyber attack on a seller i as

$$p_i = 1 - s_i, \quad i = 1, \dots, m. \quad (2.3)$$

For each seller i to be able to attain a network security level of s_i they have to invest into cybersecurity; we denote such costs as $h_i(Q_i, s_i)$. Each seller i will have distinct investment cost functions based on the size of the firm, where the firms size is shown by the volume of product sold Q_i . We use a function such that if $h_i(Q_i, 0) = 0$ then seller i invests nothing into network security and is an entirely insecure firm. If seller i has complete security against cyber attacks, then we want $h_i(Q_i, 1) = \infty$ so that complete security is impossible. It should also be structured in a way that a larger firm has a steeper investment cost function, and prices increase exponentially as security increases to 1. The cybersecurity investment function used in this thesis is found in [27] and is of the form:

$$h_i(Q_i, s_i) = \left(\sum_{j=1}^n Q_{ij} \right) \left(\frac{1}{\sqrt{1-s_i}} - 1 \right), \quad i = 1, \dots, m. \quad (2.4)$$

The buyer's demand for the product being sold is given by:

$$d_j = \sum_{i=1}^m Q_{ij}, \quad j = 1, \dots, n, \quad (2.5)$$

where

$$Q_{ij} \geq 0, \quad i = 1, \dots, m; \quad j = 1, \dots, n, \quad (2.6)$$

and

$$\sum_{j=1}^n Q_{ij} \leq M_i, \quad i = 1, \dots, m, \quad (2.7)$$

where M_i is the maximum volume of product seller i is able to produce. We combine the demand values for each buyer in a vector $d \in R_+^n$.

The price that each buyer is willing to pay for the given product is given by the price demand curve. In general, the price a buyer is willing to pay depends on the buyer's own demand for that product and the demand of all of the other buyers in the marketplace for said product. Also, in this cybersecurity marketplace game, buyers adjust their price demand by the average security level in the marketplace. The demand price function for buyer j is given by:

$$\rho_j = \rho_j(d, \bar{s}), \quad j = 1, \dots, n, \quad (2.8)$$

Each seller i for $i = 1, \dots, m$ is faced with a fixed cost c_i associated with the processing and handling of the product and a transaction cost function $c_{ij}(Q_{ij})$; $j = 1, \dots, n$ which can include the cost of transporting/shipping the product to the buyer, cost of using network services, etc., and it varies as the volume of product sold changes. The total cost of seller i is given by:

$$C_i = c_i \sum_{j=1}^n Q_{ij} + \sum_{j=1}^n c_{ij}(Q_{ij}), \quad i = 1, \dots, m. \quad (2.9)$$

The revenue for each seller i in the absence of a cyber attack is then given by the volume of product sold to each buyer at the price each buyer is willing to pay for it. Thus, we have the revenue:

$$R_i = \sum_{j=1}^n \rho_j(Q, s) Q_{ij}, \quad i = 1, \dots, m. \quad (2.10)$$

Then, the profit for each seller i denoted f_i can be calculated by taking the revenue and subtracting the cost:

$$f_i(Q, s) = R_i - C_i, \quad i = 1, \dots, m. \quad (2.11)$$

If a cyber attack occurs on a seller i , that seller has a financial loss. The expected cost of a cyber attack is represented by:

$$D_i p_i, \quad D_i > 0, \quad i = 1, \dots, m. \quad (2.12)$$

Using the above expressions (2.3), (2.10), and (2.11), we set up the expected utility or expected profit function $E(U_i)$ of seller i ; $i = 1, \dots, m$ as:

$$E(U_i) = s_i f_i(Q, s) + (1 - s_i)(f_i(Q, s) - D_i) - h_i(Q_i, s_i). \quad (2.13)$$

The expected utility functions are then grouped into an m -dimensional vector $E(U)$. We

let K^i denote the feasible set in regards to seller i , where

$$K^i \equiv \{(Q_i, s_i) \mid Q_i \geq 0, \text{ and } 0 \leq s_i \leq 1\}$$

and define the Cartesian product of these sets as

$$K \equiv \prod_{i=1}^m K^i.$$

2.2 2-Seller, 1-Buyer Game

Let us look at the particular case of a marketplace network which looks like Fig 2.2 below:

We now have that sellers index $i \in \{1, 2\}$ while the buyer index is $j = 1$. Both sellers sell 1

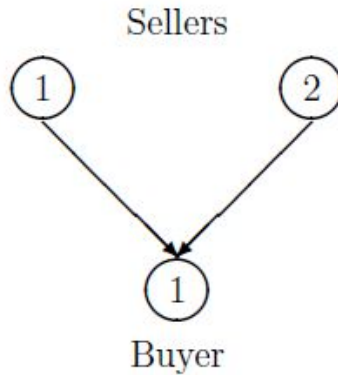


Figure 2.2: Network structure of a 2-seller 1-buyer market

type of product to the buyer. Thus:

$$s_1 \in [0, 1], s_2 \in [0, 1], \bar{s} = \frac{s_1 + s_2}{2},$$

$$Q_{11}, Q_{21} \in R_+,$$

$$p_1 = 1 - s_1, p_2 = 1 - s_2,$$

$$h_1(Q_1, s_1) = Q_{11} \left(\frac{1}{\sqrt{1-s_1}} - 1 \right),$$

$$h_2(Q_2, s_2) = Q_{21} \left(\frac{1}{\sqrt{1-s_2}} - 1 \right).$$

Demand for product by buyer $j = 1$ is d_1 and satisfies:

$$d_1 = \sum_{i=1}^2 Q_{i1} = Q_{11} + Q_{21}.$$

For $j = 1$, we have 1 demand price function:

$$\rho_1 = \rho_1(d, \bar{s}) = \rho_1(Q_{11}, s_1, Q_{21}, s_2),$$

since d depends on (Q_{11}, Q_{21}) and \bar{s} depends on s_1, s_2 .

Cost functions:

a) transaction costs:

$$c_{11}(Q_{11}), c_{21}(Q_{21}).$$

b) flat fixed cost:

$$c_1 \text{ and } c_2 \text{ which gives us the following total cost:}$$

$$C_1 = c_1 Q_{11} + c_{11}(Q_{11}) \text{ and } C_2 = c_2 Q_{21} + c_{21}(Q_{21}).$$

The Profit of the sellers is given by:

$$f_1(Q_1, s_1, Q_2, s_2) = R_1 - C_1,$$

$$f_2(Q_1, s_1, Q_2, s_2) = R_2 - C_2.$$

Each seller has an average financial damage $D_i > 0$.

Therefore, we have the following two expected utility functions:

$$E(U_1) = s_1 f_1(Q, s) + (1 - s_1)(f_1(Q, s) - D_1) - Q_{11} \left(\frac{1}{\sqrt{1 - s_1}} - 1 \right),$$

$$E(U_2) = s_2 f_2(Q, s) + (1 - s_2)(f_2(Q, s) - D_2) - Q_{21} \left(\frac{1}{\sqrt{1 - s_2}} - 1 \right).$$

Let us now describe three competitive frameworks for the 2-seller 1-buyer model. The first is that of a noncooperative competition, i.e. a Nash game, between the sellers.

1) Nash game:

P1:

$$\max_{(Q_1, s_1)} E(U_1) = s_1 f_1(Q, s) + (1 - s_1)(f_1(Q, s) - D_1) - Q_{11} \left(\frac{1}{\sqrt{1 - s_1}} - 1 \right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq Q_{11} \leq M_1, \\ 0 \leq s_1 \leq 1. \end{cases}$$

P2:

$$\max_{(Q_2, s_2)} E(U_2) = s_2 f_2(Q, s) + (1 - s_2)(f_2(Q, s) - D_2) - Q_{21} \left(\frac{1}{\sqrt{1 - s_2}} - 1 \right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq Q_{21} \leq M_2, \\ 0 \leq s_2 \leq 1. \end{cases}$$

The second framework is that of a competitive setting where all sellers draw from a

common (previously agreed upon) pool of resources. That is to say, they agreed to contribute to a common budget in order to be able to spend this jointly on security against cyber attacks. This is called a generalized Nash Game with a shared (common) constraint, described by:

$$g(Q, s) = h_1(Q_1, s_1) + h_2(Q_2, s_2) - B \leq 0.$$

Where B is the budget to be spent on cybersecurity investments agreed upon by the sellers.

The game now can be formulated as:

2) Generalized Nash Game (competitive):

P1:

$$\begin{aligned} \max_{(Q_1, s_1)} E(U_1) &= s_1 f_1(Q, s) + (1 - s_1)(f_1(Q, s) - D_1) - Q_{11} \left(\frac{1}{\sqrt{1 - s_1}} - 1 \right), \\ \text{s.t.} \quad &\begin{cases} 0 \leq Q_{11} \leq M_1, \\ 0 \leq s_1 \leq 1, \\ Q_{11} \left(\frac{1}{\sqrt{1 - s_1}} - 1 \right) + Q_{21} \left(\frac{1}{\sqrt{1 - s_2}} - 1 \right) - B \leq 0. \end{cases} \end{aligned}$$

P2:

$$\begin{aligned} \max_{(Q_2, s_2)} E(U_2) &= s_2 f_2(Q, s) + (1 - s_2)(f_2(Q, s) - D_2) - Q_{21} \left(\frac{1}{\sqrt{1 - s_2}} - 1 \right), \\ \text{s.t.} \quad &\begin{cases} 0 \leq Q_{21} \leq M_2, \\ 0 \leq s_2 \leq 1, \\ Q_{11} \left(\frac{1}{\sqrt{1 - s_1}} - 1 \right) + Q_{21} \left(\frac{1}{\sqrt{1 - s_2}} - 1 \right) - B \leq 0. \end{cases} \end{aligned}$$

The third scenario is one of cooperation, meaning that the the 2 sellers decide to maximise

the sum of their expected utilities. From [40] in a cooperative scenario, we have:

$$E(U_C) = \sum_{i=1}^2 E(U_i) = E(U_1) + E(U_2).$$

3) Cooperative Game (non-competitive):

P1 & P2:

$$\begin{aligned} \max_{(Q_1, s_1, Q_2, s_2)} \sum_{i=1}^2 E(U_i) &= \max_{(Q_1, s_1, Q_2, s_2)} E(U_1) + E(U_2), \\ \text{s.t.} \quad &\begin{cases} 0 \leq Q_{11} + Q_{21} \leq M_1 + M_2, \\ 0 \leq s_1 + s_2 \leq 2. \end{cases} \end{aligned}$$

In order to solve the above three games, a well-established approach is to solve the KKT systems associated to each of them. From our Chapter 1, KKT systems represent sufficient and necessary conditions for finding equilibria of our games. We specifically compute so-called inner equilibria of Nash games¹. In the next subsection we write explicitly the KKT systems associated to each of the three scenarios highlighted above.

2.2.1 KKT systems for the competitive and cooperative scenarios

Starting with the Nash game, we have four binding constraints for each player, thus we will have four Lagrange multipliers λ_k^i for $i \in \{1, 2\}$ and $k \in \{1, 2, 3, 4\}$.

Then the Lagrangian of each player i is given by:

$$\mathcal{L}(Q_i, s_i, \lambda_1^i, \lambda_2^i, \lambda_3^i, \lambda_4^i) = E(U_i) + \lambda_1^i Q_{i1} + \lambda_2^i (M_i - Q_{i1}) + \lambda_3^i s_i + \lambda_4^i (1 - s_i).$$

¹By inner we mean mixed player strategies which do not contain pure strategies of any of the players.

So the KKT conditions for $i \in \{1, 2\}$ are:

$$\frac{\partial \mathcal{L}}{\partial Q_{i1}} = 0 \Leftrightarrow \frac{\partial E(U_i)}{\partial Q_{i1}} + \lambda_1^i - \lambda_2^i = 0,$$

$$\frac{\partial \mathcal{L}}{\partial s_i} = 0 \Leftrightarrow \frac{\partial E(U_i)}{\partial s_i} + \lambda_3^i - \lambda_4^i = 0,$$

$$\left\{ \begin{array}{l} \lambda_1^i \geq 0, \quad Q_{i1} \geq 0, \quad \lambda_1^i Q_{i1} = 0, \\ \lambda_2^i \geq 0, \quad (M_i - Q_{i1}) \geq 0, \quad \lambda_2^i (M_i - Q_{i1}) = 0, \\ \lambda_3^i \geq 0, \quad s_i \geq 0, \quad \lambda_3^i s_i = 0, \\ \lambda_4^i \geq 0, \quad (1 - s_i) \geq 0, \quad \lambda_4^i (1 - s_i) = 0. \end{array} \right.$$

In order to obtain interior equilibrium points of the game, we need to impose corresponding conditions on the Lagrange multipliers of each player, as follows. We assume that $\lambda_1^i = 0$ in order to get $Q_{i1} > 0$ and we assume that $\lambda_3^i = 0$ in order to get $s_i > 0$.

We also can assume that $\lambda_4^i = 0$ so that we have $s_i < 1$ since if $s_i = 1$ we get a zero in the denominator of the security investment cost function since (2.4) goes to infinity as s_i approaches 1. Last, looking at $Q_{i1} \leq M_i$ in order to obtain an interior solution we set $\lambda_2^i = 0$ giving us $M_i - Q_{i1} > 0$. Thus the KKT conditions for player $i \in \{1, 2\}$ yielding interior solutions looks as follows:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} = 0, \\ \frac{\partial E(U_i)}{\partial s_i} = 0. \end{array} \right.$$

KKT conditions for the generalized Nash Game

From [6] and Chapter 1, we have that each strategy set of a player $i \in \{1, 2\}$ can be

regarded as:

$$S_i := \{x \in \mathbb{R}^{n_i} \mid g^j(x) \leq 0, j \in \{1, \dots, m_i\}\}.$$

In our case $m_1 = m_2 = 5$, $x = ((Q_{11}, s_1), (Q_{21}, s_2))$ and the constraints are given explicitly below:

$$\text{for each } i \in \{1, 2\}, g^1(x) = -Q_{i1}, \quad (2.14)$$

$$g^2(x) = Q_{i1} - M_i, \quad (2.15)$$

$$g^3(x) = -s_i, \quad (2.16)$$

$$g^4(x) = s_i - 1, \quad (2.17)$$

$$g^5(x) = h_1(Q_1, s_1) + h_2(Q_2, s_2) - B. \quad (2.18)$$

So player i has a 5-dimensional multiplier vector $\underline{\lambda}^i \in \mathbb{R}^5$ such that the KKT conditions are:

$$\nabla_{x_i} E(U_i)(x_i, x_{-i}) + \nabla_{x_i} g(x_i, x_{-i}) \underline{\lambda}^i = 0,$$

$$0 \leq \underline{\lambda}^i \perp g(x_i, x_{-i}) \leq 0,$$

where \perp represents the cross product of the two vectors. In our case $x_i = (Q_i, s_i)$. Also, when $i = 1 \Rightarrow -i = 2$, and when $i = 2$, then $-i = 1$. Thus if $x_1 = (Q_1, s_1)$ then $x_{-1} = (Q_2, s_2)$. Putting all this together we obtain the following KKT systems:

$$\begin{pmatrix} \frac{\partial E(U_i)}{\partial Q_{i1}} \\ \frac{\partial E(U_i)}{\partial s_i} \end{pmatrix} + \begin{pmatrix} \frac{\partial g^1}{\partial Q_{i1}} & \frac{\partial g^2}{\partial Q_{i1}} & \cdots & \frac{\partial g^5}{\partial Q_{i1}} \\ \frac{\partial g^1}{\partial s_i} & \frac{\partial g^2}{\partial s_i} & \cdots & \frac{\partial g^5}{\partial s_i} \end{pmatrix} (\lambda_1^i, \lambda_2^i, \dots, \lambda_5^i)^T = 0, \quad i \in \{1, 2\},$$

$$0 \leq (\underline{\lambda}_1^i, \underline{\lambda}_2^i, \dots, \underline{\lambda}_5^i) \perp (g^1(x), \dots, g^5(x)) \leq 0.$$

These are then for $i \in \{1, 2\}$

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} + \sum_{j=1}^5 \lambda_j^i \frac{\partial g^j}{\partial Q_{i1}} = 0, \\ \frac{\partial E(U_i)}{\partial s_i} + \sum_{j=1}^5 \lambda_j^i \frac{\partial g^j}{\partial s_i} = 0, \\ \lambda_j^i \geq 0, \forall j \in \{1, \dots, 5\}, \\ g^j(Q_i, s_i) \leq 0, \forall j \in \{1, \dots, 5\}, \\ \lambda_j^i g^j(Q_i, s_i) = 0, \forall j \in \{1, \dots, 5\}. \end{array} \right.$$

From equations (2.14)-(2.18), we have:

$$\frac{\partial g^1}{\partial Q_{i1}} = -1, \frac{\partial g^2}{\partial Q_{i1}} = 1, \frac{\partial g^3}{\partial Q_{i1}} = 0, \frac{\partial g^4}{\partial Q_{i1}} = 0, \frac{\partial g^5}{\partial Q_{i1}} = \frac{\partial h_i(Q_i, s_i)}{\partial Q_{i1}},$$

and:

$$\frac{\partial g^1}{\partial s_i} = 0, \frac{\partial g^2}{\partial s_i} = 0, \frac{\partial g^3}{\partial s_i} = -1, \frac{\partial g^4}{\partial s_i} = 1, \frac{\partial g^5}{\partial s_i} = \frac{\partial h_i(Q_i, s_i)}{\partial s_i}.$$

So for each player i the KKT conditions are:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} - \lambda_1^i + \lambda_2^i + \lambda_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}} = 0, \\ \frac{\partial E(U_i)}{\partial s_i} - \lambda_3^i + \lambda_4^i + \lambda_5^i \frac{\partial(h_i(Q_i, s_1))}{\partial s_i} = 0, \\ \lambda_1^i \geq 0, \quad -Q_{i1} \leq 0, \quad \lambda_1^i(-Q_{i1}) = 0, \\ \lambda_2^i \geq 0, \quad (Q_{i1} - M_i) \leq 0, \quad \lambda_2^i(Q_{i1} - M_i) = 0, \\ \lambda_3^i \geq 0, \quad -s_i \leq 0, \quad \lambda_3^i(-s_i) = 0, \\ \lambda_4^i \geq 0, \quad (s_i - 1) \leq 0, \quad \lambda_4^i(s_i - 1) = 0, \\ \lambda_5^i \geq 0, \quad (h_1 + h_2 - B) \leq 0, \quad \lambda_5^i(h_1 + h_2 - B) = 0. \end{array} \right.$$

Again if $Q_{i1} > 0$, $s_i > 0 \Rightarrow \lambda_1^i = 0 = \lambda_3^i$. Again, as in the KKT conditions for the regular

Nash game we need $s_i < 1 \Rightarrow 0 = \underline{\lambda}_4^i$ and we further consider $Q_{i1} < M_i \Rightarrow \underline{\lambda}_2^i = 0$ and we arrive at the following KKT system:

Then KKT conditions for player i are:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} = -\underline{\lambda}_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}}, \\ \frac{\partial E(U_i)}{\partial s_i} = -\underline{\lambda}_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i}, \\ \underline{\lambda}_5^i \geq 0, \\ (h_1 + h_2 - B) \leq 0, \\ \underline{\lambda}_5^i (h_1 + h_2 - B) = 0. \end{array} \right.$$

KKT conditions for the cooperative game

Looking at the cooperative game we are trying to solve the following system:

$$\max_{(Q_1, s_1, Q_2, s_2)} \sum_{i=1}^2 E(U_i) = \max_{(Q_1, s_1, Q_2, s_2)} E(U_1) + E(U_2),$$

$$\text{s.t.} \left\{ \begin{array}{l} Q_{11} \geq 0, \\ Q_{21} \geq 0, \\ Q_{11} + Q_{21} \leq M_1 + M_2, \\ s_1 \geq 0, \\ s_2 \geq 0, \\ s_1 + s_2 \leq 2. \end{array} \right.$$

Thus, the Lagrangian $\mathcal{L}(Q_1, s_1, Q_2, s_2, \lambda_1^C, \lambda_2^C, \lambda_3^C, \lambda_4^C, \lambda_5^C, \lambda_6^C)$ for the system is as follows:

$$\mathcal{L} = \left(\sum_{i=1}^2 E(U_i) + \lambda_i^C Q_{i1} \right) + \lambda_3^C (M_1 + M_2 - Q_{11} - Q_{21}) + \lambda_4^C s_1 + \lambda_5^C s_2 + \lambda_6^C (2 - s_1 - s_2).$$

So the KKT conditions for the cooperative game is for $i \in \{1, 2\}$:

$$\frac{\partial \mathcal{L}}{\partial Q_{i1}} = 0 \Leftrightarrow \frac{\partial E(U_1)}{\partial Q_{i1}} + \frac{\partial E(U_2)}{\partial Q_{i1}} + \lambda_i^C - \lambda_3^C = 0,$$

$$\frac{\partial \mathcal{L}}{\partial s_i} = 0 \Leftrightarrow \frac{\partial E(U_1)}{\partial s_i} + \frac{\partial E(U_2)}{\partial s_i} + \lambda_{i+3}^C - \lambda_6^C = 0,$$

$$\left\{ \begin{array}{l} \lambda_i^C \geq 0, \quad Q_{i1} \geq 0, \quad \lambda_i^C Q_{i1} = 0, \\ \lambda_3^C \geq 0, \quad (M_1 + M_2 - Q_{11} - Q_{21}) \geq 0, \quad \lambda_3^C (M_1 + M_2 - Q_{11} - Q_{21}) = 0, \\ \lambda_{i+3}^C \geq 0, \quad s_i \geq 0, \quad \lambda_{i+3}^C s_i = 0, \\ \lambda_6^C \geq 0, \quad (2 - s_1 - s_2) \geq 0, \quad \lambda_6^C (2 - s_1 - s_2) = 0. \end{array} \right.$$

Similar to the previous two games, we assume $\lambda_1^C = \lambda_2^C = 0$ in order to get $Q_{11} > 0$ and $Q_{21} > 0$. We again also assume $\lambda_4^C = \lambda_5^C = 0$ in order to get $s_1 > 0$ and $s_2 > 0$. The cooperative game has different upper constraints on the Quantity Sold variables and the Security Level variables since they are combined. Again looking at $2 - s_1 - s_2 \geq 0$ we see that it is impossible for $s_i = 1$ for both sellers, given the security investment cost functions (2.4). Therefore, $2 - s_1 - s_2 > 0$ which implies $\lambda_6^C = 0$. Lastly, looking at $Q_{11} + Q_{21} \leq M_1 + M_2$ in order to obtain an interior solution we set $\lambda_3^C = 0$ giving us $M_1 + M_2 - Q_{11} - Q_{21} > 0$.

Therefore, we arrive at the following KKT solution for when both sellers cooperate:

$$\text{For } i \in \{1, 2\} \text{ we have: } \left\{ \begin{array}{l} \frac{\partial E(U_1)}{\partial Q_{i1}} + \frac{\partial E(U_2)}{\partial Q_{i1}} = 0, \\ \frac{\partial E(U_1)}{\partial s_i} + \frac{\partial E(U_2)}{\partial s_i} = 0. \end{array} \right.$$

2.2.2 Analysis of equilibria in the three scenarios

Our three KKT conditions for our games are as follows: The Nash KKT conditions for player i :

$$\begin{cases} \frac{\partial E(U_i)}{\partial Q_{i1}} = 0, \\ \frac{\partial E(U_i)}{\partial s_i} = 0. \end{cases}$$

The GNE KKT conditions for player i :

$$\begin{cases} \frac{\partial E(U_i)}{\partial Q_{i1}} = -\lambda_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}}, \\ \frac{\partial E(U_i)}{\partial s_i} = -\lambda_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i}, \\ \lambda_5^i \geq 0, (h_1 + h_2 - B) \leq 0, \lambda_5^i(h_1 + h_2 - B) = 0. \end{cases}$$

The cooperative game for the combined players $i = 1, 2$:

$$\begin{cases} \frac{\partial E(U_1)}{\partial Q_{i1}} + \frac{\partial E(U_2)}{\partial Q_{i1}} = 0, \\ \frac{\partial E(U_1)}{\partial s_i} + \frac{\partial E(U_2)}{\partial s_i} = 0. \end{cases}$$

Proposition 1 *Under the assumption that $0 < Q_{i1} < M_i, \forall i = 1, 2$ and $0 < s_i < 1, \forall i = 1, 2$, then any solution $(Q_1^*, s_1^*, Q_2^*, s_2^*)$ of the GNE game is a solution of the Nash game only if $g(Q_1^*, s_1^*, Q_2^*, s_2^*) < 0$.*

Proof:

If $g(Q_1^*, s_1^*, Q_2^*, s_2^*) < 0$ then $\underline{\lambda}_5^i = 0$. Thus the KKT conditions for the GNE game becomes:

$$\begin{cases} \frac{\partial E(U_i)}{\partial Q_{i1}} = -\underline{\lambda}_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}}, \\ \frac{\partial E(U_i)}{\partial s_i} = -\underline{\lambda}_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i}, \\ \underline{\lambda}_5^i = 0, \quad (h_1 + h_2 - B) < 0, \quad \underline{\lambda}_5^i(h_1 + h_2 - B) = 0, \end{cases}$$

which then becomes:

$$\begin{cases} \frac{\partial E(U_i)}{\partial Q_{i1}} = 0, \\ \frac{\partial E(U_i)}{\partial s_i} = 0, \\ \underline{\lambda}_5^i = 0, \quad (h_1 + h_2 - B) < 0, \quad \underline{\lambda}_5^i(h_1 + h_2 - B) = 0, \end{cases}$$

where the KKT conditions for both players are exactly the same as in the Nash game. \blacksquare

Corollary: *If $g(Q_1^*, s_1^*, Q_2^*, s_2^*) = 0$ then the GNE scenario will have more equilibria than the Nash game.*

Example:

We proved in the above that if $g(Q_1^*, s_1^*, Q_2^*, s_2^*) < 0$ then the solution to the Nash game will also be a solution to the GNE game. However, if $g(Q_1^*, s_1^*, Q_2^*, s_2^*) = 0$ then: $h_1 + h_2 = B$. This implies that the shared constraint is binding and thus the KKT system for the GNE game is as follows:

$$\begin{cases} \frac{\partial E(U_i)}{\partial Q_{i1}} = -\underline{\lambda}_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}}, \\ \frac{\partial E(U_i)}{\partial s_i} = -\underline{\lambda}_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i}, \\ \underline{\lambda}_5^i \geq 0, \quad (h_1 + h_2 - B) = 0, \quad \underline{\lambda}_5^i(h_1 + h_2 - B) = 0. \end{cases}$$

Thus if $\lambda_5^i > 0$ and either $\frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}} \neq 0$ or $\frac{\partial(h_i(Q_i, s_i))}{\partial s_i} \neq 0$ for $i = 1, 2$, then the KKT system above differs from the Nash game KKT system resulting in different equilibria.

Proposition 2 *Any solution $(Q_1^*, s_1^*, Q_2^*, s_2^*)$ of the Nash game is a solution of the cooperative game if $\frac{\partial E(U_2)}{\partial Q_{11}} = 0$, $\frac{\partial E(U_2)}{\partial s_1} = 0$, $\frac{\partial E(U_1)}{\partial Q_{21}} = 0$ and $\frac{\partial E(U_1)}{\partial s_2} = 0$.*

Proof:

Setting the KKT Nash system equal to the KKT cooperative system we get the following:

$$\left\{ \begin{array}{l} \frac{\partial E(U_1)}{\partial Q_{11}} = \frac{\partial E(U_1)}{\partial Q_{11}} + \frac{\partial E(U_2)}{\partial Q_{11}}, \\ \frac{\partial E(U_2)}{\partial Q_{21}} = \frac{\partial E(U_1)}{\partial Q_{21}} + \frac{\partial E(U_2)}{\partial Q_{21}}, \\ \frac{\partial E(U_1)}{\partial s_1} = \frac{\partial E(U_1)}{\partial s_1} + \frac{\partial E(U_2)}{\partial s_1}, \\ \frac{\partial E(U_2)}{\partial s_2} = \frac{\partial E(U_1)}{\partial s_2} + \frac{\partial E(U_2)}{\partial s_2}. \end{array} \right.$$

Thus, the only way the above system of equations is satisfied is when $\frac{\partial E(U_2)}{\partial Q_{11}} = 0$, $\frac{\partial E(U_2)}{\partial s_1} = 0$, $\frac{\partial E(U_1)}{\partial Q_{21}} = 0$ and $\frac{\partial E(U_1)}{\partial s_2} = 0$. ■

When the shared constraint of the GNE game becomes binding, then the KKT systems of the GNE scenario are as follows:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} = -\lambda_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}}, \\ \frac{\partial E(U_i)}{\partial s_i} = -\lambda_5^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i}, \\ \lambda_5^i > 0, (h_1 + h_2 - B) = 0, \lambda_5^i(h_1 + h_2 - B) = 0. \end{array} \right.$$

Proposition 3 *Assume $(Q_1^*, s_1^*, Q_2^*, s_2^*)$ is a solution of the GNE game with $g(Q_1^*, s_1^*, Q_2^*, s_2^*) =$*

0. Then, this is a solution of the cooperative game if:

$$\left\{ \begin{array}{l} \frac{\partial E(U_2)}{\partial Q_{11}} = \lambda_5^1 \frac{\partial(h_1(Q_1, s_1))}{\partial Q_{11}}, \quad \frac{\partial E(U_1)}{\partial Q_{21}} = \lambda_5^2 \frac{\partial(h_2(Q_2, s_2))}{\partial Q_{21}}, \\ \frac{\partial E(U_2)}{\partial s_1} = \lambda_5^1 \frac{\partial(h_1(Q_1, s_1))}{\partial s_1}, \quad \frac{\partial E(U_1)}{\partial s_2} = \lambda_5^2 \frac{\partial(h_2(Q_2, s_2))}{\partial s_2}. \end{array} \right.$$

Proof:

Again, following the methods used in the last proof we arrive at the following system of equations:

$$\left\{ \begin{array}{l} \frac{\partial E(U_1)}{\partial Q_{11}} + \lambda_5^1 \frac{\partial(h_1(Q_1, s_1))}{\partial Q_{11}} = \frac{\partial E(U_1)}{\partial Q_{11}} + \frac{\partial E(U_2)}{\partial Q_{11}}, \\ \frac{\partial E(U_2)}{\partial Q_{21}} + \lambda_5^2 \frac{\partial(h_2(Q_2, s_2))}{\partial Q_{21}} = \frac{\partial E(U_1)}{\partial Q_{21}} + \frac{\partial E(U_2)}{\partial Q_{21}}, \\ \frac{\partial E(U_1)}{\partial s_1} + \lambda_5^1 \frac{\partial(h_1(Q_1, s_1))}{\partial s_1} = \frac{\partial E(U_1)}{\partial s_1} + \frac{\partial E(U_2)}{\partial s_1}, \\ \frac{\partial E(U_2)}{\partial s_2} + \lambda_5^2 \frac{\partial(h_2(Q_2, s_2))}{\partial s_2} = \frac{\partial E(U_1)}{\partial s_2} + \frac{\partial E(U_2)}{\partial s_2}. \end{array} \right.$$

Therefore after some rearranging we see that:

$$\left\{ \begin{array}{l} \lambda_5^1 \frac{\partial(h_1(Q_1, s_1))}{\partial Q_{11}} = \frac{\partial E(U_2)}{\partial Q_{11}}, \\ \lambda_5^2 \frac{\partial(h_2(Q_2, s_2))}{\partial Q_{21}} = \frac{\partial E(U_1)}{\partial Q_{21}}, \\ \lambda_5^1 \frac{\partial(h_1(Q_1, s_1))}{\partial s_1} = \frac{\partial E(U_2)}{\partial s_1}, \\ \lambda_5^2 \frac{\partial(h_2(Q_2, s_2))}{\partial s_2} = \frac{\partial E(U_1)}{\partial s_2}. \end{array} \right.$$

■

2.3 m -Seller, n -Buyer Game

Let us look at a general case of a marketplace network which looks like Fig 2.1 at the start of this chapter:

We now have that sellers index $i \in \{1, 2, \dots, m\}$ while the buyer index is $j \in \{1, 2, \dots, n\}$.

All sellers sell 1 type of product to the buyers. Thus:

$$\begin{aligned}
 s_1 &\in [0, 1] , s_2 \in [0, 1] , \dots , s_m \in [0, 1] , \bar{s} = \frac{s_1 + s_2 + \dots + s_m}{m} , \\
 Q_1 &= (Q_{11}, Q_{12}, \dots, Q_{1n}), \dots, Q_m = (Q_{m1}, Q_{m2}, \dots, Q_{mn}) \in R_+, \\
 p_1 &= 1 - s_1 , p_2 = 1 - s_2 , \dots , p_m = 1 - s_m , \\
 h_1(Q_1, s_1) &= (Q_{11} + Q_{12} + \dots + Q_{1n}) \left(\frac{1}{\sqrt{1-s_1}} - 1 \right) , \\
 h_2(Q_2, s_2) &= (Q_{21} + Q_{22} + \dots + Q_{2n}) \left(\frac{1}{\sqrt{1-s_2}} - 1 \right) , \\
 &\vdots \\
 h_m(Q_m, s_m) &= (Q_{m1} + Q_{m2} + \dots + Q_{mn}) \left(\frac{1}{\sqrt{1-s_m}} - 1 \right) .
 \end{aligned}$$

Demand for product by buyer $j = (1, 2, \dots, n)$ is d_j and satisfies:

$$\begin{aligned}
 d_1 &= \sum_{i=1}^m Q_{i1} = Q_{11} + Q_{21} + \dots + Q_{m1} , \\
 &\vdots \\
 d_n &= \sum_{i=1}^m Q_{in} = Q_{1n} + Q_{2n} + \dots + Q_{mn} .
 \end{aligned}$$

For $j = (1, 2, \dots, n)$, we have n demand price functions:

$$\rho_1 = \rho_1(d, \bar{s}) = \rho_1(Q_{11}, s_1, Q_{21}, s_2, \dots, Q_{m1}, s_m),$$

⋮

$$\rho_n = \rho_n(d, \bar{s}) = \rho_n(Q_{1n}, s_1, Q_{2n}, s_2, \dots, Q_{mn}, s_m),$$

since d depends on (Q_1, Q_2, \dots, Q_m) and \bar{s} depends on s_1, s_2, \dots, s_m .

Cost functions:

a) transaction costs:

$$c_{ij}(Q_{ij}), \forall i = (1, 2, \dots, m), j = (1, 2, \dots, n).$$

b) flat fixed cost:

c_1, c_2, \dots, c_m which gives us the following total cost:

$$C_1 = c_1 \left(\sum_{j=1}^n Q_{1j} \right) + \sum_{j=1}^n c_{1j}(Q_{1j}),$$

$$C_2 = c_2 \left(\sum_{j=1}^n Q_{2j} \right) + \sum_{j=1}^n c_{2j}(Q_{2j}),$$

⋮

$$C_m = c_m \left(\sum_{j=1}^n Q_{mj} \right) + \sum_{j=1}^n c_{mj}(Q_{mj}).$$

The Profit of the sellers is given by:

$$f_1(Q_1, s_1, Q_2, s_2, \dots, Q_m, s_m) = R_1 - C_1,$$

$$f_2(Q_1, s_1, Q_2, s_2, \dots, Q_m, s_m) = R_2 - C_2,$$

⋮

$$f_m(Q_1, s_1, Q_2, s_2, \dots, Q_m, s_m) = R_m - C_m.$$

Each seller has an average financial damage $D_i > 0$.

Therefore, we have the following expected utility functions:

$$\begin{aligned}
E(U_1) &= s_1 f_1(Q, s) + (1 - s_1)(f_1(Q, s) - D_1) - \left(\sum_{j=1}^n Q_{1j} \right) \left(\frac{1}{\sqrt{1 - s_1}} - 1 \right), \\
E(U_2) &= s_2 f_2(Q, s) + (1 - s_2)(f_2(Q, s) - D_2) - \left(\sum_{j=1}^n Q_{2j} \right) \left(\frac{1}{\sqrt{1 - s_2}} - 1 \right), \\
&\vdots \\
E(U_m) &= s_m f_m(Q, s) + (1 - s_m)(f_m(Q, s) - D_m) - \left(\sum_{j=1}^n Q_{mj} \right) \left(\frac{1}{\sqrt{1 - s_m}} - 1 \right).
\end{aligned}$$

Let us now describe three competitive frameworks for the m-sellers n-buyers model. The first is that of a non cooperative competition, i.e. a Nash game, between the sellers.

1) Nash game (competitive):

P1:

$$\max_{(Q_1, s_1)} E(U_1) = s_1 f_1(Q, s) + (1 - s_1)(f_1(Q, s) - D_1) - \left(\sum_{j=1}^n Q_{1j} \right) \left(\frac{1}{\sqrt{1 - s_1}} - 1 \right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq \sum_{j=1}^n Q_{1j} \leq M_1, \\ 0 \leq s_1 \leq 1. \end{cases}$$

\vdots

Pm:

$$\max_{(Q_m, s_m)} E(U_m) = s_m f_m(Q, s) + (1 - s_m)(f_m(Q, s) - D_m) - \left(\sum_{j=1}^n Q_{mj} \right) \left(\frac{1}{\sqrt{1 - s_m}} - 1 \right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq \sum_{j=1}^n Q_{mj} \leq M_m, \\ 0 \leq s_m \leq 1. \end{cases}$$

The second framework is that of a competitive setting where all sellers draw from a common (previously agreed upon) pool of resources. That is to say, they agreed to contribute to a common budget in order to be able to spend this jointly on security against cyber attacks. This is called a generalized Nash Game with a shared (common) constraint, described by:

$$g(Q, s) = \sum_{i=1}^m h_i(Q_i, s_i) - B \leq 0.$$

The game now can be formulated as:

2) Generalized Nash Game (competitive):

P1:

$$\max_{(Q_1, s_1)} E(U_1) = s_1 f_1(Q, s) + (1 - s_1)(f_1(Q, s) - D_1) - \left(\sum_{j=1}^n Q_{1j} \right) \left(\frac{1}{\sqrt{1 - s_1}} - 1 \right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq \sum_{j=1}^n Q_{1j} \leq M_1, \\ 0 \leq s_1 \leq 1, \\ \sum_{i=1}^m h_i(Q_i, s_i) - B \leq 0. \end{cases}$$

⋮

Pm:

$$\max_{(Q_m, s_m)} E(U_m) = s_m f_m(Q, s) + (1 - s_m)(f_m(Q, s) - D_m) - \left(\sum_{j=1}^n Q_{mj} \right) \left(\frac{1}{\sqrt{1 - s_m}} - 1 \right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq \sum_{j=1}^n Q_{mj} \leq M_m, \\ 0 \leq s_m \leq 1, \\ \sum_{i=1}^m h_i(Q_i, s_i) - B \leq 0. \end{cases}$$

The third scenario is one of cooperation, meaning that the m sellers decide to maximise the sum of their expected utilities. From [40] in a cooperative scenario, we have:

$$E(U_C) = \sum_{i=1}^m E(U_i).$$

3) Cooperative Game (non-competitive):

P1 , P2 , ... , Pm:

$$\begin{aligned} \max_{(Q_1, s_1, Q_2, s_2, \dots, Q_m, s_m)} \sum_{i=1}^m E(U_i) &= \max_{(Q_1, s_1, Q_2, s_2, \dots, Q_m, s_m)} E(U_1) + E(U_2) + \dots + E(U_m), \\ \text{s.t.} \quad &\begin{cases} 0 \leq \sum_{i=1}^m \sum_{j=1}^n Q_{ij} \leq \sum_{i=1}^m M_i, \\ 0 \leq \sum_{i=1}^m s_i \leq m. \end{cases} \end{aligned}$$

In order to solve the above three games, a well-established approach is to solve the KKT systems associated to each of them. It is known (see Chapter 1) that KKT systems represent sufficient and necessary conditions for finding equilibria of Nash games. In the next subsection we write explicitly the KKT systems associated to each of the three scenarios highlighted above.

2.3.1 KKT systems for the three scenarios

Starting with the Nash game, we have $n + 3$ constraints for each player, thus we will have $n + 3$ Lagrange multipliers for each i : λ_j^i for $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n + 3\}$.

Then the Lagrangian:

$$\mathcal{L}(Q_i, s_i, \lambda_1^i, \lambda_2^i, \dots, \lambda_{n+3}^i) = E(U_i) + \sum_{j=1}^n \lambda_j^i Q_{ij} + \lambda_{n+1}^i \left(M_i - \sum_{j=1}^n Q_{ij} \right) + \lambda_{n+2}^i s_i + \lambda_{n+3}^i (1 - s_i).$$

So the KKT conditions for $i \in \{1, 2, \dots, m\}$ are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial Q_{i1}} = 0 &\Leftrightarrow \frac{\partial E(U_i)}{\partial Q_{i1}} + \lambda_1^i - \lambda_{n+1}^i = 0, \\ &\vdots \\ \frac{\partial \mathcal{L}}{\partial Q_{in}} = 0 &\Leftrightarrow \frac{\partial E(U_i)}{\partial Q_{in}} + \lambda_n^i - \lambda_{n+1}^i = 0, \\ \frac{\partial \mathcal{L}}{\partial s_i} = 0 &\Leftrightarrow \frac{\partial E(U_i)}{\partial s_i} + \lambda_{n+2}^i - \lambda_{n+3}^i = 0, \\ \left\{ \begin{array}{l} \lambda_j^i \geq 0, \quad Q_{ij} \geq 0, \quad \lambda_j^i Q_{ij} = 0, \quad \forall j = (1, 2, \dots, n), \\ \lambda_{n+1}^i \geq 0, \quad \left(M_i - \sum_{j=1}^n Q_{ij} \right) \geq 0, \quad \lambda_{n+1}^i \left(M_i - \sum_{j=1}^n Q_{ij} \right) = 0, \\ \lambda_{n+2}^i \geq 0, \quad s_i \geq 0, \quad \lambda_{n+2}^i s_i = 0, \\ \lambda_{n+3}^i \geq 0, \quad (1 - s_i) \geq 0, \quad \lambda_{n+3}^i (1 - s_i) = 0. \end{array} \right. \end{aligned}$$

Therefore we can assume that $\lambda_j^i = 0$ in order to get $Q_{ij} > 0 \forall j = (1, 2, \dots, n)$ and that $\lambda_{n+2}^i = 0$ in order to get $s_i > 0$. We also can assume that $\lambda_{n+3}^i = 0$ so that we have $s_i < 1$ since if $s_i = 1$ we get a zero in the denominator of the security investment cost function since

(2.4) goes to infinity as s_i approaches 1. Last, looking at $\sum_{j=1}^n Q_{ij} \leq M_i$ in order to obtain an interior solution we set $\lambda_{n+1}^i = 0$ giving us $M_i - \sum_{j=1}^n Q_{ij} > 0$. Thus the KKT conditions for player i , for $i \in \{1, 2, \dots, m\}$ are:

$$\begin{cases} \frac{\partial E(U_i)}{\partial Q_{i1}} = 0, \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} = 0, \\ \frac{\partial E(U_i)}{\partial s_i} = 0. \end{cases}$$

KKT conditions for generalized Nash game

From Chapter 1 we have that $S_i = \{x \in \mathbb{R}^{n_i} \mid g(x) \leq 0\}$ where $g(x) = (g^1(x), \dots, g^{m_i}(x))$.

In our case $m_1 = \dots = m_i := n + 4$. Then the constraints for each player are:

$$\text{For each } i \in \{1, \dots, m\} \quad g^j(x) = -Q_{i,j}, \quad (2.19)$$

$$g^{n+1}(x) = \sum_{j=1}^n Q_{i,j} - M_i, \quad (2.20)$$

$$g^{n+2}(x) = -s_i, \quad (2.21)$$

$$g^{n+3}(x) = s_i - 1, \quad (2.22)$$

$$g^{n+4}(x) = \sum_{i=1}^m h_i(Q_i, s_i) - B. \quad (2.23)$$

So player i has an $(n + 4)$ -dimensional vector $\underline{\lambda}^i \in \mathbb{R}^{n+4}$ such that the KKT's are:

$$\nabla_{x_i} \theta_i(x_i, x_{-i}) + \nabla_{x_i} g(x_i, x_{-i}) \underline{\lambda}^i = 0, \quad 0 \leq \underline{\lambda}^i \perp g(x_i, x_{-i}) \leq 0.$$

In our case $x_i = (Q_i, s_i)$ Also, $i = 1 \Rightarrow -i = (2, 3, \dots, m)$ when $i = 2, -i = (1, 3, 4, \dots, m)$

when $i = m$, $-i = (1, 2, \dots, m - 1)$. Also we have that $\theta_i = E(U_i)$. If $x_1 = (Q_1, s_1)$ then $x_{-1} = (Q_2, s_2, Q_3, s_3, \dots, Q_m, s_m)$ etc. Therefore,

$$\begin{pmatrix} \frac{\partial E(U_i)}{\partial Q_{i1}} \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} \\ \frac{\partial E(U_i)}{\partial s_i} \end{pmatrix} + \begin{pmatrix} \frac{\partial g^1}{\partial Q_{i1}} & \frac{\partial g^2}{\partial Q_{i1}} & \cdots & \frac{\partial g^{n+4}}{\partial Q_{i1}} \\ \vdots \\ \frac{\partial g^1}{\partial Q_{in}} & \frac{\partial g^2}{\partial Q_{in}} & \cdots & \frac{\partial g^{n+4}}{\partial Q_{in}} \\ \frac{\partial g^1}{\partial s_i} & \frac{\partial g^2}{\partial s_i} & \cdots & \frac{\partial g^{n+4}}{\partial s_i} \end{pmatrix} (\lambda_1^i, \lambda_2^i, \dots, \lambda_{n+4}^i)^T = 0,$$

$$0 \leq (\lambda_1^i, \dots, \lambda_{n+4}^i) \perp (g^1(x), \dots, g^{n+4}(x)) \leq 0.$$

These are, in more condensed form: for $i \in \{1, 2, \dots, m\}$

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} + \sum_{j=1}^{n+4} \lambda_j^i \frac{\partial g^j}{\partial Q_{i1}} = 0, \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} + \sum_{j=1}^{n+4} \lambda_j^i \frac{\partial g^j}{\partial Q_{in}} = 0, \\ \frac{\partial E(U_i)}{\partial s_i} + \sum_{j=1}^{n+4} \lambda_j^i \frac{\partial g^j}{\partial s_i} = 0, \\ \lambda_j^i \geq 0, \forall j \in \{1, \dots, n+4\}, \\ g^j(x) \geq 0, \forall j \in \{1, \dots, n+4\}, \\ \lambda_j^i g^j(x) = 0, \forall j \in \{1, \dots, n+4\}. \end{array} \right.$$

From equations (2.19)-(2.23), we have:

$$\begin{aligned} \frac{\partial g^1}{\partial Q_{i1}} = -1, \frac{\partial g^j}{\partial Q_{i1}} = 0, j \neq 1, \frac{\partial g^{n+1}}{\partial Q_{i1}} = 1, \frac{\partial g^{n+2}}{\partial Q_{i1}} = 0, \frac{\partial g^{n+3}}{\partial Q_{i1}} = 0, \frac{\partial g^{n+4}}{\partial Q_{i1}} = \frac{\partial h_i(Q_i, s_i)}{\partial Q_{i1}}, \\ \vdots \end{aligned}$$

$$\frac{\partial g^j}{\partial Q_{in}} = 0, j \neq n, \frac{\partial g^n}{\partial Q_{in}} = -1, \frac{\partial g^{n+1}}{\partial Q_{in}} = 1, \frac{\partial g^{n+2}}{\partial Q_{in}} = 0, \frac{\partial g^{n+3}}{\partial Q_{in}} = 0, \frac{\partial g^{n+4}}{\partial Q_{in}} = \frac{\partial h_i(Q_i, s_i)}{\partial Q_{in}},$$

and:

$$\frac{\partial g^j}{\partial s_i} = 0, j = \{1, \dots, n+1\}, \frac{\partial g^{n+2}}{\partial s_i} = -1, \frac{\partial g^{n+3}}{\partial s_i} = 1, \frac{\partial g^{n+4}}{\partial s_i} = \frac{\partial h_i(Q_i, s_i)}{\partial s_i}.$$

Again if $Q_{i,j} > 0, s_i > 0 \Rightarrow \underline{\lambda}_1^i = \dots = \underline{\lambda}_n^i = 0 = \underline{\lambda}_{n+2}^i$.

Then KKT conditions for player i are:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} + \underline{\lambda}_{n+1}^i + \underline{\lambda}_{n+4}^i \frac{\partial (h_i(Q_i, s_i))}{\partial Q_{i1}} = 0, \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} + \underline{\lambda}_{n+1}^i + \underline{\lambda}_{n+4}^i \frac{\partial (h_i(Q_i, s_i))}{\partial Q_{in}} = 0, \\ \frac{\partial E(U_i)}{\partial s_i} + \underline{\lambda}_{n+3}^i + \underline{\lambda}_{n+4}^i \frac{\partial (h_i(Q_i, s_i))}{\partial s_i} = 0, \\ \underline{\lambda}_{n+1}^i \geq 0, \left(\sum_{j=1}^n Q_{ij} - M_i \right) \leq 0, \underline{\lambda}_{n+1}^i \left(\sum_{j=1}^n Q_{ij} - M_i \right) = 0, \\ \underline{\lambda}_{n+3}^i \geq 0, (s_i - 1) \leq 0, \underline{\lambda}_{n+3}^i (s_i - 1) = 0, \\ \underline{\lambda}_{n+4}^i \geq 0, \left(\sum_{i=1}^m h_i - B \right) \leq 0, \underline{\lambda}_{n+4}^i \left(\sum_{i=1}^m h_i - B \right) = 0. \end{array} \right.$$

Again, as in the KKT for the regular Nash game we need $s_i < 1 \Rightarrow \underline{\lambda}_{n+3}^i = 0$ and we further

consider $\sum_{j=1}^n Q_{ij} < M_i \Rightarrow \lambda_{n+1}^i = 0$ our KKT system then becomes:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} = -\lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}}, \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} = -\lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{in}}, \\ \frac{\partial E(U_i)}{\partial s_i} = -\lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i}, \\ \lambda_{n+4}^i \geq 0, \left(\sum_{i=1}^m h_i - B \right) \leq 0, \lambda_{n+4}^i \left(\sum_{i=1}^m h_i - B \right) = 0. \end{array} \right.$$

KKT conditions for cooperative game

We arrive at the following KKT solution for when all sellers cooperate:

$$\text{For } i \in \{1, \dots, m\} \text{ we have } \left\{ \begin{array}{l} \sum_{k=1}^m \frac{\partial E(U_k)}{\partial Q_{i1}} = 0, \\ \sum_{k=1}^m \frac{\partial E(U_k)}{\partial Q_{i2}} = 0, \\ \vdots \\ \sum_{k=1}^m \frac{\partial E(U_k)}{\partial Q_{in}} = 0, \\ \sum_{k=1}^m \frac{\partial E(U_k)}{\partial s_i} = 0. \end{array} \right.$$

2.3.2 Analysis of equilibria in the three scenarios

Proposition 4 *Under the assumption that $0 < \sum_{j=1}^n Q_{ij} < M_i, \forall i = (1, 2, \dots, m)$ and $0 < s_i < 1, \forall i = (1, 2, \dots, m)$ then any solution $(Q_1^*, s_1^*, Q_2^*, s_2^*, \dots, Q_m^*, s_m^*)$ of the GNE game is a solution of the Nash game only if $g(Q_1^*, s_1^*, Q_2^*, s_2^*, \dots, Q_m^*, s_m^*) < 0$.*

Proof:

If $g(Q_1^*, s_1^*, Q_2^*, s_2^*, \dots, Q_m^*, s_m^*) < 0$ then $\lambda_{n+4}^i = 0$. Thus the KKT conditions for the GNE game becomes:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} = -\lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}}, \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} = -\lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{in}}, \\ \frac{\partial E(U_i)}{\partial s_i} = -\lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i}, \\ \lambda_{n+4}^i = 0, \left(\sum_{i=1}^m h_i - B \right) \leq 0, \lambda_{n+4}^i \left(\sum_{i=1}^m h_i - B \right) = 0, \end{array} \right.$$

which then becomes:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} = 0, \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} = 0, \\ \frac{\partial E(U_i)}{\partial s_i} = 0, \\ \lambda_{n+4}^i = 0, \left(\sum_{i=1}^m h_i - B \right) = 0, \lambda_{n+4}^i \left(\sum_{i=1}^m h_i - B \right) = 0, \end{array} \right.$$

where the KKT conditions for all players are exactly the same as in the Nash game. ■

Proposition 5 *Any solution $(Q_1^*, s_1^*, Q_2^*, s_2^*, \dots, Q_m^*, s_m^*)$ of the Nash game is a solution of the cooperative game if:*

$\forall i = (1, 2, \dots, m), k = (1, 2, \dots, m)$ and $j = (1, 2, \dots, n)$ s.t. $i \neq k$ then

$$\begin{cases} \frac{\partial E(U_i)}{\partial Q_{kj}} = 0, \\ \frac{\partial E(U_i)}{\partial s_k} = 0. \end{cases}$$

Proof:

Setting the KKT Nash system equal to the KKT cooperative system we get the following:

$$\begin{cases} \frac{\partial E(U_i)}{\partial Q_{i1}} = \sum_{k=1}^m \frac{\partial E(U_k)}{\partial Q_{i1}}, \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} = \sum_{k=1}^m \frac{\partial E(U_k)}{\partial Q_{in}}, \\ \frac{\partial E(U_i)}{\partial s_i} = \sum_{k=1}^m \frac{\partial E(U_k)}{\partial s_i}. \end{cases}$$

Thus, the only way the above system of equations is satisfied is when $\sum_{k=1}^m \frac{\partial E(U_k)}{\partial Q_{ij}} = 0$ and

$$\sum_{k=1}^m \frac{\partial E(U_k)}{\partial s_i} = 0, \forall j = (1, 2, \dots, n) \text{ and } \forall i, k = (1, 2, \dots, m) \text{ when } i \neq k. \quad \blacksquare$$

In the case of a binding shared constraint the GNE game becomes:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} = -\lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}}, \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} = -\lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{in}}, \\ \frac{\partial E(U_i)}{\partial s_i} = -\lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i}, \\ \lambda_{n+4}^i > 0, \left(\sum_{i=1}^m h_i - B \right) = 0, \lambda_{n+4}^i \left(\sum_{i=1}^m h_i - B \right) = 0. \end{array} \right.$$

Proposition 6 *Assume $(Q_1^*, s_1^*, Q_2^*, s_2^*, \dots, Q_m^*, s_m^*)$ is a solution of the GNE game with $g(Q_1^*, s_1^*, Q_2^*, s_2^*, \dots, Q_m^*, s_m^*) = 0$. Then, this is a solution of the cooperative game if:*

$\forall i = (1, 2, \dots, m), k = (1, 2, \dots, m)$ and $j = (1, 2, \dots, n)$ s.t. $i \neq k$ then

$$\left\{ \begin{array}{l} \sum_{k=1}^m \frac{\partial E(U_k)}{\partial Q_{ij}} = \lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{ij}}, \\ \sum_{k=1}^m \frac{\partial E(U_k)}{\partial s_i} = \lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i}. \end{array} \right.$$

Proof:

Again, following the methods used in the last proof we arrive at the following system of

equations:

$$\left\{ \begin{array}{l} \frac{\partial E(U_i)}{\partial Q_{i1}} + \lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}} = \sum_{k=1}^m \frac{\partial E(U_k)}{\partial Q_{i1}}, \\ \vdots \\ \frac{\partial E(U_i)}{\partial Q_{in}} + \lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{in}} = \sum_{k=1}^m \frac{\partial E(U_k)}{\partial Q_{in}}, \\ \frac{\partial E(U_i)}{\partial s_i} + \lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i} = \sum_{k=1}^m \frac{\partial E(U_k)}{\partial s_i}. \end{array} \right.$$

Therefore after some rearranging we see that:

$$\left\{ \begin{array}{l} \lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{i1}} = \sum_{\substack{k=1 \\ k \neq i}}^m \frac{\partial E(U_k)}{\partial Q_{i1}}, \\ \vdots \\ \lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial Q_{in}} = \sum_{\substack{k=1 \\ k \neq i}}^m \frac{\partial E(U_k)}{\partial Q_{in}}, \\ \lambda_{n+4}^i \frac{\partial(h_i(Q_i, s_i))}{\partial s_i} = \sum_{\substack{k=1 \\ k \neq i}}^m \frac{\partial E(U_k)}{\partial s_i}. \end{array} \right.$$

■

2.4 Conclusion

In the previous section we showed that a general game can be formed for Nash, GNE, and Cooperative scenarios. It was seen that as the market system gets larger the KKT conditions slightly change. As the number of buyers increase, the KKT systems get larger in the form of having more variables to have derivatives taken with. We showed in general that if the expected utility functions for all players are of the form (2.13) then each individual game can

have the same solution as another game if certain criteria are met. We found that both Nash and generalized Nash games will have the same solutions for when the shared constraint is non-binding, and there will exist more unique solutions for the GNE game if the shared constraint is binding. In order for both the Nash game and the cooperative game to share solutions we need each seller's expected utility function to be non-changing with respect to all other seller's strategic variables. Thus, it is very unlikely that such an interior solution exists in this economic game scenario. It is even more unlikely that both cooperative game and binding GNE game will share solutions since the system of equations in (2.3.2) must be satisfied. It can be seen that as the marketplace increases in size, the chances of the games having common solutions decreases.

Chapter 3

Case studies

We conduct the following case studies in order to model how each of the three KKT systems behave with varying parameters and to show that the previous propositions are satisfied. In the case studies we will have the price demand functions given for each individual buyer as:

$$\rho_j(d, \bar{s}) = -\delta_j \left(\sum_{i=1}^n Q_{ij} \right) + \beta_j \bar{s} + \gamma_j,$$

where, δ_j represents the slope of the demand price curve of buyer j . Clearly, as δ_j increases the slope becomes more steep, resulting in the buyer valuing the product being purchased less. Here, β_j represents the buyer's sensitivity to the average security level of the marketplace, and γ_j represents how much the buyer values the product being sold. Again, \bar{s} is the average security level of the marketplace, where the security investment cost functions are described in (2.4).

For these case studies we are going to use a quadratic transaction cost function obtained

from [27] given by:

$$c_{ij}(Q_{ij}) = A_{ij}Q_{ij}^2 + B_{ij}Q_{ij},$$

where A_{ij} and B_{ij} are coefficients affecting the costs of selling the product from seller i to buyer j . A transaction cost of this form shows a quadratic increasing function of Q_{ij} . That is, the more product transferred between seller i and buyer j the more expensive it becomes to sell the product. This could be caused by increasing shipping costs per item, taxes, etc.

In this chapter we conduct sensitivity analyses focusing on several case studies, using different parameters. The following parameter values will be set as our base case for all case studies unless stated otherwise:

Base case parameter values					
Seller dependent parameter values					
$A_{11} = 0.5$	$A_{12} = 0.25$	$A_{21} = 0.5$	$A_{22} = 0.25$	$A_{31} = 0.5$	$A_{32} = 0.25$
$B_{11} = 1$	$B_{12} = 1$	$B_{21} = 1$	$B_{22} = 1$	$B_{31} = 1$	$B_{32} = 1$
$c_1 = 10$	$c_2 = 10$	$c_3 = 10$	$D_1 = 75$	$D_2 = 75$	$D_3 = 75$
Buyer dependent parameter values					
$\delta_1 = 1$	$\delta_2 = 1$	$\beta_1 = 0.1$	$\beta_2 = 0.1$	$\gamma_1 = 100$	$\gamma_2 = 100$

Table 3.1: Parameters used for the base case in the following case studies.

It is to be noted that the first three rows of parameters above are all dependent on the seller, whereas the last row of parameters are all buyer dependent.

In order to use the KKT conditions to solve the systems of equations we need to show that the expected utility function for each player is concave. To do so, we introduce the following Theorem:

Theorem 4 *For any twice differentiable function, it is strictly concave if and only if, the Hessian is negative definite. The function is concave if and only if, the Hessian is negative semi-definite.*

Therefore, looking at the 3-seller, 2-buyer scenario we have the following expected utility functions:

$$E(U_i) = \sum_{j=1}^2 \rho_j Q_{ij} - \sum_{j=1}^2 (c_i Q_{ij} + c_{ij}(Q_{ij})) - D_i + s_i D_i - \left(\sum_{j=1}^2 Q_{ij} \right) \left(\frac{1}{\sqrt{1-s_i}} - 1 \right).$$

If we compute the Hessian for seller 1 we arrive at the following matrix:

$$\begin{bmatrix} -2\delta_1 - 2A_{11} & 0 & \frac{\beta_1}{3} - \frac{1}{2}(1-s_1)^{-\frac{3}{2}} & -\delta_1 & 0 & \frac{\beta_1}{3} & -\delta_1 & 0 & \frac{\beta_1}{3} \\ 0 & -2\delta_2 - 2A_{12} & \frac{\beta_2}{3} - \frac{1}{2}(1-s_1)^{-\frac{3}{2}} & 0 & -\delta_2 & \frac{\beta_2}{3} & 0 & -\delta_2 & \frac{\beta_2}{3} \\ \frac{\beta_1}{3} - \frac{1}{2}(1-s_1)^{-\frac{3}{2}} & \frac{\beta_2}{3} - \frac{1}{2}(1-s_1)^{-\frac{3}{2}} & -\frac{3}{4}(Q_{11} + Q_{12})(1-s_1)^{-\frac{5}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\delta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_1}{3} & \frac{\beta_2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\delta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_1}{3} & \frac{\beta_2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Looking at the Hessian matrix of seller 1's expected utility function we see that the matrix is symmetric. We assume that the matrix is negative semi-definite, thus giving us concave expected utility functions as desired. Then, by using Theorem 1, we see that the cooperative game is also maximizing a concave function.

We implemented the KKT systems formulated in Chapter 2 using the `lsqnonlin` user function on Matlab. Each system was given 100 random initial points inside each variable's appropriate bounds in which `lsqnonlin` performed iterations until it converged to a solution.

3.1 Case 1: 2-Seller, 1-Buyer

While conducting the first case study we use a 2-seller, 1-buyer market as shown in Fig. 2.2.

In addition, we state the following lower and upper bounds for the following variables:

$$0 < Q_{ij} < 100, \quad 0 < s_i < 1, \quad \forall i = (1, 2), j = 1,$$

and the shared budget for the GNE game is set to 40, unless stated otherwise.

In the Nash game seller 1 is trying to maximize the following expected utility function:

$$E(U_1) :=$$

$$\left(-Q_{11} - Q_{21} + 0.1 \left(\frac{s_1 + s_2}{2}\right) + 100\right) Q_{11} - 10Q_{11} - 0.5Q_{11}^2 - Q_{11} - 75 + 75s_1 - Q_{11} \left(\frac{1}{\sqrt{1-s_1}} - 1\right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq Q_{11} \leq 100, \\ 0 \leq s_1 \leq 1. \end{cases}$$

Seller 2 is trying to maximize the following expected utility function:

$$E(U_2) :=$$

$$\left(-Q_{11} - Q_{21} + 0.1 \left(\frac{s_1 + s_2}{2}\right) + 100\right) Q_{21} - 10Q_{21} - 0.5Q_{21}^2 - Q_{21} - 75 + 75s_2 - Q_{21} \left(\frac{1}{\sqrt{1-s_2}} - 1\right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq Q_{21} \leq 100, \\ 0 \leq s_2 \leq 1. \end{cases}$$

We will now show that the constraint qualifications are satisfied for this Nash game. We

have the following four constraints for each individual seller:

$$\begin{cases} g^1(\bar{x}) = -Q_{i1} \leq 0, \\ g^2(\bar{x}) = Q_{i1} - 100 \leq 0, \\ g^3(\bar{x}) = -s_i \leq 0, \\ g^4(\bar{x}) = s_i - 1 \leq 0. \end{cases}$$

We need

$$\nabla g(\bar{x})\lambda = 0,$$

where

$$\nabla g(\bar{x}) = \begin{bmatrix} \frac{\partial g^1(\bar{x})}{\partial Q_{i1}} & \frac{\partial g^1(\bar{x})}{\partial s_i} \\ \frac{\partial g^2(\bar{x})}{\partial Q_{i1}} & \frac{\partial g^2(\bar{x})}{\partial s_i} \\ \frac{\partial g^3(\bar{x})}{\partial Q_{i1}} & \frac{\partial g^3(\bar{x})}{\partial s_i} \\ \frac{\partial g^4(\bar{x})}{\partial Q_{i1}} & \frac{\partial g^4(\bar{x})}{\partial s_i} \end{bmatrix}$$

which gives us:

$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \lambda^i = 0.$$

Clearly this is only satisfied when $\lambda^i = 0, \forall i \in (1, 2)$. Thus, satisfying the constraint qualification.

In the generalized Nash game seller 1 is trying to maximize the following expected utility function:

$$E(U_1) :=$$

$$\left(-Q_{11} - Q_{21} + 0.1 \left(\frac{s_1 + s_2}{2}\right) + 100\right) Q_{11} - 10Q_{11} - 0.5Q_{11}^2 - Q_{11} - 75 + 75s_1 - Q_{11} \left(\frac{1}{\sqrt{1-s_1}} - 1\right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq Q_{11} \leq 100, \\ 0 \leq s_1 \leq 1, \\ Q_{11} \left(\frac{1}{\sqrt{1-s_1}} - 1 \right) + Q_{21} \left(\frac{1}{\sqrt{1-s_2}} - 1 \right) - B \leq 0. \end{cases}$$

Seller 2 is trying to maximize the following expected utility function:

$$E(U_2) :=$$

$$\left(-Q_{11} - Q_{21} + 0.1 \left(\frac{s_1 + s_2}{2} \right) + 100 \right) Q_{21} - 10Q_{21} - 0.5Q_{21}^2 - Q_{21} - 75 + 75s_2 - Q_{21} \left(\frac{1}{\sqrt{1-s_2}} - 1 \right),$$

$$\text{s.t.} \quad \begin{cases} 0 \leq Q_{21} \leq 100, \\ 0 \leq s_2 \leq 1, \\ Q_{11} \left(\frac{1}{\sqrt{1-s_1}} - 1 \right) + Q_{21} \left(\frac{1}{\sqrt{1-s_2}} - 1 \right) - B \leq 0. \end{cases}$$

We will now show that a constraint qualification is satisfied for this generalized Nash game. To do so we will state the Mangasarian-Fromowitz constraint qualification obtained from [10], which states:

(i) There exist a $\bar{y} \in \mathbb{R}^n$ such that

$$\nabla g_i(\bar{x})^t \bar{y} < 0, \quad i \in I(\bar{x}) = \{i \mid g_i(\bar{x}) = 0\},$$

(ii) The gradients $\{\nabla h_j(\bar{x})\}, j = 1, \dots, p$ are linearly independent.

In the absence of equality constraints is equivalent to the Cottle constraint qualification: the system

$$\sum_{i \in I(\bar{x})} u_i \nabla g_i(\bar{x}) = 0, \quad u_i \geq 0$$

has no non-zero solution.

Thus, in our GNE game, from Chapter 2 we showed that $g^i(x) = 0, \forall i = 1, 2, 3, 4$, leaving us with only $g^5(\bar{x}) = Q_{11} \left(\frac{1}{\sqrt{1-s_1}} - 1 \right) + Q_{21} \left(\frac{1}{\sqrt{1-s_2}} - 1 \right) - B$. Therefore, in the case of a binding constraint, we have $g^5(\bar{x}) = 0$. So then we want:

$$\lambda_5^i \nabla g^5(\bar{x}) = 0 \Rightarrow \lambda_5^i = 0,$$

as desired.

3.1.1 Sensitivity analysis of seller parameters

Case 1.1: A_{11}

We first analyse the sensitivity of the game scenarios to an increase of the transaction cost coefficient for seller 1 while the rest of the parameters remain as stated as base values. The “Base” denoted in the parameter value of the tables is to show the base case scenario for the simulations.

First we note that, as shown in 2.2, the Nash game and the non-binding GNE game have the exact same equilibria.

It is seen that as A_{11} increases, the volume of product sold by seller 1 decreases, which results in the expected profit of the firm decreasing. This result holds for all three games. Also, as the volume being sold by seller 1 decreases, the volume of product sold by seller 2 increases. This is due to the buyers’ demand for the product remaining the same even as the transaction cost increases for seller 1; this could have many causes such as the firm’s shipping supplier prices increasing, increased taxes, etc. As seller 1 starts selling less product, their firm would have less production overall resulting in a lesser security investment cost function.

	Nash Game					
A_{11}	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
Base	22.04	0.72	22.04	0.72	708.09	708.09
1.0	15.97	0.78	24.09	0.71	493.21	848.25
1.5	12.51	0.81	25.25	0.70	376.77	933.68
2.0	10.27	0.83	26.00	0.69	303.98	990.99
2.5	8.71	0.85	26.53	0.69	254.27	1032.03
3.0	7.56	0.86	26.91	0.69	218.21	1062.84
3.5	6.67	0.87	27.21	0.68	190.89	1086.82
4.0	5.97	0.88	27.45	0.68	169.49	1105.99
4.5	5.40	0.89	27.64	0.68	152.28	1121.68
5.0	4.93	0.90	27.79	0.68	138.15	1134.74

Table 3.2: 2-Seller 1-Buyer Nash equilibrium solutions as A_{11} increases while implementing the quadratic transaction cost function.

	GNE Game (Non-binding)						
A_{11}	Q_{11}	s_1	Q_{21}	s_2	λ	$E(U_1)$	$E(U_2)$
Base	22.04	0.72	22.04	0.72	0	708.09	708.09
1.0	15.97	0.78	24.09	0.71	0	493.21	848.25
1.5	12.51	0.81	25.25	0.70	0	376.77	933.68
2.0	10.27	0.83	26.00	0.69	0	303.98	990.99
2.5	8.71	0.85	26.53	0.69	0	254.27	1032.03
3.0	7.56	0.86	26.91	0.69	0	218.21	1062.84
3.5	6.67	0.87	27.21	0.68	0	190.89	1086.82
4.0	5.97	0.88	27.45	0.68	0	169.49	1105.99
4.5	5.40	0.89	27.64	0.68	0	152.28	1121.68
5.0	4.93	0.90	27.79	0.68	0	138.15	1134.74

Table 3.3: 2-Seller 1-Buyer Non-binding GNE equilibrium solutions as A_{11} increases while implementing the quadratic transaction cost function.

In light of this, seller 1 starts to invest in more cybersecurity while seller 2 starts investing less in order to produce a higher profit.

Whenever the shared constraint is binding, then the firms exhaust their set budget in investments in cybersecurity. The marginal cost λ of this investment is positive and increasing as A_{11} increases. The most interesting thing here to observe is that both players invest more in security than in the previous scenarios above, at a cost, while roughly maintaining

their production and expected utility values. It seems thus that pooling resources to share in security investments does not lead to any gain for either one of the firms. We see that if the budget B is set to low, we do not obtain any separate solutions from the Nash game. However, if we set B such that we obtain more solutions for the GNE game we will see that the greater the budget is the lower the expected payoffs for each seller will be as a result. We also note that in the scenario when the transaction cost is increasing from the base case to a value such that $A_{11} = 1$ the binding GNE game has no equilibrium solutions. This is due to the fact that when A_{11} is equal to 0.5 the firms are investing in security more than what the budget is set at. Thus, the GNE game and the Nash game have the same equilibria in these two cases as shown in the corollary in 2.

	GNE Game (Binding)						
A_{11}	Q_{11}	s_1	Q_{21}	s_2	λ	$E(U_1)$	$E(U_2)$
Base	–	–	–	–	–	–	–
1.0	15.97	0.79	24.09	0.72	0.06	493.21	848.25
1.5	12.52	0.82	25.26	0.72	0.11	376.65	933.29
2.0	10.29	0.85	26.01	0.73	0.16	303.80	990.35
2.5	9.73	0.87	26.53	0.73	0.19	254.04	1031.15
3.0	7.58	0.88	26.92	0.73	0.22	217.94	1061.74
3.5	6.70	0.90	27.22	0.74	0.25	190.58	1085.51
4.0	6.00	0.91	27.46	0.74	0.27	169.14	1104.50
4.5	5.43	0.91	27.65	0.74	0.29	151.91	1120.01
5.0	4.96	0.92	27.81	0.75	0.30	137.75	1132.92

Table 3.4: 2-Seller 1-Buyer Binding GNE equilibrium solutions as A_{11} increases while implementing the quadratic transaction cost function, where “–” means that there is no solution to this type of game.

Last but not least, let us look at the cooperative scenario. We see that in this case seller 1 is at more of a disadvantage since their production investment pair (Q_{11}, s_1) describes lower production and higher security investment for a low expected utility than in all previous competitive scenarios.

We present graphs of these three games comparing each individual variable in the three

A_{11}	Cooperative Game					
	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
Base	17.60	0.76	17.60	0.76	757.02	757.02
1.0	10.83	0.83	22.17	0.72	461.80	956.80
1.5	7.79	0.86	24.22	0.71	330.01	1046.40
2.0	6.07	0.88	25.38	0.70	255.58	1097.18
2.5	4.96	0.90	26.12	0.69	207.88	1129.85
3.0	4.19	0.91	26.64	0.69	174.75	1152.60
3.5	3.62	0.92	27.02	0.69	150.43	1169.35
4.0	3.18	0.92	27.31	0.68	131.84	1182.18
4.5	2.84	0.93	27.54	0.68	117.19	1192.32
5.0	2.56	0.93	27.73	0.68	105.34	1200.54

Table 3.5: 2-Seller 1-Buyer Cooperative equilibrium solutions as A_{11} increases while implementing the quadratic transaction cost function.

games as A_{11} increases:

We see that for Q_{11} , Q_{21} , $E(U_1)$ and $E(U_2)$ the values for both the Nash game and the binding GNE game are very close to each other. However, the two games differ in the security level graphs. When A_{11} increases the security level of seller 1 increases for all games. One trait that stands out in these graphs is the security level of seller 2 decreases for both Nash and cooperative games, but increases for the binding GNE game. This result comes from the fact that the sellers must invest the budget amount prescribed.

We see that quantity sold by both sellers is lower in the cooperative game. However, when looking at the expected profits seller 2 earns more when sellers cooperate. Seller 1 on the other hand earns more when cooperating than competing when A_{11} is small.

The conclusion in this sensitivity analysis is that an increase in A_{11} is disadvantageous for seller 1 and advantageous for seller 2, in both competitive and cooperative scenarios. The opposite results can be shown when varying A_{21} .

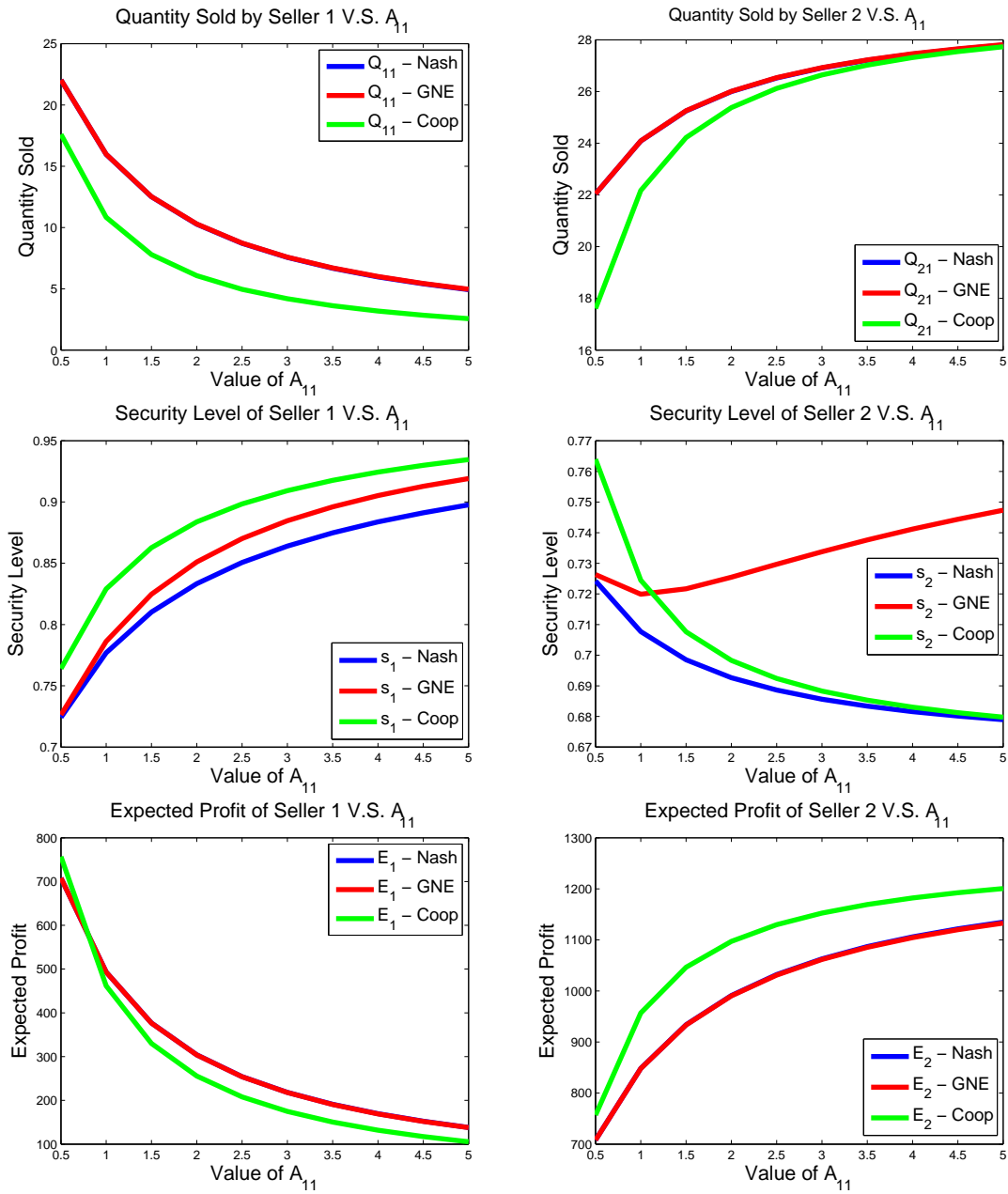


Figure 3.1: 2-Seller 1-Buyer solutions for all three games as A_{11} increases.

Case 1.2: D_1

For the next case study we look at the three games as seller 1's damage incurred during

a cyberattack changes. In this case study we increased the GNE budget to be equal to 75. As shown in the previous case study both Nash and non-binding Generalized Nash solutions are identical so we put them both into one table.

It is seen that starting at a base value such that $D_1 = 75$ the equilibria for both sellers are identical for all three games. We see that as D_1 increases seller 1 starts to invest more into cybersecurity to defend against the increasing cost in the occurrence of a successful cyberattack. As a result seller 2 starts investing less in cybersecurity since the buyer is only aware of the average security level in the marketplace. In both the Nash game and the cooperative game, as D_1 increases, it causes the average security level in the marketplace to increase, that is, seller 1 is investing more in security then seller 2 is decreasing in their investments. This is the same case for the binding GNE game until $D_1 = 105$, where the slope of s_2 becomes steeper than the slope of s_1 . This should also occur eventually in both the Nash and Cooperative games.

Nash Game and GNE (Non-binding) Game						
D_1	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
15	22.33	0.22	21.94	0.73	735.96	701.29
45	22.15	0.61	22.00	0.72	718.65	705.56
Base	22.04	0.72	22.04	0.72	708.09	708.09
105	21.96	0.78	22.07	0.72	700.09	710.01
135	21.89	0.81	22.10	0.72	693.49	711.59

Table 3.6: 2-Seller 1-Buyer Nash equilibrium solutions and GNE solutions when $\lambda = 0$ as D_1 increases while implementing the quadratic transaction cost function.

Again, looking at all three games, the trend as D_1 increases is that Q_{11} decreases while Q_{21} increases. As a result seller 1's expected profits drop while seller 2's profit increases. It is also noted that seller 1's profit declines faster than seller 2's profit increases. This makes sense since as D_1 increases the marketplace as a whole is worse off.

Again we see when comparing the cooperative game to the other two games, when both

	GNE Game (Binding)					
D_1	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
15	22.26	0.75	22.13	0.91	720.88	685.12
45	22.19	0.84	22.12	0.88	707.36	694.64
Base	22.12	0.86	22.12	0.86	699.47	699.47
105	22.06	0.88	22.13	0.85	693.46	702.81
135	22.01	0.89	22.13	0.83	688.41	705.43

Table 3.7: 2-Seller 1-Buyer Binding GNE equilibrium solutions as D_1 increases while implementing the quadratic transaction cost function.

sellers cooperate, total profit earned is higher than when both sellers compete with one another. We also see once again that in a cooperative scenario, product sold by each seller to each individual buyer is substantially lower than in both competitive games, and also security levels in a cooperative game are higher than those of a Nash game but lower than those in a binding GNE game.

	Cooperative Game					
D_1	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
15	18.10	0.34	17.26	0.77	794.79	741.79
45	17.80	0.67	17.47	0.77	771.13	751.21
Base	17.60	0.76	17.60	0.76	757.02	757.02
105	17.46	0.81	17.70	0.76	746.28	761.45
135	17.33	0.84	17.79	0.76	737.42	765.13

Table 3.8: 2-Seller 1-Buyer Cooperative equilibrium solutions as D_1 increases while implementing the quadratic transaction cost function.

We present graphs of these three games comparing each individual variable as D_1 increases:

One interesting characteristic we see in the graphs is that there is a value for D_1 for which product sold by seller 1 is the same in both Nash and GNE games, and there is another value of D_1 such that this occurs for seller 2. We see that all games follow similar trends for all variables of the game.

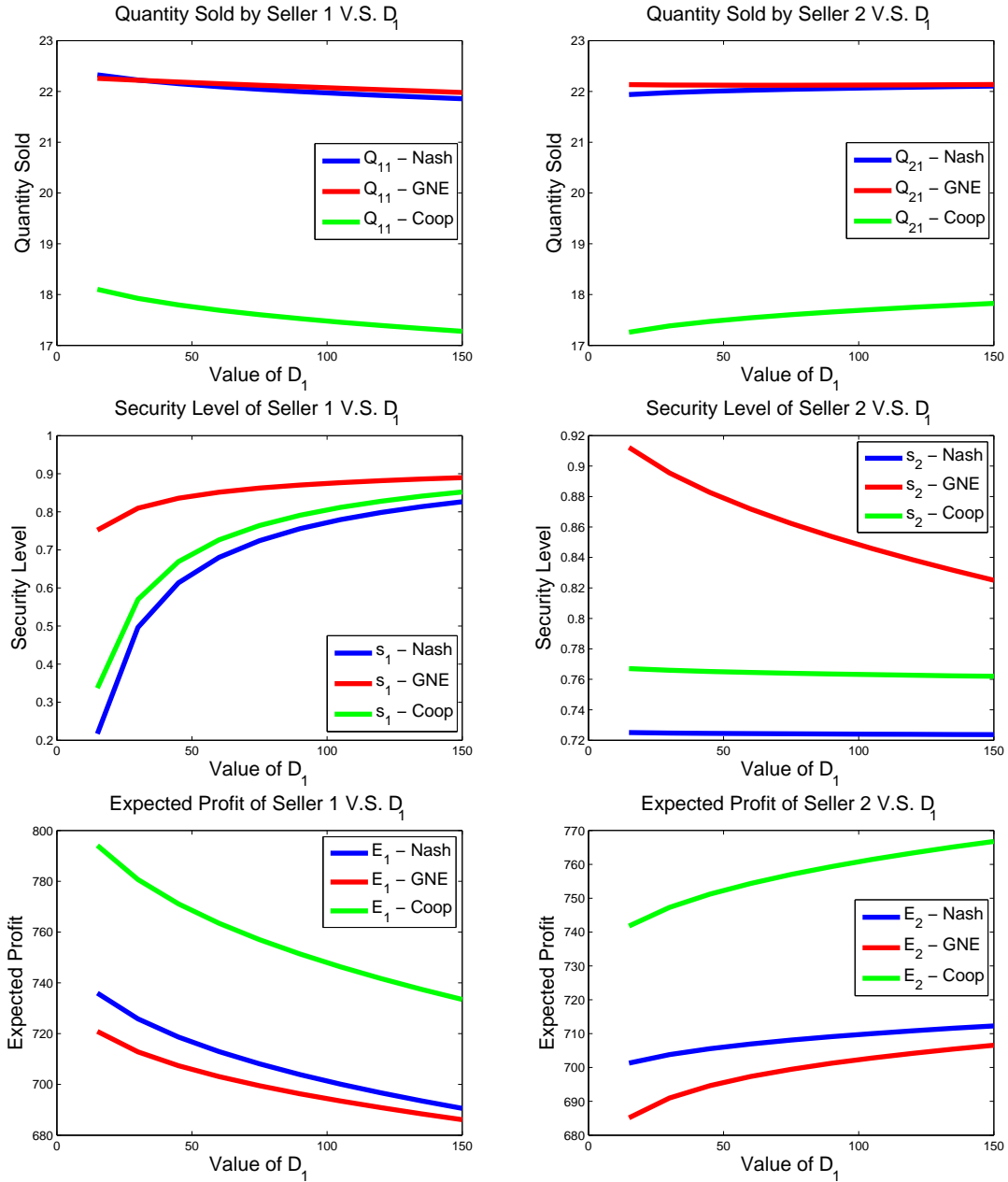


Figure 3.2: 2-Seller 1-Buyer solutions for all three games as D_1 increases.

The conclusion in this sensitivity analysis is that an increase in D_1 is disadvantageous to seller 1 but plays to seller 2's advantage. However, for the marketplace as a whole it is disadvantageous. Analogous results hold when D_2 is increased while keeping the rest the

same.

3.1.2 Sensitivity analysis of buyer parameters

Next we analyse how the games act when we vary how much the buyer values the product it is purchasing while the sellers are identical as seen for the base case parameters with the binding budget set to 75.

Case 1.3: δ_1

Our next sensitivity analysis will be conducted by varying the product demand coefficient of the buyer, δ_1 . This value affects the steepness of the slope of the price demand function, so the larger δ_1 is the less product the buyer will want to purchase.

Nash Game and GNE (Non-binding) Game						
δ_1	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
0.5	35.57	0.62	35.37	0.62	1223.03	1223.03
Base	22.04	0.72	22.04	0.72	708.09	708.09
1.5	15.99	0.78	15.99	0.78	494.81	494.81
2.0	12.54	0.81	12.54	0.81	378.94	378.94
2.5	10.31	0.83	10.31	0.83	306.37	306.37

Table 3.9: 2-Seller 1-Buyer Nash equilibrium solutions and GNE solutions when $\lambda = 0$ as δ_1 increases while implementing the quadratic transaction cost function.

Starting off we see that as δ_1 increases the volume of product sold by both sellers to the buyer decreases for all games. As δ_1 increases the change in Q_{i1} gets smaller for every iteration. A common trend for all three games is that the less the buyer values the product the more investment into cybersecurity the firms do. This is because the cybersecurity investment function is becoming cheaper. This is true for all three games.

In this case study we see the opposite trend between the binding GNE and the Nash/non-binding GNE games. That is, as the parameter δ_1 increases the binding game solutions for

GNE (Binding) Game						
δ_1	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
0.5	35.42	0.76	35.42	0.76	1218.02	1218.02
Base	22.12	0.86	22.12	0.86	699.47	699.47
1.5	16.09	0.91	16.09	0.91	483.10	483.10
2.0	12.65	0.94	12.65	0.94	364.75	364.75
2.5	10.42	0.95	10.42	0.95	290.19	290.19

Table 3.10: 2-Seller 1-Buyer Binding GNE solutions as δ_1 increases while implementing the quadratic transaction cost function.

the expected profits are becoming further apart from the non-binding solutions. However, both the Q_{i1} and s_i solutions are becoming closer together for the two games. This is due to the fact that since firms are selling less product, the investment function is becoming cheaper, but, since the budget implemented does not change, the binding game is forced to invest more than the Nash game.

Cooperative Game						
δ_1	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
0.5	29.44	0.67	29.44	0.67	1275.50	1275.50
Base	17.60	0.76	17.60	0.76	757.02	757.02
1.5	12.54	0.81	12.54	0.81	536.22	536.22
2.0	9.73	0.84	9.73	0.84	414.16	414.16
2.5	7.95	0.86	7.95	0.86	336.82	336.82

Table 3.11: 2-Seller 1-Buyer Cooperative solutions as δ_1 increases while implementing the quadratic transaction cost function.

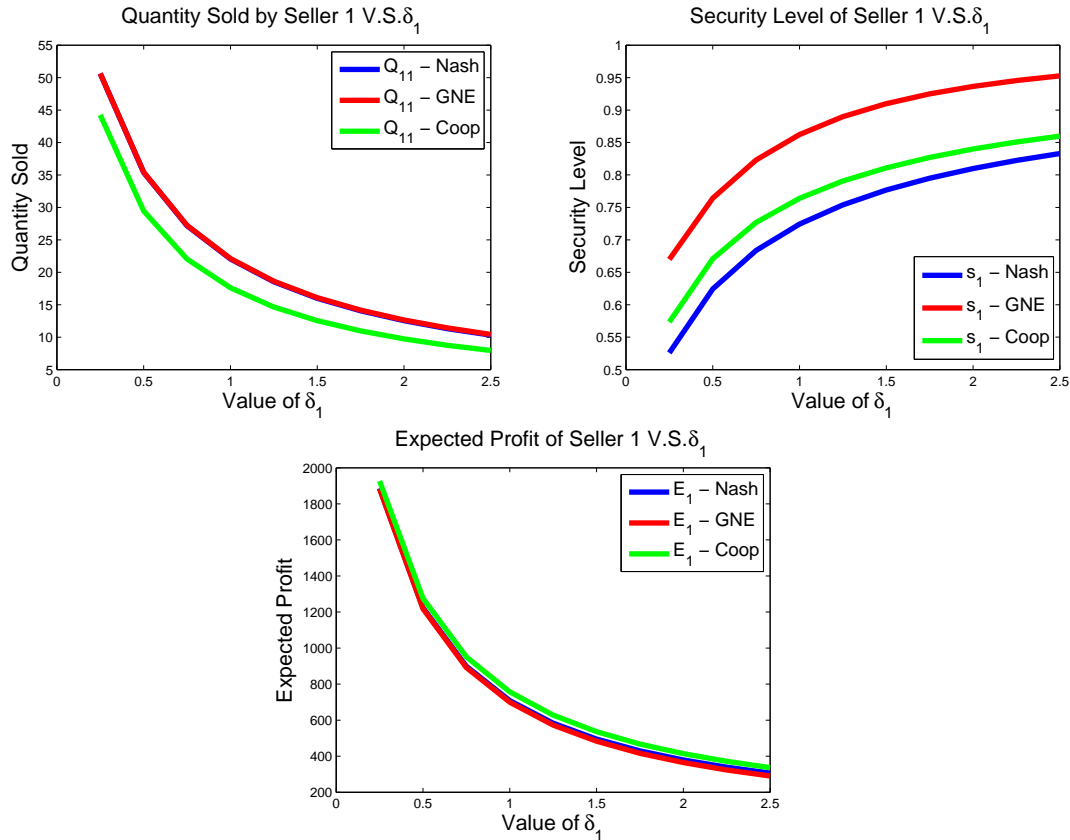


Figure 3.3: 2-Seller 1-Buyer solutions for all three games as δ_1 increases.

Again we see that when players cooperate their payoffs are greater than when they compete, even while selling less product. All three games show similar trends with little difference in the expected pay-off solutions.

The conclusion to this sensitivity analysis is that it is disadvantageous for all sellers of the marketplace if δ_1 increases. The buyer values the product being sold less and as a result purchases less from all buyers. We found this trend can be seen in all three games in both scenarios. We also conclude that when more sellers are added to the marketplace varying buyer parameters affects the marketplace as a whole instead of sellers individually as long as they have identical parameters.

Case 1.4: β_1

The next sensitivity analysis we will perform will be when we change β_1 , that is, how much the buyer values the average security level of the marketplace.

	Nash Game and GNE (Non-binding) Game					
β_1	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
Base	22.04	0.72	22.04	0.72	708.09	708.09
2	22.37	0.76	22.37	0.76	732.74	732.74
4	22.74	0.79	22.74	0.79	760.36	760.36
6	23.14	0.81	23.14	0.81	789.34	789.34
8	23.55	0.83	23.55	0.83	819.48	819.48

Table 3.12: 2-Seller 1-Buyer Nash equilibrium solutions and GNE solutions when $\lambda = 0$ as β_1 increases while implementing the quadratic transaction cost function.

As expected, as β_1 increases the security level for both sellers increases for both the Nash and Cooperative games. It is interesting to note however, the binding GNE game does the opposite. We see that in both the Nash games and cooperative games, security levels increase at roughly the same pace. As β_1 increases the overall increase in security is less for larger β_1 values. Even though the binding GNE game is showing a decreasing trend as β_1 increases, the average security level of the marketplace decreases by only 1.

	GNE (Binding) Game					
β_1	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
Base	22.12	0.86	22.12	0.86	699.47	699.47
2	22.49	0.86	22.49	0.86	727.74	727.74
4	22.87	0.86	22.87	0.86	757.90	757.90
6	23.25	0.85	23.25	0.85	788.45	788.45
8	23.62	0.85	23.62	0.85	819.40	819.40

Table 3.13: 2-Seller 1-Buyer Binding GNE solutions as β_1 increases while implementing the quadratic transaction cost function.

It is shown that as the buyer values cybersecurity more, both firms end up selling a higher volume of product. This is true for all three games. So overall the market place benefits as

the buyer becomes more aware of network security levels.

β_1	Cooperative Game					
	Q_{11}	s_1	Q_{21}	s_2	$E(U_1)$	$E(U_2)$
Base	17.60	0.76	17.60	0.76	757.02	757.02
2	17.86	0.81	17.86	0.81	783.67	783.67
4	18.17	0.84	18.17	0.84	813.57	813.57
6	18.49	0.86	18.49	0.86	844.91	844.91
8	18.83	0.88	18.83	0.88	877.48	877.48

Table 3.14: 2-Seller 1-Buyer Cooperative solutions as β_1 increases while implementing the quadratic transaction cost function.

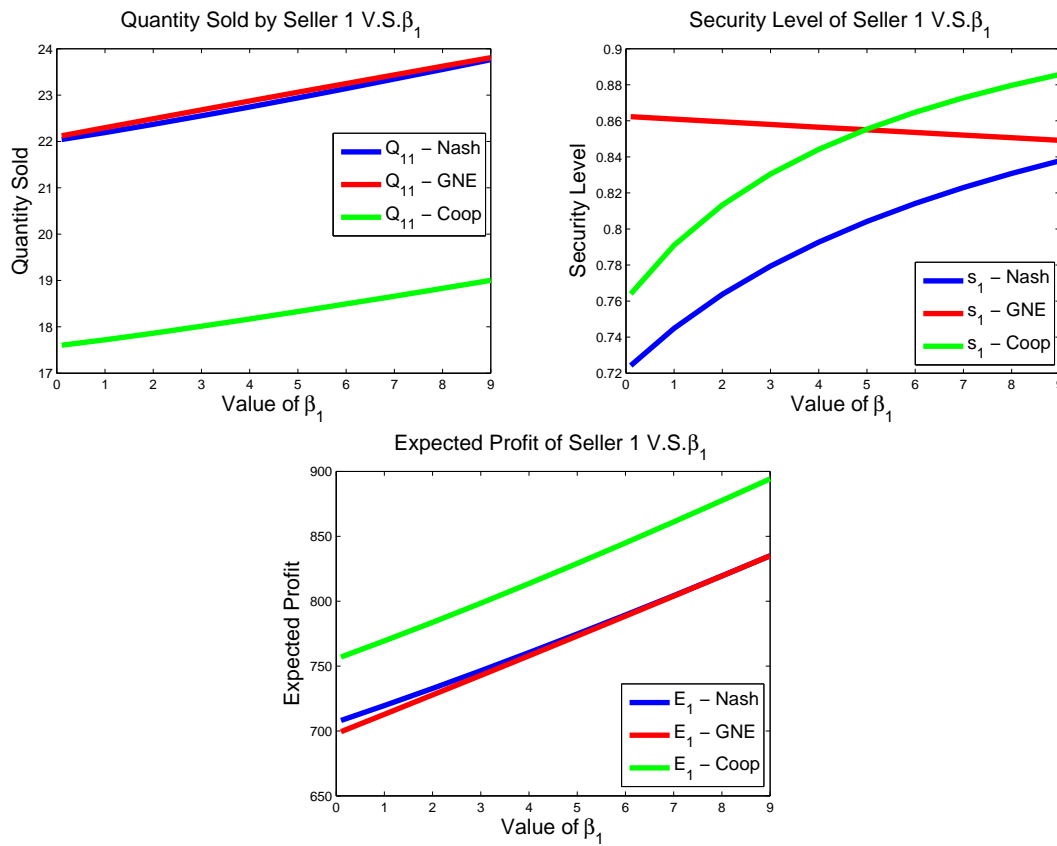


Figure 3.4: 2-Seller 1-Buyer solutions for all three games as β_1 increases.

Lastly, when comparing profits of the games we see that as β_1 increases, the binding GNE game solutions become closer to the Nash as previous examples have shown. Also for

all three games the expected profits of both firms increase as one buyer values security more. Again we see that when players cooperate their payoffs are greater than when they compete. Also, when the players are bound by a budget they are worse off.

Looking at the graphs below we see that it is true that as β_1 increases both Nash and binding GNE solutions are becoming more alike. We also see that there is a much greater difference in the cooperative solutions when compared to the competitive solutions, especially in quantity of product sold. However, it looks as though competitive solutions and cooperative solutions are increasing at the same rate.

We conclude that from this sensitivity analysis, the more the buyer values cybersecurity in the marketplace, the more sellers invest into cybersecurity resulting in a higher expected profit. It can also be seen that if the sellers are identical, then a change in the buyer's preferences results in the same changes for both sellers.

3.2 Case 2: 3-Seller, 2-Buyer

In our next case study we will conduct a sensitivity analysis on seller and buyer parameters while using a 3-seller 2-buyer marketplace. Since we have increased the number of sellers and buyers, we set the budget of the GNE game to be equal to 150. In addition, we increase the upper bounds such that:

$$0 < Q_{ij} < 200, \quad \forall i = (1, 2, 3), j = (1, 2).$$

In the Nash game seller 1 is trying to maximize the following expected utility function:

$$E(U_1) :=$$

$$\begin{aligned} & \left(- \sum_{i=1}^3 Q_{i1} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{11} + \left(- \sum_{i=1}^3 Q_{i2} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{12} \\ & - 10Q_{11} - 0.5Q_{11}^2 - Q_{11} - 10Q_{12} - 0.5Q_{12}^2 - Q_{12} - 75 + 75s_1 - (Q_{11} + Q_{12}) \left(\frac{1}{\sqrt{1-s_1}} - 1 \right), \end{aligned}$$

$$\text{s.t.} \quad \begin{cases} 0 \leq Q_{11} + Q_{12} \leq 200, \\ 0 \leq s_1 \leq 1. \end{cases}$$

Seller 2 is trying to maximize the following expected utility function:

$$E(U_2) :=$$

$$\begin{aligned} & \left(- \sum_{i=1}^3 Q_{i1} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{21} + \left(- \sum_{i=1}^3 Q_{i2} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{22} \\ & - 10Q_{21} - 0.5Q_{21}^2 - Q_{21} - 10Q_{22} - 0.5Q_{22}^2 - Q_{22} - 75 + 75s_2 - (Q_{21} + Q_{22}) \left(\frac{1}{\sqrt{1-s_2}} - 1 \right), \end{aligned}$$

$$\text{s.t.} \quad \begin{cases} 0 \leq Q_{21} + Q_{22} \leq 200, \\ 0 \leq s_2 \leq 1. \end{cases}$$

Seller 3 is trying to maximize the following expected utility function:

$$E(U_3) :=$$

$$\begin{aligned} & \left(- \sum_{i=1}^3 Q_{i1} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{31} + \left(- \sum_{i=1}^3 Q_{i2} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{32} \\ & - 10Q_{31} - 0.5Q_{31}^2 - Q_{31} - 10Q_{32} - 0.5Q_{32}^2 - Q_{32} - 75 + 75s_3 - (Q_{31} + Q_{32}) \left(\frac{1}{\sqrt{1-s_3}} - 1 \right), \end{aligned}$$

$$\text{s.t.} \quad \begin{cases} 0 \leq Q_{31} + Q_{32} \leq 200, \\ 0 \leq s_3 \leq 1. \end{cases}$$

We will now show that the constraint qualifications are satisfied for this Nash game. We have the following four constraints for each individual seller:

$$\begin{cases} g^1(\bar{x}) = -Q_{i1} \leq 0, \\ g^2(\bar{x}) = -Q_{i2} \leq 0, \\ g^3(\bar{x}) = Q_{i1} + Q_{i2} - 200 \leq 0, \\ g^4(\bar{x}) = -s_i \leq 0, \\ g^5(\bar{x}) = s_i - 1 \leq 0. \end{cases}$$

We need

$$\nabla g(\bar{x})\lambda = 0,$$

which gives us:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \lambda^i = 0.$$

Clearly this is only satisfied when $\lambda^i = 0, \forall i \in (1, 2, 3)$. Thus, satisfying the constraint qualification.

In the generalized Nash game seller 1 is trying to maximize the following expected utility function:

$$E(U_1) :=$$

$$\begin{aligned} & \left(- \sum_{i=1}^3 Q_{i1} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{11} + \left(- \sum_{i=1}^3 Q_{i2} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{12} \\ & - 10Q_{11} - 0.5Q_{11}^2 - Q_{11} - 10Q_{12} - 0.5Q_{12}^2 - Q_{12} - 75 + 75s_1 - (Q_{11} + Q_{12}) \left(\frac{1}{\sqrt{1-s_1}} - 1 \right), \end{aligned}$$

$$\text{s.t. } \begin{cases} 0 \leq Q_{11} + Q_{12} \leq 200, \\ 0 \leq s_1 \leq 1, \\ (Q_{11} + Q_{12}) \left(\frac{1}{\sqrt{1-s_1}} - 1 \right) + (Q_{21} + Q_{22}) \left(\frac{1}{\sqrt{1-s_2}} - 1 \right) + (Q_{31} + Q_{32}) \left(\frac{1}{\sqrt{1-s_3}} - 1 \right) - B. \end{cases}$$

Seller 2 is trying to maximize the following expected utility function:

$$E(U_2) :=$$

$$\begin{aligned} & \left(- \sum_{i=1}^3 Q_{i1} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{21} + \left(- \sum_{i=1}^3 Q_{i2} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{22} \\ & - 10Q_{21} - 0.5Q_{21}^2 - Q_{21} - 10Q_{22} - 0.5Q_{22}^2 - Q_{22} - 75 + 75s_2 - (Q_{21} + Q_{22}) \left(\frac{1}{\sqrt{1-s_2}} - 1 \right), \end{aligned}$$

$$\text{s.t. } \begin{cases} 0 \leq Q_{21} + Q_{22} \leq 200, \\ 0 \leq s_2 \leq 1, \\ (Q_{11} + Q_{12}) \left(\frac{1}{\sqrt{1-s_1}} - 1 \right) + (Q_{21} + Q_{22}) \left(\frac{1}{\sqrt{1-s_2}} - 1 \right) + (Q_{31} + Q_{32}) \left(\frac{1}{\sqrt{1-s_3}} - 1 \right) - B. \end{cases}$$

Seller 3 is trying to maximize the following expected utility function:

$$E(U_3) :=$$

$$\begin{aligned} & \left(- \sum_{i=1}^3 Q_{i1} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{31} + \left(- \sum_{i=1}^3 Q_{i2} + 0.1 \left(\frac{s_1 + s_2 + s_3}{3} \right) + 100 \right) Q_{32} \\ & - 10Q_{31} - 0.5Q_{31}^2 - Q_{31} - 10Q_{32} - 0.5Q_{32}^2 - Q_{32} - 75 + 75s_3 - (Q_{31} + Q_{32}) \left(\frac{1}{\sqrt{1-s_3}} - 1 \right), \end{aligned}$$

$$\text{s.t.} \begin{cases} 0 \leq Q_{31} + Q_{32} \leq 200, \\ 0 \leq s_3 \leq 1, \\ (Q_{11} + Q_{12}) \left(\frac{1}{\sqrt{1-s_1}} - 1 \right) + (Q_{21} + Q_{22}) \left(\frac{1}{\sqrt{1-s_2}} - 1 \right) + (Q_{31} + Q_{32}) \left(\frac{1}{\sqrt{1-s_3}} - 1 \right) - B. \end{cases}$$

Following the same methods used in the previous case study we see that $g^i(x) = 0$, $\forall i = 1, 2, 3, 4, 5$, leaving us with only $g^6(\bar{x}) = Q_{11} \left(\frac{1}{\sqrt{1-s_1}} - 1 \right) + Q_{21} \left(\frac{1}{\sqrt{1-s_2}} - 1 \right) - B$. Therefore, in the case of a binding constraint, we have $g^6(\bar{x}) = 0$. So then we want:

$$\lambda_6^i \nabla g^6(\bar{x}) = 0 \Rightarrow \lambda_6^i = 0,$$

as desired.

3.2.1 Sensitivity analysis of seller parameters

Case 2.1: Scaled A_{ij}

We first analyse how this new game acts when we scale two of the seller's transaction cost coefficients relative to the third. Here we set $A_{21} = 1$ and then set $A_{11} = (1 - \alpha)A_{21}$ and $A_{31} = (1 + \alpha)A_{21}$, where $\alpha \in [-1, 1]$.

Nash Game and GNE (Non-binding) Game												
α	Q_{11}	Q_{12}	s_1	Q_{21}	Q_{22}	s_2	Q_{31}	Q_{32}	s_3	$E(U_1)$	$E(U_2)$	$E(U_3)$
-1.0	6.94	19.55	0.69	11.59	19.61	0.65	35.06	19.80	0.50	598.46	723.27	1681.58
-0.5	10.59	19.60	0.66	14.14	19.64	0.63	21.26	19.71	0.58	735.00	854.61	1132.37
Base	14.74	19.65	0.63	14.74	19.65	0.63	14.74	19.65	0.63	889.14	889.14	889.14
0.5	21.26	19.71	0.58	14.14	19.64	0.63	10.59	19.60	0.66	1132.37	854.61	735.00
1.0	35.05	19.80	0.50	11.59	19.61	0.65	6.94	19.55	0.69	1681.58	723.27	598.46

Table 3.15: 3-Seller 2-Buyer Nash and Non-binding GNE equilibrium solutions as A_{11} and A_{31} are scaled values of A_{21} , while implementing the quadratic transaction cost function.

The case when $\alpha = -1$ implies the transaction cost is greatest for seller 1 and smallest for seller 3 (vice-versa for $\alpha = 1$). We denote $\alpha = 0$ the Base case since $A_{11} = A_{21} = A_{31}$ for this value. As expected as α increases seller 1 starts selling more product to buyer 1 while seller 3 starts selling less. This is true for all three games. As a result the expected profits of these two firms follow the same trends. We note that when tables contain a " \approx " the solution is approaching to its upper or lower bound. The solutions are still within the interior of the set.

GNE (Binding) Game												
α	Q_{11}	Q_{12}	s_1	Q_{21}	Q_{22}	s_2	Q_{31}	Q_{32}	s_3	$E(U_1)$	$E(U_2)$	$E(U_3)$
-1.0	6.98	19.65	0.86	11.64	19.68	0.84	35.06	19.77	0.77	587.92	710.86	1659.51
-0.5	10.63	19.68	0.85	14.18	19.70	0.83	21.29	19.73	0.81	722.10	840.20	1114.97
Base	14.77	19.70	0.83	14.77	19.70	0.83	14.77	19.70	0.83	874.19	874.19	874.19
0.5	21.29	19.73	0.81	14.18	19.70	0.83	10.63	19.68	0.85	1114.97	840.20	722.10
1.0	35.06	19.77	0.77	11.64	19.68	0.84	6.98	19.65	0.86	1659.51	710.86	587.92

Table 3.16: 3-Seller 2-Buyer Binding GNE equilibrium solutions as A_{11} and A_{31} are scaled values of A_{21} , while implementing the quadratic transaction cost function.

It is interesting to see that as α deviates from the Base case, seller 2 sells less and earns less, even though the expected utility function is staying the same. We also see that in the cooperative game when $|\alpha| = 1$ both seller 2 and either seller 1 or 3 sell close to no product to buyer 1. This result occurs since when players cooperate they benefit as a whole when the seller with the cheapest transaction cost sells to buyer 1.

Cooperative Game												
α	Q_{11}	Q_{12}	s_1	Q_{21}	Q_{22}	s_2	Q_{31}	Q_{32}	s_3	$E(U_1)$	$E(U_2)$	$E(U_3)$
-1.0	≈ 0	12.98	0.81	≈ 0	12.98	0.81	44.04	14.69	0.48	555.26	555.26	2578.89
-0.5	6.24	13.29	0.75	9.41	13.50	0.72	19.03	13.93	0.64	841.12	988.92	1430.14
Base	11.03	13.57	0.71	11.03	13.57	0.71	11.03	13.57	0.71	1063.20	1063.20	1063.20
0.5	19.03	13.93	0.64	9.41	13.50	0.72	6.24	13.29	0.75	1430.14	988.92	841.12
1.0	44.04	14.69	0.48	≈ 0	12.98	0.81	≈ 0	12.98	0.81	2578.89	555.26	555.26

Table 3.17: 3-Seller 2-Buyer Cooperative equilibrium solutions as A_{11} and A_{31} are scaled values of A_{21} , while implementing the quadratic transaction cost function.

We again see similar trends in the security levels of the sellers as in the first case study. That is, as sellers start to sell more product they invest less into cybersecurity as a result of the increasing investment function. We note that for the following graphs, those of seller 1 are the inverse of seller 3. Looking at the graphs we see that there are cases where the cooperative game's expected profits are lower than the competitive games. However, we still see that the payoffs of the Nash game are still greater than those of the Binding GNE game. We also now see cases where the cooperative game has a higher quantity of product sold over the competitive games, though only for sellers 1 and 3.

The conclusion to this case study is that increasing the market size shows similar trends, such that an increase to A_{ij} is detrimental to seller i . Overall, increasing transaction cost is not beneficial to the marketplace. We also see that when sellers are not identical, the cooperative game is not always more beneficial for all players.

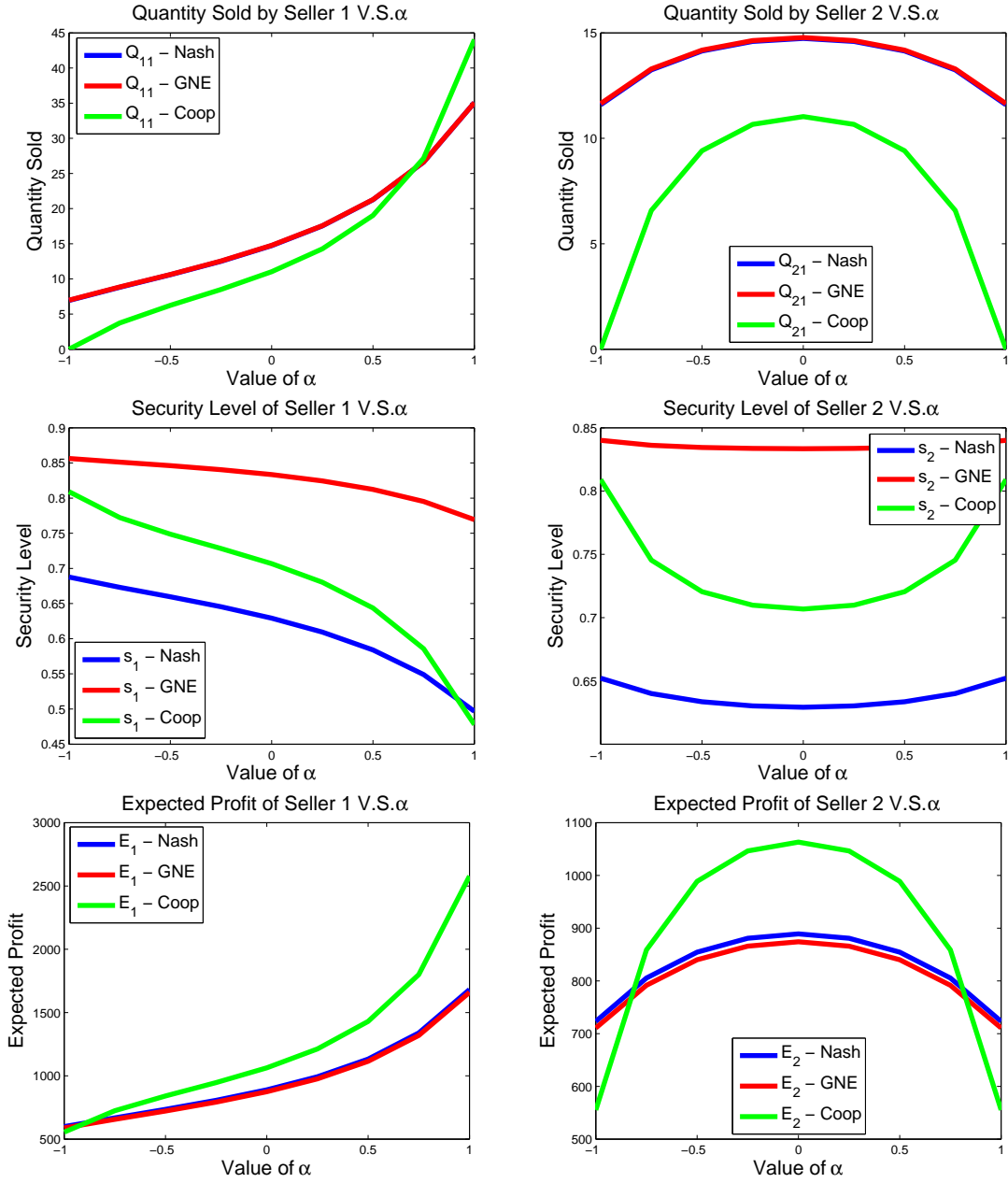


Figure 3.5: 3-Seller 2-Buyer solutions for all three games as A_{11} and A_{31} are scaled relative to A_{21} .

Case 2.2: Scaled D_i

In the next case study we conduct a sensitivity analysis similar to the previous while looking at D_i . Here, we set $D_2 = 75$ with $D_1 = D_2(1 - \alpha)$ and $D_3 = D_2(1 + \alpha)$. We set $A_{i1} = 0.5$ as it is stated in the Base case.

Nash Game and GNE (Non-binding) Game												
α	Q_{11}	Q_{12}	s_1	Q_{21}	Q_{22}	s_2	Q_{31}	Q_{32}	s_3	$E(U_1)$	$E(U_2)$	$E(U_3)$
-1.0	18.00	20.35	0.75	18.20	20.62	0.60	16.62	17.68	≈ 0	929.25	961.72	923.14
-0.5	17.57	19.49	0.70	17.68	19.64	0.61	17.85	19.86	0.38	904.03	921.96	947.60
Base	17.69	19.66	0.61	17.69	19.66	0.61	17.69	19.66	0.61	923.24	923.24	923.24
0.5	17.85	19.86	0.38	17.68	19.64	0.61	17.57	19.49	0.70	947.60	921.96	904.03
1.0	16.62	17.68	≈ 0	18.20	20.62	0.60	18.00	20.35	0.75	923.14	961.72	929.25

Table 3.18: 3-Seller 2-Buyer Nash and Non-binding GNE equilibrium solutions as D_1 and D_3 are scaled values of D_2 , while implementing the quadratic transaction cost function.

Similar to Case 2, as D_i increases, seller i invests more into cybersecurity. This is true for all three games. We see that in both Nash and cooperative games, small deviations from the base case result in seller 2's profit slightly decreasing. However, on the bounds ($|\alpha| = 1$) the profit increases to a much greater value. This is because either seller 1 or 3 take no damage from a cyberattack and therefore their security level goes to the lower bound.

GNE (Binding) Game												
α	Q_{11}	Q_{12}	s_1	Q_{21}	Q_{22}	s_2	Q_{31}	Q_{32}	s_3	$E(U_1)$	$E(U_2)$	$E(U_3)$
-1.0	17.63	19.57	0.91	17.73	19.70	0.86	17.88	19.89	≈ 0	864.41	896.78	976.58
-0.5	17.68	19.63	0.87	17.73	19.70	0.82	17.80	19.79	0.72	890.13	907.01	931.05
Base	17.73	19.70	0.82	17.73	19.70	0.82	17.73	19.70	0.82	908.98	908.98	908.98
0.5	17.80	19.79	0.72	17.73	19.70	0.82	17.68	19.63	0.87	931.05	907.01	890.13
1.0	17.88	19.89	≈ 0	17.73	19.70	0.86	17.63	19.57	0.91	976.58	896.78	864.41

Table 3.19: 3-Seller 2-Buyer Binding GNE equilibrium solutions as D_1 and D_3 are scaled values of D_2 , while implementing the quadratic transaction cost function.

The most interesting characteristic of this game we see is that there are cases where expected profits are less in the Nash game than the binding GNE game. The cause of this is again the fact that the solution is going to its lower bound. Any interior solution to this Nash game still has a higher expected profit than the binding GNE game.

Cooperative Game												
α	Q_{11}	Q_{12}	s_1	Q_{21}	Q_{22}	s_2	Q_{31}	Q_{32}	s_3	$E(U_1)$	$E(U_2)$	$E(U_3)$
-1.0	13.13	16.73	0.79	13.57	17.58	0.66	11.14	6.44	≈ 0	1257.31	1326.92	809.16
-0.5	12.30	12.96	0.77	12.57	13.52	0.70	12.96	14.27	0.51	1085.52	1128.60	1188.43
Base	12.61	13.58	0.69	12.61	13.58	0.69	12.61	13.58	0.69	1132.73	1132.73	1132.73
0.5	12.96	14.27	0.51	12.58	13.52	0.70	12.30	12.96	0.77	1188.43	1128.60	1085.52
1.0	11.14	6.44	≈ 0	13.57	17.58	0.66	13.13	16.73	0.79	809.16	1326.92	1257.31

Table 3.20: 3-Seller 2-Buyer Cooperative equilibrium solutions as D_1 and D_3 are scaled values of D_2 , while implementing the quadratic transaction cost function.

We see once again that the cooperative game sells less than the competitive games, with security levels in between while having a higher expected profit when solutions lie in the interior. Again, looking at the graphs, those of seller 3 are reciprocals of the graphs of seller 1 so we do not include them in the following figure.

The conclusion of this sensitivity analysis is that if $D_i = 0$ then seller i does not invest in any cybersecurity as expected. We see that this game follows the trends as described in Case 2 for interior solutions and as one seller invests more into cybersecurity the others tend to invest less and piggyback off the investing seller. The same results can be shown when using D_1 or D_3 as the scaling parameter.

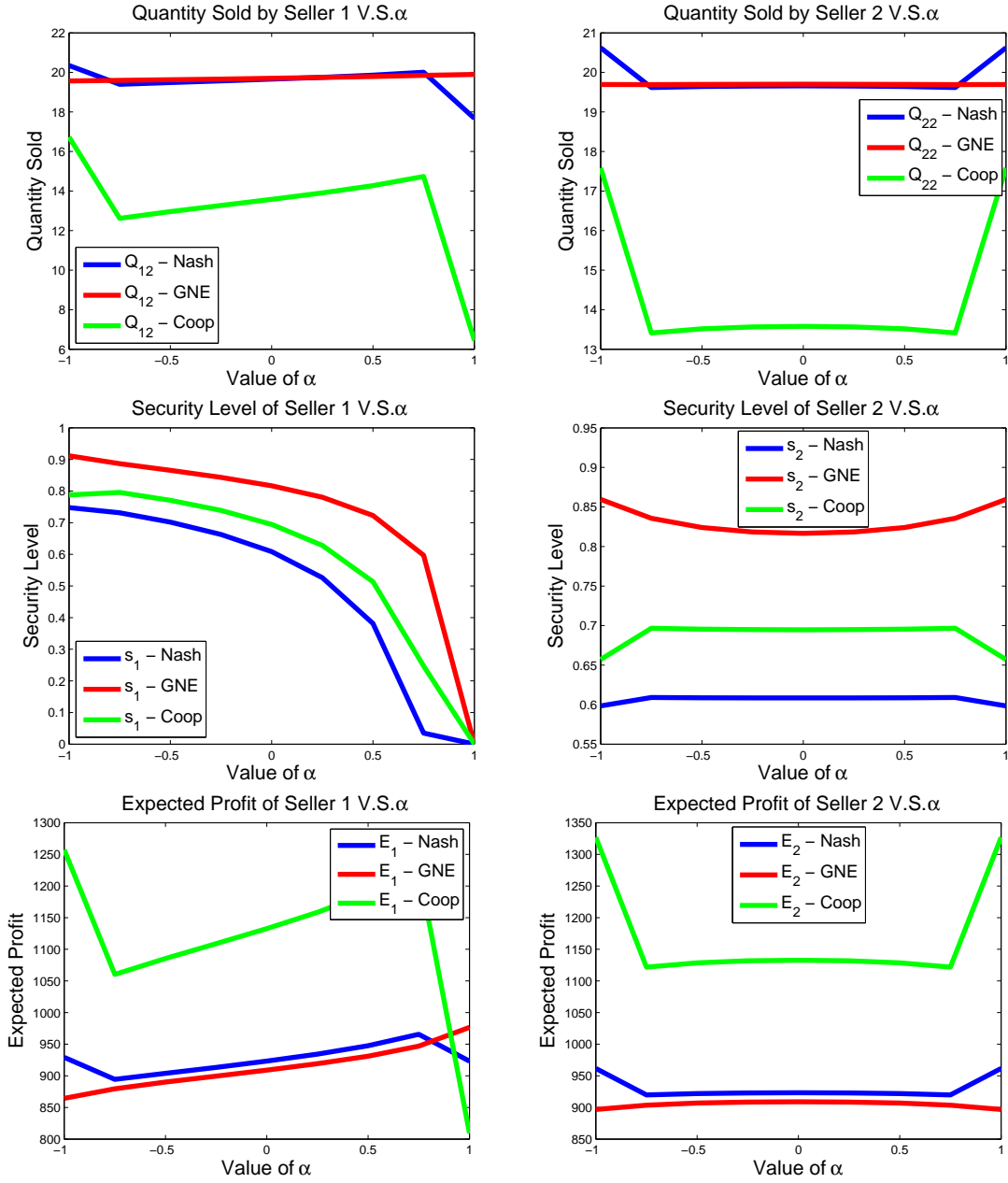


Figure 3.6: 3-Seller 2-Buyer solutions for all three games as D_1 and D_3 are scaled relative to D_2 .

3.2.2 Sensitivity analysis of buyer parameters

Case 2.3: Scaled δ_j

For this sensitivity analysis, since there are only two buyers we set the scaling parameters to be: $\delta_1 = 1(1 - \alpha)$ and $\delta_2 = 1(1 + \alpha)$ with the rest of the parameters being equal to the base case values. Since the sellers are identical in this case study, the solutions for all sellers are identical. Therefore, we only write a general form of the equilibrium values in the tables below.

Nash Game and GNE (Non-binding) Game				
α	Q_{i1}	Q_{i2}	s_i	$E(U_i)$
-1.0	9.85	165.37	≈ 0	8052.11
-0.5	12.65	35.43	0.54	1227.26
Base	17.69	19.66	0.61	923.24
0.5	29.51	13.62	0.57	1163.27
1.0	88.86	10.45	0.26	4138.80

Table 3.21: 3-Seller 2-Buyer Nash and Non-binding GNE equilibrium solutions as δ_1 and δ_2 are relatively scaled, while implementing the quadratic transaction cost function.

As the price demand curve becomes more steep for one buyer, it becomes less steep for the other buyer. Therefore, from the tables we see a trade-off between buyers as α increases. It is interesting to see that for $\alpha > 0$ the increase of Q_{i1} is less than the increase in Q_{i2} for $\alpha < 0$. This is a result of $A_{i1} > A_{i2}$.

Since it costs more for firms to sell to buyer 1 we see that the expected profits are much higher for $\alpha < 0$ for all three games. Security levels are greatest for the Base case. We note that both Nash game and binding GNE game share close equilibrium values, except for security levels.

Looking at the figure of this game we note that each of the four graphs have the identical

GNE (Binding) Game				
α	Q_{i1}	Q_{i2}	s_i	$E(U_i)$
-1.0	9.88	177.85	0.38	8076.10
-0.5	12.67	35.47	0.76	1215.58
Base	17.73	19.70	0.82	908.98
0.5	29.56	13.64	0.79	1150.78
1.0	88.82	10.45	0.56	4130.40

Table 3.22: 3-Seller 2-Buyer Binding GNE equilibrium solutions as δ_1 and δ_2 are relatively scaled, while implementing the quadratic transaction cost function.

Cooperative Game				
α	Q_{i1}	Q_{i2}	s_i	$E(U_i)$
-1.0	6.85	177.99	0.01	8150.66
-0.5	8.84	25.26	0.64	1480.00
Base	12.61	13.58	0.69	1132.73
0.5	22.09	9.30	0.66	1361.08
1.0	88.82	7.11	0.31	4208.99

Table 3.23: 3-Seller 2-Buyer Cooperative equilibrium solutions as δ_1 and δ_2 are relatively scaled, while implementing the quadratic transaction cost function.

curves for the other two sellers. We see that in this game the cooperative game follows closely to the competitive games. It still produces higher expected profits, but not by much. The security levels of the three games show a decreasing trend when deviating from the base case. Last, we see that the quantity sold graphs form opposite trends as α increases as expected.

The conclusion to this sensitivity analysis is that it is disadvantageous for all sellers of the marketplace if δ_j increases. The buyer values the product being sold less and as a result purchases less from all sellers. We found this trend can be seen in all three games in both scenarios. We also conclude that when more sellers are added to the marketplace, varying buyer parameters affects the marketplace as a whole instead of sellers individually as long as they have identical parameters. We note that although increasing δ_j is bad for all sellers, it can be beneficial if there is a trade-off such that another buyer's demand coefficient decreases, the market could end up with higher expected profits.

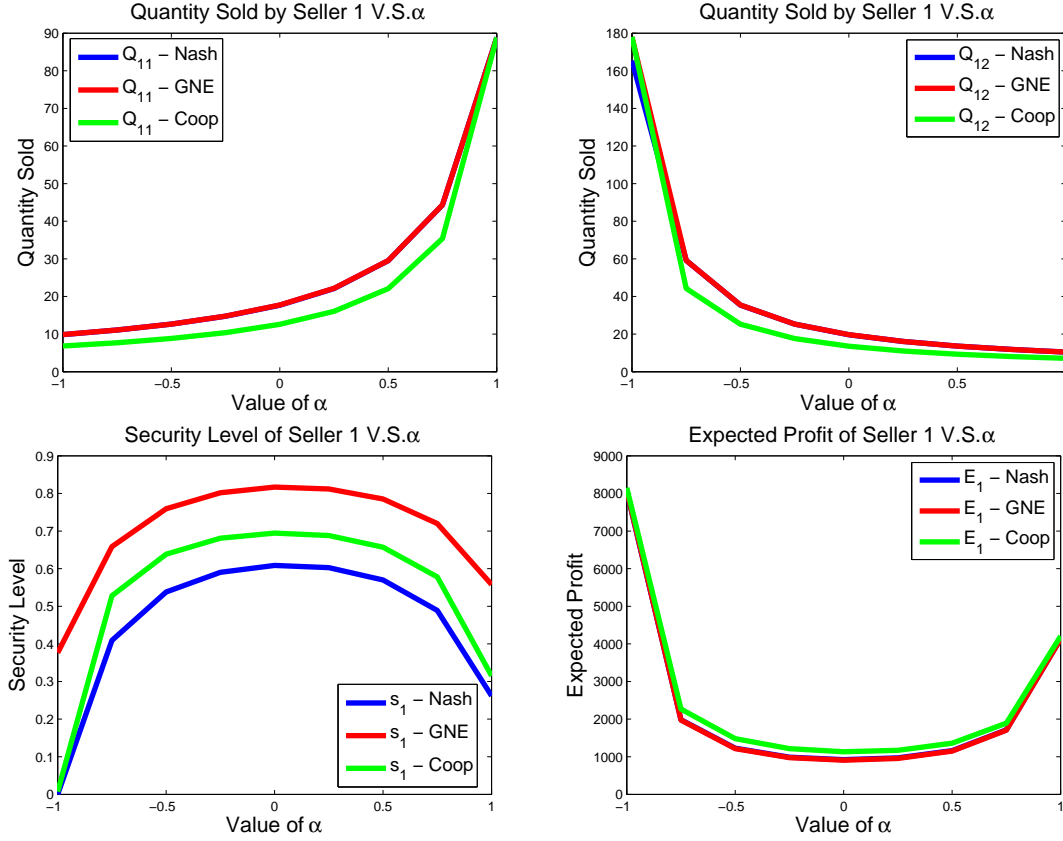


Figure 3.7: 3-Seller 2-Buyer solutions for all three games as δ_j are relatively scaled.

3.3 Multi-parameter Variation

For our last sensitivity analysis we look at the 3-seller 2-buyer game when we scale two independent parameters. We set $A_{21} = 1$ while letting $A_{11} = A_{21}(1-\alpha)$ and $A_{31} = A_{21}(1+\alpha)$ for $\alpha \in (-1, 1)$. Next, we set $\delta_1 = 1 - \sigma$ and $\delta_2 = 1 + \sigma$ for $\sigma \in (-1, 1)$.

We present the following graphs in order to compare the security levels to the profits of these games. We note that the Nash game and the GNE game share very close solutions and trends so we omit the Nash solution graphs.

From figures 3.8 and 3.9 it is seen that both GNE and cooperative solutions have similar behaviours. We see that for all sellers there is a negative relationship between the security level invested in and expected profits. All games share the fact that sellers earn a consistently higher profit for smaller values of σ , although these might not be the maximum profits for all sellers.

We see again, that as δ deviates from the base case all sellers invest less in cybersecurity, which implies they earn a greater profit. Also, as α deviates from the base case seller 2 shows a decrease in profit with a slight increase in security. Whereas, sellers 1 and 3 have a trade-off in expected profit as α increases from -1 to 1.

When comparing the GNE game to the cooperative game we see that both exhibit similar trends. However, the expected profits graph for all three sellers in the cooperative game is shifted up compared to the GNE game. Also, the security levels of the cooperative game are lower than in the GNE game, as in other case studies.

One characteristic that stands out between the two games is that the graphs for seller 2 in the cooperative game seem to differentiate more for changing values of α . That is, as α increases or decreases from the Base case, seller 2 has a greater increase in security level and a greater decrease in expected profits when compared to the competitive games.

The conclusion to this sensitivity analysis is that when there are 2 parameters being varied, one can see similar trends shown by each function as in when only one parameter is being changed. There are the cases however, when the change in one parameter may null the effects on the utility function that another parameter has, thus showing little to no change when these parameters change.

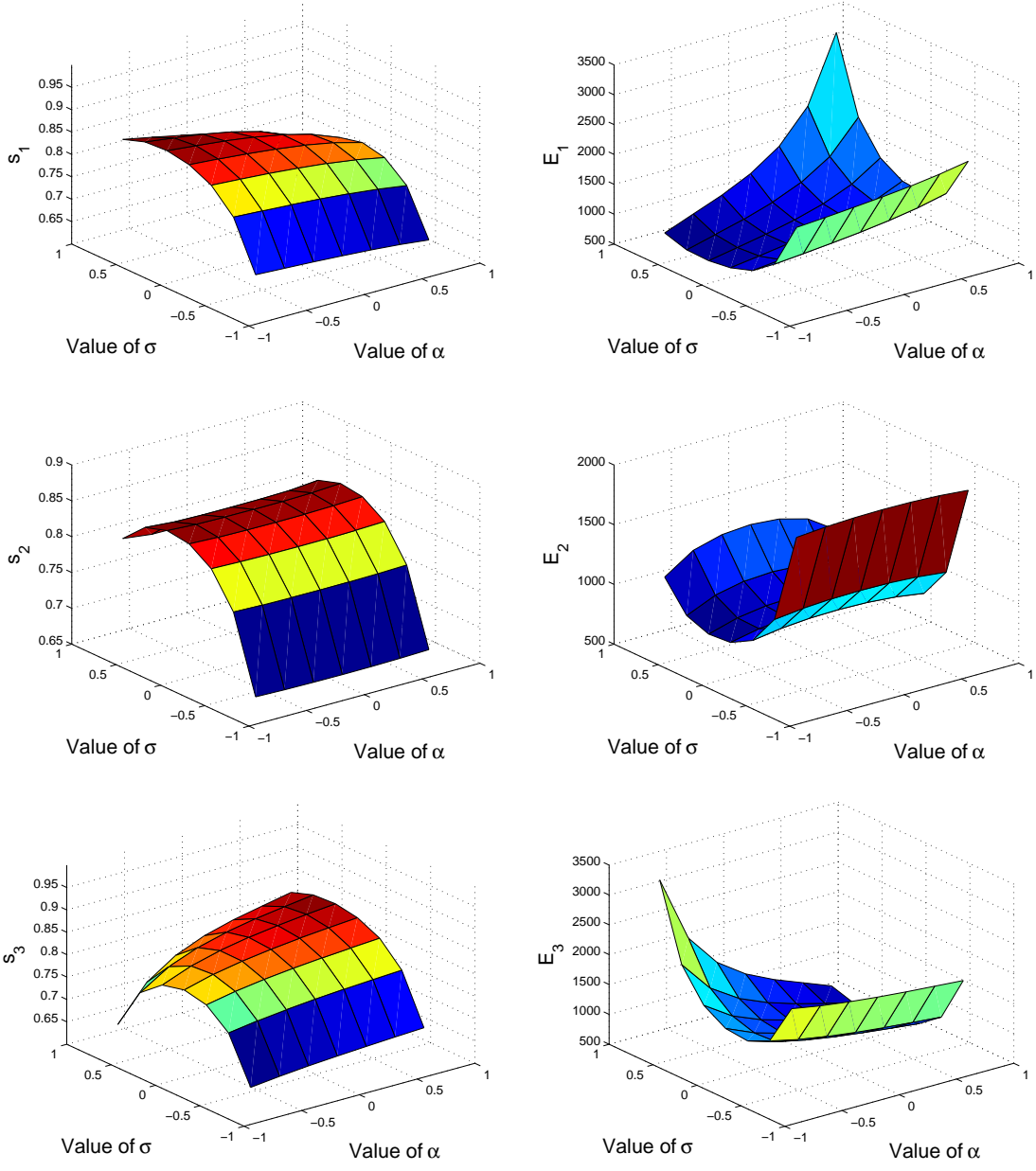


Figure 3.8: 3-Seller 2-Buyer solutions for the GNE game as α and σ scale the parameters.

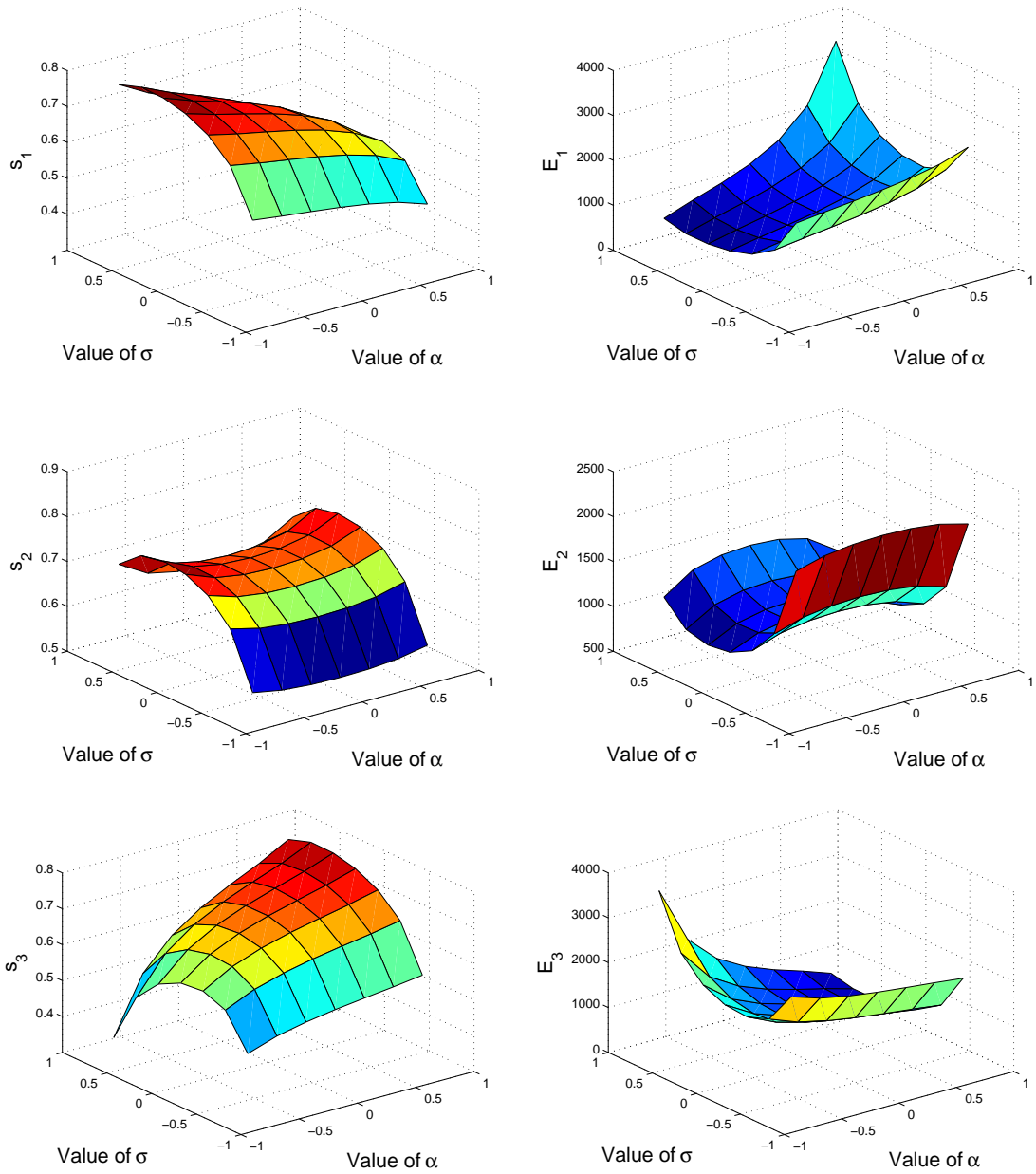


Figure 3.9: 3-Seller 2-Buyer solutions for the Cooperative game as α and σ scale the parameters.

Chapter 4

Conclusion And Further Work

4.1 Conclusion

In this thesis we developed economic games in a marketplace scenario in which we investigated interaction between sellers and buyers with increasing players and resources.

We started with a competitive Nash game and we developed a generalized Nash game and a cooperative game. After formulating the KKT system of the three games we performed analysis of these systems. After conducting numerical simulations on the three games we found that expected payoffs of a Nash game are always better than when sellers share a binding constraint. If the constraint however, is non-binding the solution will also be a solution to the Nash game. It was also shown that when players cooperate in these marketplace cybersecurity games the overall expected utility of the market is greater than when sellers compete with one another when we have interior solutions. If the game contains corner solutions then the expected payoffs for when sellers cooperate may not be greater than when

they compete. As a result security levels in a cooperative game are higher than in a Nash game but not as great as a binding GNE game where there is a constraint forcing sellers to invest in cybersecurity.

We showed that the trends seen in the case studies where one parameter is being varied can also be seen when multiple parameters are changing. When buyers are added to the game the marketplace has a greater overall expected profit, and as sellers are added to the game all sellers expected profits decrease.

4.2 Further Work

By introducing two new types of economic games our model can be further studied in different marketplace situations. More advanced models can also be formulated for cooperative games where not all sellers in the marketplace decide to cooperate with one another creating separate coalitions.

Further studies can also be conducted in this marketplace scenario by formulating non-convex/concave functions and evaluating the game. However, if this is done the method we used is not viable since in order to solve a KKT system all functions must be convex/concave. It could be modelled where buyers are aware of the security level of the marketplace, where each buyer isn't equally aware as others, or where, how if buyers are aware of the security levels of each individual seller, how that would effect how the sellers decide to invest.

Since it was shown that when the GNE game is binding, sellers invest in more security than the Nash game, one could study the effect a minimum enforced security level per buyer would have on the associated games. Another field which this opens up more work in is

developing appropriate insurance schemes in the case a cyberattack occurs on a specific market.

All of these can be studied in single games or a cross comparison of the three games described in this paper to further study if the marketplace as a whole benefits when sellers agree to cooperate with one another. More cooperative games could be formed, such that sellers cooperate in order to maximize average market security and agree to work together in furthering the technology used to prevent and defend cyberattacks.

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Appendix A

Source Code

```
lb = [0,0,0,0,0,0,0,0,0,0,0,0,0];
lb_reg=lb(1:9);
lbGNE_bind=lb(1:10);
ub = [M1,M1,1,M2,M2,1,M3,M3,1,+Inf,+Inf,+Inf,+Inf];
ub_coop = [M1+M2+M3,M1+M2+M3,1,M1+M2+M3,M1+M2+M3,1,M1+M2+M3,M1+M2+M3,1];
ub_reg=ub(1:9);
ubGNE_bind=ub(1:10);

rng default
xx = rand(13,inpts);
% inpts is the number of initial values of the KKT systems

for k=1:inpts
x0(1,k)=10*xx(1,k);
x0(2,k)=10*xx(2,k);
x0(3,k)=xx(3,k);
x0(4,k)=10*xx(4,k);
x0(5,k)=10*xx(5,k);
x0(6,k)=xx(6,k);
x0(7,k)=10*xx(7,k);
```

```
x0(8,k)=10*xx(8,k);
x0(9,k)=xx(9,k);
x0(10,k)=xx(10,k);
x0(11,k)=xx(11,k);
x0(12,k)=xx(12,k);
x0(13,k)=xx(13,k);

[y(k,:),res] = lsqnonlin(@KKT,x0(1:9,k),lb_reg,ub_reg);
[z(k,:),res] = lsqnonlin(@KKTGNE_bind,x0(1:10,k),lbGNE_bind,ubGNE_bind);
[v(k,:),res] = lsqnonlin(@KKTCoop,x0(1:9,k),lb_reg,ub_coop);
end
```