An Energy Balance Model of Climate Change
Incorporating the Roles of Carbon Dioxide and Water Vapour as Greenhouse Gases

by

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ABSTRACT

AN ENERGY BALANCE MODEL OF CLIMATE CHANGE

INCORPORATING THE ROLES OF CARBON DIOXIDE

AND WATER VAPOUR AS GREENHOUSE GASES

Brady Dortmans Advisor: University of Guelph, 2017 Dr. Allan R. Willms

The Arctic ice-caps have been melting at an alarming rate, possibly due to anthropogenic forcing. Herein an energy balance model is constructed to explore the forcing factors of Earth’s climate, particularly the greenhouse gases carbon dioxide and water vapour. The effects of greenhouse gases on the Arctic climate are explored incorporating two non-linear mechanisms: ice-albedo feedback and water vapour feedback. The roles of ocean and atmospheric heat transport are also explored. The model explores the possibility of multiple climate states in the Arctic as well as tipping points involving the forcing factors. This model is also applied to the Tropics and Antarctic. In addition, the energy balance model provides a plausible explanation to the Pliocene Paradox when the Arctic was ice-free but with similar forcing factors as today.
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Chapter 1

Introduction

The atmospheres of the Earth and of other planets have been modelled in many different ways ranging from simple slab models to complicated PDE models. Energy balance slab models are commonly used to describe the behaviour of physical properties in the atmosphere while remaining as simple and general as possible. This type of simplified model is sometimes referred to as a conceptual model. In slab models, the atmosphere is treated as a uniform slab, with the temperature throughout the slab having a single value $T_A$; see Figure 1.1. More complicated models, such as global circulation models (GCM), use more detailed descriptions of the Earth using equations that account for the physical processes that vary across the Earth. Typically, when studying a new physical process, a conceptual model is used as a starting point. This way the effect of any new feature added to the model can be ascertained with minimal work, and a general expectation of its effect in a more detailed climate model can be predicted.

In this thesis, an energy balance slab model is used to describe the effects of several feedback mechanisms on the temperature of the atmosphere and the temperature of the surface. Previous studies of slab models have ignored feedback mechanisms. This model investigates some physical processes and
parameters that exist in the current climate. The model attempts to find multi-stability in certain regions of the Earth.

The incoming solar radiation, denoted $Q$, reaches the surface as shortwave radiation and is partially absorbed by the surface. The remaining shortwave radiation not absorbed is reflected back into space. The shortwave radiation, both incoming and reflected, passes through the atmosphere and is not absorbed. Clouds over the Earth can reflect incoming solar radiation back to space before reaching the surface. For the current work, clouds are assumed to be not present. The effect of clouds can be accounted for by a small reduction of $Q$. The albedo, $\alpha$, is the fraction of shortwave radiation reflected by the surface. The albedo changes in value depending on the reflective properties of the surface and in particular the amount of ice present. The surface emits the absorbed radiation as longwave radiation acting as a black body. Since the surface acts as a black body, the power radiated by the surface $I_S$ can be expressed as

$$I_S = \epsilon_E \sigma T_S^4.$$  

(1.1)

This is known as the Stefan-Boltzmann law. The law describes the power emitted from a black body in terms of temperature from quantum mechanics. The constant of proportionality $\sigma$ is called the Stefan-Boltzmann constant and the surface temperature is denoted $T_S$. The emissivity of the surface, $\epsilon_E$, $0 < \epsilon_E \leq 1$, measures how well the surface emits radiation compared to a perfect black body at $\epsilon = 1$. The Earth is considered very close to a black body, meaning that the emissivity term $\epsilon_E$ is close to 1, reducing (1.1) to

$$I_S = \sigma T_S^4.$$  

(1.2)

After the longwave radiation leaves the surface, certain gases in the atmosphere absorb the longwave radiation. The absorbing gases in the atmosphere are known as greenhouse gases. These greenhouse gases trap heat in the atmosphere and this process is one of the main factors determining
the temperature. The main greenhouse gases in the atmosphere that are considered in this model are carbon dioxide, CO$_2$, and water vapour, H$_2$O. The heat trapped in the atmosphere rapidly disperses to the other gases present, primarily Nitrogen gas and Oxygen gas, and these gases are the primary source of emission or heat loss of the atmosphere. The gases in the atmosphere emit radiation, which either is reabsorbed by the surface or escapes to space, along with the remaining surface radiation not absorbed by the atmosphere. The power radiated by the atmosphere, $I_A$, is described by the Stefan Boltzmann law for the atmospheric temperature, $T_A$, as

$$I_A = \epsilon \sigma T_A^4.$$ (1.3)

The atmosphere is not a perfect black body. Thus the emissivity of the atmosphere is $\epsilon < 1$. Typically, the emissivity changes with wavelength and temperature for a given material. For our model, the emissivity will be held constant. There are heat storages within the Earth. However, we assume there is negligible transfer of heat from the storages to the surface or atmosphere.

Next, we introduce symmetry. That is, we will assume that our model is independent of longitude $\theta$. For example, any point on the equator will be taken as equivalent. The North and South poles are not assumed to be the same. In terms of latitude, $\phi$, we will investigate three points, 0, $\pi/2$ and $-\pi/2$ radians, which are the equator, the north pole, and the south pole, respectively. Since this is a model of long-term behaviour, we will take the time frame to be over multiple years. This means that we will ignore all daily and seasonal variations and look at an averaged climate over multiple years. Seasonal variations will have small changes in temperature values but the overall behaviour is unchanged. This also includes no variation in the radiation from the sun. Ice is present if the temperature of the surface, $T_S$, is below zero, and ice is absent if $T_S$ is above zero. This change is immediate and the albedo, $\alpha$, will have two values to choose from and will be in the form of a step function in $T_S$. Later on, we will consider a gradual change
in albedo from ice-covered to ice-free.

1.1 Model by Payne et al. 2015

In the work of Payne et al. [26], a slab model was designed and can be visualized in Figure 1.1. In a slab model, the atmosphere is assumed to be a uniform slab at a single uniform temperature $T_A$. The state variables of the model are $T_A$ and $T_S$. Their model assumes the solar radiation, $F_S$, absorbed by the surface; atmospheric heat transport, $F_A$, from the equator to the polar region; and convection, $F_C$, from the surface to the slab atmosphere to be constants. The emissivity, $\epsilon$, is the emissivity of the atmosphere.

Following Payne et al. [26], from Figure 1.1 we can produce two energy balance equations for the surface and the atmosphere. The atmosphere is assumed to be a uniform gas. The emissivity, $\epsilon$, is taken to have a value of 0.90. Radiation is emitted in all directions. However, in the work of Payne et al. [26], a value of $\epsilon \sigma T_A^4$ is emitted up towards space and $\epsilon \sigma T_A^4$ is emitted down towards the surface. This leads to a total of $2\epsilon$ as the fraction of black
body radiation emitted from the atmosphere, which is greater than 1. This appears to be an error of their work and will be corrected in the current work. In Payne et al. 2015 [26], the surface energy balance equation is

\[ 0 = F_S + \epsilon \sigma T_A^4 - \sigma T_S^4 - F_C. \]  

(1.4)

For the atmosphere, the energy balance equation in Payne et al. 2015 [26] is

\[ 0 = F_A + F_C - 2 \epsilon \sigma T_A^4 + \epsilon \sigma T_S^4. \]  

(1.5)

The atmospheric heat transport, \( F_A \), is an exchange of heat from the warm tropics to the Arctic. The last term on the right hand side of (1.5) describes the fraction of radiation emitted by the surface that is absorbed by the atmosphere. The slab model from Payne et al. [26] considers a thermodynamic equilibrium and assumes the atmosphere to be a uniform body of gas that satisfies Kirchhoff’s law. Kirchhoff’s law states that a body at thermodynamic equilibrium that is emitting and absorbing radiation has the absorptivity equal to the emissivity. Therefore, in Payne et al. 2015 [26], both are assumed equal to \( \epsilon \).
Chapter 2

Energy Balance Slab Model

In the model of this thesis, the atmosphere is taken to be a mixture of different gases and does not satisfy Kirchhoff’s law. Rather, we assume the absorption of radiation is done by the greenhouse gases and the heat is distributed throughout the atmosphere and is then emitted as blackbody radiation by other gases, primarily nitrogen and oxygen. Hence, the absorptivity and emissivity are not equal here. The absorption rate of the atmosphere will be denoted by $\eta$, defined as the fraction of the surface radiation absorbed by the greenhouse gases. The emissivity of the atmosphere is denoted $\epsilon$ and, unlike Payne et al. [26], totals $\epsilon \sigma T_A^4$ for all emitted radiation both up and down. We will take a closer look into the absorption factors of CO$_2$ and H$_2$O that will comprise $\eta$.

In the atmosphere, the conduction term $F_C$ is assumed zero for the time being in our slab model. Then $F_C$ can be omitted from the atmosphere and surface energy balance equations. The variables coupling the atmosphere and surface energy balance equations are $T_S$ and $T_A$.

The atmosphere emits radiation both upward and downward. The downward radiation, $\beta$ is reabsorbed by the surface. Then 1-$\beta$ is lost to space by the upward radiation. The value of $\beta$ is determined later using Figure 2.2.
Figure 2.1: A slab style visualization of the atmosphere drawn similarly to Figure 1.1 [26], but enhanced for the present study. Emitted radiation from the atmosphere, $\beta$ is not 50/50 suggested from satellite data. The total IR radiated from the atmosphere is now $\epsilon \sigma T_A^4$, not $2 \epsilon \sigma T_A^4$ as in Payne et al. [26]. This corrects the error in their total atmospheric emission. In Figure 2.1, energy balance equations can be constructed in the same way as for Figure 1.1. The new surface energy balance equation is written as

$$0 = F_S + F_O + \beta \epsilon \sigma T_A^4 - \sigma T_S^4.$$  \hspace{1cm} (2.1)

The factor of $\beta$ on $\epsilon \sigma T_A^4$ takes into account that the total radiation emitted by the atmosphere is not equal upward and downward. The difference comes from the change in density with height in the atmosphere. Although emission is a 50/50 split at any point in the atmosphere, downward emission from higher levels of the atmosphere makes its way toward the surface. The accumulation of downward emission causes the emission of the total atmosphere to be a 37/67 split. This comes from data used by Martin Wild in 2012 [40] and is also referenced in the IPCC 2013 report [36]. Figure 2.2 shows the globally averaged values used in the energy balance of the Earth. From Figure 2.2, the radiation emitted from the atmosphere is about 340.3 W/m$^2$ downward and 199.8 W/m$^2$ upward. The radiation passing through
the atmospheric window, about 40.1 W/m$^2$, is already accounted for in $(1-\eta)\sigma T_3^4$ and has been deducted from 239.9 W/m$^2$. Of the total radiation emitted from the atmosphere only, the split comes out to approximately 37% upward and 63% downward, respectively. Therefore, the value of $\beta$ is chosen as 0.63.

A previous version of this model, Dortmans 2017 [10], used $\beta = 0.5$ to represent the upward and downward radiation from the atmosphere.

The solar radiation absorbed, $F_S$, depends on the solar insolation, $Q$, and on the surface albedo, $\alpha$. The amount of insolation reaching the surface at a given point depends on the solar constant, $S_0 = 1372$ W/m$^2$, and depends on the current latitude due to the curvature of the Earth. The solar constant is divided by 4 to represent ratio of the surface area of the Earth to the Earth’s disk silhouette. The model assumes the solar insolation reflected or
absorbed by the atmosphere is zero. The difference can be shown later on as a reduction in $Q$. The obliquity of the Earth is defined as the angle between the equatorial plane and the plane of the Earth’s orbit around the sun. The obliquity of the Earth is approximately $\gamma = 23.5$ degrees [23]. From the work of McGehee and Lehman [23], the distribution of solar radiation is described as $s(y)$, where $y = \sin(\theta)$ and $\theta$ is the latitude of the Earth. With symmetry of the latitudes, $y$ takes on a value between 0 and 1. The equation for $s(\theta)$ is integrated around the longitude, $\phi$, and is given as

$$s(y) = \frac{2}{\pi^2} \int_0^{2\pi} \sqrt{1 - (1 - y^2 \sin \gamma \cos \phi - y \cos \gamma)^2} \, d\phi,$$

$$s(\theta) = \frac{2}{\pi^2} \int_0^{2\pi} \sqrt{1 - (1 - \sin^2 \theta \sin \gamma \cos \phi - \sin \theta \cos \gamma)^2} \, d\phi. \tag{2.3}$$

Since we are considering the two polar regions and the equator, we can use $s(\pi/2) = 0.505$ for the poles and $s(0) = 1.221$ for the tropics [23], [17]. The value of $s(\theta)$ should have a max of 4 since the Earth’s disk silhouette is represented by a division by 4. This is not the case since the Earth does not face the Sun head on and the tilt/obliquity of the Earth deflects some of the sunlight. We consider first the North Pole (Arctic) region. The Antarctic and tropical region will be considered later. Then $F_S$ is expressed as

$$F_S = (1 - \alpha)Q = (1 - \alpha)\frac{S_0}{4}s(\theta). \tag{2.4}$$

Here, $\alpha$ is the albedo of the region. The albedo can change depending on the composition of the surface, mainly ice or no ice. When ice is present, a larger fraction of the solar radiation is reflected back to space. Without ice, a larger fraction of the solar radiation is absorbed. The Arctic is covered mainly by sea ice with a layer of snow on top, this increases the albedo below freezing. This creates a step function around the freezing point of water and
can be replaced with

$$\alpha = \begin{cases} 
\alpha_c & \text{if } T_S \leq 273.15 \text{K}, \\
\alpha_w & \text{if } T_S > 273.15 \text{K}.
\end{cases}$$

If $T_S < 273.15 \text{K}$ then $\alpha_c$ (cold) is used to represent the albedo when ice is present. Otherwise, if $T_S > 273.15 \text{K}$ then $\alpha_w$ (warm) is used when ice is not present. A typical value of $\alpha_c$ for ocean ice that is covered in snow is around 0.9 [22] and ocean ice is between 0.5 and 0.7 [22]. Glaciers that have travelled will have picked up sediments along the way. As the ice melts, the debris and sediments contained will become exposed, reducing $\alpha_c$. For dirty ice, the value of $\alpha_c$ can be as low as 0.4 [8]. A typical value of $\alpha_w$ for open ocean is around 0.04 [15]. A region covered mostly by grass or soil will have an albedo in the range of 0.17 to 0.25 [22]. The albedo for the Arctic will be chosen as $\alpha_w = 0.04$ and $\alpha_c = 0.4$. Since the Arctic has been losing ice coverage over the years, the lower value of 0.4 is chosen for $\alpha_c$ to better describe the Arctic’s current state.

The ocean heat transport and albedo are both step functions. This adds a discontinuity to the model. In Chapter 3, a $\tanh(\cdot)$ function will be used to connect the warm and cold surface equations. A $\tanh(\cdot)$ switch function is written as

$$\text{Switch} = 0.5 + 0.5 \tanh \left( \frac{T_S - a}{b} \right).$$  \hspace{1cm} (2.5)

The location of the switch, $a$, will be at the freezing line, $a = T_R$. The smoothness, $b$ is how quickly the switch function transitions from 0 to 1. The chosen smoothness is $b = 0.01 \text{ K}$. This value is difficult to determine, as the transition between ice-free and ice-covered can vary. The step functions were used in the model of Dortmans 2017 [10]. The incorporation of this switch function can make the model behave more like nature. Discontinuities, like the albedo described earlier, would not exist in nature. The smoothed transition is more indicative of nature.

The new term $F_O$ is defined as the ocean water heat transport term. On
the surface, the oceans circulate heat from the warm tropics to the Arctic. $F_O$ increases in value as the surface temperature rises above 273.15K. This is due to the rising ocean levels from the melting ice caps. With higher ocean levels and less ice, there is greater flow of water into the Arctic from the warm tropics. Current ocean water heat transport to the Arctic is around 10.7 W/m$^2$ [4], whereas a warm ice-free Arctic has an estimated ocean water heat transport of around 36.2 W/m$^2$ [4]. Similar to $\alpha$, $F_O$ can be replaced with

$$F_O = \begin{cases} F_{O,c} & \text{if } T_S \leq 273.15 \text{K}, \\ F_{O,w} & \text{if } T_S > 273.15 \text{K.} \end{cases}$$

Rearranging for $\sigma T_S^4$ and replacing $F_S$, the surface energy balance equation for the Arctic, (2.1), now becomes

$$\sigma T_S^4 = (1 - \alpha) S_0 \frac{S}{4}(\pi/2) + 0.63 \epsilon \sigma T_A^4 + F_O. \quad (2.6)$$

The new atmosphere energy balance equation from Figure 2.1 is

$$0 = F_A - \epsilon \sigma T_A^4 + \eta \sigma T_S^4. \quad (2.7)$$

With the changes made, Equation (2.7) can be rearranged as

$$\eta \sigma T_S^4 = \epsilon \sigma T_A^4 - F_A. \quad (2.8)$$

The value of atmospheric heat transport, $F_A$, will be chosen from Barron 1981 [4]. Barron 1981 [4] simulates heat transport for both an ice-free and a frozen Earth. The values of the ice-free and frozen atmospheric heat transports are $F_A = 126.56$ W/m$^2$ (frozen) and $F_A = 71.75$ W/m$^2$ (ice-free) [4]. An average of the two will be used for the model at $F_A = 99.16$ W/m$^2$ instead of implementing a step function as the atmosphere temperature of the Arctic crosses the freezing point. The absorptivity by greenhouse gases in the atmospheric energy balance equation is the focus of the current study and $F_A$ will remain a constant. A main contribution of this thesis is the
development of appropriate expressions for the absorptivity $\eta$. In order to analyse the absorptivity of the atmosphere, more information on the greenhouse gases, CO$_2$ and H$_2$O, will be needed.

### 2.1 Absorptivity of Greenhouse Gases

Here we develop expressions for the absorptivity, $\eta$, for the greenhouse gases CO$_2$ and water vapour. Our model is built on the assumption that the gases of the atmosphere satisfy the ideal gas law. The ideal gas law is

$$PV = nRT, \quad (2.9)$$

where $P$ is pressure in Pa, $V$ is volume of the gas in m$^3$, $n$ is the number of moles of gas, $T$ is the current temperature of the gas in K, and $R$ is the universal gas constant in units of kg m$^2$/s$^2$/K. If we rearrange for $P$ and replace $n = \frac{m}{M}$, where $m$ is the mass of gas in kg and $M$ is the molar mass of the gas in kg/mol, the law now becomes

$$P = \frac{mR}{MV}T. \quad (2.10)$$

The ratio $\frac{m}{V}$ can be replaced with $\rho_i$, the density of the $i$th gas in kg/m$^3$, and $\frac{R}{M}$ can be replaced with $R_i$, a gas constant specific to the $i$th gas species. Let $P_i$ be the partial pressure of the $i$th gas. Then the ideal gas law for the $i$th gas is

$$P_i = \rho_iR_iT. \quad (2.11)$$

We assume the slab model satisfies a thermodynamic equilibrium. As mentioned before, the energy absorbed is rapidly dispersed through the atmosphere as heat. The greenhouse gases in the atmosphere will absorb long wave radiation from the surface according to Beer’s law. For a single
greenhouse gas, Beer’s law is

$$\frac{dI}{dz} = -k_i \rho_i I(z). \quad (2.12)$$

In Equation (2.12), $I$ is the intensity of the radiation; the parameter $k_i$ denotes the absorption coefficient of the $i$th gas species in units of $m^2/kg$, the parameter $\rho_i$ denotes the density of the $i$th gas species in units of $kg/m^3$, and $z$ is altitude, in m, in the slab. Beer’s law can break down and become less accurate if the density of the gas becomes too large or if the radiation passing through the atmosphere is limited. In the slab model, $\rho_i$ is independent of $z$. Therefore, this is a linear differential equation in terms of $I$ and can then be integrated directly from 0 to $z$ to give

$$I(z) = I_S e^{-k_i \rho_i z}. \quad (2.13)$$

At surface level, $z = 0$, $I(z = 0)$ is written as $I_S$. Equation (2.13) describes the intensity of radiation at an altitude $z$ as an exponentially decaying function. At the top of the atmosphere, $z = Z$, (2.13) determines how much radiation is escaping to space. Then we can determine the amount of radiation trapped by the atmosphere,

$$I_S - I(Z) = I_S (1 - e^{-k_i \rho_i Z}) = \eta_i I_S. \quad (2.14)$$

The absorptivity is then

$$\eta_i = 1 - e^{-k_i \rho_i Z}. \quad (2.15)$$

In the case of a single gas, $\eta_i$ is the expression given by (2.15). When there are two greenhouse gases, equation (2.12) becomes

$$\frac{dI}{dz} = -k_i \rho_i I(z) - k_j \rho_j I(z), \quad (2.16)$$

which integrates to

$$I(z) = I_S e^{-k_i \rho_i z - k_j \rho_j z} = I_S e^{-k_i \rho_i z} e^{-k_j \rho_j z}. \quad (2.17)$$
At $z = Z$, the radiation trapped by the two greenhouse gases is

$$I_S - I(Z) = I_S(1 - e^{-k_i \rho_i Z} - k_j \rho_j Z) = I_S(1 - e^{-k_i \rho_i Z} e^{-k_j \rho_j Z}). \quad (2.18)$$

Then the effective emissivity, $\eta_{\text{eff}}$ is expressed as

$$\eta_{\text{eff}} = 1 - e^{-k_i \rho_i Z} - k_j \rho_j Z = 1 - (1 - \eta_i)(1 - \eta_j). \quad (2.19)$$

This generalizes the expression of $\eta$ to more greenhouse gases as

$$\eta_{\text{eff}} = 1 - \Pi_{i=1}^n e^{-k_i \rho_i Z} = 1 - \Pi_{i=1}^n (1 - \eta_i). \quad (2.20)$$

### 2.1.1 Carbon Dioxide

Observations suggest that carbon dioxide gas rapidly spreads uniformly throughout the atmosphere. The amount of CO$_2$ gas in the atmosphere is typically expressed in parts per million, ppm, $\mu$, molar fraction of dry air itself

$$\zeta_C = \frac{\mu}{10^6} \zeta_A. \quad (2.21)$$

Here $\zeta_C$ and $\zeta_A$ are the number of moles of CO$_2$ and dry air in the atmosphere, respectively. These may also be expressed as $\zeta_C = \frac{m_C}{M_C}$ where $m_C$ is the mass of CO$_2$ and $M_C$ is the molar mass of CO$_2$, and similarly $\zeta_A = \frac{m_A}{M_A}$. Equation (2.21) becomes

$$\frac{m_C}{M_C} = \frac{\mu}{10^6} \frac{m_A}{M_A}. \quad (2.22)$$

Dividing by the volume of the atmosphere on the left and right hand side of Equation (2.22), the density of CO$_2$ and the density of the atmosphere can be related in the expression

$$\rho_C = \frac{\mu}{10^6} \frac{M_C}{M_A \rho_A}. \quad (2.23)$$

Here, $\rho_C$ is the density of CO$_2$, in kg/m$^3$, in the atmosphere and $\mu$ is the molar concentration in ppm of CO$_2$ to dry air. The density of dry air is
given as $\rho_A$. The composition of the dry air in the atmosphere is made up of 78% Nitrogen, 21% Oxygen, and 0.9% Argon with molar masses 28 g/mol, 32 g/mol, and 40 g/mol, respectively [16]. The other trace gases in the atmosphere are negligible in the molar mass of the atmosphere. Using $M_C \approx 44 \times 10^{-3}$ kg/mol and $M_A \approx 29 \times 10^{-3}$ kg/mol, a ratio of

$$\frac{M_C}{M_A} = \frac{44 \times 10^{-3}}{29 \times 10^{-3}} \approx 1.52$$

(2.24)

can be used in (2.23) to relate the density of CO$_2$ to the density of dry air as

$$\rho_C = \frac{1.52 \mu}{10^6} \rho_A.$$  

(2.25)

Today, CO$_2$ levels in the atmosphere are around 400 ppm and rising [1]. We can vary the $\mu$ parameter to see how changing CO$_2$ levels affects the behaviour of the model. In the atmosphere, the density of air varies with altitude. For the slab model, an average is taken for a column of mass in the atmosphere. The term $\rho_iZ$, for the $i$th gas, represents the mass per unit cross-section of atmosphere containing that $i$th gas. That is,

$$\gamma_i = \rho_iZ.$$  

(2.26)

This will result in Equation (2.15) to be written for the atmosphere as

$$\eta_A = 1 - e^{-k_A \gamma_A}.$$  

(2.27)

In Equation (2.27), the parameter $\gamma_A$ is expressed in terms of the mass of the atmosphere, $m_A$, and the radius of the Earth, $R_E$, resulting in

$$\gamma_A = \frac{m_A}{4\pi R_E^2} \approx 10296 \approx 1.03 \times 10^4 kg/m^2.$$  

(2.28)

The values used in Equation (2.28) are $m_A = 5.25 \times 10^{18}$ [38] and $R_E = 6.37 \times 10^6$ [16]. Equation (2.28) describes the mass of the atmosphere acting on a unit cross-sectional area, or column, of surface.
A column of atmosphere has the weight of dry air pushing down onto an area of surface. The pressure of the atmosphere, \( P_A = 101,325 \text{ Pa} \), divided by the gravitational constant at the surface, \( g = 9.81 \text{ m/s}^2 \), can represent the amount of air mass pushing down on a unit cross-sectional area in the same way as Equation (2.28). The resultant column of atmosphere is given as

\[
\gamma_A = \frac{P_A}{g} \approx 10339 \approx 1.03 \times 10^4 \text{kg/m}^2.
\]  

(2.29)

Equations (2.28) and (2.29) differ by a value of 43, or 0.4%. This difference is negligible in the current work and we are free to choose either equation to use for our calculation of \( \eta \). Equation (2.29) will be used. The density of \( \text{CO}_2 \) is given as a molar fraction to dry air. A column of \( \text{CO}_2 \) is represented as

\[
\gamma_C = \frac{1.52 \mu}{10^6} \rho_A Z = \frac{1.52 \mu P_A}{10^6 g} = 1.57 \times 10^{-2} \mu.
\]  

(2.30)

Carbon dioxide absorbs infrared radiation emitted by the surface. The radiation comes in at different wavelengths and certain windows of these wavelengths are readily absorbed by the \( \text{CO}_2 \) gas and some windows are not. A common approach is to use a so-called Grey gas model [15], [7]. A Grey atmosphere is an approximation in terms of a weighted sum using the wavelengths and corresponding absorption coefficients. The result is a ‘smoothed’ value of \( k_i \) for some \( i \)th gas and is used in Beer’s law, Equation (2.27). This ‘smoothed’ absorption coefficient is useful for simpler models or to save on computational time [9]. This Grey gas approximation is used throughout the thesis.

If we assume \( \eta \) to be composed of the contribution from carbon dioxide only using (2.25) in (2.27) and (2.29) to give

\[
\eta = 1 - \exp \left( -k_C \frac{1.52 \mu}{10^6} \gamma_C \right),
\]  

(2.31)
then we can work backwards using the emissivity of the atmosphere to find an approximate value for the absorption coefficient of CO$_2$, $k_C$, to use in our Grey atmosphere. We can compare the range of absorption coefficients found by working backwards with those of the literature to determine how well the approximation of $\epsilon = \eta$ works for models that assume Kirchhoff’s law is satisfied.

Some slab models use the value of emissivity, $\epsilon$, for the absorption of infrared radiation since those models satisfied a thermodynamic equilibrium. Knowing this, and other known parameters, we can work backwards using $\epsilon = \eta$ to determine a range of possible absorption coefficients that can be used in (2.27). Doing this, we can recreate previous slab models, such as Payne et al. [26], and analyse the effect of varying certain parameters. In the case of CO$_2$ as the only greenhouse gas, a range of values for the absorption coefficient can be found. We can use the approximation $\eta = \epsilon$ to give

$$\epsilon = \eta = 1 - \exp \left( -\frac{1.52 \mu}{10^6} k_C \gamma_A \right),$$  \hspace{1cm} (2.32)

which can be rearranged to estimate $k_C$ as

$$k_C = -\frac{\ln(1 - \epsilon)}{\frac{1.52 \mu}{10^6} \gamma_A}.$$  \hspace{1cm} (2.33)

Using this, and a varied $\epsilon$ between 0 and 1, a range of possible absorption coefficients can be found for carbon dioxide. This allows us to play with the absorption coefficient of carbon dioxide. For example, if the emissivity of carbon dioxide was 0.2, we can estimate a corresponding absorption coefficient at that point to be $k_C = 0.0474$ m$^2$/kg from Figure 2.3. This depends on equation (2.27).

The total emissivity of carbon dioxide was found in the literature to be between 0.15 and 0.2 [6]. The paper by Byun and Chen [6], uses several different models with spectral data in higher temperatures to determine the total emissivity of carbon dioxide. An extrapolation is done to approximate
Figure 2.3: An approximate value for an absorption coefficient of CO$_2$ is found by working backwards using $\mu = 300$ ppm, $\gamma_A = 10339$ kg/m$^2$. Using a known emissivity for CO$_2$, a corresponding value of $k_C$ can be found.

an emissivity of carbon dioxide for atmospheric temperatures, around 280K - 310K. The ppm concentration of carbon dioxide was taken at 300ppm [6].

We can use Figure 2.3 to choose an absorption coefficient based on the total emissivity, $\epsilon$, of carbon dioxide. If we assume that carbon dioxide is the only greenhouse gas that absorbs infrared radiation, we can use this as our absorption term, $\eta$. From Figure 2.3, the literature values of the emissivity for carbon dioxide, 0.15 and 0.2, will have a corresponding $k_C$ of 0.0345 m$^2$/kg and 0.0474 m$^2$/kg, respectively.

In the literature, an approximate value of $k_C$ is determined by Roberts et al. [31] to be 0.037 m$^2$/kg in the 8-12$\mu$m window. This is not the main window of absorption but still contributes to the absorption for CO$_2$. Byun and Chen [6] study the emissivity of CO$_2$ in the 15$\mu$m window, the most
important absorption window for CO$_2$. Seeing how the value from Roberts et al. [31] is within the range determined from Byun and Chen [6], the estimated range of $k_C$ will be a good range to choose from for the current work. The upper value of $k_C = 0.0474$ m$^2$/kg will be chosen.

### 2.1.2 Water Vapour

An estimate for a column of water vapour, $\gamma_W$ in Equation (2.27), can also be established. However, the concentration of water vapour does not stay as consistent as CO$_2$ does in the atmosphere. The pressure/density of H$_2$O varies strongly with temperature. The approximate change of vapour pressure is expressed using the saturated vapour pressure. The saturated vapour pressure does not refer to saturation of water vapour in the atmosphere, but as an equilibrium between liquid and vapour. At a given pressure and temperature, the amount of molecules leaving as liquid (evaporating) is the same as the amount returning (condensing). For higher values of pressure, or lower temperatures, an excess of water vapour molecules will condense into liquid water. This is what causes rain [16]. The relationship of saturated water vapour pressure to temperature is governed by the Clausius-Clapeyron equation

\[
\frac{dP_{sat}}{dT} = \frac{1}{T} \frac{L}{\rho_v^{-1} - \rho_c^{-1}}.
\] (2.34)

Here, $P_{sat}$ is the saturated vapour pressure, and $\rho_{v,c}$ are the saturated density of water in the gas state and condensed state, respectively. The Clausius-Clapeyron equation is valid for the temperature range of the Earth. The Clausius-Clapeyron equation becomes invalid for temperatures above boiling, 100°C. Since the saturated density of water vapour is quite small in comparison, that is $\rho_c >> \rho_v$, the term $\rho_c^{-1}$ becomes negligible in (2.34). Since the density relies on temperature, $\rho_v$ can be replaced using the ideal gas law (2.11). Then (2.34) becomes

\[
\frac{dP_{sat}}{dT} = \frac{1}{T^2} \frac{L P_{sat}}{R_W}.
\] (2.35)
The gas constant, $R_W$, is specific to water vapour. The latent heat, $L$, is assumed constant with respect to temperature in the current work. In the atmosphere, the Clausius-Clapeyron equation acts between the water vapour and the ‘surface’ of water droplets or ice crystals present. Since the temperature of the atmosphere is below freezing, the ‘surface’ acted on will be the ice crystals and the latent heat of sublimation, $L_S$, will be used for $L$.

Now (2.35) is ready to integrate. Bringing $P_{sat}$ to the left hand side and $dT$ to the right hand side,

$$\frac{1}{P_{sat}} dP_{sat} = \frac{1}{T^2 R_W} L_s dT,$$

we can integrate both sides from some reference temperature $T_R$ to $T_A$

$$\int_{P_{sat}(T_R)}^{P_{sat}(T_A)} \frac{1}{P_{sat}} dP_{sat} = \int_{T_R}^{T_A} \frac{1}{T^2 R_W} L_s dT.$$  

(2.37)

Resulting in

$$\ln\left(\frac{P_{sat}(T_A)}{P_{sat}(T_R)}\right) = \frac{L_s}{R_W} \left[ \frac{1}{T_R} - \frac{1}{T_A} \right].$$

(2.38)

From this we obtain

$$P_{sat}(T_A) = P_{sat}(T_R) \exp\left[ \frac{L_s}{R_W} \left[ \frac{1}{T_R} - \frac{1}{T_A} \right] \right].$$

(2.39)

The partial pressure of water vapour in the atmosphere is less than that of the saturated vapour pressure. The relative humidity is defined as the ratio of the current vapour pressure to saturated vapour pressure, or $\delta = \frac{P_W(T_A)}{P_{sat}(T_A)}$. Since the vapour pressure is less than the saturated vapour pressure, the relative humidity, $\delta$, will be between 0 and 1. If the vapour pressure reaches saturation, precipitation occurs to remove the excess water vapour in the air. Multiplying both sides by $\delta$ and making the substitution
for $\delta$ on the left will give

$$P_W(T_A) = \delta P_{sat}(T_R) \exp \left[ \frac{L_s}{R_W} \left( \frac{1}{T_R} - \frac{1}{T_A} \right) \right]. \quad (2.40)$$

The ratio $\delta$ is held constant, and this parameter represents the relative humidity in the atmosphere. At any two given temperatures of $T_A$, the relative humidity can be the same. The remaining parameters of the Clausius-Clapeyron account for the change in the quantity of water vapour. The Clausius-Clapeyron relation can be expressed in terms of density instead of pressure using the ideal gas law (2.11) to give

$$\rho_W(T_A) = \delta \rho_{sat}(T_R) \frac{T_R}{T_A} \exp \left[ \frac{L_s}{R_W} \left( \frac{1}{T_R} - \frac{1}{T_A} \right) \right], \quad (2.41)$$

where $\rho_W$ is the density of water vapour in kg/m$^3$, and $\rho_{sat}(T_R)$ is the density, in kg/m$^3$, at a specific reference temperature $T_R$. We will define $\gamma_W(T_A)$ as

$$\gamma_W(T_A) = \rho_W(T_A)Z, \quad (2.42)$$

and $\gamma_{W_{sat}}(T_R)$ as

$$\gamma_{W_{sat}}(T_R) = \rho_{sat}(T_R)Z. \quad (2.43)$$

We can not estimate $\gamma_W(T_A)$ in the same way as carbon dioxide in Equation (2.28) since the density of water vapour in the atmosphere varies locally and is dependent on temperature. Similar to Equation (2.29), $\gamma_{W_{sat}}(T_R)$ can be calculated by using the pressure of water vapour at the reference temperature, $P_{sat}(T_R)$, with a value of 611.2 Pa at $T_R = 273.15K$, [20], divided by the acceleration due to gravity,

$$\gamma_{W_{sat}}(T_R) = \frac{P_{sat}(T_R)}{g} \approx 62.4 kg/m^2. \quad (2.44)$$

An estimated value of $k_W$ will need to be found. We can work backwards with $\eta = \epsilon$ as before to find an approximate range of absorption coefficients. We are considering a constant value of the absorption coefficient for our grey
atmosphere. Using Equation (2.41) in Equations (2.27) and (2.44) to give
\[ \epsilon = \eta = 1 - \exp(-k_W \gamma_W(T_A)) \]  
\[ \epsilon = \eta = 1 - \exp\left(-k_W \frac{P_{\text{sat}}(T_R) T_R}{g T_A} \exp\left[ \frac{L_s}{R_W} \left( \frac{1}{T_R} - \frac{1}{T_A} \right) \right] \right), \]

we can find an estimate of \( k_W \) by rearranging Equation (2.45) and using Equation (2.44) to give
\[ k_W = \frac{-\ln(1 - \epsilon)}{\delta \frac{P_{\text{sat}}(T_R) T_R}{g T_A} \exp\left[ \frac{L_s}{R_W} \left( \frac{1}{T_R} - \frac{1}{T_A} \right) \right]}. \]

However, the concentration of water vapour varies strongly with temperature. Instead of plotting a single curve, multiple curves are done at varying temperatures. Then an average can be taken. Using the parameters; reference temperature, \( T_R = 273\text{K} \), relative humidity, \( \delta = 0.6 \), and \( P_{\text{sat}}(T_R)/g = 62.4 \text{ kg/m}^2 \), a temperature range of \( T_A \) from 265K to 280K can be used to find a value for the absorption coefficient in Figure 2.4.

From Figure 2.4, we take an average of the four curves based on a given emissivity. From the literature, [24] and [29], water vapour has an emissivity of 0.4. Using this, an average was found for \( k_W = 0.016 \text{ m}^2/\text{kg} \) by working backwards.

The value used by Roberts et al. 1976 [31] is calculated to be in the range \( k_W = 0.0053 \) to \( 0.0189 \text{ m}^2/\text{kg} \) for the 8-12\( \mu \text{m} \) window. The 8-12\( \mu \text{m} \) window is considered the most important window for water vapour [31]. This range was calculated using Equation (5) in Roberts et al. 1976 [31] and corrected to the reference temperature, \( T_R = 273.15 \text{ K} \). The lower end corresponds to a water vapour pressure of 4 torr and the upper end corresponds to a water vapour pressure of 14 torr. At 273.15 K, the water vapour pressure is 4.5851 torr, or 0.6112 kPa. The corresponds to \( k_W = 0.0062 \text{ m}^2/\text{kg} \). This value is chosen as it corresponds to the water vapour pressure at the reference temperature, rather than the estimates made in Figure 2.4.
2.2 Model Summary

In this section, we give the final equations compiling all the parameters and substitutions made. A table of parameter values is summarized here as well. It is important to note that the emissivity values used in the literature for CO₂ and water vapour were used to find estimates of the absorption coefficients and do not affect the emissivity value used for dry air, $\epsilon = 0.9$.

In the original atmosphere equation, Equation (2.8), $\eta$ can have the proper substitutions made to give an expression in terms of surface temperature $T_S$. This will give

$$\sigma T_S^4 = \frac{\epsilon \sigma T_A^4 - F_A}{\eta}$$

and

$$= \frac{\epsilon \sigma T_A^4 - F_A}{1 - \exp \left[-k_C \gamma_C - k_W \gamma_W\right]}.$$
Finally, adding in the pieces constructed in Equations (2.25) and (2.41) will produce

\[
\sigma T_S^4 = \frac{\epsilon \sigma T_A^4 - F_A}{1 - \exp \left( -1.52 \frac{\mu}{10^6} k_{\text{CO}} \frac{P_A}{g} - \delta k \frac{P_{\text{sat}}(T_R)}{T_A} \frac{T_R}{T_A} \exp \left[ \frac{L_s}{R_W} \frac{1}{T_R} - \frac{1}{T_A} \right] \right)}.
\] (2.48)

The energy balance equation for the surface remains as

\[
\sigma T_S^4 = (1 - \alpha) \frac{S_0}{4} s(\pi/2) + 0.63 \epsilon \sigma T_A^4 + F_O
\] (2.49)

with \(\alpha\) and \(F_O\) as step functions at \(T_S = 273\) K. The step functions are replaced with the switch function later in Chapter 3. Table 2.1 and 2.2 is a collection of the parameters used thus far.

The energy balance equations of the atmosphere and surface, Equations (2.48) and (2.49), can be non-dimensionalized using \(\tau_A = \frac{T_A}{T_R}\) and \(\tau_S = \frac{T_S}{T_R}\). The freezing point of water is now \(\tau = 1\). New non-dimensional constants \(G_i\) can simplify the atmosphere energy balance equation to

<table>
<thead>
<tr>
<th>Parameters and Constants</th>
<th>Symbol</th>
<th>Value (units)</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stefan-Boltzmann</td>
<td>(\sigma)</td>
<td>(5.67 \times 10^{-8}) (\text{Wm}^{-2}\text{K}^{-4})</td>
<td>[16]</td>
</tr>
<tr>
<td>Emissivity of dry air</td>
<td>(\epsilon)</td>
<td>0.90</td>
<td>[26]</td>
</tr>
<tr>
<td>Albedo of ocean ice</td>
<td>(\alpha_c)</td>
<td>0.5</td>
<td>[22]</td>
</tr>
<tr>
<td>Albedo of open ocean</td>
<td>(\alpha_w)</td>
<td>0.04</td>
<td>[15]</td>
</tr>
<tr>
<td>Solar constant</td>
<td>(S_0)</td>
<td>1372 (\text{Wm}^{-2})</td>
<td>[23]</td>
</tr>
<tr>
<td>Solar distr. at (\theta = 0, \pm \frac{\pi}{2})</td>
<td>(s(\theta))</td>
<td>1.221, 0.505</td>
<td>[23]</td>
</tr>
<tr>
<td>Ocean heat transport</td>
<td>(F_{O,c})</td>
<td>10.7 (\text{Wm}^{-2})</td>
<td>[4]</td>
</tr>
<tr>
<td></td>
<td>(F_{O,w})</td>
<td>36.2 (\text{Wm}^{-2})</td>
<td>[4]</td>
</tr>
<tr>
<td>Reference Temperature</td>
<td>(T_R)</td>
<td>273.15 K</td>
<td>Author</td>
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Table 2.1: Summary of parameters used in the Arctic energy balance model.
<table>
<thead>
<tr>
<th>Parameters and Constants</th>
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<th>Value (units)</th>
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<td>99.16 W m$^{-2}$</td>
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<tr>
<td>Earth’s gravity</td>
<td>$g$</td>
<td>9.81 m s$^{-2}$</td>
<td>[16]</td>
</tr>
<tr>
<td>Emissivity value of CO$_2$</td>
<td>$\epsilon_C$</td>
<td>0.15-0.2</td>
<td>[6]</td>
</tr>
<tr>
<td>Concentration of CO$_2$</td>
<td>$\mu$</td>
<td>200-800 ppm</td>
<td>[1]</td>
</tr>
<tr>
<td>Mean mass of dry air</td>
<td>$M_A$</td>
<td>5.135×10$^{18}$ kg</td>
<td>[16]</td>
</tr>
<tr>
<td>Radius of the Earth</td>
<td>$R_E$</td>
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<td>[16]</td>
</tr>
<tr>
<td>Pressure of dry air</td>
<td>$P_A$</td>
<td>101.325 Pa</td>
<td>[16]</td>
</tr>
<tr>
<td>Density of dry air</td>
<td>$\rho_A$</td>
<td>1.225 kg m$^{-3}$</td>
<td>[16]</td>
</tr>
<tr>
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<td>$\gamma_A$</td>
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<td>Density of water vapour at $T_R$</td>
<td>$\rho_W(T_R)$</td>
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</tr>
<tr>
<td>Pressure of water vapour at $T_R$</td>
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<td>Column of water vapour</td>
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<td>Relative humidity</td>
<td>$\delta$</td>
<td>0-1</td>
<td>Author</td>
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<tr>
<td>Emissivity value of water vapour</td>
<td>$\epsilon_W$</td>
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<td>[24], [29]</td>
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<td>Latent heat of sublimation/vapourization</td>
<td>$L_s$</td>
<td>2.8373×10$^6$ m$^2$ s$^{-2}$</td>
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<td>Universal gas constant for CO$_2$</td>
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<td>Specific gas constant for water vapour</td>
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<td>Absorption coefficient for CO$_2$</td>
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<tr>
<td>Absorption coefficient for water vapour</td>
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<td>[31]</td>
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<tr>
<td></td>
<td></td>
<td>0.016 m$^2$ kg$^{-1}$</td>
<td>Author</td>
</tr>
</tbody>
</table>

Table 2.2: Summary of parameters used in the Arctic energy balance model.
\[ \tau_S^4 = \frac{\epsilon \tau_A^4 - f_A}{\left(1 - \exp\left(-\kappa G_1 - \delta G_2 \tau_A \exp\left[G_3 \left(1 - \frac{1}{\tau_A}\right)\right]\right)} \]  
\text{and the surface energy balance equation to}

\[ \tau_S^4 = (1 - \alpha)G_S + 0.63\epsilon \tau_A^4 + f_O. \]

Table 2.3 summarizes the non-dimensional constants and parameters of Equations (2.50) and (2.51). The parameter \( f_O \) is a step function similar to \( F_O \) expressed as

\[ f_O = \frac{F_O}{\sigma T_R^4} = \begin{cases} 
      f_{O,c} & \text{if } \tau_S \leq 1, \\
      f_{O,w} & \text{if } \tau_S > 1.
\end{cases} \]

The step function for \( \alpha \) is the same and has the transition at \( \tau_S = 1 \). This is expressed as

\[ \alpha = \begin{cases} 
      \alpha_c & \text{if } \tau_S \leq 1, \\
      \alpha_w & \text{if } \tau_S > 1.
\end{cases} \]

The remaining non-dimensional constants in Equations (2.50) and (2.51) are given as

\[ G_S = \frac{S_0 s(\pi)}{4\sigma T_R^4}, \]
\[ G_1 = \frac{1.52k_C P_A}{10^6 g}, \]
\[ G_2 = \frac{k_W P_{sat}(T_R)}{g}, \]
\[ G_3 = \frac{L_s}{T_R R_W}, \]
\[ f_A = \frac{F_A}{\sigma T_R^4}. \]
<table>
<thead>
<tr>
<th>Parameters and Constants</th>
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<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Emissivity of dry air</td>
<td>$\epsilon$</td>
<td>0.90</td>
<td>[26]</td>
</tr>
<tr>
<td>Albedo of ocean ice</td>
<td>$\alpha_c$</td>
<td>0.5</td>
<td>[22]</td>
</tr>
<tr>
<td>Albedo of open ocean</td>
<td>$\alpha_w$</td>
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<td>Surface Constant</td>
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<td>Ocean heat transport</td>
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<td></td>
<td>$f_w$</td>
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<td>Atmosphere heat transport</td>
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<td>CO$_2$ Concentration</td>
<td>$\mu$</td>
<td>200-800 ppm</td>
<td>Author</td>
</tr>
<tr>
<td>CO$_2$ Constant 1</td>
<td>$G_1$</td>
<td>$7.4417 \times 10^{-4}$</td>
<td>Author</td>
</tr>
<tr>
<td>Relative Humidity</td>
<td>$\delta$</td>
<td>0-1</td>
<td>Author</td>
</tr>
<tr>
<td>Water Vapour Constant 2</td>
<td>$G_2$</td>
<td>0.3863</td>
<td>Author</td>
</tr>
<tr>
<td>Water Vapour Constant 3</td>
<td>$G_3$</td>
<td>22.5126</td>
<td>Author</td>
</tr>
<tr>
<td>Constant 3 After Corner</td>
<td>$G_3$</td>
<td>17.8989</td>
<td>Author</td>
</tr>
</tbody>
</table>

Table 2.3: Summary of nondimensional parameters used in the Arctic model.
Chapter 3

Arctic Model

The energy balance slab model will first be applied to the Arctic. Plots of surface temperature, $\tau_S$, with the atmosphere temperature, $\tau_A$, can be made using Equations (2.50) and (2.51). We will break down the discussion into multiple parts: the first section will consider the carbon dioxide contribution only, the second section will consider water vapour contribution only, the third section will be a combination of the two, the fourth section will consider variations in ocean heat transport, and the fifth section will consider changes in albedo. The plots shown here will have intersections between Equations (2.50) and (2.51). The energy balance equations of the model are the nullclines for the atmosphere and surface. A nullcline for the atmosphere represents zero change in the atmospheric temperature. A similar case made for the surface temperature. An intersection between the nullclines is a fixed point of the model. These fixed points are the equilibrium solutions or climate states for the model. The stability of these fixed points are determined in Appendix B. Curves in the bifurcation plots will be solid for a stable solution and dashed for an unstable solution.
Figure 3.1: Arctic energy balance model for carbon dioxide only. At each level of $k_C$, $\mu$ was varied from 200 ppm (left blue line) to 400 to 600 to 800 ppm (right blue line). The left plot has an absorptivity of 0.15 and the right plot has an absorptivity of 0.2 with corresponding $k_C$ values of 0.0345 m$^2$/kg and 0.0474 m$^2$/kg, respectively. Here $\delta = 0$ and the remaining parameters have values as in Table 2.3.

3.1 Carbon Dioxide Contribution

In this section, the effects of carbon dioxide, as if it were the only greenhouse gas in the atmosphere, will be analysed by setting $\delta$ to 0. For the different values of $k_C$ found, we can vary the ppm concentration of CO$_2$, $\mu$. For the general picture, a plot of Equations (2.50) and (2.51) was done at two levels of $k_C$, 0.0345 m$^2$/kg and 0.0474 m$^2$/kg, in Figure 3.1. In both cases, the atmospheric heat transport, $F_A$, is held constant at $F_A = 99.16$ W/m$^2$ as in Table 2.2.

Looking at Figure 3.1, the atmosphere equation Equation (2.50) (blue) intersects with the surface Equation (2.51) (red and magenta) at least once. Each intersection is an equilibrium of the system. The equilibrium position moves as we vary $\mu$ and $k_C$. In Equation (2.50), the numerator is non-negative when $\varepsilon \tau_A^4$ is greater than or equal to $f_A$. As the value of $\mu$ increases, the intersection moves to the right, or a warmer atmospheric temperature. As the value of $\mu$ decreases, the equilibrium moves to the left, or a colder
atmospheric temperature. This is to be expected since adding more gas that can trap heat in the atmosphere will give a higher atmosphere temperature at equilibrium and vice versa.

The absorption coefficient, $k_C$, varies with temperature. In this model, we consider that value to be constant over the current temperature range. From Byun and Chen [6], the emissivity of carbon dioxide varies between 0.15 and 0.2. When we calculate an absorption coefficient from these two values, 0.0345 m$^2$/kg and 0.0474 m$^2$/kg respectively, we create a range of potential absorption coefficients. In a plot, varying the value of $k_C$ will produce curves similar to those in Figure 3.1. The lower absorption coefficients are represented by the left set of blue lines and the larger absorption coefficients are represented by the right set of blue lines. The value of $k_C$ will be taken as a constant at 0.0474 m$^2$/kg from this point on.

In Figure 3.1, carbon dioxide is the only acting greenhouse gas. In both cases, left and right set of blue lines produce a stable equilibrium point that has an ice-covered and an ice-free Arctic for certain values of $\mu$. However, this situation can change if the ocean heat transport $F_O$ or the atmospheric heat transport, $F_A$, are changed. If either are reduced sufficiently, then for the range of CO$_2$ concentrations considered here, the ice-free equilibrium (on the magenta line) ceases to exist. The concentration of CO$_2$ would need to be at a value of at least $\mu = 400$ ppm to have an ice-free equilibrium at $k_C = 0.0474$ m$^2$/kg and current Earth-like conditions.

An important note from Figure 3.1 is that for large enough CO$_2$ values, there are two possible equilibrium points. With the two points, we now have bi-stability in the Arctic. In order to “jump” from the lower branch, the surface temperature would need to rise above 273.15 K. In the case of carbon dioxide only, the cold equilibrium always exists for realistic values of $\mu$. This is seen following the red and magenta curves in Figure 3.1 until an intersection with the blue curve occurs.

If the climate were at the ice-free equilibrium in the Arctic and $\mu$ de-
creased sufficiently, this equilibrium point would disappear and the climate
would jump down to the cold equilibrium below freezing.

For the ice-covered Arctic, the warming done by doubling CO$_2$ from 270
ppm to 540 ppm increases the equilibrium surface temperature of the Arctic
by 6.938°C. This is used in part for the Equilibrium Climate Sensitivity
(ECS) and will be covered later in Chapter 8.

The temperature increase from CO$_2$ is not linear since the contribution
comes from an exponential equation. However, small changes in µ can make
the change in temperature seem linear. In order for the ice-covered Arctic
equilibrium point to disappear, leaving only the ice-free equilibrium, the
model would require an increase in carbon dioxide concentration to above
1600 ppm, which is not realistic. However, carbon dioxide is not the only
acting greenhouse gas in the atmosphere.

3.2 Water Vapour Contribution

In this section, we consider η to be composed of the contribution from water
vapour only from Equation (2.50) by setting µ to 0. A plot of water vapour
acting as the only greenhouse gas is shown in Figure 3.2. Similar to carbon
dioxide, we can see how the curves differ as we vary the relative humidity.
Adjusting the value of parameters $k_W$ and $\gamma_{W\text{sat}}(T_R)$ will have a similar
impact on Figure 3.2 compared to changes in $\delta$.

Here we see that there are multiple intersections in Figure 3.2. This is
due to the non-linear dependence on temperature of the saturated vapour
pressure of water. The curve representing the atmosphere energy balance,
the blue curve, will cross the curves representing the surface energy balance,
the red and/or magenta curves, at least once under Earth-like conditions.

In our model, the intersections of the blue curve with the red (frozen)
curve are known as the frozen equilibrium points. The left intersection
Figure 3.2: Arctic energy balance model for water vapour only. Water vapour at varying relative humidity from 10% (top blue line) to 30% to 50% to 70% (bottom blue line). Here, $k_W = 0.0062 \text{ m}^2/\text{kg}$, $\gamma_{W\text{sat}}(T_R) = 62.4 \text{ kg/m}^2$, and $\mu = 0$. 

---

$k_W = 0.0062 \text{ m}^2/\text{kg}$, $\gamma_{W\text{sat}}(T_R) = 62.4 \text{ kg/m}^2$, and $\mu = 0$. 

---
represents a stable equilibrium point. An unstable equilibrium point can exist for certain conditions.

When the value of $\delta$ is large enough, there can exist an unstable equilibrium point close to freezing on the red curve. If the red curve was to extend past $\tau_S = 1$, there would be another intersection with the blue curve. Since this intersection does not happen below $\tau_S = 1$, no stable equilibrium on the frozen curve near freezing can exist in Figure 3.2.

In the case of the leftmost equilibrium, the temperature of the atmosphere is low and the amount of water vapour in the atmosphere is low as well. Without enough water vapour to trap radiation, the equilibrium changes only slightly for all values of $\delta$ between 0 and 1.

The intersections of the blue curve with the magenta (ice-free) curve are known as the ice-free equilibrium points. When the value of $\delta$ is large enough, a stable equilibrium exists on the far right hand side for high temperatures of the surface and atmosphere. This is called the hot stable equilibrium. This is the case of large amounts of water vapour in the atmosphere. At this point, $\tau_A = 1.3$ so $T_A = 355$ K. This temperature is below boiling and the Clausius-Clapeyron equation still applies. Higher temperatures increase the density and partial pressure of water vapour in the atmosphere even while the relative humidity is unchanged. When the relative humidity is close to 0, the hot equilibrium disappears in a saddle-node bifurcation with the nearby unstable ice-free equilibrium point on the magenta curve. This suggests a lack of water vapour can result in no more ice-free equilibrium points. This right fixed point does not exist between $\delta = 0$ and 0.085.

Due to the step up from the ice albedo feedback, there is only one ice-free intersection before the right stable equilibria. This intersection is unstable and exists at $\delta = 0.085$. There should be an ice-free stable equilibrium near the freezing point. However, the ice-free surface curve stops before any intersection can occur.
When the relative humidity, $\delta$, drops below 0.08, any intersections between the blue and magenta curves do not occur above $T_S = 273.15$ K. This would suggest that the amount of water vapour in the Arctic has dropped to a point where an ice-free climate is no longer sustainable. Then the surface temperature would drop until the surface freezes over, at $T_S = 273.15$ K, and we have dropped on to the frozen surface curve (red). For values above $\delta = 0.08$, two ice-free equilibrium points exist, the equilibrium can be stable or unstable. This would suggest that the climate state would cool down below freezing if starting to the left of the unstable equilibrium point. Otherwise the climate state would heat up to the hot, stable equilibrium.

The hot equilibrium would suggest that there is an abundance of water vapour in the atmosphere that the escaping radiation from the surface can not move through the atmosphere unabsorbed, or $\eta$ increasing close to 1. That is to say, the atmosphere may not be saturated with water vapour, but the equilibrium does not move at higher values of $\delta$. This suggests that the warming of the Earth becomes limited by the amount of radiation leaving the surface. Increasing solar radiation and energy flow into the Arctic would increase radiation from the surface, allowing for a higher intersection.

It is possible for the temperature of the surface and atmosphere to rise past the boiling point of water, 100°C. At this point the Clausius-Clapeyron equation breaks down and is no longer valid. The surface water then begins to boil and evaporate off the surface. However, the breakdown of the model does not mean that there can not be any hot equilibrium or that this equilibrium should be ignored. If the water were to boil off, the energy transport terms $F_A$ and $F_O$ would greatly reduce in value.

With enough heat transport from the atmosphere and ocean currents, there could exist a stable ice-free equilibrium at a surface temperature above freezing and below that of the right stable equilibrium. To do this, the magenta curve would need to be raised. The magenta curve can be raised by increasing $F_{O,w}$ to 90 W/m$^2$. Changes in ocean heat transport is covered in Section 3.4. The equilibrium points will be called cold for equilibrium
points with $T_S \leq 1$ and warm for equilibrium points with $T_S > 1$. These points are unstable and stable accordingly.

The inner ice-free equilibrium is considered unstable (see Appendix B) since a perturbation in the system can move from the unstable equilibrium point to either of the stable equilibria on the left or right in Figure 3.2. The same can be said for the unstable cold equilibria. With a small increase in temperature, the system now has a higher saturation point meaning more water can be evaporated. This can escalate until the warmer equilibrium is reached. Here, temperatures are high enough to have a high saturation point of water vapour and a low relative humidity. Similarly, a small decrease in temperature limits the amount of water vapour and decreases the saturation point. The unstable equilibrium suggests a dividing line for water vapour behaviour. A perturbation would need to perturb the system past this point in order to change the state of the climate.

A saddle node bifurcation occurs at $\delta = 0.085$ to create a stable and an unstable equilibrium point on the ice-free line of Figure 3.2. The two equilibrium points move apart as $\delta$ increases as seen in Figure 3.3.

The saddle node bifurcation seen in Figure 3.3 suggest tipping points of the climate state. When $\delta$ has dropped too low to sustain a hot ice-free equilibrium, the system will fall below the freezing point and jump onto the frozen climate state. For values of $\delta$ above this tipping point, multiple stable equilibria exist, warm and cold. The model can sustain multiple climate states at the same parameters and conditions.

### 3.2.1 Non-linearity in the Clausius-Clapeyron Equation

Multiple states of climate exist in our model and are due to the non-linearity of the Clausius-Clapeyron equation. The non-linear term

$$NL = \frac{1}{\tau_A} \exp \left[ G_3 \left( \frac{\tau_A - 1}{\tau_A} \right) \right]$$

(3.1)
Figure 3.3: Bifurcation plot for water vapour only in the Arctic. A saddle node bifurcation occurs at a value of $\delta = 0.085$. The only greenhouse gas present is water vapour.
Figure 3.4: The non-linearity of water vapour is strongly influenced by the exponential term compared to the inverse $\tau_A$. The value of $\delta$ varies from 10% (top blue line) to 30% to 50% to 70% (bottom blue line).

is what causes the hill and valley to occur in Figure 3.2. Separating each term by setting $1/\tau_A = 1$ or setting $\exp[\cdot] = 1$ can show which non-linear term has the stronger influence.

Figure 3.4 suggests that the exponential component of Equation (3.1) is the dominating non-linear factor. When inverse $\tau_A$ is set to 1, shown as the cyan curves, the hill and valley still appear and stretch further apart. When $\exp[\cdot]$ is set to 1, shown as the green curves, the atmosphere curve now looks similar to Figure 3.1 but bends further to the right. The $\exp[\cdot]$ dominates the Clausius-Clapeyron equation with a large value of $G_3 = 22.5126$ multiplying into $1-1/\tau_A$. Any reduction to $G_3$ would reduce the non-linear effect, shown by the S shaped curve.

When the atmosphere temperature passes $\tau_A = 1$, the atmosphere is now above freezing and the ‘surface’ acted on by the Clausius-Clapeyron equation
Figure 3.5: The change in the atmosphere equation as $\tau_A$ increases past 1. The drop from $L_s = 2.8373 \times 10^6 \text{ m}^2 \text{s}^{-2}$ to $L_v = 2.2558 \times 10^6 \text{ m}^2 \text{s}^{-2}$ reduces the valley caused by the Clausius-Clapeyron equation. The value of $\delta$ varies from 10% (top blue line) to 30% to 50% to 70% (bottom blue line).

is now the water droplets in the atmosphere instead of ice crystals. This changes the latent heat of sublimation to the latent heat of vapourization, $L_v = 2.2558 \times 10^6 \text{ m}^2 \text{s}^{-2}$. This change occurs at $\tau_A = 1$ and the exponential term of Equation (3.1) is 1. This does not cause a discontinuity but does cause a corner to occur in the atmosphere equation. This change reduces the valley created by the Clausius-Clapeyron equation and moves the unstable fixed point to the right. In the hot equilibrium case, it does not change location between latent heats used but does change the value of $\delta$ needed to cause a saddle node bifurcation. The effect of the corner on the atmosphere equation is shown in Figure 3.5. The constant $G_3$ drops in value from 22.5126 to 17.8989 and will be incorporated into the model.
3.3 Combination of Greenhouse Gases

In this section, $\eta$ represents a combination of the carbon dioxide and water vapour greenhouse gases. The values of $\mu$ and $\delta$ are non-zero. Using the denominator of Equation (2.50), $\eta$ can be written as

$$
\eta = 1 - \exp\left(-\mu G_1 - \delta \frac{G_2}{\tau_A} \exp\left[G_3\left(\frac{\tau_A - 1}{\tau_A}\right)\right]\right). \tag{3.2}
$$

Using the parameters found in Sections 3.1 and 3.2, we can combine the effects of the greenhouse gas into the atmosphere energy balance equation, Equation (2.50). Following the process of Sections 3.1 and 3.2, we produce Figures 3.6 and 3.8 to show the effects of varying $\mu$ and $\delta$ on the energy balance equations.

In Figure 3.6, the combination of both carbon dioxide and water vapour has an S-curve similar to Figure 3.2. With the inclusion of carbon dioxide, the ice-covered stable equilibrium is at a warmer temperature than the case of water vapour only. Increasing the ppm concentration of $\text{CO}_2$ greatly impacts this equilibrium point compared to that of $\delta$ in Figure 3.8. Similar to Figure 3.1, values below $\mu = 380$ ppm do not allow the existence of a stable ice-free equilibrium near freezing temperatures for the surface. After the concentration of $\text{CO}_2$ has increased past 380 ppm, a stable ice-free equilibrium exists near the freezing point. An ice-free unstable equilibrium exists between the left and right stable equilibrium points. Similar to Figure 3.3, a saddle node bifurcation occurs for low values of $\delta$. As before, the equilibrium points will be called cold for the ice-covered stable equilibrium on the left, hot for the ice-free equilibrium on the right, and warm for any stable or unstable ice-free equilibrium points.

Increasing the concentration of $\text{CO}_2$ further causes the warm unstable equilibrium to collide with the warm stable equilibrium point. A saddle node bifurcation occurs in Figure 3.7 and these two equilibrium points disappear at $\mu = 726$ ppm. The only ice-free stable equilibrium remaining is
Figure 3.6: Arctic energy balance model with the combination of both greenhouse gases at constant $\delta = 40\%$. The value of $\mu$ varies from 200 ppm (top blue line) to 400 ppm to 600 ppm to 800 ppm (bottom blue line).

on the right hand side of the plot. With relative humidity constant at 40%, the value of $\mu$ would need to be around 1266 ppm to move across the freezing point from the frozen surface line (red). Increasing $\delta$ will decrease the amount of CO$_2$ needed to jump off the frozen surface line. The value of $\delta$ is the main driver for crossing the freezing point from below and both $\mu$ and $\delta$ are the main drivers for crossing the freezing point from above.

In both Figure 3.6 and Figure 3.8, increasing to large values of $\mu$ or $\delta$ will not change the position of the far right stable equilibrium point. With the presence of water vapour, doubling CO$_2$ from 270 ppm to 540 ppm increases the ice-covered surface temperature by 7.157°C. The difference between this increase and the CO$_2$ only case is 0.219°C. This is expected as increasing the amount of water vapour does increase the temperature of the stable ice-covered surface. In the water vapour only case, the warming is minimal but noticeable when combined with CO$_2$. Doubling CO$_2$ at today’s level of 400
Figure 3.7: The saddle node bifurcation for the combined greenhouse gases in the Arctic. This occurs at $\mu = 726$ ppm as the stable and unstable equilibria collapse and disappear. The stable equilibrium exists after $\mu = 380$ ppm. The value of $\delta$ is constant at 40%.
Figure 3.8: Arctic energy balance model with the combination of both greenhouse gases at constant $\mu = 400$ ppm. The value of $\delta$ varies from 10% (top blue line) to 30% to 50% to 70% (bottom blue line).

pm to 800 ppm increases the surface temperature by 10.653°C.

The shape of Figure 3.8 is similar to Figure 3.6 as $\delta$ is varied instead of $\mu$. Due to the existence of CO$_2$, a stable ice-free equilibrium can exist. Due to the value of $\mu = 400$ ppm, the stable warm equilibrium exists at $\delta = 0$. Similar to the case of water vapour only, a stable hot equilibrium exists as the amount of water vapour in the atmosphere increases. As the value of $\delta$ increases from 0, a bifurcation occurs at $\delta = 0.208$ and the stable hot and unstable warm equilibrium exist.

The cold unstable fixed point exists after $\delta = 0.443$. The cold unstable fixed point and frozen stable fixed point (far left) collide and disappear at the saddle point bifurcation after $\delta = 1$. This is shown in Figure 3.9 for both the ice-free and frozen fixed points.
Figure 3.9: The saddle node bifurcation for the combined greenhouse gas case in the Arctic. A bifurcation occurs to produce the warm fixed points at $\delta = 0.208$. The value of $\mu$ is constant at 400 ppm.
An interesting side note to Figures 3.6 and 3.8 has to do with crossing the freezing line from below. In order to jump from the cold surface to the warm surface, $\mu$ had to increase past 1300 ppm at $\delta = 0.4$. The value of $\mu$ required for the jump is less when $\delta$ is higher. At this level of carbon dioxide, a bifurcation occurred on the warm surface equation and the warm stable and unstable solutions collided and disappeared.

The model suggests that melting the Arctic would take the “long way around” to reach an ice-free state as it was in the past. This means that an ice-free Arctic is sustainable at 400 ppm but in order to achieve the ice-free state, $\mu$ needs to increase past 1300 ppm. There is a possibility that the tipping point for the warm solution is crossed while in the frozen state of the Arctic. This means that once the jump up occurs, the hot solution is the only remaining solution above freezing. This is the end of the side note.
Figure 3.11: The saddle node bifurcations for the combined greenhouse gas case in the Arctic. The bifurcations occur in the same way as Figure 3.9. However, the value of $\mu$ is now 270 ppm.
When carbon dioxide is reduced to 270 ppm, Figure 3.10 shows that the warm equilibrium point does not exist as we vary $\delta$ between 0 and 1. The only bifurcation that occurs in Figure 3.11 is the creation of the hot and warm unstable fixed points at $\delta = 0.215$ again since $\mu$ has little effect at high temperatures compared to water vapour. The bifurcation that occurs at the collision of the stable frozen and unstable cold fixed points does not occur before $\delta = 1$. Figure 3.10 shows the atmosphere and surface equations for $\mu = 270$ ppm.

The two different states of climate are separated by a discontinuity from the change in albedo and in ocean heat transport. If an intermediate value was used for both equations, the discontinuity would disappear. Figure 3.12 suggests that the multiple climate states can exist along this intermediate line. An albedo of $\alpha = 0.3$ and an ocean heat transport of $F_O = 25$ W/m$^2$ are used. As $\mu$ varies between 200 ppm and 800 ppm. There are two stable equilibrium points, one cold and one hot, and one unstable equilibrium point. In order to move from one stable solution to another, either a sufficiently large perturbation occurs or a parameter varies sufficiently far to cause the stable solution to undergo a saddle-node bifurcation with the unstable solution. As shown in Figure 3.12, a bifurcation can occur as $\mu$ increases. A similar bifurcation can occur for increasing $\delta$. The hot equilibrium can only be removed by decreasing $\delta$ until a bifurcation occurs as the hot and unstable equilibrium collide and disappear.

### 3.4 Ocean Heat Transport Contribution

In this section, both $\mu$ and $\delta$ will be constant and ocean heat transport, $F_O$, will vary. Ice that covers the surface of the Arctic can prevent the transfer of heat between the ocean current and the surface of the ice. This can be seen as a reduction in energy transfer from the warm tropical waters to the cooler Arctic ocean. This appears to be suggested in Barron 1981 [4] as the energy from ocean heat transport is different depending on whether or not
Figure 3.12: Arctic energy balance model with the combination of both greenhouse gases at constant $\delta = 0.4$. The value of $\mu$ varies from 200 (top blue line) to 400 to 600 to 800 (bottom blue line).
Figure 3.13: Arctic energy balance model for the combined greenhouse gas case. Ocean heat transport parameters, $F_O$ and $F_O$, vary from 60 W/m$^2$ (left magenta/red line) to 40 to 20 to 0 W/m$^2$ (right magenta/red line). The values of $\mu$ and $\delta$ are 400 ppm and 0.4, respectively.

the Arctic is covered in ice. In the earlier Sections, $F_O$ was considered a step function. Recall that

$$F_O = \begin{cases} F_{O,c} & \text{if } T_S \leq 273.15 \text{K or } \tau_S \leq 1, \\ F_{O,w} & \text{if } T_S > 273.15 \text{K or } \tau_S > 1. \end{cases}$$

It will remain a step function much like the albedo except the values of $F_{O,w}$ and $F_{O,c}$ will be varied for each step. Each will increase from 0 to 60 W/m$^2$ in Figure 3.13. The net effect is to raise the magenta and red curves. The values of $\mu$ and $\delta$ will be 400 ppm and 0.4, respectively.

In Figure 3.14, the warm and cold surface equations vary $F_{O,w}$ and $F_{O,c}$ from $-50$ to 200 W/m$^2$. Both the warm and cold parameters are taken as the same range and are referred to as $F_O$ in Figures 3.13 and 3.14. The
Figure 3.14: The saddle node bifurcations for the combined greenhouse gas case as $F_O$ and $F_O$ vary in the Arctic. The bifurcations occur at $F_O = -18 \text{ W/m}^2$ and $167 \text{ W/m}^2$. The values of $\mu$ and $\delta$ are 400 ppm and 0.4, respectively.
large range greatly exceeds the estimates found in Barron 1981 [4]. The purpose was to show where the saddle-node bifurcations occur in Figure 3.14. A negative value of $F_O$ means that energy is leaving the Arctic by ocean currents. A situation where this could occur would be when the Arctic is much warmer than the surrounding regions. The bifurcation that creates the unstable and hot equilibrium solutions begin at $F_{O,w} = -18 \text{ W/m}^2$. For values greater than $-18 \text{ W/m}^2$, an ice-free Arctic will have high temperature values. In Section 3.2, the ‘hot’ equilibrium was seen as the greenhouse gas abundant atmosphere. With an abundance of greenhouse gas, the amount of long wave radiation from the surface is the limiting factor for warming the Arctic. In this Section, the result is the same. However, for low enough $F_{O,w}$, a hot equilibrium can exist while the Clausius-Clapeyron equation is still valid.

Multiple warm stable climate states occur after $F_{O,w} = 32.5 \text{ W/m}^2$. This is the point where the stable ice-free equilibrium exists near freezing. A second bifurcation occurs at $F_{O,w} = 167 \text{ W/m}^2$ and the stable warm and unstable warm equilibrium points collide and disappear. The only option past this value is the hot equilibrium.

The stable frozen solution remains as the only equilibrium below freezing. The temperature of the surface increases gradually as $F_{O,c}$ increases. With enough $F_{O,c}$, the stable frozen solution disappears as the surface temperature moves above $\tau_S = 1$ at $F_{O,c} = 95 \text{ W/m}^2$. After this, there are no sustainable cold climate states.

In the previous Sections, there has been temperature inversion in many of the plots and Figures. In this thesis, temperature inversion is when the temperature of the atmosphere is warmer than the temperature of the surface. This type of situation hasn’t occurred in Earth’s known climate history, with the exception of local and temporary “weather” inversions. In Figure 3.13, we see temperature inversion for the hot equilibrium and for a short period for the frozen equilibrium. During these instances, the atmosphere has absorbed enough energy to maintain a higher temperature value than
the surface. Since the Arctic surface gets a portion of its energy from ocean heat transport, $F_O$, and incoming solar radiation, $F_S$, a reduction in either value would reduce the temperature of the surface while the atmosphere stays warm with the presence of greenhouse gases. The hot equilibrium has temperature inversion due to high density of water vapour and the frozen equilibrium has temperature inversion due to a high albedo, $\alpha_c$, and low $F_{O,c}$ alongside a high ppm of CO$_2$, $\mu$. The warm stable equilibrium near freezing is the middle ground, where the density of water vapour is not too high and enough energy is coming into the surface for a warmer surface.

### 3.5 Changes in Arctic Albedo

In the Arctic, the albedo changes as the ice melts, partially or completely. As ice melts, any sediments collected from moving glaciers may come to the surface, reducing the albedo, $\alpha_c$. In the other direction, as snow and ice accumulate, more sunlight is reflected as $\alpha_c$ increases. In Chapter 2, the albedo of the current Arctic was chosen as $\alpha_c = 0.4$. In the past, it had been higher when the ice was cleaner and more reflective. Recall that

$$\alpha = \begin{cases} 
\alpha_c & \text{if } T_S \leq 273.15 \text{K or } \tau_S \leq 1, \\
\alpha_w & \text{if } T_S > 273.15 \text{K or } \tau_S > 1.
\end{cases}$$

Varying $\alpha$ in our model gives a similar plot as Figure 3.13 and a similar bifurcation plot between 0 and 1 in Figure 3.15. Varying $\alpha_c$ between 0 and 1 represents melting ice and increased solar absorption. Varying $\alpha_w$ between 0 and 1 represents a change in solar absorption for an ice-free surface. An increase of $\alpha_w$ would be an increase in reflectivity. For example, if land mass were to replace the Arctic, the albedo would increase.

A bifurcation occurs at $\alpha_w = 0.35$ as the hot and unstable equilibrium solutions disappear. The value of $\alpha_{w,c} = 0$ corresponds to total absorption of
Figure 3.15: The saddle node bifurcations for the combined greenhouse gas case as $\alpha_w$ and $\alpha_c$ vary in the Arctic. The bifurcation occurs at $\alpha_w = 0.35$. The values of $\mu$ and $\delta$ are 400 ppm and 0.4, respectively, and $F_{O,w} = 36.2$ W/m$^2$ and $F_{O,c} = 10.7$ W/m$^2$.

sunlight and $\alpha_{w,c} = 1$ corresponds to totally reflected sunlight. As the warm surface begins to reflect more sunlight, there is not enough sunlight absorbed to re-emit into the atmosphere to sustain an ice-free equilibrium. Even increasing $\alpha_w$ past 0.06 is enough to eliminate the warm ice-free equilibrium near freezing.

Temperature inversion occurs here again, the ice-free Arctic was able to absorb enough sunlight and emit radiation for the atmosphere to absorb such that the atmosphere could maintain a higher temperature. In the frozen Arctic, the more sunlight that was reflected prevented the surface from gaining enough energy to sustain a warmer temperature than the atmosphere. Increasing the albedo has a similar impact to reducing incoming solar radiation or reducing the ocean heat transport.
Instead of a step function for $\alpha$ and $F_O$, a switch function can be used to eliminate the discontinuity. A typical switch function used is $\tanh(\cdot)$. A switch function can be implemented for $\alpha$ as

$$\alpha = (0.5 + 0.5 \tanh(\frac{\tau S - 1}{b}))\alpha_w + (0.5 - 0.5 \tanh(\frac{\tau S - 1}{b}))\alpha_c. \quad (3.3)$$

A similar switch function can be made for $F_O$. The smoothness, $b$, determines how quickly the $\tanh(\cdot)$ function switches from $\alpha_c$ to $\alpha_w$ as $\tau_S$ increases from below 1 to above 1. The switch is half accomplished at $\tau_S = 1$. A value of $b = 0.01$ will be used. The cold and warm surface labels will remain to distinguish the different fixed points/solutions and indicate the transition from warm to cold surface equations.

Increasing $\mu$ with the switch function allows for a jump up to occur with less carbon dioxide. A concentration of 1100 ppm is needed to jump from the cold surface to warm surface. Figure 3.16 incorporates the switch function (curved surface equation) alongside the ice-albedo discontinuity (broken curves). Doing so will add another fixed point on the transition curve between the warm and cold surface equations. Stability of the fixed points in the new surface curve is found in Appendix B. An intersection occurring on the transition line will be unstable. The stability of the remaining fixed points are unchanged.

Figure 3.17 shows multiple bifurcations from the inclusion of the switch function. The collision of the warm stable and unstable fixed points are unchanged at 726 ppm. With the switch function, more CO$_2$ is required to sustain an ice-free Arctic. Before the switch function, only $\mu = 380$ ppm was needed for an ice-free Arctic. There are two new bifurcations that occur here. These are from the inclusion of the switch function, as opposed to jumps from the discontinuity. The stable warm solution and unstable transition are created at 535 ppm. The unstable transition solution collides and disappears with the stable frozen solution at 1100 ppm. After 1100 ppm, there is a jump up to the hot solution.
Figure 3.16: Arctic energy balance model for the combined greenhouse gas case. The value of $\mu$ varies from 200 ppm (top blue line) to 400 to 600 to 800 ppm (bottom blue line). The value $\delta$ is and 0.4 and a switch function is used to remove any discontinuities.

The warming done by doubling CO$_2$ to the stable frozen solution is unchanged. The frozen stable solution does not come close enough to the transition curve to be influenced by it.

3.5.1 Variability in the Water Vapour Contribution

The inclusion of water vapour forces a hot only solution after the jump from the cold surface equation occurs past 1100 ppm. On the real Earth, there should be a jump up to a warm solution. The absorption term for water vapour has some speculation with regards to the absorption coefficient $k_w$. The method of estimation may be over estimating this parameter. Increasing this value increases $G_2$ and will flatten the S-curve in a similar way as increasing $\delta$. If $k_w$ was estimated too high, a reduction in this value could
Figure 3.17: The saddle node bifurcations for the combined greenhouse gas case as $\mu$ varies in the Arctic. Bifurcations occur at $\mu = 573$ ppm, 506 ppm, and 980 ppm. The value of $\delta$ is 0.4 and a switch function is used to remove any discontinuities.
allow for a warm solution to exist after the jump from the cold surface equation to the warm surface equation.

A reduction to $G_2$, $k_w$ specifically, by a factor of five will allow for a stable warm solution to exist after the transition from the cold surface to the warm surface at $\mu = 1260$ ppm. This may be more realistic for the current Earth. However, more investigation into the absorption done by water vapour will be needed to determine this behaviour. Increasing this reduction raises the right hand side of the plot, near the hot equilibrium, off of the warm surface equation. This is seen in Figure 3.18.

The other constant of the water vapour absorption term, $G_3$, does not have the same level of speculation. The values of latent heat of sublimation and vaporization vary little in the literature at the temperature ranges studied in this model. An increase in these values will lower the valley of the atmosphere equation and a decrease will raise the valley and eventually eliminate it. Decreasing $G_3$ by two will allow for a warm solution to exist in a similar way as reducing $G_2$. Since $R_W$ is in the denominator of $G_3$, the opposite will happen when $R_W$ is increased or decreased.

Altering $G_3$ is unlikely to improve the accuracy of the model since these parameters have little uncertainty from the literature. Altering $G_2$ may improve the accuracy of the model since $k_w$ has a greater deal of uncertainty. Further investigation will be needed for the value of $k_w$, this will be added to the future work portion. For the rest of the Thesis, the values of $G_2$ and $G_3$ will be unchanged.
Figure 3.18: Arctic energy balance model for the combined greenhouse gas case. The value of $\mu$ varies from 800 ppm (top blue line) to 1000 to 1200 to 1400 ppm (bottom blue line). The value $\delta$ is and 0.4 and the value of $k_w$ is reduced by a factor of 5.
Chapter 4

Antarctic Model

In this Section we will apply the model to the Antarctic instead of the Arctic. Table 4.1 contains the parameters that change from Arctic to Antarctic. The cold albedo, $\alpha_c$, is increased to 0.7 to represent the ice and snow accumulation on the Antarctic surface [22]. The snow on Antarctica is much cleaner than in the Arctic, thus the albedo is higher. If the snow and ice were to melt off of Antarctica, sea levels would rise and the continent would also be exposed. Since exposed land mass would be present, the warm albedo, $\alpha_w$, will increase to 0.21, an average between 0.17 and 0.25 [22].

The Antarctic continent moved over the South Pole and disrupted the ocean current that circulated through the region. After the Antarctic moved across the South Pole, the Antarctic Circumpolar Circulation (ACC) began. The ACC is an ocean current that circulates the continent of Antarctica. The ACC conflicted with other ocean currents attempting to bring energy into the region by ocean transport. With warm ocean currents cut off from the Antarctic, cooling began to occur and accumulation of snow and ice began to take place. Similarly, circumpolar winds restrict the atmospheric heat transport into the Antarctic region. The climate of Antarctica was unable to sustain an ice-free equilibrium and has now become completely
ice-covered. With the accumulation of ice and snow on Antarctica, sea levels dropped by approximately 60 m [36].

On the other hand, during the Cretaceous Period, 65-145 million years ago, the South Pole was in open ocean waters and the continent of Antarctica was surrounded by the warm South Pacific Ocean. At that time there was no polar ice-cap in the Antarctic. Therefore, in this Chapter, we consider two Antarctic models, one for the modern frozen Antarctic in Section 4.1 and the other for the ice-free Cretaceous Antarctic in Section 4.2.

The ocean heat transport will be investigated as another bifurcation parameter in Section 4.2. As $F_O$ increases, more heat will be added to the surface. This can represent the gain or loss in ocean transport as the continent of Antarctica moved to the South Pole. We can use Cretaceous values for the ocean transport as an upper limit at $F_{O,w} = 103.96 \text{ W/m}^2$, since the position of Antarctica was not covering the South Pole in earlier time periods, such as the Cretaceous [4]. There is no data for an ice-free Antarctica on the South Pole. However, the ACC would still be present, so $F_{O,w} = 0 \text{ W/m}^2$ is chosen to represent an ice-free, modern day Antarctic. Atmospheric heat transport will be taken as a constant similar to the Arctic. This Chapter will cover the contributions of CO$_2$ and water vapour in the same fashion as before.

4.1 Modern Day Antarctic

In this Section, a modern day Antarctic will be considered. That is, $\alpha_w = 0.21$ and $F_{O,w} = 0 \text{ W/m}^2$. With the presence of the ACC, an ice-free equilibrium will be difficult to achieve.
### Parameter Values Used

<table>
<thead>
<tr>
<th>Parameters and Constants</th>
<th>Symbol</th>
<th>Value (units)</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Albedo of snow on ice</td>
<td>$\alpha_c$</td>
<td>0.7</td>
<td>[22]</td>
</tr>
<tr>
<td>Albedo of grass/soil</td>
<td>$\alpha_g$</td>
<td>0.21</td>
<td>[22]</td>
</tr>
<tr>
<td>Albedo of open ocean</td>
<td>$\alpha_w$</td>
<td>0.04</td>
<td>[22]</td>
</tr>
<tr>
<td>Solar distr. at $\theta = \frac{-\pi}{2}$</td>
<td>$s(\theta)$</td>
<td>0.505</td>
<td>[23]</td>
</tr>
<tr>
<td>Ocean heat transport</td>
<td>$F_{O,c}$</td>
<td>0 W/m$^2$</td>
<td></td>
</tr>
<tr>
<td>(Modern)</td>
<td>$F_{O,w}$</td>
<td>103.96 W/m$^2$</td>
<td>[4]</td>
</tr>
<tr>
<td>(Cretaceous)</td>
<td>$F_{A}$</td>
<td>60.805 W/m$^2$</td>
<td>[4]</td>
</tr>
</tbody>
</table>

Table 4.1: Parameters that will change in the Antarctic model. The corresponding non-dimensional values of the heat transports are $[\cdot]/\sigma T_R^4$.

#### 4.1.1 Carbon Dioxide Contribution

In this Section we consider the only greenhouse gas to be CO$_2$ and set $\delta = 0$. The only parameters that will change are listed in Table 4.1. The ocean heat transport will be unchanged since the continent of Antarctica covers the South Pole and the Antarctic Circumpolar Current (ACC) prevents ocean circulation into the region.

The resultant plot in Figure 4.1 suggests that current CO$_2$ levels alone can not sustain an ice-free equilibrium. Given the geographical circumstances, carbon dioxide would need to increase to a concentration above 2000 ppm to sustain a stable ice-free equilibrium. Doubling the concentration of CO$_2$ from 270 to 540 increases the frozen surface equilibrium temperature by 5.845°C.

If ocean currents were able to bring more heat into the region or if the South Pole was mainly ocean instead of land, the level of CO$_2$ required for an ice-free equilibrium would reduce. Since Antarctica covers the South Pole, an ice-free equilibrium is not easily attained.
Figure 4.1: Modern Antarctic energy balance model as $\mu$ varies from 500 ppm (left blue line) to 1000 ppm to 1500 ppm to 2000 ppm (right blue line). Here, $\delta = 0$. 
4.1.2 Water Vapour Contribution

In this Section we will consider the only greenhouse gas to be water vapour and set $\mu = 0$ ppm. Using the parameters listed in Table 4.1, we will see the impact of water vapour only on the Antarctic model. Figure 4.2 shows the energy balance equations for water vapour only. Here, $\delta$ varies from 0.3 to 0.9 and only the frozen equilibrium exists. Even at 99-100% relative humidity, the blue curve does not intersect the magenta curve. This may be due to the lack of ocean heat transport. Without the extra energy flowing into the Antarctic, an ice-free equilibrium is not attainable. Water vapour alone is not enough to reach the ice-free surface equation (magenta) from below freezing on the frozen surface equation (red).
4.1.3 Combination of Greenhouse Gases

In this Section both greenhouse gases will be considered. We will use $\mu$ and $\delta$ as the bifurcation parameters. Here, $\mu$ is varied in Figure 4.3. The relative humidity is constant at 0.2 since Antarctica is drier than the Arctic. Due to the presence of Antarctica, no energy can come from ocean heat transport, $F_O$. With less energy available, Figure 4.3 does not show an ice-free equilibrium at $\delta = 0.2$ with a carbon dioxide concentration up to 800 ppm. If $\mu$ is increased above 1800 ppm, then an ice-free equilibrium would appear. At $\delta = 0.2$, a jump up to the warm surface equation is not possible. The cold stable equilibrium always exists.

Since water vapour has little to no effect on the stable frozen equilibrium, the warming caused by doubling CO$_2$ is the same as if water vapour was not present. Doubling CO$_2$ for the frozen solution increases the surface temperature by 5.845°C. Doubling the current day CO$_2$ (400 ppm to 800 ppm) for the frozen solution increases the surface temperature by 8.249°C.

Figure 4.4 shows the energy balance model as $\delta$ varies. At high levels of relative humidity, an ice-free solution was attained. For a modern day Antarctic, the only ice-free climate states are the unstable and hot equilibria. The frozen stable equilibrium always exists. The current cold albedo, $\alpha_c$, there is no transition from the cold to warm surface. Figure 4.5 shows the bifurcation plot for the modern day Antarctic. For values above $\delta = 0.91$, an unstable and stable equilibrium are possible. The modern Antarctic is considered a dessert and quite dry. A high relative humidity here would not be realistic for the Earth.

Combining the greenhouse gases, water vapour and CO$_2$, a stable, ice-free equilibrium near the freezing point is unrealistic to achieve. This is due to the lack of ocean heat transport, $F_O$, by the presence of the continent of Antarctica. The Antarctic would need more energy entering the region by ocean currents to allow for an ice-free climate near freezing.
Figure 4.3: Modern Antarctic energy balance model with the combination of greenhouse gases. The value of $\mu$ is varied from 200 ppm (top blue line) to 400 ppm to 600 ppm to 800 ppm (bottom blue line). The value of $\delta$ is constant at 0.2.
Figure 4.4: Antarctic energy balance model the combination of greenhouse gases. The value of $\delta$ is varied from 40\% (top blue line) to 60\% to 80\% to 100\% (bottom blue line). The value of $\mu$ is constant at 400 ppm.
Figure 4.5: The saddle node bifurcation for the combined greenhouse gas case in the modern Antarctic. The bifurcation occurs at $\delta = 0.91$ to produce an unstable and stable fixed point. The value of $\mu$ is constant at 400 ppm.
4.2 Cretaceous Antarctic

In this Section we will consider Barron’s problem of a globally ice-free Earth during the Cretaceous, 65-145 million years ago. As stated earlier, the continent of Antarctica was not covering the South Pole. Without Antarctica and the ACC, ocean currents could freely circulate through the south pole. Using the Cretaceous values of $\alpha_w = 0.04$ and $F_{O,w} = 103.96 \text{ W/m}^2$ [4], an ice-free South Pole will be easier to obtain.

4.2.1 Combination of Greenhouse Gases

Both greenhouse gases will be present in the Cretaceous Antarctic. The first case will consider $\delta$ constant at 0.2 as $\mu$ varies. With the increase of energy from a higher $F_{O,w}$ and lower $\alpha_c$, a stable warm equilibrium is possible. Figure 4.6 shows the existence of two ice-free stable equilibria for the Cretaceous Antarctic. The unstable and hot equilibrium points persist with the inclusion of water vapour. Figure 4.7 shows the bifurcations as $\mu$ increases in the Cretaceous Antarctic. A bifurcation occurs at 250 ppm to create a warm stable solution and an unstable solution on the transition curve. The warm stable solution collides with the warm unstable solution at another bifurcation occurring at $\mu = 998$ ppm. The frozen albedo of 0.7 prevents a jump from the cold surface to the warm surface.

There is no tipping point to leave the frozen Antarctic, but there is a tipping point at 998 ppm where the warm solution disappears and the only ice-free solution is the hot equilibrium. Doubling CO$_2$ from 270 ppm to 540 ppm will warm the frozen surface by 5.845°C, same as the CO$_2$ only case, and warm the ice-free surface by 8.713°C.

Figure 4.8 varies in $\delta$ and remains constant in $\mu$ at 400 ppm. With the higher ocean heat transport and lower albedo, an ice-free warm equilibrium is possible at $\mu = 400$ ppm. Figure 4.9 shows one bifurcation as $\delta$ increases.
Figure 4.6: Cretaceous Antarctic energy balance model with the combination of greenhouse gases. With a higher value of $F_{O_{2}} = 103.96$, a stable ice-free equilibrium is achievable. The value of $\mu$ is varied from 200 ppm (top blue line) to 400 ppm to 600 ppm to 800 ppm (bottom blue line). The value of $\delta$ is constant at 0.2.
Figure 4.7: The saddle node bifurcation for the combined greenhouse gas case in the Cretaceous Antarctic. The bifurcations occur at $\mu = 245$ and 678 ppm. The value of $\delta$ is constant at 0.2.
Figure 4.8: Cretaceous Antarctic energy balance model with the combination of greenhouse gases. With a higher value of $F_{O_w} = 103.96$, a stable ice-free equilibrium is achievable. The value of $\delta$ is varied from 10\% (top blue line) to 30\% to 50\% to 70\% (bottom blue line). The value of $\mu$ is constant at 400 ppm.

from 0 to 1. The bifurcation occurs at $\delta = 0.158$ as the stable and unstable hot solutions are created. The stable warm solution, the unstable transition solution, and the stable frozen solution exist for all values of $\delta$ between 0 and 1.

A cold surface albedo, $\alpha_c$, of 0.7, makes it impossible to jump up to the warm surface equation at the current parameters. In both the modern day and Cretaceous cases, the cold albedo would need to reduce to 0.5 before a jump up can occur. Another option is to increase the energy entering the cold Antarctic by $F_{O_c}$, or a mix of both options.
Figure 4.9: The saddle node bifurcation for the combined greenhouse gas case in the Antarctic. A bifurcation occurs at $\delta = 0.158$ as the unstable and stable hot equilibrium points are created. The value of $\mu$ is constant at 400 ppm.
4.2.2 Ocean Heat Transport Contribution

In this section we will use $F_O$ as the bifurcation parameter. Both $\delta$ and $\mu$ will be constant at 0.2 and 400 ppm, respectively. A higher value of $\mu$ can be used to represent a warmer Cretaceous and extend the range of the warm ice-free equilibrium. A value of 400 ppm will be used to remain consistent with today’s values and the other Chapters. The ice-free albedo will be constant at $\alpha_w = 0.04$. Figure 4.10 shows the energy balance equations for the Cretaceous Antarctic. For the Cretaceous, a lower bound of $F_O$ is 0 W/m$^2$ and an upper bound is 103.96 W/m$^2$ are chosen to represent the possibility of an ice-free Antarctic. If any equilibrium points exist in this boundary, then the model would suggest an ice-free Antarctic would be feasible for Earth during the Cretaceous. Increasing $F_O$ will represent an increase of heat entering the region through ocean circulation, or as if Antarctica were to drift away to where it was in the Cretaceous. Decreasing $F_O$ will represent Antarctica drifting towards the South Pole, cutting off ocean currents. A smaller, similar result can be found as $\alpha_w$ varies between an albedo of open ocean and an albedo of grass/soil and as $\alpha_c$ varies between clean snow and dirty ice.

Figure 4.10 shows energy balance equations as $F_O$ increases. Figure 4.11 shows the bifurcations of varying $F_O$. The hot unstable and stable equilibrium points are created at the first saddle node bifurcation at $F_O = 80$ W/m$^2$. The stable solution near freezing is created by a bifurcation at $F_O = 81$ W/m$^2$ alongside an unstable solution on the transition curve. The stable and unstable fixed points above freezing collide and disappear at $F_O = 382$ W/m$^2$ as another bifurcation occurs. The frozen stable solution collides with the unstable solution at $F_O = 153.5$ W/m$^2$ as the last bifurcation occurs. At this point, a jump from the cold surface to the warm surface is possible. Figure 4.11 suggests that a large amount of heat by $F_O$ is required to force a hot equilibrium only. If the albedo were to change from grass/soil to open ocean, the ice-free surface equation would shift to the left. This change would result in the bifurcations in Figure 4.11 to occur earlier. Two
Figure 4.10: Cretaceous Antarctic energy balance model with the combination of greenhouse gases. The value of $F_{O,w}$ is varied from 100 W/m$^2$ (left magenta/red line) to 70 to 40 to 10 W/m$^2$ (right magenta/red line). The value of $\mu$ is constant at 400 ppm and $\delta$ is constant at 0.2.

Bifurcations occur within the bounds of 0 and 103.96 W/m$^2$. This suggests that within 0 and 103.96 W/m$^2$, we will have at least one stable ice-free equilibrium at $\mu = 400$ ppm and $\delta = 0.2$. Our energy balance model shows an ice-free Cretaceous before Antarctica covered the South Pole, agreeing with the geologic record.

There are no stable frozen fixed points close to freezing in Figure 4.10. This is due to the ocean heat transport and albedo for the frozen surface equation (red). The frozen equilibrium does not jump up to the warm surface under the parameters used. If the ocean heat transport was increased above $F_O = 153.5$ W/m$^2$, the frozen equilibrium would pass the freezing line and jump to the warm surface equation. Any reduction in $\alpha_c$ would lower the value of $F_O$ required for this jump to occur.
Figure 4.11: The saddle node bifurcations for the combined greenhouse gas case in the Cretaceous Antarctic. The bifurcations occur at $F_O = 80, 81, 153.5, \text{ and } 382 \text{ W/m}^2$. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.2, respectively.
Barron 1983 [2] attempted to transition from frozen poles to a globally warm and equable climate state. However, after investigating the Antarctic, this would be difficult to achieve by changes in greenhouse gas activity alone. This model suggests that the albedo and ocean heat transport are key factors in the Antarctic transition from frozen to ice-free.
Chapter 5

Tropics Model

In this chapter we will apply the model to the Tropics region instead of the Arctic and Antarctic. Table 5.1 contains the parameters that change from Arctic to the Tropics. The cold albedo, $\alpha_c$, is set to 0.5 to represent the ocean ice [22]. Any ice coverage in the tropics will be weak due to the amount of sunlight reaching the surface and have a surface albedo on the lower end of 0.5-0.7 range for ocean ice. The warm albedo, $\alpha_w$, will be set to 0.04 [15] since the lower latitudes are above 75% water. Some models consider a water world where no land is considered. Some water world models consider a ridge of land for ocean circulation; this is done in Ferreira 2011 [12].

In our model, the Arctic and Antarctic were considered as single columns at the North and South poles, respectively. The Tropics is considered at the 0° latitude belt, which is a disc around the Earth. For this model, the 0° latitude belt has rotational symmetry and any square meter column is identical at any point of the disc. Any flux values used for the Tropics will show the energy leaving this square meter area of the Tropics, either north or south.

Instead of energy being transported to the region by ocean currents and
Table 5.1: Parameters that will change in the Tropics model. The corresponding non-dimensional values of the heat transports are \([\cdot]/\sigma T^4_R\).

Atmospheric transport, energy will now leave the region towards the polar regions. The value of atmospheric transport is found in Table 3 from Barron 1981 [4]. The value of energy leaving the tropics will be taken at the equator, or the 0 latitude belt. The table includes atmosphere and total transport values, finding the difference would be the ocean heat transport term. These values are listed in Table 5.1 below. If the tropics were to freeze over, there would be little ocean circulation and little heat would be transported out of the region. So an estimated 10 W m\(^{-2}\) will leave the tropics when it is in a frozen state.

### 5.1 Carbon Dioxide Contribution

In this section we will consider CO\(_2\) as the only greenhouse gas and set \(\delta = 0\). Figure 5.1 shows the energy balance equations. The atmosphere curves plotted in Figure 5.1 will appear different than in earlier chapters. The parameters \(F_A\) and \(F_{O}\) are negative and change the shape of the curves.

Figure 5.1 suggests that the tropics can always sustain a stable ice-free surface. Due to the low value of \(\alpha_w\) and negative \(F_{O,w}\), the warm surface equation does not fall below freezing, even at 0 K. It appears that, under
Figure 5.1: Tropics Energy balance model with carbon dioxide as the only greenhouse gas. The value of $\mu$ is varied from 200 ppm (top blue line) to 400 ppm to 600 ppm to 800 ppm (bottom blue line).

In a frozen state, the Tropics would only need $\mu = 1075$ ppm to melt the ice and jump to an ice-free equilibrium. A doubling of CO$_2$ from 270 to 540 would increase a frozen tropics by 7.020°C and an ice-free tropics by 8.195°C. Doubling current day CO$_2$ (400 ppm to 800 ppm) would result in a surface warming of 9.943°C for a frozen Tropics and 11.582°C for an ice-free Tropics.
5.2 Water Vapour Contribution

In this section we will consider water vapour as the only greenhouse gas and set $\mu = 0$. In earlier chapters, water vapour had a blue line (atmosphere energy balance equation) on the far left hand side increasing rapidly. This was due to the positive value of $F_A$. With a negative value of $F_A$, this line does not exist above $\tau_A = 0$. At $\tau_A = 0$, the slope of Equation (2.50) is undefined.

Figure 5.2 shows the energy balance equations for water vapour only. Only the unstable and hot ice-free equilibrium points exist. Figure 5.3 shows the bifurcation plot for water vapour only. These equilibria exist after the saddle node bifurcation at $\delta = 0.16$. Even though the warm surface equation does not reach $\tau_S = 1$ before $\tau_A = 0$, there is no intersection around that point. Therefore, Figure 5.2 suggests there are no frozen equilibria for the tropics region.

5.3 Combination of Greenhouse Gases

In this section we will consider carbon dioxide and water vapour as the acting greenhouse gases and have $\mu$ and $\delta$ non-zero. Figure 5.4 shows the energy balance equations for the Tropics and $\mu$ will vary while $\delta$ is constant at 0.6. The hot equilibrium exists for this range of $\mu$. Figure 5.5 shows the bifurcations as $\mu$ increases from 200 ppm to 1200 ppm. The warm stable and unstable equilibrium points collide and disappear at $\mu = 831$ ppm. After this point, only the hot equilibrium remains. Higher values of $\delta$ will cause the bifurcation to occur sooner. The frozen equilibrium does not exist after $\mu = 1065$ ppm as the stable frozen and unstable transition solutions collide and disappear. At this point a frozen Tropics will jump up to an ice-free Tropics.

In the case of a frozen tropics, doubling CO$_2$ will increase the surface
Figure 5.2: Tropics energy balance model with water vapour as the only greenhouse gas. The value of $\delta$ is varied from 10% (top blue line) to 30% to 50% to 70% (bottom blue line). Here, $\mu = 0$. 
Figure 5.3: Bifurcation plot with water vapour as the only greenhouse gas in the Tropics. A saddle node bifurcation occurs at $\delta = 0.16$ to create the only two possible equilibria, an unstable and stable solution. There are no equilibria below freezing and $\mu = 0$. 
temperature by 7.020°C, the same as if no water vapour was present. For an ice-free tropics, doubling CO$_2$ will increase the surface temperature by 8.276°C. The inclusion of water vapour increased the ice-free warming by 0.081°C. Doubling the current day CO$_2$ (400 to 800 ppm) will increase the surface temperature of a frozen Tropics by 9.942°C and 13.630°C for an ice-free Tropics.

Figure 5.6 shows the energy balance equations for the Tropics as $\delta$ varies. Figure 5.7 shows the bifurcations as $\delta$ increases from 0 to 1. Varying $\delta$ as $\mu$ remains constant at 400 ppm gives the same equilibria in Figure 5.4. The frozen, transition, warm and hot equilibrium points do not change much in position as $\delta$ varies. The unstable equilibrium changes position as $\delta$ increases from 0 to 1. The only bifurcation in Figure 5.7 occurs at $\delta = 0.145$ to create the hot and unstable equilibrium fixed points. Figure 5.6 suggests that water vapour does not contribute much to the stable equilibria.
Figure 5.5: Bifurcation plot with combined greenhouse gases in the Tropics. The saddle-node bifurcations occur at $\mu = 831$ and $1065$ ppm. The value of $\delta$ is constant at 0.6.
The value of $\delta$ is varied from 10% (top blue line) to 30% to 50% to 70% (bottom blue line). The value of $\mu$ is constant at 400 ppm.
Figure 5.7: Bifurcation plot with combined greenhouse gases in the Tropics. A saddle node bifurcation occurs at $\delta = 0.145$ as the unstable and hot stable ice-free solutions are created. The value of $\mu$ is constant at 400 ppm.
5.4 Ocean Heat Transport Contribution

In this section, we will vary $F_O$ while both $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively. The Tropics have warmer temperatures than the Arctic or Antarctic. This way, ocean currents bring energy from the Tropics to each polar region to increase surface temperatures. The tropics region spans the latitude belt at $0^\circ$, and for this reason, the estimates from Barron 1981 [4] may not accurately represent the heat lost in the entire Tropics. This model considers the tropics as a single square meter column, similar to the Arctic and Antarctic while the area of the Tropics is much larger. Figure 5.8 shows the energy balance equations as $F_{O,w}$ and $F_{O,c}$ vary from losing energy at 150 W/m² to 0 W/m². Figure 5.9 shows the corresponding bifurcations for a range of $-200$ to $+300$ W/m². The purpose is to show change in the equilibrium positions with increased or decreased energy leaving the Tropics.

Figure 5.9 suggests that the more energy that flows out of the tropics can reduce the hot equilibrium to near freezing. The hot and unstable equilibrium solutions exist after $F_O$ increases above $-160$ W/m². The warm stable solution appears after $F_O$ increases past $-102$ W/m².

In an ice-free Tropics, starting from $F_O = -160$ W/m², a saddle node bifurcation occurs as the hot and warm unstable are created. The warm stable solution is created at another bifurcation at $F_O = -102$ W/m² alongside the unstable transition solution. The last ice-free bifurcation occurs as the warm stable and unstable solutions collide and disappear at $F_O = +260$ W/m². The unstable solution crosses below freezing to collide with the frozen stable solution at the only cold bifurcation at $F_O = +43$ W/m².

Increasing ocean heat transport past 43 W/m² will cause a jump from a frozen Tropics to an ice-free Tropics. Increasing the energy out of the Tropics by ocean heat transport past $-102$ W/m² will cause a jump from an ice-free Tropics to a frozen Tropics.

An interesting side note from this section and the previous section is
Figure 5.8: Tropics energy balance model with combined greenhouse gases. The value of $F_O$ is varied from 0 W/m$^2$ (left red/magenta line) to $-50$ to $-100$ to $-150$ W/m$^2$ (right red/magenta line). The value of $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively.
Figure 5.9: Bifurcation plot with combined greenhouse gases in the Tropics. The saddle node bifurcations occur at $F_O = -160, -102, 43,$ and $260 \text{ W/m}^2$. The value of $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively.
that the temperature of the atmosphere is well below freezing for the stable warm solutions. Using a larger negative value of $F_O$ may suggest that the tropics region is actually at the hot equilibrium today rather than the warm solution. Perhaps the Tropics region is losing more energy than previously thought, since the model considers the Tropics as a column, rather than a disc around the Earth. In the case of either the warm or hot equilibrium, our model suggests water vapour has little effect. Our model suggests that CO$_2$ concentration has little impact on the hot equilibrium but does impact the warm solution. However, since global warming has been progressing and carbon emissions have been increasing, the surface temperature of the Tropics should have been increasing steadily since $\mu = 270$ until now. Our model suggests a change in surface temperature when it has increased little since then, comparatively. Our model does not give exact detailing of the climate but could give a possible explanation or ballpark estimation.
Chapter 6

Paradox Application

Over the course of the Earth’s history, there have been times where the Earth had a climate that did not seem to agree with known drivers of the climate at those times. One of those paradoxes is known as the Pliocene Paradox 3-5 million years ago. At the beginning of that time period, the Earth had an ice-free Arctic, a frozen Antarctic, and a warm equable climate between the Tropics and the Arctic. However, the drivers of the climate and most parameters were the same or similar to today. The concentration of carbon dioxide and the amount of incoming solar radiation were the main similarities between today and in the Pliocene [11].

6.1 Pliocene Paradox

The Arctic was ice-free at the beginning of the Pliocene Epoch (5.3 to 2.6 mya) even though the forcing factors and mechanisms were similar to those of today. However, there was an abrupt cooling that caused the Arctic to transition from ice-free to ice-covered, without a corresponding abrupt change in forcing factors. This is known as the Pliocene Paradox [11]. Some global circulation models (GCMs) have had difficulty reproducing an ice-free
Arctic that transitioned to an ice-covered Arctic [4], [2]. This is a benefit of using an energy balance model; using the core physical processes may reveal behaviour that GCMs have difficulty showing. In this Section, the model will be applied to the Pliocene Paradox and provide a plausible explanation as to how the Arctic went from ice-free to ice-covered. The combination of ocean heat transport, $F_O$, and carbon dioxide concentration, $\mu$, will be considered as the key factors in the transition.

In the Arctic model, Chapter 3, both $\mu$ and $F_O$ were varied separately and tipping points associated with melting or freezing of the Arctic were found. Here, a linear time-line, $\nu$, will be used in place of $\mu$ and $F_O$. That is, $\mu$ and $F_O$ will vary from Eocene values to pre-industrial modern values, as linear functions of the bifurcation parameter $\nu$, while $\nu$ varies from 0 to 1. The concentration of carbon dioxide during the Eocene was estimated to be 490 ppm [41] and the pre-industrial modern concentration of carbon dioxide is 270 ppm [1]. Therefore, the value of $\mu$ will vary from 490 ppm to 270 ppm as $\nu$ varies from 0 to 1. The time-line function for CO$_2$ is

$$\mu = \mu_e(1 - \nu) + \mu_p\nu,$$

where $\mu_e$ is the Eocene level of CO$_2$, 490 ppm, and $\mu_p$ is the pre-industrial level of CO$_2$, 270 ppm. The similar function for $F_O$ is

$$F_O = F_{O,e}(1 - \nu) + F_{O,m}\nu,$$

where $F_{O,e}$ is the Eocene level of $F_O$, 60 W/m$^2$, and $F_{O,p}$ is the pre-industrial level of $F_O$, 20 W/m$^2$.

In the case of $F_O$, energy transport during the Eocene was difficult to estimate. However, from the work of Barron 1981 [4], we can provide a range of values for the Eocene, 33.9-56 million years ago. Bounding this range from above and below the Cretaceous values, 65-145 million years ago, $F_O$ will vary from 60 W/m$^2$ to 20 W/m$^2$ for both the ice-free and ice-covered Arctic as $\nu$ varies from 0 to 1. The purpose of using the same $F_O$ for the ice-free
and ice-covered Arctic will show that decreasing ocean heat transport will cause the warm solution to disappear while increasing ocean heat transport will not allow the frozen Arctic to return to an ice-free state for $0 \leq \nu \leq 1$.

As the Arctic began to cool and freeze, snow and ice began to accumulate. Fresh ice and snow are highly reflective and have a high albedo, $\alpha_c = 0.6$ [22], for the frozen Arctic. With a high albedo, this may be a reason why some GCMs have had difficulty returning to an ice-free Arctic [2]. A paper by Greenwood 2010 [14] uses data for the early geologic record to suggest that the Arctic could have supported a rainforest. A rainforest has a relative humidity around 60%. For the Pliocene, $\delta = 0.6$ will be used. Other than $\mu$, $F_O$, $\delta$, and $\alpha_c$, the parameters used in the Arctic model, Chapter 3, will be used here.

Figure 6.1 shows the changes in the carbon dioxide concentration and the ocean heat transport around $\tau_S = 1$. Figure 6.2 shows the corresponding bifurcation as the Arctic cools. The decrease in both $\mu$ and $F_O$ are represented by $\nu$ in Figure 6.2. This suggests that the Arctic suddenly cooled as $\nu$ increased above 0.285. The corresponding value of $\mu$ and $F_O$ are 427 ppm and 48.6 W/m$^2$. At this point the warm stable solution has fallen to the cold surface curve. Reversing back in time, a possible jump up is available for the frozen Arctic. However, this does occur for $0 \leq \nu \leq 1$. A jump up would agree with Chapter 3 as the Arctic model suggested a transition from a frozen surface to ice-free with enough ocean heat transport. With enough CO$_2$, that transition can occur sooner.

The Figures 6.1 and 6.2 suggest a possible explanation to the Pliocene paradox. A frozen Antarctic, which contributed to changes in ocean currents, along with reductions of atmospheric CO$_2$ may have together caused cooling of the Arctic. With the loss of energy coming in by the ocean and less energy trapped by CO$_2$, the Arctic was not able to sustain an ice-free Arctic. The results from this Section agree with the Earth’s geological record.

Even with this sudden cooling, a return to an ice-free Arctic is possible.
Figure 6.1: Arctic energy balance model for the Pliocene. The value of $F_O$ is varied from 60 W/m$^2$ (left magenta line) to 40 to 20 to 0 W/m$^2$ (right magenta line). The value of $\mu$ is varied from 800 W/m$^2$ (top blue line) to 600 to 400 to 200 W/m$^2$ (bottom blue line). The value of $\delta$ and $\alpha_c$ are constant at 0.6.
Figure 6.2: Bifurcation plot for the Pliocene paradox. The saddle-node bifurcation occurs at $\nu = 0.285$. The sudden cooling of the Arctic corresponds to $\mu = 427$ ppm and $F_O = 48.6$ W/m$^2$. The value of $\delta$ and $\alpha_c$ are constant at 0.6.
Many GCMs have attempted this. Barron 1983 [2] attempted to find a transition from an ice-covered Arctic to an ice-free Arctic through increased CO$_2$ levels. However, this was not achieved and a possible reason may be that there was not enough energy coming in to the Arctic by ocean heat transport alongside higher CO$_2$ levels. A GCM by Ferreira 2011 [12] used a water world to describe the Earth. An ice-free Arctic was attainable through changes in the ocean heat transport to the Arctic. Similar results of an ice-free Arctic were found in our energy balance model in Chapter 3, Sections 3.3 and 3.4.
Chapter 7

Generalizations of the Model

Earlier in the Thesis several parameters were kept constant, replaced with a step function, or zero. This chapter will investigate possible changes to these parameters, namely convection/conduction, $F_C$, the atmospheric heat transport, $F_A$, and the ocean heat transport, $F_O$. The changes will be applied to the Arctic model from Chapter 3 and the Tropics model from Chapter 5 with both greenhouse gases present.

7.1 Convection

Convection/conduction are physical processes that effectively move heat between the surface and the atmosphere. Typically, the surface is warmer than the atmosphere. Conduction occurs at the surface to heat up the nearby air molecules. Then, convection moves the warm air up to disperse the energy higher up in the atmosphere. In Chapter 2, this term was taken as $F_C = 0$ W/m$^2$. This time, we will consider a non-zero $F_C$ and consider energy from the air near the surface to mix with the air in the atmosphere. Latent heat is another physical process adding heat to the atmosphere. Energy added by latent heat is given by liquid water evaporating and rising to the clouds and
then condensing back into liquid/rain, releasing the stored energy. Referring back to the NASA energy budget, Figure 2.2 in Chapter 2, the conduction and latent heat contributions added 16.4 W/m$^2$ and 66.4 W/m$^2$ to the atmosphere, respectively. For the purpose of this section, both of these values will be combined as one. That is, $F_C = 16.4 \text{W/m}^2 + 66.4 \text{W/m}^2 = 82.8 \text{W/m}^2$.

Conduction/convection, $F_C$, will be added to the Arctic and Tropics model. Figure 2.2 gave values as a globally averaged number. Little evaporation occurs in the Arctic compared to the tropics, so the latent heat portion will be much lower in value for the ice-covered Arctic. When the Arctic is ice-free, more energy may be available by conduction and latent heat. In each case, both greenhouse gases will be present at $\mu = 400 \text{ppm}$ and $\delta = 0.4$ and 0.6 for the Arctic and Tropics, respectively. Figure 7.1 varies $F_C$ in the Arctic from 100 W/m$^2$ to 10 W/m$^2$. Since $F_C$ is present in both the surface and atmosphere equations, multiple lines appear for each colour. There are no bifurcations that occur.

In the Tropics, the atmosphere and surface curves behave in a similar way as the Arctic. Since the atmosphere had a lower temperature than the surface, adding convection will increase the temperature of the atmosphere and reduce the temperature of the surface. Figure 7.2 shows the change in equations as $F_C$ varies from 100 W/m$^2$ to 10 W/m$^2$. No bifurcations occur here between $F_C = 100 \text{W/m}^2$ to 10 W/m$^2$.

Temperature inversion occurs in both the Arctic and Tropic regions for the hot equilibrium points. Adding convection affects the difference between the surface and atmosphere temperatures. Convection brings the warm ice-free surface and atmosphere temperatures closer together. With enough convection added, the frozen surface temperature decreases below the temperature of the atmosphere. In the case of inverted temperatures, convection should work in the opposite direction and not add energy to the higher temperature. Next, the model will assume a linear, temperature dependant
Figure 7.1: Arctic energy balance model with convection. The value of $F_C$ is varied from 10 W/m$^2$ (top/left blue/magenta/red lines) to 40 to 70 to 100 W/m$^2$ (bottom/right blue/magenta/red lines). The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.4, respectively.
Figure 7.2: Tropics energy balance model with convection. The value of $F_C$ is varied from 10 W/m$^2$ (top/left blue/magenta/red lines) to 40 to 70 to 100 W/m$^2$ (bottom/right blue/magenta/red lines). The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively.
equation for convection. That is,

$$F_C = C_C(T_S - T_A)$$  \hspace{1cm} (7.1)

and $C_C$ is a positive constant in units of [W/m$^2$/K]. Again, $F_C$ incorporates heat added due to latent heat. Using the NASA values from Figure 2.2, an estimate of $C$ can be found and varied when applied to the model.

Convection as a linear temperature dependant function in the Arctic doesn’t suggest much interesting behaviour. Figure 7.3 shows changes in the convection constant for the energy balance model. Convection has little to no impact on the surface or atmosphere equations above freezing, $\tau_S$ and $\tau_A$ above 1. This is due to the small difference in temperatures of the atmosphere and surface. As $C_C$ increases, the cold surface curve rises as well. Looking at the top right hand corner of Figure 7.3, The bottom magenta/blue curves corresponds to $C_C = 0$ W/m$^2$/K and the top magenta/blue curves corresponds to $C_C = 6$ W/m$^2$/K. The orientation is reversed as the atmosphere temperature falls below $\tau_A = 1.117$. The orientation reverses again as the atmosphere temperature falls below $\tau_A = 0.835$. A similar change in orientation for the surface curve occurs as the surface temperature falls below $\tau_S = 1.103$ and again for $\tau_S = 0.957$.

Similarly in the Tropics, adding convection has little impact on the atmosphere and surface equations above $\tau_S$ and $\tau_A = 1$. Figure 7.4 shows the energy balance equations with the convection function incorporated into the model. The cold surface equation in Figure 7.4 has a higher surface temperature for higher values of $C_C$. This allows for a warmer surface temperature than the atmosphere temperature for values below $\tau_A = 1$.

Adding convection to the model had little impact in both the Arctic and Tropics for temperatures above freezing. Below freezing, the cold surface equation was raised to a higher temperature and reduced the temperature gap in the frozen case. A constant value of $F_C$ may be more useful than a linear function in Section 7.4. The value of $F_C$ is chosen to be 10 W/m$^2$.
Figure 7.3: Arctic energy balance model with convection as a function. The value of $C_C$ is varied from 6 W/m$^2$/K (top/right blue/magenta lines) to 4 to 2 to 0 W/m$^2$/K (bottom/right blue/red/magenta lines). The orientation if for the top right-hand corner of the plot. The orientation reverses for the blue curve as $\tau_A$ falls below 1.117 and again for 0.835. The orientation reverses for the magenta/red curve as $\tau_S$ falls below 1.103 and again for 0.957. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.4, respectively.
Figure 7.4: Tropics energy balance model with convection as a function. The value of $C_C$ is varied from 0 W/m$^2$/K (top/left blue/red/magenta lines) to 2 to 4 to 6 W/m$^2$/K (bottom/left blue/red/magenta lines). The orientation is reversed for the atmosphere equation when $\tau_A$ increases above 1.161. The orientation is reversed for the surface equation when $\tau_S$ increases above 1.273. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively.
for the Arctic and 60 W/m$^2$ for the Tropics. These are chosen because the
warm stable equilibrium exists in both regions.

### 7.2 Atmospheric Heat transport

Atmospheric heat transport is a key process for providing or removing energy
from the atmosphere of a region. In this section, we will vary the value of
atmospheric heat transport, $F_A$, in the Arctic and Tropics model. The value
of $\mu$ will be 400 ppm for both regions and $\delta = 0.4$ and 0.6 for the Arctic and
the Tropics, respectively.

In the Arctic, a portion of energy in the atmosphere comes from the
atmospheric heat transport, $F_A$. The atmospheric heat transport brings
energy from the lower latitudes to the higher latitudes. If the Arctic were
to increase in temperature, it seems reasonable that less energy would come
from the south. Figure 7.5 varies the $F_A$ from zero to large positive values.
Figure 7.6 shows the corresponding bifurcations.

There are three bifurcations for the ice-free fixed points and one bifur-
cation for the frozen fixed points in Figure 7.6. The first ice-free bifurcation
occurs at 48 W/m$^2$ to create the hot and unstable fixed point. The stable
warm and unstable transition solutions are created at the second bifurcation
at 122.5 W/m$^2$. The third ice-free bifurcation occurs at 148.5 W/m$^2$ as the
warm stable and unstable fixed points collide and disappear, leaving the hot
fixed point as the only ice-free equilibrium.

The stable frozen solution collides with the unstable transition solution
at the last bifurcation at $F_A = 176$ W/m$^2$. With enough energy from $F_A$,
the frozen equilibrium points disappear. Then a jump up occurs to the hot
equilibrium point.

The atmospheric heat transport plays a slightly different role in the
Tropics. Figure 7.7 shows the energy balance equations for the Tropics as
Figure 7.5: Arctic energy balance model with atmospheric heat transport, $F_A$. The value of $F_A$ is varied from 150 W/m$^2$ (bottom blue line) to 100 to 50 to 0 W/m$^2$ (top blue line). The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.4, respectively.
Figure 7.6: Arctic bifurcation plot with atmospheric heat transport. The bifurcations occur at $F_A = 48, 122.5, 148.5, \text{ and } 176 \text{ W/m}^2$. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.4, respectively.
$F_A$ varies. Since the value is negative, energy leaves the Tropics to go north towards the Arctic or south towards the Antarctic. The blue curves in Figure 7.7 may not intersect with the cold surface equation at all depending on $F_A$. Any positive value of $F_A$ allows the blue curve to intersect the red or magenta curves early on. However, once large enough negative, the blue curve will not intersect with the cold surface equation and eventually the warm surface equation as well. Then there are no equilibria in the Tropics in the current model. In this case, temperatures would continue to drop in both the surface and atmosphere. The values needed to do so are outside realistic values for today’s climate. The values of $F_A$ in Figure 7.7 vary from $-150$ W/m$^2$ to 0 W/m$^2$.

Figure 7.8 shows the corresponding bifurcations for the Tropics. The first bifurcation occurs at $F_A = -131$ W/m$^2$ where the unstable and hot solutions are created. The second bifurcation occurs at $F_A = 76$ W/m$^2$ as the unstable and warm solutions collide and disappear. There is no connecting curve between the warm surface equation and the transition curve before $\tau_A = 0$. The stable warm solution and unstable transition solution do not appear from a saddle-node bifurcation. The stable warm solution exists after $F_A = -92$ W/m$^2$.

The frozen solution does not exist for the same reason as the warm solution before $F_A = -51$ W/m$^2$. There is a bifurcation as the stable frozen and transition solutions collide and disappear at $F_A = 67$ W/m$^2$. After this, there are no cold solutions and a jump up to the warm surface equation will occur. Any positive value of $F_A$ represents energy flowing into the Tropics from the surrounding regions, suggesting that the Tropics has a lower atmospheric temperature than the surrounding areas.

The amount of energy brought in by atmospheric heat transport, $F_A$, is dependant on the temperature of the atmosphere and the temperature of the surrounding regions. As the atmosphere temperature of the Arctic increases towards that of the Tropics, the energy brought in by $F_A$ will decrease.
The value of $F_A$ is varied from $-150$ W/m$^2$ (top blue line) to $-100$ to $-50$ to 0 W/m$^2$ (bottom blue line). The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively.

Figure 7.7: Tropics energy balance model with atmospheric heat transport.
Figure 7.8: Tropics bifurcation plot with atmospheric heat transport. The bifurcations occur at $F_A = -131, 67, \text{ and } 76 \text{ W/m}^2$. The values of $\mu$ and $\delta$ are constant at $400 \text{ ppm}$ and $0.6$, respectively.
Similar to Section 7.1, the model will be altered to assume that the atmospheric heat transport depends linearly on the temperature difference between the region and the global average. That is,

\[ F_A = C_A (\overline{T}_A - T_A) \]  \hspace{1cm} (7.2)

where \( C_A \) is a positive constant in units of [W/m\(^2\)/K] and \( \overline{T}_A \) is the average atmospheric temperature of the Earth in [K]. A globally averaged surface temperature from a few months in 2017 was suggested to be around 13°C from the website https://www.ncdc.noaa.gov/sotc/global/201706 [13]. The temperature found from the web page is for surface temperature. The atmosphere temperature should be a few degrees less. A value of \( T_A = 5^\circ C \) or 278.15 K or \( \tau_A = 1.018 \) will be used as the average temperature, \( T_A \). The value does not need to be exact, it is used to examine changes in the constant \( C_A \) and the corresponding impact on the model.

Introducing the linear temperature function from Equation (7.2) into the model adjusts the equilibrium points. Figure 7.9 shows the energy balance equations with varying \( C_A \). As the magnitude of \( C_A \) increases, the amount of energy from the temperature difference increases. The value of \( C_A \) is varied from 0 W/m\(^2\)/K to 9 W/m\(^2\)/K. Since the function is linear with respect to temperature, there will be a point shared by all variations of \( C_A \) in Equation (7.2) at \( \overline{T}_A = T_A \). The blue curve corresponding to 9 W/m\(^2\)/K is the furthest left at the bottom of the Figure and 0 W/m\(^2\)/K is the top curve. The orientation reverses for values of \( \tau_A \) above \( \overline{\tau}_A = 1.018 \). A positive value of \( F_A \) determines the amount of energy flowing into the Arctic before \( \tau_A = \overline{\tau}_A \). Once \( \tau_A \) increases past \( \overline{\tau}_A \), \( F_A \) given by Equation (7.2) is now negative and energy is flowing out of the Arctic.

Figure 7.10 shows the corresponding bifurcations by varying \( C_A \). Adjusting the value of \( C_A \) can still have bifurcations occurring. With the current parameters, a bifurcation between the stable warm and unstable warm solutions can not happen due to the position of \( \overline{\tau}_A \). A bifurcation occurs to create a stable warm solution and the unstable transition solution at \( C_A \).
= 5.28 W/m²/K and exists from that point on. The stable frozen solution does not collide with the unstable transition solution between $C_A = 0$ and 10 W/m²/K. This bifurcation would occur at 32.5 W/m²/K. This value is large because the frozen solution is close to freezing and the temperature difference is small.

There is a portion of $C_A$ values where no ice-free equilibrium exists. This can occur if there is not enough energy entering the Arctic to sustain an ice-free Arctic. Another possibility is due to the convergence of curves from Equation (7.2). Either energy flows into the Arctic or out of the Arctic depending on whether $T_A$ is below or above $T_{A}$. Figure 7.9 suggests that linear dependence on temperature is unable to sustain a hot equilibrium at all values of $C_A$.

Similar to the Arctic, energy flows into the Tropics when the temperature of the atmosphere is less than the global average. Figure 7.11 shows the balance equations for the Tropics and it is similar to Figure 7.9. The difference between them is that atmosphere curves are all shifted down and to the right. Figure 7.12 shows bifurcations as $C_A$ varies. In this case, the hot equilibrium is the only fixed point remaining.

The stable hot solution disappears as it collides with the unstable solution at $C_A = 3.15$ W/m²/K. Another bifurcation occurs as the frozen solution collides with the unstable transition solution at $C_A = 1.38$ W/m²/K. Above $C_A = 1.38$ W/m²/K, a jump up to the warm equilibrium occurs.

A value of $C_A$ will be chosen for the Arctic for Section 7.4. This value is estimated using the averaged value of $F_A = 99.16$ W/m². The value used for $\tau_A$ is taken from the temperature of the atmosphere corresponding to the stable frozen fixed point at $\tau_A = 0.851$. A corresponding value of $C_A$ was found to be 2.174 W/m²/K.

In the case of the Tropics, the value used for $\tau_A$ are taken from the temperature of the atmosphere corresponding to the stable warm fixed point.
Figure 7.9: Arctic energy balance model with atmospheric heat transport as a function, $C_A(T_A - T_A)$. The value of $C_A$ is varied from 0 W/m$^2$/K (top/left blue line) to 5 to 10 to 15 W/m$^2$/K (bottom/left blue line). For $T_A$ below $T_A = 1.018$, the blue curves from left to right correspond to values of $C_A$ from 0 to 5 to 10 to 15 W/m$^2$/K. For $T_A$ above $T_A$, the orientation is reversed. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.4, respectively.
Figure 7.10: Arctic bifurcation plot with atmospheric heat transport as a function. A bifurcation occurs at $C_A = 5.28 \text{ W/m}^2/\text{K}$ as the stable warm and unstable transition solutions appear. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.4, respectively.
Figure 7.11: Tropics energy balance model with atmospheric heat transport as a function. The value of $C_A$ is varied from 0 W/m$^2$/K (top/left blue line) to 2 to 4 to 6 W/m$^2$/K (bottom/left blue line). For values of $T_A$ above $T_A$, the orientation is reversed. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively.
Figure 7.12: Tropics bifurcation plot with atmospheric heat transport as a function. The bifurcations occur at $C_A = 1.38$ and $3.15 \text{ W/m}^2/\text{K}$. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively.
7.3 Ocean Heat Transport

Ocean heat transport played a strong role in each of the regions to which our model was applied. Adjusting $F_O$ as a constant value was done in Chapters 3, 4, and 5. Similar to atmospheric transport, we assume that the ocean heat transport, $F_O$, will depend linearly on the temperature difference between the region and the global average. That is,

$$F_O = C_O(T_S - T_{S\text{ave}})$$ (7.3)

where $C_O$ is a positive constant in units of [W/m$^2$/K] and $T_S$ is the average surface temperature of the Earth in [K]. A globally averaged surface temperature for a few months of 2017 was suggested to be around 13°C from the website https://www.ncdc.noaa.gov/sotc/global/201706 [13]. This surface temperature does not need to be exact for the model.

Positive values of $C_O$ have little impact on the warm surface curves. The surface temperature of the ice-covered Arctic increases close to freezing as $C_O$ increases. Doing so increases the surface temperature for the hot equilibrium. For the cold surface, a negative $C_O$ shifts the cold surface curve away from the atmosphere curve. A negative $C_O$ is considered to represent a temperature inversion between the Tropics and the global mean temperature. This prevents any intersection between them. Figure 7.13 shows the energy balance equations for a change in ocean heat transport.

Figure 7.14 shows the corresponding bifurcations. A bifurcation at $C_O = 0.75$ W/m$^2$/K as the hot and unstable equilibrium collide and disappear. Between $C_O = 0.75$ and 4.77 W/m$^2$/K, the only equilibrium is the stable frozen equilibrium. A bifurcation occurs as the stable warm and unstable
transition solutions are created. The last bifurcation occurs as the unstable transition and stable frozen solutions collide and disappear at $C_O = 5.18 \text{ W/m}^2/\text{K}$. Above 5.18 W/m$^2$/K, the only solution remaining is the stable warm solution. This corresponds to a value of $F_O = 74.425 \text{ W/M}^2$. In the original Arctic model, $F_O = 95 \text{ W/m}^2$ was needed to jump from the cold surface equation to the warm surface equation. This suggests that a non-constant function for ocean heat transport allows for a transition from ice-covered to ice-free with less energy.

Figure 7.15 shows the energy balance equations for the Tropics. The hot equilibrium decreases in surface temperature as $C_O$ increases. Figure 7.16 shows the bifurcation diagram for varying $C_O$. The hot solution and unstable solution collide at a bifurcation and disappear at $C_O = 4.96 \text{ W/m}^2/\text{K}$. 

Figure 7.13: Arctic energy balance model with ocean heat transport as a function. The value of $C_O$ is varied from 6 W/m$^2$/K (top/left magenta/red lines) to 4 to 2 to 0 W/m$^2$/K bottom/left magenta/red line). For $T_S$ above $\overline{T}_S$, the positions reverse. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.4, respectively.
Figure 7.14: Arctic bifurcation plot with ocean heat transport as a function. The bifurcations occur at $C_O = 0.75$, 4.77, and 5.18 W/m$^2$/K. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.4, respectively.
This corresponds to $F_O = -153.64 \text{ W/m}^2$ compared to the $-160 \text{ W/m}^2$ needed for the same bifurcation in the Tropics model with $F_O$ constant. This suggests that the hot solution would need to lose a large amount of energy in order to disappear, even with a temperature dependant function for $F_O$.

The stable warm solution always exists. The unstable transition solution collides with the stable frozen solution at another bifurcation at $C_O = 2.45 \text{ W/m}^2/K$. This corresponds to $F_O = 42.03 \text{ W/m}^2$ compared to the $43 \text{ W/m}^2$ needed for the same bifurcation in the Tropics model with $F_O$ as a constant. This suggests that the frozen solution would need to gain a large amount of energy in order to disappear, even with a temperature dependant function for $F_O$. The warm and hot solutions can exist for a wide range of $F_O$ values. This may suggest that today’s climate can exist in either state at higher $C_O$ as the difference in surface temperature gets smaller as $C_O$ increases.

The bifurcations and frozen solution jump are similar in value to that of the constant case. This may suggest that constant values of $F_O$ used in the Tropics model have a similar impact as a linear function. To use a linear function of $F_O$, a cold and warm value of $C_O$ will be used. For the Tropics, using $\tau_S = 1.071$ for $C_{O,w}$ and $\tau_S = 0.611$ for $C_{O,c}$ gives a corresponding value of $C_{O,w} = 6.687 \text{ W/m}^2/K$ and $C_{O,c} = -0.0839 \text{ W/m}^2/K$. Since $C_{O,c}$ is found to be negative, this term will be chosen as $0 \text{ W/m}^2/K$.

The frozen Arctic value was found using $\tau_S = 0.8586$ for $C_{O,c} = 0.207 \text{ W/m}^2/K$. For the chosen parameters, a warm solution does not exist in the Arctic. A bifurcation occurs in Chapter 3 where the warm solution is created. The surface temperature of this point will be used instead. Using $\tau_S = 1.013$, a corresponding $C_{O,c}$ is found to be $3.383 \text{ W/m}^2/K$. These values will be used in Section 7.4.
Figure 7.15: Tropics energy balance model with ocean heat transport as a function. The value of $C_O$ is varied from 3 W/m$^2$/K (top red and bottom magenta line) to 2 to 1 to 0 W/m$^2$/K (top magenta and bottom red line). The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively.
Figure 7.16: Tropics bifurcation plot with ocean heat transport as a function. The bifurcations occur at $C_O = 2.45$ and $4.96$ W/m$^2$/K. The values of $\mu$ and $\delta$ are constant at 400 ppm and 0.6, respectively.
Table 7.1: Parameters used in Chapter 7. The corresponding non-dimensional values of the linear functions are $[-]/\sigma T^4_R$. GA stands for globally averaged.

7.4 Combination of Supplementary Work

In this section we will combine the linear functions from Sections 7.1, 7.2, and 7.3 and use this in the Arctic and Tropics model with both greenhouse gases present. The value of $\delta$ is constant at 0.4 for the Arctic and 0.6 for the Tropics. The value of $\mu$ will vary to investigate the change in warming done by adding the linear functions found in Sections 7.2 and 7.3. A summary of the values obtained are listed in Table 7.1.

Figure 7.17 shows the energy balance equations for the combination model using linear functions for ocean and atmospheric heat transport. The overall effect, compared to using constant values for $F_O$ and $F_A$ is that the atmosphere balance curve is shifted upwards for large $T_A$ values. This removes any previous unstable or hot equilibrium points obtained by the model. However, this does allow for an increase in relative humidity, $\delta$. Greenwood et al. 2010 [14] wrote a paper suggesting that an ice-free Arctic would have a high relative humidity. Geological data found that a rainforest may have existed in the Arctic region during the Eocene [14]. This would
Figure 7.17: Arctic energy balance model with the varied functions. The value of $\mu$ is varied from 200 ppm (top blue line) to 400 to 600 to 800 ppm (bottom blue line). The extra functions prevent other ice-free equilibria. The value of $\delta$ is constant at 0.4.

suggest a relative humidity above 0.6.

Doubling the CO$_2$ from 270 ppm to 540 ppm increases the surface temperature of the frozen equilibrium by 3.278°C. This is lower than the warming in Chapter 3. A possible explanation is due to the generalized functions added to the model. At higher temperatures, the surface and atmosphere were still receiving energy by the constants $F_A$ and $F_O$ whereas the added functions reduced the incoming energy as the surface and atmosphere warmed up. A doubling of CO$_2$ from 400 ppm to 800 ppm increases the surface temperature of the frozen equilibrium by 4.562°C.

Figure 7.18 shows the energy balance equations for the Tropics model when $F_O$ and $F_A$ are linear functions of the temperature. The atmosphere and surface curves of Figure 7.18 appear similar to that of the Arctic. Figure
Figure 7.18: Tropics energy balance model with varied functions. The value of $\mu$ is varied from 200 ppm (top blue line) to 400 to 600 to 800 ppm (bottom blue line). The value of $\delta$ is constant at 0.6.

7.19 shows a saddle-node bifurcation occurring as $\mu$ varies. As $\mu$ increases past 689 ppm, the stable frozen and unstable transition solutions collide and disappear at a bifurcation.

The stable warm equilibrium is the only ice-free equilibrium. Increasing CO$_2$ from 270 ppm to 540 ppm increases the surface temperature of the ice-free Tropics by 3.551 °C. The reduction of warming compared to Chapter 5 is due to the inclusion of linear functions, as in the Arctic model.

Warming in the Tropics is greater than the Arctic due to the ice covering the Arctic region. However, the Arctic has been noted to have increased in temperature at a faster rate than the Tropics in the real world. A possible explanation for this may be the climate state of the Tropics. If the Tropics are in a hot climate, any increase in CO$_2$ would provide little to no warming as suggested by the energy balance model. This would suggest that the
Figure 7.19: Tropics bifurcation plot with varied functions. A bifurcation occurs at $\mu = 689$ ppm as the stable frozen and unstable transition solutions collide and disappear. The value of $\delta$ is constant at 0.6.
warming in the Arctic is greater than that in the Tropics. Increasing greenhouse gases would then give the most warming to regions away from the equator. This may be related to the side note made in Chapter 5, Section 5.3.
Equilibrium climate sensitivity, ECS, is a characterization of the long-term, global temperature response to an increase of atmospheric carbon dioxide concentration on Earth. This single number is often used to describe the severity of climate change [18]. This ECS value is not a physical quantity but a type of estimation that can be done to measure surface warming. The ECS value is for a doubling of carbon dioxide from 270 to 540 ppm, where 270 ppm is the pre-industrial concentration of atmospheric carbon dioxide. Anthropogenic activity is suspected to be the cause of increasing CO₂, so 270 ppm acts as the starting point for ECS.

The Intergovernmental Panel on Climate Change, IPCC, recently released a report that widened the ECS range from 2-4.5°C to 1.5-4.5°C [36]. The range widened due to widening estimates of ECS from mathematical climate models and historical data and observations [18] [30]. Historical models are on the lower end of ECS as they assume a linear radiative response to warming. Mathematical modelling of the atmosphere tended to the high end of the ECS range [32]. The papers by Rogelj et al. 2012 [32],
Knutti et al. 2017 [18], and Priostosescu and Huybers 2017 [30] all discuss the variability in the range of ECS values. A mathematical model that incorporates the non-linearity of nature can often describe the Earth more accurately than a statistical model that uses a linear regression of data.

For the various situations modelled in the Thesis, carbon dioxide was doubled from the pre-industrial value of 270 ppm [1] to a value of 540 ppm. The warming done by that doubling was recorded. Table 8.1 summarizes the surface temperature increases by doubling CO$_2$ concentration, $\mu$.

### 8.1 Geometric Approximation

In the case of our model, the ECS values are local to a region, while the IPCC values are global. A weighted average, global ECS can be found from the local values found. We can find a geometrical approximation of the Arctic and the Tropics to find a weighted average of the surface temperature increase. The Arctic can be represented by the dome (upper section) shown in Figure 8.1. The surface area of the dome is determined by

$$S.A._{\text{dome}} = 2\pi R h,$$  \hspace{1cm} (8.1)

where $h$ can be substituted using trigonometry as

$$\frac{R - h}{R} = \cos(\theta)$$  \hspace{1cm} (8.2a)

$$h = R (1 - \cos(\theta)).$$  \hspace{1cm} (8.2b)

Thus, the surface area of the Arctic is

$$S.A._{\text{dome}} = 2\pi R^2 (1 - \cos(\theta)).$$  \hspace{1cm} (8.3)

The value of $\theta$ is found by taking $90^\circ$ and subtracting the latitude belt. Using the radius of the earth, $R = R_E = 6371$ km, and the latitude of the
<table>
<thead>
<tr>
<th>Location of Model</th>
<th>$T_S$ (K)</th>
<th>WV</th>
<th>$\Delta T_S$ (°C)</th>
<th>$\mu$ (ppm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arctic Model</td>
<td>233.704</td>
<td>No</td>
<td>6.938°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>247.091</td>
<td>No</td>
<td>9.779°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>243.786</td>
<td>Yes</td>
<td>7.157°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>247.255</td>
<td>Yes</td>
<td>10.653°C</td>
<td>400-800</td>
</tr>
<tr>
<td>Antarctic Model</td>
<td>205.928</td>
<td>No</td>
<td>5.845°C</td>
<td>270-540</td>
</tr>
<tr>
<td>(Modern)</td>
<td>208.796</td>
<td>No</td>
<td>8.249°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>205.928</td>
<td>Yes</td>
<td>5.845°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>208.796</td>
<td>Yes</td>
<td>8.249°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>283.967</td>
<td>Yes</td>
<td>13.466°C</td>
<td>400-800</td>
</tr>
<tr>
<td>(Cretaceous)</td>
<td>205.928</td>
<td>Yes</td>
<td>5.845°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>279.569</td>
<td>Yes</td>
<td>8.713°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>208.796</td>
<td>Yes</td>
<td>8.249°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>283.967</td>
<td>Yes</td>
<td>13.466°C</td>
<td>400-800</td>
</tr>
<tr>
<td>Tropics Model</td>
<td>247.365</td>
<td>No</td>
<td>7.020°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>288.501</td>
<td>No</td>
<td>8.195°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>250.779</td>
<td>No</td>
<td>9.943°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>292.516</td>
<td>No</td>
<td>11.582°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>247.365</td>
<td>Yes</td>
<td>7.020°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>288.501</td>
<td>Yes</td>
<td>8.276°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>250.779</td>
<td>Yes</td>
<td>9.942°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>292.516</td>
<td>Yes</td>
<td>13.630°C</td>
<td>400-800</td>
</tr>
<tr>
<td>Variations in the</td>
<td>242.366</td>
<td>Yes</td>
<td>3.278°C</td>
<td>270-540</td>
</tr>
<tr>
<td>Arctic Model</td>
<td>243.978</td>
<td>Yes</td>
<td>4.562°C</td>
<td>400-800</td>
</tr>
<tr>
<td>Variations in the</td>
<td>258.072</td>
<td>Yes</td>
<td>5.163°C</td>
<td>270-540</td>
</tr>
<tr>
<td>Tropics Model</td>
<td>292.653</td>
<td>Yes</td>
<td>3.551°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>294.128</td>
<td>Yes</td>
<td>12.182°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>294.128</td>
<td>Yes</td>
<td>13.630°C</td>
<td>400-800</td>
</tr>
</tbody>
</table>

Table 8.1: Summary of surface temperature increases from the doubling and increasing of CO$_2$ concentration, $\mu$. In column 2, WV, represents the presence of water vapour.
Figure 8.1: A sphere representing the Earth can be divided up into sections. These sections can represent the Arctic or the Tropics, depending on $\theta$.

Arctic, chosen as $\theta = 10^\circ$, an estimated area of the Arctic is $3.8745 \times 10^6$ km$^2$. The Tropics is represented as the disc (lower section) shown in Figure 8.1. The surface area of the disc is determined by

$$S.A._{disc} = 2\pi R^2 - S.A.\_dome$$  \hspace{1cm} (8.4a)

$$S.A._{disc} = 2\pi R^2 - 2\pi R^2 (1 - \cos(\theta))$$ \hspace{1cm} (8.4b)

$$S.A._{disc} = 2\pi R^2 \cos(\theta).$$ \hspace{1cm} (8.4c)

The area of the Tropics chosen here ranges from $10^\circ$N to $10^\circ$S. This is accounted for by doubling the value Equation (8.4) Using the radius of the earth, $R = R_E = 6371$ km, and the latitude of the Tropics, chosen as $\theta = 80^\circ$, the surface area of the Tropics is estimated as $8.8572 \times 10^7$ km$^2$.

Of the total area of the two regions are taken, the Arctic contributes about 0.0419 of the area and the rest is the area of the Tropics. This is close to the area of $1/21$ used by Thorndike 2011 [37]. The Antarctic is larger than the Arctic, however the latitude belt considered is $80^\circ$S to $90^\circ$S and so the size will be the same. The ECS will be evaluated including and excluding the Antarctic.
### Table 8.2: Summary of the averaged surface temperature increases from the doubling of CO₂ concentration, \( \mu \).

<table>
<thead>
<tr>
<th>Location of Model</th>
<th>Ant</th>
<th>WV</th>
<th>( \Delta T_S )</th>
<th>( \mu ) (ppm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frozen Arctic and Tropics</td>
<td>No</td>
<td>No</td>
<td>8.142°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
<td>8.050°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>Yes</td>
<td>8.229°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>Yes</td>
<td>Yes</td>
<td>8.133°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>No</td>
<td>11.506°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
<td>11.375°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>Yes</td>
<td>13.505°C</td>
<td>400-800</td>
</tr>
<tr>
<td></td>
<td>Yes</td>
<td>Yes</td>
<td>13.294°C</td>
<td>400-800</td>
</tr>
<tr>
<td>Ice-free Arctic and Tropics (extras)</td>
<td>No</td>
<td>Yes</td>
<td>3.540°C</td>
<td>270-540</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>Yes</td>
<td>11.863°C</td>
<td>400-800</td>
</tr>
</tbody>
</table>

Table 8.2: Summary of the averaged surface temperature increases from the doubling of CO₂ concentration, \( \mu \). Column 2, Ant, represents the inclusion of Antarctica in the weighted average. The frozen Antarctic is the only climate state considered. Column 3, WV, represents the presence of water vapour.

### 8.2 Weighted Averaged Temperature

A weighted average between the Arctic and Tropics has been made using the geometry of the Earth. An average warming from a CO₂ doubling is summarized in Table 8.2. Different ranges of \( \mu \) are considered and the warming contribution from a frozen Antarctic is included in some ECS values.
Chapter 9

Conclusions and Future Work

With any thesis, there is always more to be done. This Chapter will outline some ideas for future work and summarize the conclusions obtained throughout this Thesis.

9.1 Conclusion

The energy balance model developed in this Thesis found bi-stability in a simplified climate system of the Earth. In Chapter 3, an ice-free Arctic and a frozen Arctic were able to coexist with the same greenhouse gas parameters. This energy balance model is a qualitative result, and the results found here do not quantitatively reflect the Earth. Increasing CO$_2$ and water vapour can cause a transition between the current frozen state and a potentially ice-free Arctic. However, the requirement of $\mu = 1100$ ppm should not be taken as an exact tipping point.

The transition between frozen and ice-free is difficult to model and is
approximated here. In this model, we’ve seen these tipping points as saddle-node bifurcations. A jump upward or downward occurs as these tipping points are passed. The real Earth transitioning from frozen to ice-free won’t be as simple as the transition presented here. With a simplified model, climate change is inevitably here. When this change occurs and how severe is still in question.

The energy balance model was applied to the Pliocene Paradox. A possible explanation was found for the sudden cooling of the Arctic during the Pliocene. Carbon dioxide and ocean current levels were decreasing throughout the Eocene and Pliocene, represented by a linear reduction in $\mu$ and $F_O$, the reduction of these parameters passed a tipping point and the Arctic began to cool suddenly. The tipping point from a frozen to an ice-free Arctic is not the same so increasing $\mu$ and $F_O$ to the original values was unable to recreate the Pliocene. The tipping point back to an ice-free Arctic required a large increase in both $\mu$ and $F_O$. This may be a reason why some GCMs have had difficulty recreating the Cretaceous [2] or recreating an ice-free Arctic.

In Chapter 4, the current day Antarctic does not see as much change after increasing greenhouse gases. Total melting of the Antarctic is not seen for realistic values of $\mu$. A possible Cretaceous Antarctic was presented in this Thesis. An ice-free Antarctic was possible through high values of $\mu$ and $F_O$. Without the continent of Antarctica covering the South Pole, ocean currents were able to bring energy into the region that today’s currents could not due to the ACC. A transitioning $F_O$ was presented to provide a possible explanation of how the Antarctic cooled. Without ocean currents, an ice-free Antarctic was not possible.

The Tropics model in Chapter 5 had bi-stability for realistic values of the Earth. It was found that a snowball Earth was possible for today’s parameters but could not be reached by falling off a tipping point. A fall off from the ice-free Tropics was possible through a increase in ocean currents leaving the Tropics or a reduction in the amount of solar radiation absorbed.
A frozen Tropics is possible in the model. However, it is unlikely to occur without a dramatic change to the current Earth’s climate.

The warming in the Arctic and Tropics were compared. The ECS typically ranged between 2 to 4.5°C for a CO₂ doubling, 270-540 ppm, whereas the model presented an ECS value above this range at 8.229°C without the Antarctic and 8.133°C with the Antarctic. A few variations to the model were made and the warming done by a CO₂ doubling reduced to 3.54°C. As the model improved, the value of ECS was found to be more realistic. However, the ECS does not say much about the warming of the Arctic as the area ratio of the Arctic to the Tropics is small.

This model was applied to the Pliocene paradox. Another paradox, known as the Faint-Young-Sun paradox, may be an interest to this model. This paradox was introduced by Carl Sagan in 1972 [34]. In this paradox, the amount of incoming solar radiation was 25% less than today. However, there is evidence to suggest that the world had surface temperatures above freezing [34]. Possible simulations, such as some run by Hart 1978 [15], suggest that a greenhouse dense atmosphere was the explanation for this. Trace greenhouse gases that were negligible in this model may have existed in larger quantities in the distant past. Those greenhouse gases would need to be incorporated into the model. A good start to this application would lie in Sagan 1972 [34] and Hart 1978 [15].

9.2 Future Work

Further development of this energy balance model should start with the choice of parameter values. Some of the parameters were referenced from sources earlier than 1990, such as Barron 1981 [4] for the values of atmospheric and ocean heat transports. The absorption coefficients for carbon dioxide and water vapour were taken as a single number, known as a Grey Gas approximation [7]. In reality, the absorption by a greenhouse gas is
based off of the spectral lines of specific wavelengths of the longwave radiation and the structure of the molecule itself [27]. Finding up-to-date data and values on these parameters would help to improve the accuracy of the model and better describe the physical processes active.

In the news, the Arctic should be warming and the Tropics should be staying relatively the same. A possible explanation as to why our model estimates a large warming in the Tropics may be due to the inaccuracy of the absorption coefficients. Further investigation into the accuracy of the absorption coefficients should improve the results to coincide with that seen on Earth.

The next step would be to improve the modelling of the physical processes. A step function for the ocean heat transport and the albedo of the Earth were replaced with a \( \tanh(\cdot) \) function to remove the discontinuities found in the model. The smoothness was estimated and further refinement of the switch function can be made. Some physical processes, such as the ocean heat transport and atmospheric heat transport were taken as constant when each depends on temperature and other factors in a complicated way. These were changed to depend linearly on temperature in Sections 7.2 and 7.3. In Chapter 7, a conduction type of equation was used whereas a convective equation may have been a better fit. The purpose of Chapter 7 was to show that these improvements were a step in the right direction. However, these processes do not simply depend linearly on temperature differences between two regions, and further work could be done to more accurately model these processes.

The energy balance model uses two state variables representing the atmosphere and the surface. The model can be improved using differential equations to follow solutions as they move through time. Another option is to use a 1-D boundary value problem from surface to atmosphere as the atmosphere changes with height. Eventually, this can be applied to a global circulation model (GCM) using a 2-D map of the Earth. The 2-D map can consist of anything from a water based planet similar to one in Fer-
reira 2011 [12] to a geographical map of the Earth similar to one in Barron 1989 [3].

The model only considers energy transport between two regions. When considering the Arctic, the Tropics region is considered constant, and vice versa for the Tropics. Improving the model would include coupling the two models by replacing the constant temperature with another set of variables, surface and atmosphere temperature for the coupled region. This can be difficult as the Tropics region is a disc around the Earth while a meter squared area was considered for the current work. This can also be extended to include the Antarctic region as well.

The model developed here focused on using physical laws as opposed to data. Weather stations and satellites collect large amounts of data on the changes in temperature on Earth. A statistical analysis of the temperature changes can be done and compared to the results of the physical model here.
Bibliography


Appendix A

Literature Review

The climate of Earth has always been a complex system. Many studies have shown that the climate of today is very different from the climate hundreds of millions of years ago. In today’s climate, we see that ice caps cover the north and south poles of the Earth. In the Cretaceous period, 65-145 million years ago, evidence suggested that the climate was equable and warmer than today, resulting in ice-free poles. The term equable refers to a lack of extreme temperature difference between the equatorial (Tropics) and polar regions or a low meridional temperature gradient. In the Cretaceous, the physical processes that drove the climate were essentially the same as today [2], [11]. These physical processes impact the temperature of the Earth and some of these processes are known as climate or temperature feedbacks, specifically ice-albedo feedback, greenhouse gas feedback, etc. Key differences between the Cretaceous and today’s climate are the ocean levels and equilibrium temperatures of the surface and atmosphere. Other key differences are the forcing factors of the climate. These forcing factors include the solar insolation and the concentration of greenhouse gases in the atmosphere. Evidence suggests that the greenhouse gases and ocean levels were larger in the Cretaceous period [2].
The paper written by Eric Barron 1983 [2] focused on three main problems related to the climate during the Cretaceous period. These problems were focused on determining the past climate state (through data), determining the control factors of the climate (physical processes), and the climate response to these control factors. Based on data from flora and sediments, Barron estimated the temperature of the tropics to be between 32°C and 27°C, and the poles to be between 15°C and 0°C [2]. Within these limits, the poles are considered free of permanent ice.

The control factors considered by Barron 1983 [2] are based on solar radiation, composition of the atmosphere, and the geography of the surface. Barron concluded that the variation of solar radiation may have played a role in climate change but was not the primary factor. The solar radiation would need to decrease up to 4-5% to cause an ice age on its own, where solar output has not varied more than 1% over the course of 100 million years [2]. The amount of solar radiation reaching the Earth’s surface has a distribution between the tropics and the poles. The intensity is largest in the equator and smallest at the poles due to the curvature of the Earth and the angle between regions of equatorial and the orbital plane [23]. With a larger amount of solar radiation coming into the tropics, the temperature of the tropics is warmer than the temperature of the poles. Ice coverage of the tropics only is not feasible.

The composition of the atmosphere plays a primary role in the climate. The role of greenhouse gases in the atmosphere control the amount of infrared radiation that can be trapped. The most well studied greenhouse gas, Carbon Dioxide, has a primary role in determining the temperature. Levels of Carbon Dioxide, CO₂, can vary greatly. Volcanic activity and carbon storage/reservoirs impact atmospheric CO₂ leaving the concentration of CO₂ to be highly speculative [4], [2]. However, the general consensus is that CO₂ levels in the atmosphere were higher in the Cretaceous than today [2]. The record of volcanic activity suggests that no unusually large activity could have caused a drastic change in the Earth’s climate. Even
though volcanic activity increases atmospheric $\text{CO}_2$, volcanic activity is not considered a primary factor in controlling the climate [2].

Over the course of the last 100 million years, the Earth’s surface has changed considerably. Key changes in surface geography were a drop in sea level, exposing more land, and raising the albedo, or reflectivity of a surface. The albedo of ocean water is much lower than that of land. With a higher albedo, more sunlight would have been reflected into space leaving less radiation absorbed at the surface. With more land, seasonal snow and ice was more likely to accumulate and remain for longer periods of time. The albedo of snow and ice are much higher than land or ocean water, allowing for more sunlight to be reflected to space. The combination of seasonal snow accumulation and increase of exposed land area may have played a role in the decrease of global temperatures, especially in the polar regions [2].

Based on temperature estimates of the Cretaceous period, a polar ice-free climate was only attainable when the initial conditions of the model were above freezing, i.e. no ice present in the pole regions. Many of the physical processes were difficult to specify in great detail (as of 1983), so model predictions would be easier and possibly more precise as improvements are made to the model and drivers of climate [2]. Barron [2] tried and failed to reproduce Cretaceous climate using the global circulation models (GCM) of the day (1983).

Two authors, Budyko and Sellers, were among the first to describe the climate using a type of slab model known as an energy balance model (EBM). The purpose of these models was to show the behaviour of the climate using as few physical processes as possible to avoid complexity. The main result was that the climate of the poles had to impact the rest of the world in some way. The focus of Sellers 1969 [35], was ice-albedo feedback. The reflectivity of the Earth controlled the amount of solar radiation that would be absorbed into the surface. With the removal of permanent ice in the Arctic, the temperature would increase up to $14^\circ\text{C}$. Any reduction in solar radiation would increase the ice coverage on the Earth. In the case of Sellers
[35], an estimated reduction in solar radiation up to 5% can suggest an ice age. Sellers [35] uses a model of ice-albedo feedback that incorporates atmospheric and ocean transport to the each of the poles from the tropics. With the removal of the ice caps at either pole, changing the albedo, the globe would see warming in every region instead of just the Arctic or tropics. Sellers [35] uses a distribution of energy throughout each of the latitudes to model the transport of heat into each region. At each latitude, the model simulates the exchange of heat to the next latitude and the heat coming in from solar insolation or from the previous latitude. The dependent variable of the model is the annual sea level temperature near the tropics. Using albedo manipulation, Sellers [35] was able to find an increase in temperature as the albedo was lowered.

Budyko 1968 [5], who wrote a paper around the same time, found similar results with variations in solar radiation and ice-albedo feedback. Ocean heat transport leading into the poles was blocked or reduced by the ice line. Then the main source of energy came from absorbed solar radiation due to the albedo [5]. The model developed by Budyko [5] focuses on solar variation. Reduction in solar radiation reduced the temperature distribution between tropics and the Arctic. The resultant temperature decrease by reduced solar radiation becomes non-linear as the ice-line boundary extends further south from the Arctic. Planetary albedo was held constant as solar radiation was varied. However, similar variations in planetary albedo will have similar effects on the temperature. In an ice-free pole, ocean heat circulation was much stronger without any interference due to ice. The climate steady states found here, ice-free Earth and ice-covered poles, were considered unstable as a perturbation on solar radiation or albedo could result in the production or melting of ice resulting in a change from one climate state to the other [5].

Both authors claim that man’s activity can result in a new climate state from the addition of dust or greenhouse gases reducing the transparency of the atmosphere [5], [35]. Widiasih 2013 [39] used Budyko’s energy balance
model and improved on the ice line dynamics of the model. With the use of Budyko 1968 [5], Widiasih [39] was able to run a similar simulation that found three steady states of the Earth to be ice-free, large ice coverage in the poles, and small ice coverage in the poles. Stability was found for the last of the three whereas the first two states are unstable [39].

An EBM has rich structure for the climate. This structuring allows for multiple solutions to be found and explored [25]. Although drastic simplifying can reduce accuracy, an EBM can give insight on different types of climatic states. North 1990 [25] has written several papers using EBMs for the climate. Similar to Budyko 1968 [5] and Sellers 1969 [35], North found several climate states, including a deep freeze, an ice-covered pole, and an ice-free state [25]. The state of the climate depends on the tipping points, or bifurcations, found in the EBM. Tipping past these points can cause a change in the Earth’s climate, either at the poles or globally, and may be the cause of some major extinction events throughout the Earth’s history [25].

Milankovitch cycles are known to affect Earth’s climate as well. The cycles relate Earth’s orbit around the sun along with Earth’s rotation on its own axis. The Milankovitch cycles influence ice-albedo feedback as the Milankovitch cycles can vary the amount of solar radiation reaching the Earth through time. McGehee 2012 [23] constructed a model to combine the effects of Milankovitch cycles with ice-albedo feedback [23]. The model was comparable to known historical data over the time frame of 5.32 million years [23]. However, discrepancies between the model and data suggest that parameters were not specific enough or other mechanisms not included could have played a role in the climate of the Earth. One discrepancy was that ice coverage lagged behind historical data. The lag suggests that large amounts of ice took more time to respond to changes in solar insolation [23].

Aside from ice-albedo feedback, another important climate factor is the concentration of greenhouse gases. Callendar 1938 [33] suggested that the increasing concentration of carbon dioxide, CO₂, contributed to an increase in global temperature [33]. Based on data found in 1938, Callendar suggested
that 150 tons of CO$_2$ was added to the atmosphere pre year by combustion and about 3/4 of it remained in the atmosphere. He estimated to an average warming of 0.003 °C/year to temperature. Based on data collected by meteorological stations, the average warming was closer to 0.005°C/year [33]. Further in the report, Callendar [33] found the radiation absorption due the water vapour to be stronger than the radiation absorption of CO$_2$ suggesting that the impact of CO$_2$ could be ignored. In the next 80 years, CO$_2$ has increased to much higher levels from 275, [33], to 400 ppm, [1]. With the increase of CO$_2$ concentration, its impact on climate may no longer be ignored.

In 1977, a simulation was constructed on the evolution of the atmosphere by Hart 1977, [15]. This simulation took the composition of atmosphere and the processes driving an increase or decrease of certain gases. To name a few, O$_2$, CO$_2$, and N$_2$ were some gases that varied throughout the simulation of the atmosphere. The simulation started at 4.5 billion years ago with no oxygen gas and the atmosphere consisted of carbon dioxide, methane, and water vapour. The simulations end target was the conditions of the current climate. This simulation is one possible description of the atmosphere in its history. According to the results in Table 2 of [15] on page 30, levels of greenhouse gases consisted of the bulk of the atmosphere for the first two billion years. The greenhouse affect caused by these values caused a warming of 50 °C. With such a high temperature, surface water would evaporate causing an increase in cloud cover and albedo. While water vapour evaporated, oxygen levels increased to oxidize methane and ammonia. By two billion years ago, nitrogen began to fill the atmosphere, allowing oxygen levels to rise while levels of methane and ammonia reduced. These situations reduced the average temperature of the surface enough to the point where ice caps had formed. Eventually these values levelled out to the values they are today and the dominating greenhouse gases are water vapour and carbon dioxide [15].

The simulation by Hart [15] studied was a best fit to today’s (1977)
conditions. It does not detail specifically what happened throughout the history of the Earth but provides a reasonable guess over billions of years. It does share with Barron in 1983 the fact that the levels of CO$_2$ and other greenhouse gases were higher in the past to produce a warm equable climate or higher averaged surface temperatures to be globally ice-free. Warm temperatures caused the evaporation of water resulting in increased cloud cover and albedo to the point of possible glaciation [15]. Reduced solar radiation absorption has been seen in previous papers. These papers suggest that reduction of solar radiation has the potential to cause an ice age [35], [5].

Specifically for the Cretaceous period, 65-145 million years ago, many simulations and GCM models have not had great success in re-creating the climate of the Cretaceous. The main obstacle for these models are the constraints placed on the simulation, whether estimates of Cretaceous’ conditions or physical feedbacks were well known or defined enough to recreate. In 1981, Barron et al. [4] attempted to re-create the conditions of the Cretaceous in a GCM simulation. Within data and known physical processes, Barron was unable to recreate an ice-free Cretaceous starting from today’s conditions. A possible reason why the model failed was due to multiple steady states for the climate at the same parameters. With the same parameters and climate drivers, the Earth could have ice-free or ice-covered poles depending on the starting values. This would suggest that the data provided a range of temperature and other parameters to be too broad and inconclusive or that there are missing temperature feedbacks that drive the climate. At the time, (1981), it was possible that other feedbacks that were unclear, or could not be specified in more detail, may have a larger, or smaller, impact on the climate [4]. However, the temperature feedback from greenhouse gases had success in increasing the surface temperature. Therefore, one of the main parameters under focus was the concentration of CO$_2$ and other greenhouse gases in the atmosphere. Doubling concentration of CO$_2$ in the atmosphere led to an increase of 2-3°C. Data on these values are difficult to specify in the distant past due to the environmental buffers put in place, such as weathering of carbon, organic carbon storage, and deep
ocean carbon storage [4].

Doubling CO\textsubscript{2} not only increases surface temperature, but can also impact the moisture content. Manabe and Wetherald 1980 [21] developed a model to reduce the meridional temperature gradient through additional CO\textsubscript{2} warming. The model was run with increasing levels of CO\textsubscript{2} to monitor the increase in polar temperature. The ice caps in the polar region would reduce in size and water vapour content would increase. This caused a large impact on polar climate. Smaller, more uniform changes throughout the higher latitudes were from the poleward transport of latent heat from the tropics. With the retreat of ice and snow, ocean water was able to bring more heat through the ocean currents further into the higher latitudes [21]. With these types of large and small changes, it is possible that the increase of CO\textsubscript{2} over a certain threshold could have non-uniform impacts on climate [21].

When ice caps melt, sea and ocean levels rise allowing for ocean currents to transport heat further poleward. Combined with ice-albedo feedback, ocean heat transport can result in multiple climate states. This is seen in the model simulation by Ferreira et al. 2011 [12]. The three states resulting from the combination of ice-albedo feedback with ocean heat transport match the climate states the Earth had within the past billion years. In the Neoproterozoic, 640-730 million years ago, the climate was a snowball Earth where ice and snow coverage existed from the poles into the tropics. In the Cretaceous, Earth’s climate was equable and ice-free across the globe [2]. Lastly, today’s climate features ice caps at the poles and warm tropics [12]. From Paleo-climate records, changes from state to state happened abruptly, relative to the length of time within each state.

The simulation by Ferreira et al. 2011 [12] takes place on two types of planets. Both are Earth like and are either covered entirely in ocean water or have a ridge of land from the north to south pole. Having a ridge in the simulation impacts ocean currents and therefore the climate. In both models, these Earth like planets were shown to have the 3 states of climate under certain conditions. Altering the energy flux in the form of ocean (and
atmospheric) transport or solar absorption from ice albedo feedback controlled how the states would move from snowball Earth to ice-capped pole to a equable ice-free climate [12]. An important result of the simulations was that multiple steady states of the climate was found. This is what Barron [4], [2] had difficulty in finding.

There are several regions on the Earth that will be impacted by climate change in some way, shape, or form. These regions will change at different times as the climate passes a certain threshold in that region. For example, the Greenland ice sheet has been decreasing in mass over the years and will be unable to recover lost mass at some point. A paper written by Lenton et al. 2008, [19], discusses tipping points of regions susceptible to climate change. In this case, a tipping point is a threshold where a small perturbation can cause a large change in the current state of a system. The threshold of each region cannot be exactly determined with the number of active physical processes. However, an estimate of each threshold can be made [19].

With the Greenland ice sheet, adding enough greenhouse gases cause a temperature increase leading into ice albedo feedback (and other feedbacks as well) that can move the region into an ice-free state [19]. The same can be said for other regions, such as the Arctic sea ice, permafrost, Antarctic ice sheets, etc. Forested regions are not at risk of melting, but warming temperatures can dry available water near these forests preventing growth and coverage of trees in regions like the Boreal forest and Amazon rain-forests. Other risks of climate change in the region are side effects ranging from forest fires to disease and mortality rates, [19].

The tipping points for any of the regions above is related to the warming of surface temperatures. The increase of CO₂ from anthropogenic forcing may be the driver of these tipping points [19]. These transitions will not be smooth and may not be happen at the same time. That being said, the impacts on climate will not be independent between regions, [19].
The El-Niño Southern Oscillation (ENSO) is an important driver for Earth's climate. ENSO is known as an irregular, aperiodical variation in winds and average temperature of the sea surface level over the tropical eastern Pacific Ocean. Two main phases of ENSO are El-Niño, the warm phase consisting of warm sea-surface temperatures, and La-Niña, the cold phase consisting of cooler sea-surface temperatures. In the Pliocene, climate forcing parameters were essentially the same as today but produced higher temperatures than temperatures today [11]. Data relating to the Cenozoic era (65 million years ago) indicate that the temperatures of the polar regions were around 10°C.

The shift towards global cooling is suspected to be from the cumulation of continental drifting, changes in ocean basin geometry, mountain forming, volcanic activity, etc. making the Pliocene (23-5.3 million years ago) of interest due to these feedbacks [11]. The Earth has seen prolonged glaciations, ice ages, and brief warm interglacial periods throughout the past million years. Studies have changed focus from the high latitude regions to the tropics, specifically to ENSO. Given warmer temperatures in the Pacific ocean, a permanent El-Niño (warm phase) may be possible. The main outcomes from a permanent El-Niño are decreased albedo from the lack of stratus clouds, increased atmospheric water vapour, and changes to the energy flux from ocean heat transport [11].

Freshening surface waters in the extra tropics, from rising sea levels, can alter the meridional density gradient between the low and high latitudes and reduce the ocean heat transport into the extra tropics [11]. With enough freshening of the extra tropics, the sea-surface temperature gradient vanishes and warm conditions become a possible mechanism for a permanent El-Niño. This is explained in a review paper by Federov et al. 2006 [11]. Anthropogenic forcing may play a role in driving these warm conditions, directly by trapping heat in the atmosphere and indirectly by melting ice-covered regions to increase sea levels around the Earth [11].

When it comes to climate change, greenhouse gas temperature feedback
is a recurring topic and can influence other climate drivers. Other drivers of climate, such as ice-albedo feedback and heat transport, are not influenced by human activity as much as temperature change from greenhouse gases. Shortwave radiation absorption and longwave radiation emission were recognized in 1827 by Joseph Fourier, mentioned by Pierrehumbert 2011 [27]. Sunlight reaches the Earth as shortwave radiation and is absorbed by the surface. Longwave radiation in the infrared spectrum is emitted upwards from the surface. Molecules in the atmosphere, namely CO\textsubscript{2} and water vapour, absorb the infrared radiation (IR) as the radiation moves upwards to space. Once absorbed, the molecules moving in the atmosphere collide and exchange energy through momentum or emit IR. IR of particular frequencies, or absorption bands (windows), are absorbed based on the molecule itself and are emitted at the same frequency. Windows of IR absorption depend on the molecule itself. The windows of CO\textsubscript{2} and water vapour span most of the IR spectrum that is emitted from the surface of the Earth. Other molecules of the atmosphere, such as Nitrogen gas, do not absorb the IR at Earth-like conditions [27].

Due to the structure of absorption bands for CO\textsubscript{2} and water vapour, there is little overlap between the two molecules, leaving little competition between them. The levels of CO\textsubscript{2} and water vapour are far from saturation in Earth’s atmosphere. Adding additional levels of either greenhouse gas would therefore allow for more heat to be trapped in the atmosphere. On Venus, CO\textsubscript{2} fills a large portion of the atmosphere, trapping most to all of the heat that radiates from the surface. However, the CO\textsubscript{2} level is not saturated, the upper atmosphere of Venus is less dense and is where most of the heat trapped in the atmosphere escapes to space. If one were to add CO\textsubscript{2} to the upper atmosphere of Venus, more heat would be trapped and the temperature would rise [27]. Similar to Earth, CO\textsubscript{2} and water vapour levels can be added to trap more heat. Since these levels are far from saturation, competition to absorb IR on overlapping absorption bands will not increase dramatically [27].
If CO$_2$ was to be removed from the atmosphere, the Earth would cool. A cool Earth will limit the amount of water vapour that can be held at the current temperature. This would also increase precipitation and further cool the Earth. With additional CO$_2$, more water vapour can be held in the atmosphere to hold more heat and then a positive feedback loop can occur. The presence of CO$_2$, as an important greenhouse gas, allows for larger quantities of water vapour in the atmosphere [27].

Once heat, in the form of absorbed IR, is trapped within the atmosphere, convection and other fluid processes (such as wind) control the movement of molecules by stirring the atmosphere. These processes allow for heat to disperse throughout the atmosphere, otherwise the air close to the surface would have a temperature much greater than the upper troposphere compared to what we see today. This helps to balance the temperature between the surface and the atmosphere [27].

Classic models of the atmosphere make assumptions and simplifications to construct a model to illustrate the fundamental points of physical processes. This method is often used for physical processes to describe the behaviour and build complexity from that point onward. A simple toy model by Thorndike 2011 [37] reduced the model of Budyko, [5], to find states of equilibria for climate. A single cell is introduced to represent the surface of the equator. This cell is used to determine a range for the solar insolation to produce two steady states. In the toy model, this range is the ice albedo feedback. A second cell is introduced to represent the surface of the polar region. Both cells are coupled using ocean and atmospheric heat transports as a simple diffusion term [37]. The coupling, alongside ice-albedo feedback, is used to produce 4 possible steady states. The steady states found represent ice-free Earth, snowball Earth, ice caps at the poles only, and ice only at the equator [37]. The last steady state is not found in other models as the equator region is warmer than the poles due to stronger solar insolation at the equator.

The impact of greenhouse gases on the toy model was reduced to a single
constant. This parameter produced a hysteresis between the steady states. However, Figure (3) of Thorndike 2011 [37] shows that the transition to other states depend on this greenhouse parameter and the initial conditions. Similar to other models, increasing the greenhouse parameter allowed for warmer temperatures and ice-free conditions [37].

An increase in greenhouse gas can lead to warmer global temperatures. However, this warming may be more dramatic in some regions compared to others. This is called polar amplification when a parameter change causes more warming in the Arctic compared to the tropics. The main feedbacks that contribute to Arctic warming and possibly Arctic Amplification are greenhouse gas warming, ice-albedo feedback, Planck feedback, and lapse rate feedback. These feedbacks are studied by Pithan and Mauritsen 2014 [28]. The warming from greenhouse gases and ice-albedo feedback have already been discussed earlier in this chapter.

Outgoing longwave radiation (OLR) from the surface is described using the Stefan-Boltzmann law. This law is

\[ OLR = \sigma T_S^4. \] (A.1)

The Stefan-Boltzmann law is often approximated in simplified models, such as Budyko 1968 [5] and Thorndike 2011 [37], as

\[ OLR = A + BT_S, \] (A.2)

where \( A \) and \( B \) are constants derived from empirical data. Often values are given as \( A = 320 \text{W/m}^2 \) and \( B = 4.6 \text{W/m}^2/\text{K} \), [37], and \( T_S \) is the temperature of the surface in Kelvin, K. This approximation is used for simplifying to a linear version of the Stefan-Boltzmann law used in several papers [5], [15], and [37].
Planck feedback is a response to climate forcing terms using the Stefan-Boltzmann law. For a forcing of 1 W/m$^2$ at a surface temperature of 30°C, the value of the surface temperature, $T_S$, increases by 0.16°C. This response is weaker when the surface temperature is colder [28].

Lapse rate is a response to the difference in temperature between the height of the atmosphere and the surface. The lapse rate is the negative gradient of the temperature of the atmosphere, $T_A$, at some height $z$ in the atmosphere, written as

$$L.R. = -\frac{dT_A}{dz} \quad (A.3)$$

The system is stable when the lapse rate is positive or close to zero. Positive values occur when the temperatures of the atmosphere are smaller as we increase in height. Unstable conditions occur when temperatures are colder closer to the surface and warmer in the atmosphere, or when the lapse rate is negative. This is seen in our current climate. When the lapse rate is negative, processes such as convection occur to mix the molecules in atmosphere to redistribute the heat towards the surface [28]. Convection can also occur to redistribute heat upwards into the atmosphere if the lapse rate is large and positive.

Arctic amplification was found without the addition of ice-albedo feedback in a paper by Pithan and Mauritsen 2014, [28]. This would suggest that warming in the Arctic is caused by other feedbacks alongside ice-albedo feedback. Based on contributions of these feedbacks, Pithan found the contributions to Arctic amplification to be ordered as lapse rate feedback, ice-albedo feedback, and Planck feedback from greatest to least. Warming from greenhouse gases were found to warm the tropics more than they were found to warm the Arctic, resulting in tropical amplification as opposed to Arctic amplification. This can be seen in Figure 2 of Pithan and Mauritsen 2014 [28]. This is a surprising conclusion that we challenge and investigate in this thesis.
Figure (2a), from Pithan and Mauritsen 2014 [28], uses a top of atmosphere (TOA) perspective to evaluate temperature feedback. In the tropics, air mixes between near surface and upper atmosphere to create a coupling between the two areas. The Arctic has little mixing between the near surface and the upper atmosphere. In these conditions, warming of the surface can cause a larger impact to the TOA in the tropics compared to the Arctic. Figure (2c) uses a surface perspective to evaluate temperature feedbacks [28]. Figure (3) of [28] compares the contributions of several mechanisms across several simulations and models. In the case of Arctic warming, water vapour feedback produces more warming compared to CO$_2$.

Lapse rate feedback is meant to reflect changes in temperature due to physical processes as opposed to being a physical process itself. This would require warming or cooling from other feedbacks to cause any change in lapse rate instead of surface temperature alone [26], [28]. Physical process can interact with each other, this can prove difficult in pinning down exact values for Arctic amplification. The models from Payne et al. 2015 [26] and Pithan and Mauritsen 2014 [28] attempt to isolate the feedbacks and produce estimates for each feedback.

The current work starts with the energy balance model from Payne et al. 2015 [26]. The work of Payne et al. [26] attempts to isolate lapse rate feedback for the high latitude and low latitude regions, or the Arctic and the tropics. In the higher latitude, lapse rate is strong enough to cause a decrease in atmospheric temperature for increasing optical depth, possibly providing extra warming to the surface. Here, optical depth is the thickness of the atmosphere in terms of greenhouse gases [26]. The model suggests that polar warming occurs at the surface and tropical warming occurs in the upper troposphere. Resulting in the temperature gradient at the top of the atmosphere to be large between the tropics and the Arctic. The surface temperature gradient is decreased after warming [26].

An increase in concentration of CO$_2$ leads to a decrease in lapse rate for the Arctic as the atmospheric temperature increases. When water vapour
was included, warming of the tropics was greater than the warming in the Arctic, cancelling Arctic amplification. However, changes in atmospheric water vapour can impact atmospheric heat transport, ice-albedo feedback, or cloud presence. Any impact to these feedbacks could result in a restoration of Arctic amplification [26]. Water vapour feedback can influence and be influenced by many physical processes. When it comes to an analysis, physical processes will need to be taken into consideration making it difficult to study the effects of water vapour feedback or the lapse rate.
Appendix B

Stability of Fixed Points

The equations developed in the model are taken at equilibrium. The papers of Widiasih [39] and Ferreira [12] instead use ODE’s to model the atmosphere and surface. The ODE’s represent the imbalance of the surface and atmosphere and are called the rate of storage [25]. The right hand side of each equation is the same or similar to the current work. An ODE for the atmosphere energy balance equation can be written as

\[ C_A \frac{dT_A}{dt} = \sigma T_A^4 - \epsilon \sigma T_A^4 + F_A - \sigma T_S^4 \times \exp \left( -1.52k_C \frac{\mu}{10^6} \frac{P_A}{g} - k_W \frac{P_{\text{sat}}(T_R)}{g} \frac{T_R}{T_A} \exp \left[ \frac{L_s}{R_W} \left[ \frac{1}{T_R} - \frac{1}{T_A} \right] \right] \right), \]  

(B.1)

or

\[ C_A \frac{dT_A}{dt} = \eta \sigma T_S^4 - \epsilon \sigma T_A^4 + F_A \]  

(B.2)

for short. To save space, \( \eta \) will be used from here on. The surface energy balance equation, as an ODE, can be written as

\[ C_S \frac{dT_S}{dt} = (1 - \alpha) \frac{S_0}{4} s(\pi/2) + 0.63 \epsilon \sigma T_A^4 + F_O - \sigma T_S^4. \]  

(B.3)
The parameters $C_{A,S}$ are positive constants and represent the specific heat capacity of the atmosphere and surface, respectively. Converting these ODE’s into non-dimensional ODE’s will have the same right hand side as the equilibrium equations and any constants moved to the left hand side are positive constants. The values of these parameters are not needed here since we only need to determine stability.

Setting the left hand side of Equations (B.2) and (B.3) equal to zero gives us the nullclines of the system. The nullcline is a set of points on a phase plane where there is no change in that variable. The fixed points are the intersections of each nullcline when Equations (B.2) and (B.3) are zero. Figure B.1 shows a phase plane portrait of the ice-free fixed points.

The ice-free fixed points are shown in Figure B.1. The frozen fixed points will have the same form as the ice-free fixed points for any stable or unstable equilibria. For this section, the parameters chosen are the same as the Arctic model, Chapter 3, with a few changes. The changed parameters are in Table B.1. For the ice-free fixed points, the concentration of CO$_2$ was chosen from the Eocene time period to be 490 ppm [41]. Relative humidity was set to 0.4 similar to the Arctic model. The atmospheric transport was chosen to as the Arctic model at a value of 99.16 W/m$^2$. The values of these parameters will not affect the stability of the frozen fixed point, only the location and therefore the only parameter to change will be the value of latent heat.

There is a corner in the atmosphere curve due to the changing values of latent heat. For the ice-free stability analysis, the latent heat of vaporization will be used and the latent heat of sublimation will be used for the frozen fixed point. In the case of either value, this will effect the unstable fixed point’s location. If the fixed point were to coincide with the position of the corner, the stability should not change.
Parameter Values Used

<table>
<thead>
<tr>
<th>Parameters and Constants</th>
<th>Symbol</th>
<th>Value (units)</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eocene CO₂ ppm</td>
<td>( \mu )</td>
<td>490 ppm</td>
<td>[41]</td>
</tr>
<tr>
<td>Relative humidity</td>
<td>( \delta )</td>
<td>0.4</td>
<td>Author</td>
</tr>
<tr>
<td>Atm. heat transport</td>
<td>( F_A )</td>
<td>99.16 W m⁻²</td>
<td>[4]</td>
</tr>
<tr>
<td>Latent heat of Vapouriza-</td>
<td>( L_v )</td>
<td>( 2.2558 \times 10^6 )</td>
<td>[16]</td>
</tr>
<tr>
<td>tion</td>
<td>( L_s )</td>
<td>( 2.8373 \times 10^6 )</td>
<td>[16]</td>
</tr>
</tbody>
</table>

Table B.1: Parameters that will change in the Stability Analysis.

Figure B.1: The phase plane of the atmosphere and ice-free surface equations. The curve of the atmosphere has \( \mu = 490 \) ppm and \( \delta = 40\% \). The vector field shows the direction that the green solutions will travel through time.
Figure B.2: The phase plane of the atmosphere and frozen surface equations near the stable fixed point. The curve of the atmosphere has $\mu = 490$ ppm and $\delta = 40\%$. The green curves represent solutions travelling through time. The latent heat of sublimation is used here.

B.1 Phase Portraits

The vector field suggests that the ice-free fixed points located at $(0.9287, 1.016)$ and $(1.301, 1.227)$ are stable and the fixed point at $(1.114, 1.109)$ is an unstable saddle. Any frozen fixed points are in similar form to the ice-free fixed points. Figure B.2 shows the phase portrait around the frozen fixed point at $(0.914, 0.876)$. The frozen fixed point is determined to be stable. Other frozen fixed points can exist for particular parameters. In the case that any exist, they will follow the same order as the ice-free fixed points. With the first fixed point being stable, then the next is an unstable saddle and the last is stable.

Figures B.3, B.4, and B.5 show close up phase portraits of each ice-free
fixed point. In the case of CO$_2$ only, there are at most two fixed points, one that is frozen and the other that is ice-free. A phase plane of these fixed points will be the same as Figures B.2 and B.3. In the case of water vapour only, there are only three fixed points present for the chosen parameters. One fixed point is frozen, another is ice-free at high temperatures, and the last is ice-free and unstable. A phase plane of the unstable and the hot fixed points are similar to those seen in Figures B.4 and B.5. A phase plane of the frozen fixed point is similar to that seen in Figure B.2. The cold fixed points near freezing are similar to Figures B.4 and B.5, except the surface temperature is below freezing.

In certain situations, an ice-free (or frozen) fixed point exists between the atmosphere curve and warm (or cold) surface equation. However, since the ice-free curve stops below freezing at $T_S = 273.15$ K, or $\tau_S = 1$, any solution travelling towards this fixed point will fall over the warm surface. Once below freezing, the solutions will travel to fixed points with the cold surface equation. In other words, falling off the warm surface equation will force us to the nearby stable equilibrium. A similar case can be made for a frozen fixed point moving above freezing.

Once the tanh(·) was incorporated into the model, a fixed point existed on this transition curve. The fixed point is an unstable saddle, similar to Figure B.4.

B.2 Jacobian Analysis

Using the location of the fixed points found, we can linearize around those points to determine local stability. That is, we can use the Jacobian,

$$J(T_A, T_S) = \begin{bmatrix} \frac{\partial f_A}{\partial T_A} & \frac{\partial f_A}{\partial T_S} \\ \frac{\partial f_S}{\partial T_A} & \frac{\partial f_S}{\partial T_S} \end{bmatrix},$$
Figure B.3: The phase plane of the atmosphere and ice-free surface equations near the stable fixed point close to freezing. The curve of the atmosphere has $\mu = 490$ ppm and $\delta = 40\%$. The green curves represent solutions travelling through time.
Figure B.4: The phase plane of the atmosphere and ice-free surface equations near the unstable saddle fixed point. The curve of the atmosphere has $\mu = 490$ ppm and $\delta = 40\%$. The green curves represent solutions travelling through time.
Figure B.5: The phase plane of the atmosphere and ice-free surface equations near the stable fixed point at high temperatures. The curve of the atmosphere has $\mu = 490$ ppm and $\delta = 40\%$. The green curves represent solutions travelling through time.
to linearize around each fixed point with $f_A$ and $f_S$ representing Equations (B.2) and (B.3), respectively.

$$J(T_A, T_S) = \begin{bmatrix} \frac{d\eta}{dT_A} \sigma T_S^4 - 4\epsilon \sigma T_A^3 & 4\eta \sigma T_S^3 \\ 2.52 \epsilon \sigma T_A^3 & -4\sigma T_S^3 \end{bmatrix}$$

In the case of CO$_2$ only, $\eta$ is independent of $T_A$ and would result in

$$J(T_A, T_S) = \begin{bmatrix} -4\epsilon \sigma T_A^3 & 4\eta \sigma T_S^3 \\ 2.52 \epsilon \sigma T_A^3 & -4\sigma T_S^3 \end{bmatrix}.$$

The expression for $\frac{d\eta}{dT_A}$ is

$$\frac{d\eta}{dT_A} = \exp \left( -\mu k_C \frac{1.52}{10^6} \frac{P_A}{g} - k_W \delta \frac{P_{sat}(T_R)}{g} \frac{T_R}{T_A} \exp \left[ \frac{L_s}{R_W} \left( \frac{1}{T_R} - \frac{1}{T_A} \right) \right] \right) \times \left( \delta k_W T_R \frac{P_{sat}}{g} \exp \left[ \frac{L_s}{R_W} \left( \frac{1}{T_R} - \frac{1}{T_A} \right) \right] \right) \left( \frac{T_R L_s}{R_W T_A} - \frac{1}{T_A^2} \right). \quad (B.4)$$

Using Matlab routines for the two ODE’s, the stability of the fixed points agree with those found in the phase planes of Figures B.1 and B.2. The Jacobian for the switch function is a similar process to the one shown here, the stability of the fixed points are the same. The fixed point on the transition curve is an unstable saddle similar to the fixed point in Figure B.4.

Another approach is to calculate the eigenvalues of the Jacobian matrix $J(T_A, T_S)$ around each fixed point. The name of each fixed point discussed in the Thesis replace $(T_A, T_S)$. The exact number is not necessary, only the sign is required to tell the stability of the fixed point. The following eigenvalues are for the fixed points in the Arctic model with the switch function incorporated. The generalized functions are not included here.
The eigenvalues of $J(T_A, T_S)$ are

$$
eig[J\text{(Frozen)}] = \begin{bmatrix} -1.38 & 0 \\ 0 & -4.90 \end{bmatrix},$$

$$
eig[J\text{(Transition)}] = \begin{bmatrix} -1.38 & 0 \\ 0 & -6.00 \end{bmatrix},$$

$$
eig[J\text{(Warm)}] = \begin{bmatrix} -1.09 & 0 \\ 0 & -6.29 \end{bmatrix},$$

$$
eig[J\text{(Unstable)}] = \begin{bmatrix} +3.14 & 0 \\ 0 & -7.61 \end{bmatrix},$$

$$
eig[J\text{(Hot)}] = \begin{bmatrix} -1.81 & 0 \\ 0 & -15.81 \end{bmatrix}.$$

### B.3 Algebraic Analysis

In this Section we will look at the slopes of each equation in the model and find stability. The Jacobian around a fixed point is

$$J(T_A, T_S) = \begin{bmatrix} \frac{\partial f_A}{\partial T_A} & \frac{\partial f_A}{\partial T_S} \\ \frac{\partial f_S}{\partial T_A} & \frac{\partial f_S}{\partial T_S} \end{bmatrix}.$$

The nullclines for the atmosphere and surface equations implicitly define two curves giving $T_S$ as a function of $T_A$. Differentiating these nullcline equations with respect to $T_A$ gives

$$0 = \frac{\partial f_A}{\partial T_A} + \frac{\partial f_A}{\partial T_S} S_A,$$  \hspace{1cm} \text{(B.5)}

$$0 = \frac{\partial f_S}{\partial T_A} + \frac{\partial f_S}{\partial T_S} S_S,$$  \hspace{1cm} \text{(B.6)}

where $S_A$ and $S_S$ are the slopes of the curves $T_S(T_A)$ for the atmosphere and surface nullclines, respectively. In order to have stability, the trace of
the Jacobian needs to be negative and the determinant needs to be positive,

\[ \text{tr}(J) = \frac{\partial f_A}{\partial T_A} + \frac{\partial f_S}{\partial T_S} < 0 \quad \text{and} \quad \text{det}(J) = \frac{\partial f_A}{\partial T_A} \frac{\partial f_S}{\partial T_S} \frac{\partial f_A}{\partial T_A} \frac{\partial f_S}{\partial T_S} > 0. \quad (B.7) \]

In the model in this Thesis, \( \frac{\partial f_A}{\partial T_S} \) and \( \frac{\partial f_S}{\partial T_A} \) are both positive. The trace condition can be expressed in terms of the nullcline slopes as

\[ \text{tr}(J) = -\frac{\partial f_A}{\partial T_S} S_A - \frac{\partial f_S}{\partial T_A} \frac{1}{S_S} < 0. \quad (B.9) \]

The following cases will consider \( \frac{\partial f_A}{\partial T_S} \) and \( \frac{\partial f_S}{\partial T_A} \) as both positive. The trace condition (B.9) becomes:

\[ S_A > -\frac{\partial f_S}{\partial T_A} \frac{1}{S_S}. \quad (B.10) \]

Case 1: If \( S_A > 0 \) and \( S_S > 0 \) \( \implies \text{tr}(J) < 0. \)

Case 2: If \( S_A > 0 \) and \( S_S < 0 \) and \((B.10) \implies \text{tr}(J) < 0. \)

Case 3: If \( S_A < 0 \) and \( S_S > 0 \) and \((B.10) \implies \text{tr}(J) < 0. \)

Case 4: If \( S_A < 0 \) and \( S_S < 0 \) \( \nRightarrow \text{tr}(J) < 0. \)

A similar approach is done for the determinant of \( J(T_A, T_S) \). The determinant condition can be expressed in terms of the nullcline slopes as
\[
\text{det}(J) = \frac{\partial f_A}{\partial T_A} \frac{\partial f_S}{\partial T_S} - \frac{\partial f_A}{\partial T_S} \frac{\partial f_S}{\partial T_A} > 0, \tag{B.11}
\]

\[
\text{det}(J) = \left( -\frac{\partial f_A}{\partial T_S} S_A \right) \left( -\frac{\partial f_S}{\partial T_A} \frac{1}{S_A} \right) - \frac{\partial f_A}{\partial T_S} \frac{\partial f_S}{\partial T_A} > 0, \tag{B.12}
\]

\[
\text{det}(J) = \frac{\partial f_A}{\partial T_S} \frac{\partial f_S}{\partial T_A} \left( \frac{S_A}{S_S} - 1 \right) > 0. \tag{B.13}
\]

The sign of the determinant depends on \(\frac{S_A}{S_S} - 1\) and \(\frac{\partial f_A}{\partial T_S} \frac{\partial f_S}{\partial T_A}\). The following cases consider \(\frac{\partial f_A}{\partial T_S}\) and \(\frac{\partial f_S}{\partial T_A}\) as both positive. The condition on the determinant becomes

\[
\frac{S_A}{S_S} - 1 > 0. \tag{B.14}
\]

Case 5: If \(S_A > 0\) and \(S_S > 0\) and (B.14) \(\implies\) \(\text{det}(J) > 0\).

Case 6: If \(S_A > 0\) and \(S_S < 0\) \(\implies\) \(\text{det}(J) < 0\).

Case 7: If \(S_A < 0\) and \(S_S > 0\) \(\implies\) \(\text{det}(J) < 0\).

Case 8: If \(S_A < 0\) and \(S_S < 0\) and (B.14) \(\implies\) \(\text{det}(J) > 0\).

In order to have stability, we need to satisfy one case of Cases 1-3 for a negative trace and one case of Cases 5 and 8 for a positive determinant. The slopes of each nullcline can determine the stability as long as the conditions are met.
Appendix C

Matlab Code

edit TauArctic

%%%%%%%%%%%%%%
% Ta and Ts are temperatures of atmosphere and surface, respectively.
% Equations are (in the simplest form):
% SURF: sigma*Ts^4 = 0.63*eps*sigma*Ta^4 + (1-alpha)*S + Fo - Fc
% ATM: sigma*Ts^4 = (eps*sigma*Ta^4 - Fa - Fc)/(CO2 + H2O absorbance)
% Ts and Ta are our state variables
%%%%%%%%%%%%%

for multiline = 1:4 % varying levels of parameters

% tau guesses for root2d.m [Ta,Ts]
x0 = [1.12,1.35]; % follow unstable fixed point
v0 = [2.,2.]; % follow hot stable fixed point
z0 = [0.85,0.91]; % follow cold stable fixed point
u0 = [0.94,1.04]; % follow warm stable fixed point
q0 = [0.9,1.0]; % follow cold stable fixed point

C = 200; % Number of points in plot
% mu increasing effect
allmux = linspace(200,726,C); % unstable % ends with stable
allmuz = linspace(400,800,C); % cold stable % all the time
allmuv = linspace(200,1500,C); % hot stable % all the time
allmuu = linspace(535,726,C); % warm stable % ends with unstable
allmuq = linspace(5,5,C); % unstable cold % on switch
allmudash = linspace(0,1500,C);% dotted line
% delta increasing effect
allhumidx = linspace(0.215,1.,C); % unstable
allhumidu = linspace(0.,0,C); % warm stable
allhumidq = linspace(1.0,1.,C); % cold stable
allhumidv = linspace(0.215,1,C); % hot stable
allhumidz = linspace(0.0,1,C); % cold stable

% ocean increasing effect
allHx = linspace(-18,167,C); % unstable warm
allHz = linspace(-60,95,C); % stable frozen
allHv = linspace(-18,200,C); % stable hot
allHu = linspace(32.5,167,C); % stable warm
allHq = linspace(0,116.2,C); % stable cold
alldash = linspace(-60,200,C); % Dashed line
% ocean linear constant
callHx = linspace(0,0.75,C); % unstable warm
callHz = linspace(0,5.18,C); % stable frozen
callHv = linspace(0,0.75,C); % stable hot
callHu = linspace(4.77,10,C); % stable warm
callHq = linspace(4.77,5.18,C); % stable cold
callH = linspace(0,10,C); % Dashed line
% alpha increasing effect 0 to 1
allalphax = linspace(0,0.35,C); % unstable warm
allalphaz = linspace(0,1,C); % stable frozen
allalphav = linspace(0,.35,C); % stable hot
allalphau = linspace(0,0.06,C); % stable warm
allalphaq = linspace(0,116.2,C); % stable cold
allalpha = linspace(0,1,C); % Dashed line

% Fc increasing effect
allFcx = linspace(-50,150,C); % unstable warm
allFcz = linspace(-50,150,C); % stable frozen
allFcv = linspace(-50,150,C); % stable hot
allFcu = linspace(200,200,C); % stable warm
allFcq = linspace(200,200,C); % stable cold
allFc = linspace(-50,200,C); % Dashed line
% Fc linear constant
callFcx = linspace(-15,10,C); % unstable warm
callFcz = linspace(-15,10,C); % stable frozen
callFcv = linspace(-4.9,10,C); % stable hot
callFcu = linspace(-1.4,10,C); % stable warm
callFcq = linspace(-10,-10,C); % stable cold
callFc = linspace(-105,105,C); % Dashed line

% Fa increasing effect
allFax = linspace(48,148.5,C); % unstable warm
allFaz = linspace(-50,176,C); % stable frozen
allFav = linspace(48,200,C); % stable hot
allFau = linspace(122.5,148.5,C); % stable warm
allFaq = linspace(122.5,176,C); % stable cold
allFa = linspace(-50,200,C); % Dashed line

% Fa linear constant
callFax = linspace(10,10,C); % unstable warm
callFaz = linspace(0,10,C); % stable frozen
callFav = linspace(10,10,C); % stable hot
callFau = linspace(5.28,10,C); % stable warm
callFaq = linspace(5.28,10,C); % stable cold
callFa = linspace(-100,100,C); % Dashed line

% Pliocene timeline parameters
alltx = linspace(0,1,C);
alltv = linspace(0,1,C);
alltz = linspace(0,1,C);
alltu = linspace(0,0.285,C);
alltq = linspace(0,0,C);
tdash = linspace(0,1,C);
Hmax = 60;
Hmin = 20;
Hcmax = 60;
Hcmin = 20;
co2max = 490;
co2min = 270;
mu = 600;%200+200*(multiline-1); % [None] % ppm concentration
humid = 0.4;%+.2*(multiline-1); % [None] % relative humidity
Fa = 99.16;%0+50*(multiline-1); % [W/m^2] % atmospheric heat transport
cFa = 2.174;%9-3*(multiline-1); % Atmosphere in linear equation
H = 36.2; % 60-20*(multiline-1); % [W/m^2] % average ocean heat transport
Hc = 10.7; % 60-20*(multiline-1); % Cold
cH = 3.383; % Warm ocean in linear equation
cHc = 0.2073; % Cold ocean in linear equation
Fc = 10;%100 - 30*(multiline-1); % [W/m^2] % convection transport
cFc = 0; % Convection in linear equation
Tbara = (273.15+5); % kelvin conversion for global avg temp air
Tbars = (273.15+13); % kelvin conversion for global avg temp surface

% Main constants and their dimensions
sigma = 5.67*10^(-8); % [W/m^2*K^4] % Stefan Boltzmann constant
eps = 0.90; % [None] % emissivity of the atmosphere
alphai = 0.4; % [None] % albedo of ocean ice
alphani = 0.04; % [None] % albedo of no ice
So = 1372; % [W/m^2] % solar constant
sttheta = 0.505; % 0.505 for pole and 1.221 for equator
S = So/4*sttheta; % [W/m^2] % solar radiation of the region
g = 9.81; % [m/s^2] % gravitational constant of Earth
Tref = 273.15; % [K] % reference temperature
Qs = S/(sigma*(Tref^4)); % 0.5488; % Surface constant

% CO2 contribution
MassE = 5.1352*10^(18); % [kg] % mean mass of dry air
RadiusE = 6371000.0; % [m] % mean radius of the Earth
PA = 101325; % [N/m^2] % density of dry air
RHO2 = PA/g; % [kg/m^2] % dry air column
kc = 0.0474; % [m^2/kg] % absorption coefficient CO2
RCO2 = 188.9; % [m/(s^2*K)] % specific gas constant CO2
G1 = kc*RHO2*1.52/(10.^(6)); % 7.5359*10^-4; % CO2 constant

% H2O contribution
kw = 0.0062; % [m^2/kg] % absorption coefficient H2O
L = 40650.0/0.01802; % [m^2/s^2] % above zero latent heat
Ls = 2.8373*10^-6; % [m^2/s^2] % below zero latent heat
RH2O = 461.4; % [m^2/(s^2*K)] % specific gas constant H2O
PH2Oref = 611.2; % [Pa] % pressure of H2O at Tref
RHOzh2o = PH2Oref/g; % [kg/m^2] % water vapour column
G2 = kw*RHOzh2o; % 1.2461; % H2O constant pre exp
G3 = L/(Tref*RH2O); % 22.6126; % H2O constant in exp
G3L = L/(Tref*RH2O); % 17.8989; % H2O constant in exp

% Surface area
SAT = 4*pi*(RadiusE^2)*(cos(pi/180*80)); % Tropics
SAA = 2*pi*(RadiusE^2)*(1-cos(pi/180*10)); % Arctic
SAAn= 2*pi*(RadiusE^2)*(1-cos(pi/180*10)); % Antarctic

%%%%% FUNCTIONS FOR Tau_A AND Tau_S %%%%%%

figure(1) % Main equation(s)
hold on % Keep previous plots! Includes contour and regular plots!

% Non dimensionalize where tau = temp/ref. temp

% EXTRA WORKS Section
% Functions are written in directly for each piece
% A --> tauA and B --> tauB
% Replace Fc = cFc.*(-A*Tref+B*Tref)
% Replace Fa = cFa.*(-A*Tref+B*Tref)
% Replace H = cH.*(Tbars-B*Tref)
% Replace Hc = cHc.*(Tbars-B*Tref)
% Replace alpha = (0.04+(0.5-0.04)*tanh((B-1)/b))
% switch = 0.5+0.5*tanh((B-1)/b) <-- 0 to 1
% need switch*alphani + (1-switch)*alphai
% This says that alphai is on until we switch at B=1 to alphani
% b is the smoothness

sC = linspace(0,1,200); % For sections of atmosphere
sW = linspace(1,2,200); % or surface equations that need
s1 = linspace(0,3,200); % special attention e.g. the kink

[A,B] = meshgrid(sC,s1);
AlineW = (1-exp(-mu*G1).*exp(-humid*G2*1./(A) ...  
    .*exp((G3).*((1.0 - 1./(A))))).*((B).^4 ...  
    - eps*(A).^4 + (Fa+Fc)/(sigma*(Tref).^4);

p1 = contour(A,B, AlineW, [0 0], 'b');

% Clausius Clapeyron Play area
start = ((Fa+Fc)/(eps*sigma*(Tref^4)))^(0.25)+0.0001;
Atau = linspace(start,2,200);
etatauCC1 = 1 - exp(-mu*G1 - ...  
    humid*(G2./1.).*exp(G3*(1-1./Atau))); % no inverse
etatauCC2 = 1 - exp(-mu*G1 - humid*(G2./Atau)*exp(0)); % no exp
StauCC1 = ((eps*(Atau.^(4)) - (Fa+Fc)/(sigma*(Tref.^4)))./etatauCC1).^(0.25);
StauCC2 = ((eps*(Atau.^(4)) - (Fa+Fc)/(sigma*(Tref.^4)))./etatauCC2).^(0.25);

%pCC1 = plot(Atau,StauCC1,'c');
%pCC2 = plot(Atau,StauCC2,'g');

% Surface contour
s2 = linspace(1,2,400);
[A,B] = meshgrid(s1,s2);
b = 0.01;
sswitch = 0.5+0.5*tanh((B-1)/b);
alphamain = sswitch*alphani + (1-sswitch)*alphai;
Hmain = sswitch*H+(1-sswitch)*Hc;
Bline = -1*(B).^4+(Hmain - Fc)/(sigma*(Tref).^4) ...  
    +0.63*eps*(A).^4+(1-(alphamain))*Qs;
p3 = contour(A,B, Bline, [0 0], 'm');
alphain = sswitch*alphani + (1-sswitch)*alphai;
Hmain = sswitch*H+ (1-sswitch)*Hc;
Bline = -1*(B).^4+(Hmain -Fc)/(sigma*(Tref)^4) ... 
+0.63*eps*(A).^4+(1-(alphain))*Qs;
p4 = contour(A,B, Bline, [0 0], 'color', [0.9 0 0.1]);
dotline = linspace(0,2,200);
contourdash = 1.*dotline+dotline;
p5 = plot(dotline,contourdash,'k:');
s2 = linspace(0,3,200);
[A,D] = meshgrid(sw,s2);
AlineC = (1-exp(-mu*G1).*(-humid*G2*1./(A) ... 
.*exp((G3L).*(1.0 - 1./(A))))).*D.*4 ... 
-eps*(A).^4 + (Fa+Fc)/(sigma*(Tref)^4);
p2 = contour(A,D, AlineC, [0 0], 'b');

%%%%%% For the step function (not switch)
Atau = linspace(0.,2,100);
Staui = (0.63*eps*Atau.*(4) + (1-alphai)*Qs + Hc/(sigma*(Tref)^4)) ... 
- Fc/(sigma*(Tref^4)).*(0.25);
Stauni = (0.63*eps*Atau.*(4) + (1-alphani)*Qs + H/(sigma*(Tref^4)) ... 
- Fc/(sigma*(Tref^4)).*(0.25);
Stauref = Atau - Atau + 1.0;
ind1 = Staui<=Stauref;
ind2 = Stauni>=Stauref;

%plot3 = plot(Atau(ind2),Stauni(ind2), 'm');
%plot4 = plot(Atau(ind1),Staui(ind1), 'r');

%%%%%%
end
xlim([0.8 1.3])
ylim([0.8 1.3])
xlabel('Atmosphere Temperature \tau_A')
ylabel('Surface Temperature \tau_S')

%title('Arctic Energy Balance Model: Carbon Dioxide only \mu ')
%title('Arctic Energy Balance Model: Water Vapour only \delta ')
%title('Arctic Energy Balance Model: Clausius-Clapeyron \delta ')
%title('Arctic Energy Balance Model: Latent Heat Adjustment')
%title('Arctic Energy Balance Model: Combination with Varied \delta ' ... 
'\chi' and \mu = 270 ppm')
%title('Arctic Energy Balance Model: Combination with Varied \mu')
title('Arctic Energy Balance Model: Combination')
%title('Arctic Energy Balance Model: Combination with Varied F_O')
%title('Arctic Energy Balance Model: Combination with Adjusted G_2')
%title('Arctic Energy Balance Model: Pliocene Paradox')
%title('Arctic Energy Balance Model: Convection Function F_C')
%title('Arctic Energy Balance Model: Atmosphere Transport F_A')
%title('Arctic Energy Balance Model: Atmosphere Transport Function F_A')
%title('Arctic Energy Balance Model: Ocean Transport Function F_O')
%title('Arctic Energy Balance Model: Generalized Functions')

legend('Atmosphere Equation','Warm Surface Equation', ... 
   'Cold Surface Equation','Surface Freezing Point',...
   'location','southeast')

% CC legend
%legend('Atmosphere Equation','Atmosphere Equation no 1/\tau_A',...
%   'Atmosphere Equation no exp(*)','Warm Surface Equation',...
%   'Cold Surface Equation','Surface Freezing Point',...
%   'location','southeast')

hold off
% Call a function for our model
fun = @root2d;   % ice-free function
fun1 = @root2df; % frozen function
xrec = ones(C,2); % unstable
vrec = ones(C,2); % stable hot
zrec = ones(C,2); % stable cold
urec = ones(C,2); % stable warm
qrec = ones(C,2); % stable cold
wrec = ones(C,2); % unstable cold

for i = 1:C

    mux = allmux(i);
    muz = allmuz(i);
    muv = allmuv(i);
    muu = allmuu(i);
    muq = allmuq(i);

    humidx = allhumidx(i); % warm unstable
    humidv = allhumidv(i); % hot stable
    humidz = allhumidz(i); % frozen stable
    humidu = allhumidu(i); % warm stable
    humidq = allhumidq(i); % cold unstable

end
Hx = allHx(i); % warm unstable
Hv = allHv(i); % hot stable
Hz = allHz(i); % frozen stable
Hu = allHu(i); % warm stable
Hz = allHz(i); % cold unstable

cHx = callHx(i); % warm unstable
cHv = callHv(i); % hot stable
cHz = callHz(i); % frozen stable
cHu = callHu(i); % warm stable
cHz = callHz(i); % cold unstable

alphax = allalphax(i); % warm unstable
alphav = allalphav(i); % hot stable
alphaz = allalphaz(i); % frozen stable
alphau = allalphau(i); % warm stable
alphaq = allalphaq(i); % cold unstable

Fcx = allFcx(i); % warm unstable
Fcv = allFcv(i); % hot stable
Fcz = allFcz(i); % frozen stable
Fcu = allFcu(i); % warm stable
Fcq = allFcq(i); % cold unstable

cFcx = callFcx(i); % warm unstable
cFcv = callFcv(i); % hot stable
cFcz = callFcz(i); % frozen stable
cFcu = callFcu(i); % warm stable
cFcq = callFcq(i); % cold unstable

Fax = allFax(i); % warm unstable
Fav = allFav(i); % hot stable
Faz = allFaz(i); % frozen stable
Fau = allFau(i); % warm stable
Faq = allFaq(i); % cold unstable

cFax = callFax(i); % warm unstable
cFav = callFav(i); % hot stable
cFaz = callFaz(i); % frozen stable
cFau = callFau(i); % warm stable
cFaq = callFaq(i); % cold unstable

tx = alltx(i); % Timeline
tv = alltv(i);
tz = alltz(i);
tu = alltu(i);
tq = alltq(i);
Hnux = Hmax*(1-tx)+Hmin*tx; % Straight line ice-free
Hnuv = Hmax*(1-tv)+Hmin*tv;
Hnuz = Hmax*(1-tz)+Hmin*tz;
Hnuu = Hmax*(1-tu)+Hmin*tu;
Hnuq = Hmax*(1-tq)+Hmin*tq;

Hcnux = Hcmax*(1-tx)+Hcmin*tx; % Straight line frozen
Hcnuv = Hcmax*(1-tv)+Hcmin*tv;
Hcnuz = Hcmax*(1-tz)+Hcmin*tz;
Hcnuu = Hcmax*(1-tu)+Hcmin*tu;
Hcnuq = Hcmax*(1-tq)+Hcmin*tq;

co2nux = co2max*(1-tx)+co2min*tx; % Straight line CO2
co2nuv = co2max*(1-tv)+co2min*tv;
co2nuz = co2max*(1-tz)+co2min*tz;
co2nuu = co2max*(1-tu)+co2min*tu;
co2nuq = co2max*(1-tq)+co2min*tq;

x = fsolve(@(x)fun(x,mux,humid,cFa,Fc,cH,cHc,stheta,alphani,alphai),x0);
v = fsolve(@(v)fun(v,muv,humid,cFa,Fc,cH,cHc,stheta,alphani,alphai),v0);
z = fsolve(@(z)fun(z,muz,humid,cFa,Fc,cH,cHc,stheta,alphani,alphai),z0);
u = fsolve(@(u)fun(u,muu,humid,cFa,Fc,cH,cHc,stheta,alphani,alphai),u0);
q = fsolve(@(q)fun(q,muq,humid,cFa,Fc,cH,cHc,stheta,alphani,alphai),q0);

xrec(i,:) = x; % records x = [Ta,Ts] into a matrix
vrec(i,:) = v;
zrec(i,:) = z;
urec(i,:) = u;
qrec(i,:) = q;

end
figure(2)

\%clf

% Track water vapour fixed points
delta = 1-allhumidz+allhumidz;
%plot(allhumidv,vrec(:,2),'m', ... % allhumidz,delta,'k:', ... % allhumidx,xrec(:,2),'m--', ... % allhumidq,qrec(:,2),'m--', ... % allhumidu,urec(:,2),'m')

% Track co2 fixed points
mudash = 1 - allmudash + allmudash;
plot(allmuv,vrec(:,2),'m', ...
allmuz,zrec(:,2),'m', ...
allmudash,mudash,'k:', ...
allmux,xrec(:,2),'m--', ...
allmuu,urec(:,2),'m', ...
allmuq,qrec(:,2),'m--')

% Track \nu fixed points
dash = 1-tdash+tdash;
%plot(alltu,urec(:,2),'m', ...
% alltz,zrec(:,2),'m', ...
% tdash,dash,'k:', ...
% alltq,qrec(:,2),'m--')
% alltx,xrec(:,2),'m--', ...
% alltv,vrec(:,2),'m', ...

% Track OHT fixed points
Hdash = 1-alldash+alldash;
%plot(allHv,vrec(:,2),'m', ...
% allHz,zrec(:,2),'m', ...
% alldash,Hdash,'k:', ...
% allHx,xrec(:,2),'m--', ...
% allHu,urec(:,2),'m')

% Track alpha fixed points
alphadash = 1 - allalpha + allalpha;
%plot(allalphav,vrec(:,2),'m', ...
% allalphaz,zrec(:,2),'m', ...
% allalpha,alphadash,'k:', ...
% allalphax,xrec(:,2),'m--', ...
% allalphau,urec(:,2),'m')

% Track Fc fixed points
Fcdash = 1.0 - allFc + allFc;
%plot(allFcv,vrec(:,2),'m', ...
% allFcz,zrec(:,2),'m', ...
% allFc,Fcdash,'k:', ...
% allFcx,xrec(:,2),'m--', ...
% allFcu,urec(:,2),'m', ...
% allFcq,qrec(:,2),'m--')

% Track Fa fixed points
Fadash = 1.0 - allFc + allFc;
%plot(allFav,vrec(:,2),'m', ...
% allFaz,zrec(:,2),'m', ...
% allFa,Fadash,'k:', ...
% allFax,xrec(:,2),'m--', ...
% allFau,urec(:,2),'m', ...

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ylabel('Surface Temperature \(\tau_S\)')
\%xlabel('Relative Humidity \(\delta\)')
xlabel('Carbon Dioxide ppm \(\mu\)')
\%xlabel('Eocene to Pre-industrial Parameter \(\nu\)')
xlabel('Ocean Heat Transport \(F_{O,w}, F_{O,c}\) [W/m^2]')
\%xlabel('Albedo \(\alpha_w, \alpha_c\)')
xlabel('Convection \(F_C\) [W/m^2]')
xlabel('Atmospheric Heat Transport \(F_A\) [W/m^2]')
xlabel('Atmospheric Heat Transport Constant \(C_{A}\) [W/m^2/K]')
xlabel('Ocean Heat Transport Constant \(C_{O}\) [W/m^2/K]')
xlabel('Convection Constant \(C_{C}\) [W/m^2/K]')
\% axes limits
ylim([0.8 1.2])
\%xlim([0 1]) \% humid and \(\nu\)
xlim([200 1200]) \% \(\mu\)
\%xlim([-50 200]) \% oht
\%xlim([30 200]) \% \(F_a\)
xlim([0 6]) \% constants

function F = root2d(x,mu,humid,Fa,Fc,Fo,Foc,stheta,alphani,alphai)
% pass along these parameters so I don't have to write them twice!

% % Remember to change parameters that change with region.
% The regions are Arctic, Antarctic, and Tropics.
% %

% Setup parameters that don't change
sigma = 5.67*10^(-8); % [W/m^2*K^4] \% Stefan Boltzmann constant
eps = 0.90; % [None] \% emissivity of the atmosphere
So = 1372; % [W/m^2] \% solar constant
S = So/4*stheta; % [W/m^2] \% solar radiation of the region
g = 9.81; % [m/s^2] \% gravitational constant of Earth
Tbara = (273.15+5); % kelvin + celsius for global avg temp air
Tbars = (273.15+13); % kelvin + celsius for global avg temp air

% CO2 section
PA = 101325; % [N/m^2] \% pressure of dry air
RH02 = PA/g; % [kg/m^2] \% air column using Pressure and gravity
k1 = 0.0474; % [m^-2/kg] \% absorption coefficient for CO2

% Water vapour section
k2 = 0.0062;
\[ \text{PH2Oref} = 611.2; \quad \% \text{[Pa]} \quad \% \text{density of H2O at reference temp.} \\
\text{Tref} = 273.15; \quad \% \text{[K]} \quad \% \text{reference temperature} \\
\text{L} = 40650.0/0.01802; \quad \% \text{[m}^2/\text{s}^2\text{]} \quad \% \text{above zero latent heat} \\
\text{Ls} = 2.8373\times10^6; \quad \% \text{[m}^2/\text{s}^2\text{]} \quad \% \text{below zero latent heat} \\
\text{RH2O} = 461.4; \quad \% \text{[m}^2/(\text{s}^2\text{K})\text{]} \quad \% \text{specific gas constant for H2O} \\
\text{RHOzh2o} = \text{PH2Oref}/g; \quad \% \text{[kg/m}^2\text{]} \quad \% \text{column on cross sectional area} \\
\% \text{Non dimensional section where \( \tau = \text{temp/ref. temp} \)} \\
\text{Qs} = S/(\text{sigma}\times(T\text{ref}^4)); \quad \% \text{0.5488; \% Surface constant} \\
\text{G1} = k1\times\text{RHOz}^2/(10.^(6)); \quad \% \text{7.5359\times10}^{-4}; \% \text{CO2 constant} \\
\text{G2} = k2\times\text{RHOzh2o}; \quad \% \text{1.2461}; \% \text{H2O constant} \\
\text{G3} = \text{Ls}/(\text{Tref}\times\text{RH2O}); \quad \% \text{22.5126}; \% \text{H2O constant in exp} \\
\text{G3L} = \text{L}/(\text{Tref}\times\text{RH2O}); \\
\% \text{Atmosphere energy balance equation w/o new latent} \\
\text{etatau} = 1 - \exp(-\mu\times G1 - \text{humid}\times(\text{G2}/x(1))\times\exp(\text{G3}\times(1-1/x(1)))); \\
\text{etatauL} = 1 - \exp(-\mu\times G1 - \text{humid}\times(\text{G2}/x(1))\times\exp(\text{G3L}\times(1-1/x(1)))); \\
\% \text{For EXTRA WORK portion: \%\%\%} \\
\% \text{Fa, H, and Fc, are imported as the constants "C_a,s,c"\%\%\%} \\
\% \text{Atmosphere Transport} \\
\text{cFa} = (\text{Fa}\times(\text{Tbara}-x(1))\times\text{Tref}); \\
\% \text{Ocean Transport warm} \\
\text{cFo} = (\text{Fo}\times(\text{Tbars}-x(2))\times\text{Tref}); \\
\% \text{Ocean Transport cold} \\
\text{cFoc} = (\text{Foc}\times(\text{Tbars}-x(2))\times\text{Tref}); \\
\% \text{Convection} \\
\text{cFc} = (\text{Fc}\times(x(2))\times\text{Tref} - x(1))\times\text{Tref}); \\
\% \text{Switch function for albedo and OHT} \\
b = 0.01; \\
\text{sswitch} = 0.5\times0.5\times\text{tanh}((x(2)-1)/b); \\
\text{alphamain} = \text{sswitch}\times\text{alphani} + (1-\text{sswitch})\times\text{alphai}; \\
\text{Hmain} = \text{sswitch}\times\text{Fo} + (1-\text{sswitch})\times\text{Foc}; \\
\text{if } x(1)\text{<=1.0} \\
\text{F(1) = (x(2))}^4\times\text{etatau} - \text{eps}\times(x(1))^-4\times(\text{Fa}+\text{Fc})/((\text{sigma}\times(\text{Tref}^4))); \\
\text{else} \\
\text{F(1) = (x(2))}^4\times\text{etatauL} -((\text{eps}\times(x(1))^-4) - (\text{Fa}+\text{Fc})/((\text{sigma}\times(\text{Tref}^4)))); \\
\text{end} \\
\text{F(2) = 0.63}\times\text{eps}\times(x(1))^-4\times(1-\text{alphamain})\times\text{Qs} + \ldots \\
\text{Hmain}/((\text{sigma}\times(\text{Tref}^4)) - \text{Fc}/((\text{sigma}\times(\text{Tref}^4)) - (x(2))^4);