Tidal Response of a Rotating Neutron Star in General Relativity

by

Philippe Landry

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Abstract

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Philippe Landry
University of Guelph, 2017

Advisor:
Professor Eric Poisson

Internal-structure-dependent tidal deformations in inspiralling neutron star binaries alter the phase of the gravitational waves generated by these systems’ orbital motion. Measurement of the tidal phase shift could serve as a probe of the neutron star equation of state, which is poorly constrained above nuclear density. Motivated by this prospect, we extend the general-relativistic theory of tidal deformations to the astrophysically relevant case of spinning bodies. Working in a perturbative framework of weak, slowly varying tides and slow rotation, we find that the familiar gravitational Love numbers $K_{2}^{el}$ and $K_{2}^{mag}$, which fully describe the external geometry of a deformed nonrotating body, must be supplemented by rotational-tidal Love numbers to account for couplings between the body’s spin and the applied tidal field. By integrating the Einstein field equations inside the body, we compute the rotational-tidal Love numbers explicitly for polytropes, and we find that they vanish identically for black holes. The field equations also reveal that the tidal field generically induces time-dependent fluid motions within the rotating body; these tidal currents are dynamical even if the tidal field is stationary. We calculate the amplitude of the currents for a typical neutron star in an equal-mass binary system, and find that it is on the order of kilometers per second.
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# Table of Contents

1. **Introduction**
   1.1 Context and motivation ........................................... 1
   1.2 Overview .......................................................... 5

2. **Tidal environment** .................................................. 8
   2.1 Tidal moments ....................................................... 8
   2.2 Tidal potentials ................................................... 10
   2.3 Bilinear moments and potentials ................................. 12

3. **Spacetime outside a tidally deformed, slowly rotating body** ............. 16
   3.1 Exterior metric .................................................... 16
   3.2 Exterior field equations .......................................... 19

4. **Spacetime inside a tidally deformed, slowly rotating body** .................. 24
   4.1 Interior metric .................................................... 24
   4.2 Energy-momentum tensor ......................................... 26
      4.2.1 Background .................................................... 26
      4.2.2 Perturbed fluid .............................................. 28
   4.3 Interior field equations ........................................... 31
      4.3.1 Zero frequency modes ...................................... 31
      4.3.2 Field equations: Tidal perturbation ........................ 32
      4.3.3 Field equations: Bilinear perturbation ...................... 33
   4.4 Discussion: Static fluid state .................................... 41

5. **Tidal response** ..................................................... 43
   5.1 Equation of state .................................................. 43
   5.2 Love numbers ...................................................... 44
      5.2.1 Gravitoelectric Love number ................................ 44
      5.2.2 Gravitomagnetic Love number ................................. 45
      5.2.3 Rotational-tidal Love number $\delta^r$ .................. 47
      5.2.4 Rotational-tidal Love number $\alpha^r$ .................. 49
      5.2.5 Discussion: Pani, Gualtieri and Ferrari .................. 51
   5.3 Tidal currents ..................................................... 54
      5.3.1 Discussion: Dynamical response ............................ 54
      5.3.2 Perturbed velocity field .................................... 56
# List of Tables

<table>
<thead>
<tr>
<th>Table Number</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Spherical-harmonic functions $Y_{\ell m}$</td>
<td>11</td>
</tr>
<tr>
<td>2.2</td>
<td>Spherical-harmonic coefficients of STF tensors</td>
<td>12</td>
</tr>
<tr>
<td>3.1</td>
<td>Nonzero radial functions appearing in the exterior metric</td>
<td>21</td>
</tr>
<tr>
<td>5.1</td>
<td>The dimensionless, equation-of-state-dependent parameter $\sigma$ for polytropic models of index $n$ and compactness $M/R$</td>
<td>63</td>
</tr>
<tr>
<td>B.1</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 0.5$ polytropes</td>
<td>72</td>
</tr>
<tr>
<td>B.2</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 0.75$ polytropes</td>
<td>73</td>
</tr>
<tr>
<td>B.3</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 1$ polytropes</td>
<td>73</td>
</tr>
<tr>
<td>B.4</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 1.5$ polytropes</td>
<td>73</td>
</tr>
<tr>
<td>B.5</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 2$ polytropes</td>
<td>74</td>
</tr>
<tr>
<td>B.6</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 2.5$ polytropes</td>
<td>74</td>
</tr>
<tr>
<td>B.7</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 0.5$ polytropes</td>
<td>74</td>
</tr>
<tr>
<td>B.8</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 0.75$ polytropes</td>
<td>75</td>
</tr>
<tr>
<td>B.9</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 1$ polytropes</td>
<td>75</td>
</tr>
<tr>
<td>B.10</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 1.5$ polytropes</td>
<td>75</td>
</tr>
<tr>
<td>B.11</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 2$ polytropes</td>
<td>76</td>
</tr>
<tr>
<td>B.12</td>
<td>Rotational-tidal Love number $\tilde{f}_n$ for $n = 2.5$ polytropes</td>
<td>76</td>
</tr>
</tbody>
</table>
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>The radial function $e_q^n$ for various polytropes</td>
<td>45</td>
</tr>
<tr>
<td>5.2</td>
<td>Scale-free gravitoelectric Love number $k_{el}^2 := (2M/R)^6 K_{el}^2$ as a function of compactness $M/R$ for polytropes of index $n$</td>
<td>46</td>
</tr>
<tr>
<td>5.3</td>
<td>The radial function $b_{q1}^n$ for various polytropes</td>
<td>47</td>
</tr>
<tr>
<td>5.4</td>
<td>Scale-free gravitomagnetic Love number $k_{mag}^2 := (2M/R)^4 K_{mag}^2$ as a function of compactness $M/R$ for polytropes of index $n$</td>
<td>48</td>
</tr>
<tr>
<td>5.5</td>
<td>The radial function $f_{o}^n$ for various polytropes</td>
<td>49</td>
</tr>
<tr>
<td>5.6</td>
<td>Scale-free gravitoelectric rotational-tidal Love number $f_{o}^n := -(2M/R)^5 \tilde{f}_{o}$ as a function of compactness $M/R$ for polytropes of index $n$</td>
<td>50</td>
</tr>
<tr>
<td>5.7</td>
<td>The radial function $k_{tr}^{n}$ for various polytropes</td>
<td>51</td>
</tr>
<tr>
<td>5.8</td>
<td>The radial function $k_{tr}^{n}$ for various polytropes</td>
<td>52</td>
</tr>
<tr>
<td>5.9</td>
<td>Scale-free gravitomagnetic rotational-tidal Love number $f_{o}^n := -(2M/R)^5 \tilde{f}_{o}$ as a function of compactness $M/R$ for polytropes of index $n$</td>
<td>53</td>
</tr>
<tr>
<td>5.10</td>
<td>Radial profile of the fluid variable $v_{a}$ for several polytropes</td>
<td>57</td>
</tr>
<tr>
<td>5.11</td>
<td>The radial function $b_{q1}^n$ for various polytropes</td>
<td>57</td>
</tr>
<tr>
<td>5.12</td>
<td>Radial profile of the fluid variable $v_{a}$ for several polytropes</td>
<td>58</td>
</tr>
<tr>
<td>5.13</td>
<td>Radial profile of the fluid variable $v_{a}$ for several polytropes</td>
<td>59</td>
</tr>
<tr>
<td>5.14</td>
<td>The radial function $k_{tr}^{n}$ for various polytropes</td>
<td>60</td>
</tr>
<tr>
<td>5.15</td>
<td>Radial profile of the fluid variable $v_{a}$ for several polytropes</td>
<td>61</td>
</tr>
<tr>
<td>5.16</td>
<td>Radial profile of the fluid variable $v_{a}$ for several polytropes</td>
<td>62</td>
</tr>
<tr>
<td>5.17</td>
<td>Equatorial projection of the internal fluid currents</td>
<td>63</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

1.1 CONTEXT AND MOTIVATION

Neutron stars are the most compact configurations of matter supported against gravitational collapse known to exist in the universe. Observations of binary pulsars place a conservative lower bound of $2M_\odot$ on the maximum neutron star mass \cite{1,2}, and theoretical constraints on the mass-radius relation imply a minimum radius of roughly 10 km for a $1.4M_\odot$ neutron star \cite{3} (the minimum radius may even be slightly smaller for a higher-mass star \cite{4}). The neutron star compactness $GM/c^2R \lesssim 0.30$ therefore approaches the absolute incompressible limit of 0.44 for material bodies in general relativity \cite{5}. The extreme compactness of neutron stars implies average densities which significantly exceed the density of nuclei $\rho_{\text{nuc}} = 2.8 \times 10^{17}$ kg/m$^3$. Since this is far denser than any naturally occurring (or laboratory-created) matter on Earth, it is perhaps unsurprising that the internal structure of neutron stars remains uncertain. Observational probes of neutron star structure are thus essential for constraining competing theoretical nuclear physics models, which predict different equations of state above nuclear density.

Traditionally, astronomers have sought to constrain the mass-radius relation – the main output of the theoretical models – with independent measurements of a neutron star’s mass and radius. However, the radius measurements in particular have proven to be a challenge: at present, no reliable radius measurements exist for any neutron stars with a precise mass determination \cite{4}. The most accurate mass measurements come from binary pulsars. Pulsar timing can be used to infer the Keplerian parameters which specify the pulsar’s mass function $f_M(M,M',i)$, where $M$ is the pulsar mass, $M'$ is the companion mass and $i$ is the system’s inclination; a separate measurement of a general-relativistic effect (such as Shapiro delay) can be used to extricate $M$ from $f_M$ \cite{4}. Radius measurements, on the other hand, usually rely on thermal X-ray emission from the neutron star’s surface. At the most basic level, one can approximate the neutron star as a blackbody and estimate its surface area with the Stefan-Boltzmann law, provided its distance is known; in practice, one must account for the neutron star’s emissivity, atmosphere and magnetosphere, as well as systematic uncertainties in the distance measurement and the interstellar medium \cite{4,6}. Given the dearth of systems suited for simultaneous mass-radius determinations, and the difficulties inherent to the radius measurement, conventional astronomical observations have placed limited constraints on neutron star structure to date.

The recent advent of gravitational-wave astronomy provides a different means of probing in-
ternal structure. Ground-based gravitational-wave detectors, both extant (Advanced LIGO, Virgo) and proposed (Einstein Telescope), should be sensitive to the gravitational waves emitted at frequencies $f_{GW} \sim 100 \text{ Hz}$ by coalescing neutron star binaries – Advanced LIGO is expected to detect them at a rate of $\sim 1/\text{yr}$ [7]. The neutron star equation of state may be revealed through tidal interactions which impact the orbital and radiative dynamics of these systems [10, 11]. One possibility envisioned by a number of authors [12–15] involves tidal disruption. As the binary’s orbit shrinks because of gravitational-wave emission, the companion’s nonuniform gravitational field raises a tide on the neutron star; by the time the binary is closer to merger, the tide may be so large that the neutron star is disrupted. The tidal disruption abruptly damps the amplitude of the system’s gravitational-wave emission, producing a signal that is strongly suppressed above a cut-off frequency $f_{\text{cut}}$ which depends sensitively on the neutron star’s internal structure [11, 13, 14]. The observation of tens of disruptive coalescences, or even a handful of especially loud events, may be sufficient to begin placing constraints on the equation of state [12].

However, although late-inspiral effects dominate the tidal contribution to the waveform, this approach presents a number of challenges: the tidal disruption occurs at frequencies outside Advanced LIGO’s most sensitive band; the extraction of the constraints relies on match-filtering the signal with gravitational-wave templates that depend on unknown binary parameters, such as the individual spins of the compact objects; and the template generation is computationally expensive, as it requires solving the field equations of general relativity and relativistic hydrodynamics in the strong-field regime [16]. An alternate proposal was articulated by Flanagan & Hinderer to overcome these obstacles. In Ref. [16], they suggested using the early inspiral portion of the signal, when the neutron star and its companion are well-separated and $f_{GW} \lesssim 400 \text{ Hz}$. During this stage, the fundamental oscillation modes ($f$-modes) of the neutron star are driven by the tidal field of the orbiting companion. The orbital motion occurs over a timescale that is long compared to the $f$-modes’ period of oscillation, so the oscillator is driven at a frequency well below resonance and the tidal bulge raised on the neutron star tracks the companion’s motion. Since the strength of the tidal field grows only over the very long radiation reaction timescale, the oscillator evolves adiabatically, steadily absorbing energy from the tidal field. The neutron star’s induced, time-dependent quadrupole moment affects the orbital dynamics of the inspiral, leaving an imprint on the gravitational-wave signal from the binary as a whole. Specifically, the gravitational-wave phase $\Psi$ acquires a tidal correction [16]

$$
\delta\Psi \propto \left( \frac{f_{GW}}{GM(M+M')} \right)^{5/3} \left[ (12q + 1)(2GM/c^2)^5 K_2^{\text{el}} + (12q^{-1} + 1)(2GM'/c^2)^5 K_2'^{\text{el}} \right],
$$

(1.1.1)

where $M := (MM')^{3/5}/(M+M')^{1/5}$ is the binary’s chirp mass and $q := M'/M$ is the mass ratio. Remarkably, the tidal phase shift depends on internal structure only through the dimensionless constants $K_2^{\text{el}}$ and $K_2'^{\text{el}}$, the tidal Love numbers of the neutron star and its companion, respectively.

The tidal Love numbers measure a body’s tidal deformability, and encode all of the internal structure dependence of its response to an applied tidal field. They have their origin in Newtonian gravity [17], where they appear as constants of proportionality between the applied tidal field and the mass multipole moments raised on the body (see e.g. Ch. 2 of Ref. [18]). For instance, a body

\[\text{An earlier rate estimate of } \sim 40 \text{ yr by Refs. [8, 9] has been revised downwards because no neutron star mergers were observed during Advanced LIGO’s first observing run.}\]
immersed in the external quadrupolar tidal field $\mathcal{E}_{ab}(t) := -\partial_{ab} U_{\text{id}}$ sourced by a binary companion ($U_{\text{id}}$ is the companion’s gravitational potential; it is evaluated at the reference body’s centre of mass after differentiation) will acquire an induced mass quadrupole moment

$$Q_{ab} = -\frac{2}{3G} R^5 k_2 \mathcal{E}_{ab}$$

To linear order in the deformation. The numerical factors in this equation are conventional, and the factor of $R^5$ is required by dimensional analysis since $Q_{ab} := \int \delta \rho(x_a x_b - \frac{r^2 \delta_{ab}}{3}) d^3 x$ in mass-centred Cartesian coordinates $x^a$. Here, $\delta \rho$ is the density perturbation created by the tide, and $r := \sqrt{\delta_{ab} x^a x^b}$ is the distance from the centre of mass. The total gravitational potential outside the tidally deformed body is thus

$$U = \frac{G M}{r} - \frac{1}{2} \left[ 1 + 2k_2 \left( \frac{R}{r} \right)^5 \right] \mathcal{E}_{ab} x^a x^b.$$  

(1.1.3)

The first term is the potential of the isolated, spherical reference body; the constant term in square brackets represents the applied tidal field, and the last term represents the body’s tidal response, characterized by the Love number $k_2$. The expression is independent of the body’s internal structure, except for the numerical value of $k_2$: the Love number provides a complete description of the tidal deformation when the body is weakly perturbed by the tidal field.

The strong gravitational fields in compact binaries naturally require a general-relativistic treatment and, with this application in mind, several authors have contributed to the development of a relativistic theory of tidal deformations [19–23]. A precise definition of the tidal Love numbers in general relativity was introduced by Damour & Nagar [21] and Binnington & Poisson [22] in terms of tidal perturbations of the spacetime metric outside the reference body, rather than the Newtonian gravitational potential. They showed that the quadrupolar tidal deformation of a nonrotating compact object is characterized by two tidal Love numbers, $K^\text{el}_2$ and $K^\text{mag}_2$. The gravitoelectric Love number $K^\text{el}_2$ measures the size of the mass quadrupole moment raised on the body by the gravitoelectric field $\mathcal{E}_{ab}$ sourced by the companion’s mass distribution. The gravitomagnetic Love number $K^\text{mag}_2$ characterizes the mass current quadrupole induced by the gravitomagnetic field $B_{ab}$ generated by this distribution’s currents. The gravitoelectric Love number is the analogue of the (rescaled) Newtonian gravitational Love number $K_2 := (2G M/c^2 R)^{-5} k_2$, while the gravitomagnetic Love number has no analogue in Newtonian gravity. Binnington & Poisson demonstrated that the gravitational Love numbers (as $K^\text{el}_2$ and $K^\text{mag}_2$ are known collectively) possess gauge-invariant significance, in the usual (restricted) sense of perturbation theory, and further showed that they vanish identically for black holes.

Flanagan’s & Hinderer’s analysis of the tidal phase shift, which claimed that a single detection of an exceptionally loud double neutron star coalescence would suffice to place an upper bound on $K^\text{el}_2$ at the 90% confidence level, triggered a surge of activity to better understand the tidal deformation of compact bodies and its impact on the gravitational-wave profile of inspiralling binaries. The gravitational Love numbers have been computed for model neutron stars with polytropic and realistic equations of state, and they have been incorporated in analytically and numerically constructed neutron star binary waveforms [19, 21, 22, 24–38]. Tidal invariants constructed from

\footnote{We find it convenient to work in terms of the scale-dependent Love number $K_2$, rather than the scale-free version $k_2$, since the scalings of the relativistic Love numbers with compactness are not known a priori.}
\( \mathcal{E}_{ab} \) and \( B_{ab} \) have been inserted into point-particle actions to provide an effective description of the tidal response of an extended body \[39\]–\[43\]. The gravitational Love numbers have also been implicated in the remarkable I-Love-Q relations for neutron stars \[44\]–\[55\], which assert that certain combinations of the moment of inertia \( I \), the Love numbers and the rotational quadrupole moment are (nearly) independent of the neutron star equation of state; an observation of a single member of the I-Love-Q trio can therefore tightly constrain the other two, provided independent mass and spin measurements are available. Moreover, the gravitoelectric Love number is involved in the phenomenon of dynamical tides, which dominate the late stage of the inspiral when the orbital frequency approaches resonance with the \( f \)-modes (or other oscillation modes) of the neutron star and the adiabatic approximation breaks down \[41\], \[56\]–\[59\].

Despite this burst of activity, the optimistic outlook of Ref. \[16\] has come under scrutiny since its publication. Revised estimates by Hinderer \textit{et al.} \[25\] predict that only with a very loud signal from an especially low-mass binary can Advanced LIGO place even marginal constraints on \( K_{el}^2 \), although the prospects are somewhat improved when using the LIGO-Virgo detector network and including the late-inspiral \( f_{GW} \gtrsim 400 \text{ Hz} \) waveform in the analysis – Damour, Nagar & Villain \[31\] claim that an upper bound on \( K_{el}^2 \) can be established at the 95% confidence level with a few dozen reasonably loud events. In any case, the factor of \( \sim 10 \) sensitivity gain of the planned Einstein Telescope over Advanced LIGO will ensure that the early part of the inspiral waveform can be used to place meaningful constraints on the neutron star equation of state in the foreseeable future \[25\].

The aforementioned studies of tidal deformability share one important limitation, however: they are restricted to nonrotating bodies. Since compact objects in nature generally have nonzero spin, the investigation of tides on rotating bodies is of astrophysical relevance. The incorporation of rotation complicates the tidal response, introducing qualitatively new effects associated with the dragging of inertial frames relative to the nonrotating case. These effects are captured by couplings between the tidal field and the body’s spin angular momentum which arise because of the nonlinearity of the Einstein field equations. The couplings produce novel kinds of deformations which are characterized by an additional class of internal-structure-dependent constants, the \textit{rotational-tidal Love numbers}, and they have dramatic consequences for a rotating neutron star’s internal state: dynamical tidal currents, characterized by time-dependent fluid and interior metric variables which vary on the timescale of the orbital period in a binary setting, are induced inside the rotating body even if the tidal field is stationary. The rotational-tidal Love numbers are expected to introduce spin corrections to the tidal phase shift of Eq. (1.1.1) \[60\], and tidal currents induced in the neutron star interior may themselves contribute to gravitational-wave phasing, as well as dynamical tides \[61\]. There are indications that I-Love-Q universality may partially extend to the rotational-tidal Love numbers \[60\], and effective point-particle actions for extended-body dynamics may be further enhanced by the inclusion of tidal invariants associated with the tidal deformation of rotating bodies.

A program to calculate the tidal deformation of slowly rotating bodies in general relativity was initiated by Poisson in Ref. \[62\], which focused on the specific case of a black hole. Subsequent work by Landry & Poisson replaced the black hole with a material body in Ref. \[63\] (Paper I) and determined its deformed exterior geometry. The interior geometry of the material body was investigated in Ref. \[61\] (Paper II), with important insight coming from earlier work \[24\], and the rotational-tidal Love numbers were computed for polytropic neutron star models by Landry in Ref. \[64\]. A similar program was simultaneously pursued by Pani, Gualtieri, Maselli and Ferrari.
[60] [65], but these authors reached starkly different conclusions. The differences between their approach and that taken in Papers I-III are reviewed in detail in Secs. 4.4 and 5.2.5.

The main goal of this thesis is to present a coherent account of the developments and results spread across Papers I-III. This task consists of three parts. First, we present the construction of the spacetime metric outside a tidally deformed, slowly rotating neutron star; second, we calculate the Love numbers which appear as integration constants in this metric; and third, we investigate the tides’ effect on fluid motions in the neutron star interior. Accordingly, this dissertation aspires to be a reasonably self-contained treatment of the tidal deformation of a rotating neutron star. It is the author’s hope that it will serve as a useful review of the formalism developed in collaboration with Poisson in his graduate research work.

1.2 Overview

We now proceed to describe the setup of the problem of interest and outline the plan of the thesis. We use geometrized units hereafter, unless otherwise stated. We consider a material body of mass $M$, radius $R$ and spin angular momentum vector $S^a$ in a vacuum region of spacetime; lowercase latin indices $a, b, ...$ label spatial components in the natural basis associated with asymptotically mass-centred coordinates $x^a$. The spacetime in a neighbourhood of the body is pervaded by the tidal influence of matter (or black holes) far removed from the vacuum region. The distant mass distribution could consist, for example, of a binary companion of mass $M'$ at a large orbital separation $b$; in any case, we take $M'$ and $b$ to be generic mass and length scales for the external matter (or collection of black holes). The tidal environment created by this mass distribution is characterized by gravitoelectric tidal multipole moments $E_{ab}, E_{abc}, ...$ and gravitomagnetic tidal multipole moments $B_{ab}, B_{abc}, ...$ to be introduced in Ch. 2. We construct the metric for the tidal environment out of these tidal multipole moments. The construction ignores the presence of the material body, and views the tidal environment metric as a perturbation of the vacuum region’s Minkowski background [62, 66]. The tidal quadrupole moments scale like $E_{ab} \sim M'/b^3$ and $B_{ab} \sim M'v/b^5$, where $v \sim b/T$ is a velocity scale related to the timescale $T \sim \sqrt{b^3/(M + M')}$ for changes in the tidal environment, and they contribute terms proportional to $r^2E_{ab} \sim (M'/b)(r/b)^2$ and $r^2B_{ab} \sim (M'/b)v(r/b)^2$ to the metric. These terms are small, since $r \ll b$ in the neighbourhood of the body. The terms contributed by the tidal octupole moments are suppressed by a factor of $(r/b)$ relative to the quadrupole terms, and the higher multipole tidal terms are further suppressed; we therefore omit all but the leading-order quadrupole terms in the metric. We also neglect terms involving the time derivative of the tidal quadrupole moments, which are suppressed by a factor of $v(r/b)$.

Having specified the tidal environment, we restore the reference body to its place in the spacetime. The tidal field embodied by $E_{ab}$ and $B_{ab}$ exerts a nonuniform force across the body which results in a deformation from its unperturbed equilibrium shape. This tidal deformation is incorporated as a perturbation of the body’s background linearized Kerr metric – we assume that the body’s rotation is slow, so that its exterior metric is identical to the Kerr metric for a rotating black hole to linear order in the parameter $\chi \ll 1$, the magnitude of the dimensionless spin vector $\chi^a := S^a/M^2$. The body’s equilibrium shape is thus approximately spherical, since rotational deformations enter at $O(\chi^2)$, and the exterior metric is independent of its composition. The slow-rotation assumption
is motivated by observational evidence that compact objects in binary systems spin at rates well below the causal limit $\chi = 1$ [67]; for example, the millisecond pulsar in the double pulsar system PSR J0737-3039 rotates with a period of only 22 ms [68], yet this corresponds to no more than $\chi \sim 0.05$ [31, 69]. At leading order in the tidal interaction, the metric outside the deformed body is characterized by terms linear in the tidal quadrupole moments, including spin-coupled terms proportional to $\chi^a E_{bc}$ and $\chi^a B_{bc}$. We assume that these terms, which are suppressed relative to the leading-order tidal terms by a factor of $\chi$, nevertheless dominate over the next-to-leading order terms. This requires $\chi \gg r/b$, and the further assumption that $M + M'$ is of the same order of magnitude as $r$ in the neighbourhood of the body implies that the spin must satisfy $v^2 \ll \chi \ll 1$. This condition is naturally fulfilled in a setting where the mutual gravity between the body and the distant mass distribution is weak [62].

The tidal deformation establishes itself over the body’s hydrodynamical timescale $T \sim \sqrt{R^3/M}$, which is much shorter than the timescale $T$ for changes in the tidal environment by virtue of the separation of scales $R \sim r \ll b$. The timescale $T$ is also much longer than the body’s rotational period, which is comparable to $T$. The variation in the tidal environment is therefore negligible over the timescale of the body’s tidal response, and as a consequence we take the tidal quadrupole moments to be independent of time. This amounts to idealizing the tidal field as stationary.

The assumptions $r \ll b$, $\chi \ll 1$ and their corollaries define the regime of stationary tides that is characteristic, for instance, of the early inspiral stage of a compact binary. We construct the metric outside the tidally deformed, slowly rotating body in Sec. 3.1 subject to these approximations. The construction involves a number of irreducible potentials, introduced in Sec. 2.2 and Sec. 2.3, that are built from the tidal quadrupole moments and the dimensionless spin vector. By solving the vacuum field equations, we find in Sec. 3.2 that the exterior metric is universal up to gravitational and rotational-tidal Love numbers. Specifically, it contains gravitational Love numbers $K_{el}^q$ and $K_{mag}^q$ which measure the quadrupolar deformations respectively induced by $E_{ab}$ and $B_{ab}$; rotational-tidal Love numbers $E^q$ and $F^q$ which measure the respective quadrupolar and octupolar deformations induced by the coupling between $\chi^a$ and $E_{ab}$; and rotational-tidal Love numbers $B^q$ and $K^q$ which measure the respective quadrupolar and octupolar deformations that result from the coupling between $\chi^a$ and $B_{ab}$. All of these Love numbers appear as integration constants in the exterior metric, and must be determined by matching the external solution to the complete internal solution.

3The matching conditions are understood to be continuity of the functions and their first derivatives across $r = R$. The matching conditions are understood to be continuity of the functions and their first derivatives across $r = R$. The matching conditions are understood to be continuity of the functions and their first derivatives across $r = R$. The matching conditions are understood to be continuity of the functions and their first derivatives across $r = R$. The matching conditions are understood to be continuity of the functions and their first derivatives across $r = R$. The matching conditions are understood to be continuity of the functions and their first derivatives across $r = R$. To determine the solution in the interior, one must solve the Einstein field equations for the metric together with the relativistic Euler equation for the matter which makes up the body. In order to model a neutron star, we assume in Sec. 4.2 that the body is made up of a perfect fluid of mass density $\rho$ and pressure $p$ which satisfy a barotropic equation of state $p = p(\rho)$, $\epsilon = \epsilon(\rho)$. Because neutron stars cool rapidly after formation, their temperature is well below the Fermi temperature of their constituent particles, and consequently they are accurately described by a one-parameter zero-temperature equation of state of this kind [70]. We further assume that the equation of state remains unchanged by the tidal perturbation, which implies that the fluid’s vorticity tensor is conserved throughout the evolution of the fluid, in accordance with the differential statement of the circulation theorem [71, 72]. This is the natural fluid state that arises if one assumes that the barotropic body was isolated in the remote past, before the tidal field was switched on adiabatically. We
shall refer to this state as “irrotational” since, at zeroth order in spin, it permits the establishment of vorticity-free internal fluid motions via gravitomagnetic induction [24]. The irrotational state stands in contrast to the “static” state traditionally employed in work on tidal deformations (e.g. Refs. [19,21,22,60,65]), in which the fluid is held in a strict hydrostatic equilibrium that prevents any motion. We discuss the differences between the two fluid states in Sec. 4.4 and in Ch. 4 we will show that a subset of the metric and fluid variables associated with the gravitomagnetic response of a fluid body in the irrotational state must be time dependent, even when $B_{ab}$ is stationary. In contrast, the gravitoelectric response bears no trace of internal dynamics.

In Sec. 4.1 we write down an ansatz for the interior metric, and we determine the field equations which govern it in Sec. 4.3. Then, after specializing the equation of state to the form of a polytrope in Sec. 5.1 we integrate the interior field equations and match the internal solution to its external counterpart across the body’s surface. This procedure specifies the numerical values of the Love numbers; we compute them explicitly for polytropes in Sec. 5.2. We also calculate the perturbed velocity field which arises due to the influence of the tides on the neutron star interior in Sec. 5.3 and specialize to a compact binary setting to estimate the amplitude of the induced tidal currents. Finally, we summarize the main conclusions of this thesis and point out some opportunities for future inquiry in Ch. 6.
CHAPTER 2

TIDAL ENVIRONMENT

2.1 TIDAL MOMENTS

In this chapter, we characterize the tidal environment in which the reference body is immersed, and introduce the irreducible potentials which form the main building blocks of the metric outside a tidally deformed, slowly rotating body. The local construction of the tidal environment metric was first performed by Zhang \[73\]. The methods of Ref. \[73\] were adapted to the context of the tidal deformation problem in the definitive treatment of Poisson & Vlasov \[66\], and these results were specialized to the relevant regime of weak, stationary tides by Poisson \[62\].

We begin by considering a smooth timelike geodesic $\lambda$ in a vacuum region of an arbitrary spacetime; $\lambda$ will become the wordline of the body upon its insertion into the spacetime. The metric in a neighbourhood $\mathcal{N}$ of the geodesic can be expanded in powers of distance away from $\lambda$, and Zhang has shown that it is totally characterized by two sets of tidal multipole moments, defined in terms of the Weyl tensor $C_{\alpha\gamma\beta\delta}$ of the external spacetime evaluated on $\lambda$. The tidal moments reflect the tidal influence of matter (or black holes) far removed from the vacuum region; the Weyl tensor formally depends on the details of the distant mass distribution, but we take it to be arbitrary. The tidal moments are defined in terms of a vectorial basis on the geodesic. Supposing that $\lambda$ is parameterized by proper time $\tau$, and described by parametric relations $z^\alpha(\tau)$ in an arbitrary coordinate system $x^\alpha$, the basis consists of the tangent $u^\alpha := dz^\alpha/d\tau$ to the geodesic, together with an orthonormal triad of vectors $e^\alpha_a(\tau)$. The members of the triad are orthogonal to $u^\alpha$, are parallel-transported along $\lambda$, and are labelled by lowercase latin indices $a, b, ...$ which are raised with the Kronecker delta $\delta^{ab}$, or lowered with its inverse $\delta_{ab}$. Greek indices $\alpha, \beta, ...$ range from zero to three, and are raised and lowered with the spacetime metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$.

Using the basis $(u^\alpha, e^\alpha_a)$, one can define the tidal quadrupole moments

$$E_{ab} := C_{a0b0}, \quad B_{ab} := \frac{1}{2} \epsilon_{acd} C^{cd}_{\ 00}$$

(2.1.1)

to encode the ten algebraically independent components of the Weyl tensor; the quantities

$$C_{a0b0}(\tau) = C_{\alpha\gamma\beta\delta} e^\alpha_a u^\gamma e^\beta_b u^\delta, \quad C_{abcd}(\tau) = C_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d$$

(2.1.2)
are the frame components of the Weyl tensor evaluated on \( \lambda \). Similarly, one can define dual tidal octupole moments \( \mathcal{E}_{abc} \) and \( B_{abc} \) to encode the 24 independent components of the first covariant derivative of \( C_{\alpha\gamma\beta\delta} \) evaluated on \( \lambda \), and higher tidal multipole moments for higher derivatives of \( C_{\alpha\gamma\beta\delta} \). The tidal quadrupole moments scale like \( \mathcal{E}_{ab} \sim M'/b^3 \) and \( B_{ab} \sim M'v/b^3 \), where \( M' \) and \( v \) are respectively the characteristic mass and velocity scales for the tidal field; \( b \) is the distance from \( \lambda \) to the mass distribution giving rise to the tides [66]. The tidal octupole moments are suppressed by a factor of \( 1/b \) relative to the quadrupole moments, and the higher multipole moments are further suppressed. We therefore omit the \( \ell > 2 \) multipole moments from our description of the tidal environment and idealize the tidal field as a pure quadrupole.

The dual tensors \( \mathcal{E}_{ab} \) and \( B_{ab} \) which characterize the tidal field are known respectively as the gravitoelectric and gravitomagnetic tidal quadrupole moments. They are spatially constant but time dependent by Eq. (2.1.1); however, we assume that they vary so slowly that they are effectively stationary. The tidal quadrupole moments are symmetric and tracefree (STF) by virtue of the symmetries of \( C_{\alpha\gamma\beta\delta} \), and they have opposite parities, in the sense that

\[
\mathcal{E}_{ab} \to \mathcal{E}_{ab}, \quad B_{ab} \to -B_{ab}
\]

(2.1.3)

under a parity transformation \( u^\alpha \to u^\alpha, e^\alpha_a \to -e^\alpha_a \) of the frame. Accordingly, the gravitoelectric tidal moment \( \mathcal{E}_{ab} \) is said to have even (or polar) parity, while the gravitomagnetic tidal moment \( B_{ab} \) has odd (or axial) parity.

The metric describing a generic tidal environment in terms of tidal multipole moments has been constructed as a curvature expansion about a timelike geodesic in a vacuum region of an arbitrary spacetime by Poisson & Vlasov, following the methods of Zhang. Here, we specialize Eq. (3.4) of Ref. [66] to stationary quadrupolar tides, and obtain

\[
\begin{align*}
g_{tt} &= -1 - r^2 \mathcal{E}^q, \\
g_{tr} &= 0, \\
g_{rr} &= 1 - r^2 \mathcal{E}^q, \\
g_{tA} &= \frac{2}{3} r^3 B^q_A, \\
g_{rA} &= 0, \\
g_{AB} &= r^2 \Omega_{AB} - r^4 \Omega_{AB} \mathcal{E}^q
\end{align*}
\]

(2.1.4a)(2.1.4b)(2.1.4c)(2.1.4d)(2.1.4e)(2.1.4f)

after a transformation to Minkowski-space spherical coordinates \( (t, r, \theta, \phi) \) and the Regge-Wheeler gauge; these are the coordinates and gauge in which we choose to work throughout this dissertation. The details of the coordinate and gauge transformations are provided in Appendix A. Uppercase latin indices \( A, B, ... \) label the angular variables \( \theta^A = (\theta, \phi) \), and are raised and lowered with the unit two-sphere metric \( \Omega_{AB} := \text{diag}(1, \sin^2 \theta) \) and its inverse \( \Omega^{AB} \). The metric of Eq. (2.1.4) describes the tidal environment in the neighbourhood \( \mathcal{N} \) of the geodesic \( \lambda \), and it depends on tidal potentials \( \mathcal{E}^q \) and \( B^q_A \) formed from the tidal quadrupole moments. The precise definitions of the tidal potentials are introduced below.
2.2 TIDAL POTENTIALS

Tidal potentials constructed from $\mathcal{E}_{ab}$ and $B_{ab}$ appear in the tidal environment metric of Eq. (2.1.4). They will also be implicated in the reference body’s tidal response in Ch. [3] the potentials repack-age the algebraically independent components of $\mathcal{E}_{ab}$ and $B_{ab}$ in irreducible scalar, vector and tensor combinations that are inserted in the metric describing the external spacetime after the body’s placement on the geodesic $\lambda$. The complete set of irreducible potentials linear in $\mathcal{E}_{ab}$ and $B_{ab}$ is constructed in this section, first in Cartesian coordinates $x^a$, and then in spherical coordinates $(r, \theta, \phi)$. The Cartesian coordinates are centred on a point of $\lambda$, and are orthogonal to the geodesic at that point.

We follow the seminal treatment of Poisson & Vlasov [66], and take as basic ingredients of the construction the tidal moments $\mathcal{E}_{ab}$ and $B_{ab}$, as well as the radial unit vector

$$n^a := x^a / r,$$

(2.2.1)

where $r = \sqrt{\delta_{ab} x^a x^b}$ is the Euclidean distance from $\lambda$. We wish to combine these three ingredients into scalar, vector and (rank-2, STF) tensor potentials, each of which is an element of an irreducible representation of the rotation group of multipole order $\ell = 2$. This implies that each potential satisfies an eigenvalue equation – these are listed in Appendix A of Ref. [66]. The (scalar, vector and tensor) potentials must moreover transform appropriately under a parity transformation, and the vector and tensor potentials must be orthogonal to $n^a$.

The quantities which result from the construction are the gravitoelectric potentials

$$E^a := \mathcal{E}_{ab} n^a n^b,$$  
$$E^a_a := \gamma_a b \mathcal{E}_{bc} n^c,$$  
$$E^a_{ab} := 2 \gamma^c_a \gamma^d_b \mathcal{E}_{cd} + \gamma_{ab} E^a $$

(2.2.2)

built from $\mathcal{E}_{ab}$ and $n^a$, and the gravitomagnetic potentials

$$B^a := \epsilon_{acd} n^c B_{bd} n^b,$$  
$$B^a_{ab} := \epsilon_{acd} n^c B_{de} \gamma^e_b + \epsilon_{bcd} n^c B_{de} \gamma^e_a$$

(2.2.3)

built from $B_{ab}$ and $n^a$. The tensor $\gamma^a_b := \delta^a_b - n^a n^b$ is a projector into the subspace transverse to $n_a$.

The conversion of the Cartesian potentials to spherical coordinates is accomplished with the Jacobian matrix $n^a_A := \partial n^a / \partial \theta^A$. In spherical coordinates, the radial unit vector is explicitly $n^a = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]$. The transformed potentials are

$$E^a_A := E^a_{ab} n^a_A n^b,$$  
$$E^a_{AB} := E^a_{ab} n^a_A n^b_B$$

(2.2.4)

and

$$B^a_A := B^a_{ab} n^a_A n^b,$$  
$$B^a_{AB} := B^a_{ab} n^a_A n^b_B$$

(2.2.5)

the scalar potential $\mathcal{E}^a$ is unchanged.

The spherical-coordinate potentials are scalar, vector and tensor fields on the unit two-sphere (they depend only on $\theta^A$), and they admit a convenient decomposition in (scalar, vector and tensor) spherical harmonics. The decomposition of the gravitoelectric potentials involves the usual scalar
\[
Y^{10} = C \\
Y^{11c} = S \cos \phi \\
Y^{11s} = S \sin \phi \\
Y^{20} = 1 - 3C^2 \\
Y^{21c} = 2SC \cos \phi \\
Y^{21s} = 2SC \sin \phi \\
Y^{22c} = S^2 \cos 2\phi \\
Y^{22s} = S^2 \sin 2\phi \\
Y^{30} = C(3 - 5C^2) \\
Y^{31c} = \frac{3}{2}S(1 - 5C^2) \cos \phi \\
Y^{31s} = \frac{3}{2}S(1 - 5C^2) \sin \phi \\
Y^{32c} = 3S^2C \cos 2\phi \\
Y^{32s} = 3S^2C \sin 2\phi \\
Y^{33c} = S^3 \cos 3\phi \\
Y^{33s} = S^3 \sin 3\phi
\]

Table 2.1: Spherical-harmonic functions \(Y^{\ell m}\). The functions are real, and they are listed for the relevant modes \(\ell = 1\) (dipole), \(\ell = 2\) (quadrupole), and \(\ell = 3\) (octupole). The abstract index \(m\) describes the dependence of these functions on the angle \(\phi\); for example, \(Y^{\ell 2s}\) is proportional to \(\sin 2\phi\). To simplify the expressions, we write \(C := \cos \theta\) and \(S := \sin \theta\).

spherical harmonics \(Y^{\ell m}\), as well as the even-parity vector and tensor harmonics

\[
Y_A^{\ell m} := D_A Y^{\ell m}, \quad Y_{AB}^{\ell m} := \left[D_A D_B + \frac{1}{2} \ell(\ell + 1)\Omega_{AB}\right] Y^{\ell m}.
\] (2.2.6)

The decomposition of the gravitomagnetic potentials involves the odd-parity vector and tensor harmonics

\[
X_A^{\ell m} := -\epsilon^B_A D_B Y^{\ell m}, \quad X_{AB}^{\ell m} := -\frac{1}{2} \left(\epsilon_A^C D_B + \epsilon_B^C D_A\right) D_C Y^{\ell m}.
\] (2.2.7)

Here, \(D_A\) is the covariant derivative operator associated with the unit two-sphere metric \(\Omega_{AB}\), and \(\epsilon_{AB}\) is the Levi-Civita tensor on the unit two-sphere (\(\epsilon_{\theta\phi} = \sin \theta\)).

The starting point for the decomposition is the correspondence between spherical harmonics of multipole order \(\ell\) and products of rank-\(\ell\) STF tensors with \(\ell\) copies of the radial unit vector (see e.g. Ch. 1 of Ref. [18]):

\[
\mathcal{E}_{ab}^{m}Y^{2m} = \sum_m \mathcal{E}^{q}_m Y^{2m}, \quad B_{ab}^{m}Y^{2m} = \sum_m B^{q}_m Y^{2m}.
\] (2.2.8)

These identities repackage the five independent components of each tidal quadrupole moment in the spherical-harmonic coefficients \(\mathcal{E}^{q}_m\) and \(B^{q}_m\). The precise values of the coefficients depend on the normalization adopted for the spherical harmonics. We follow the convention of Poisson & Vlasov, which we reproduce in Table 2.1. The corresponding coefficients \(\mathcal{E}^{q}_m\) and \(B^{q}_m\) are listed in Table 2.2. By differentiating the identities of Eq. (2.2.8) in the manner described in Appendix A of Ref. [66], one obtains the spherical-harmonic decompositions of the vector and tensor potentials. The complete set of decompositions is

\[
\mathcal{E}^{q} = \sum_m \mathcal{E}^{q}_m Y^{2m}, \quad \mathcal{E}_A^{q} = \frac{1}{2} \sum_m \mathcal{E}^{q}_m Y_A^{2m}, \quad \mathcal{E}_{AB}^{q} = \sum_m \mathcal{E}^{q}_m Y_{AB}^{2m}
\] (2.2.9a)
the leading-order tidal terms, and can thus be safely neglected. Similarly, the body’s spin angular vector \( \chi \) is small, so that the body is rotating slowly. This assumption implies that couplings of \( \mathcal{E}_m \) to those between \( \mathcal{E}^q_{1m} \) and \( \mathcal{E}_{ab} \). Similarly, the relations between \( \mathcal{K}_m \), \( \kappa_a \) and \( \chi \mathcal{B}^q_m \) are identical to those between \( \mathcal{F}_m \), \( \mathcal{F}_a \) and \( \chi \mathcal{E}^q_m \), the relations between \( \mathcal{B}^q_m \), \( \mathcal{B}_{ab} \) and \( \chi \mathcal{B}^q_m \) are identical to those between \( \mathcal{E}^q_m \), \( \mathcal{E}_{ab} \) and \( \chi \mathcal{E}^q_m \); and the relations between \( \mathcal{K}_m \), \( \kappa_{abc} \) and \( \chi \mathcal{B}^q_m \) are identical to those between \( \mathcal{F}_m \), \( \mathcal{F}_{abc} \) and \( \chi \mathcal{E}^q_m \).

\[
B^q_A = \frac{1}{2} \sum_m B^q_m \chi^{2m}_A, \quad B^q_{AB} = \sum_m B^q_m \chi^{2m}_{AB}.
\]  

(2.2.9b)

### 2.3 Bilinear Moments and Potentials

When the reference body of mass \( M \), radius \( R \) and spin angular momentum tensor \( S^{\alpha\beta} \) is placed on \( \lambda \), the tidal field provokes deformations which are reflected in the metric of the external spacetime. In addition to the linear response which is captured by the tidal-potential terms in the metric, the nonlinearity of the field equations of general relativity implies that the tidal quadrupole moments couple to themselves to produce gravitoelectric, gravitomagnetic and mixed responses of quadratic order. However, quadratic tidal terms in the metric are suppressed by a factor of \( (r/b)^2 \) relative to the leading-order tidal terms, and can thus be safely neglected. Similarly, the body’s spin angular momentum \( S_a := \frac{1}{2} \varepsilon_{apq} \varepsilon_{\mu\nu}^{pq} S^\mu \) couples to the tidal field to contribute spin-coupled tidal terms to the metric. The basic ingredients for these couplings are \( \mathcal{E}_{ab} \), \( \mathcal{B}_{ab} \) and the body’s dimensionless spin vector \( \chi_a = S_a / M^2 \). The magnitude \( \chi \) of the dimensionless spin vector is taken to be numerically small, so that the body is rotating slowly. This assumption implies that couplings of \( \chi_a \) with itself are negligible; only couplings linear in both \( \chi_a \) and the tidal moments are significant. In other words, we work in a regime of slow rotation and weak tides in which the only important nonlinear contributions to the metric are these bilinear couplings.
The body’s slow rotation is incorporated in the metric by way of a dipole vector potential

$$\chi^d = \epsilon_{abc} n^b \chi^c.$$  

(2.3.1)

The rotational potential has odd parity, since

$$\chi_a \rightarrow -\chi_a$$  

(2.3.2)

under a parity transformation. The spherical-coordinate version of the Cartesian expression in Eq. (2.3.1) is

$$\chi^d_A := \chi^d an^a.$$  

(2.3.3)

Spin-coupled tidal terms are incorporated in the metric via irreducible potentials of their own. The bilinear potentials involve STF spin-coupled tidal moments constructed from $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$ and $\chi_a$. The couplings reflect the composition of $\ell = 1$ and $\ell = 2$ spherical harmonics. The coupling of the dipole spin pseudovector and the gravitoelectric tidal quadrupole moment produces the rank-1 and rank-3 odd-parity tensors

$$\mathcal{F}_a := \mathcal{E}_{ab} \chi^b, \quad \mathcal{F}_{abc} := \mathcal{E}_{(ab} \chi^{c)};$$  

(2.3.4)

and the rank-2 even-parity tensor

$$\mathcal{E}_{ab} := 2 \chi^c \epsilon_{cd(a} \mathcal{E}^d_{b)}.$$  

(2.3.5)

The angular brackets in Eq. (2.3.4) indicate the STF operation (symmetrize and remove all traces), and the brackets in Eq. (2.3.5) denote symmetrization. Similarly, the coupling of the dipole spin pseudovector with the gravitomagnetic tidal quadrupole moment produces the rank-1 and rank-3 even-parity tensors

$$\mathcal{K}_a := \mathcal{B}_{ab} \chi^b, \quad \mathcal{K}_{abc} := \mathcal{B}_{(ab} \chi^{c)};$$  

(2.3.6)

and the odd-parity rank-2 tensor

$$\mathcal{B}_{ab} := 2 \chi^c \epsilon_{cd(a} \mathcal{B}^d_{b)}.$$  

(2.3.7)

These spin-coupled tidal moments exhaust all the possible distinct STF combinations of $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$ and $\chi_a$ that are linear in both the spin pseudovector and the tidal moments; accordingly, we shall refer to them as bilinear moments.

The algebraically independent components of the bilinear moments are repackaged in irreducible bilinear potentials for insertion in the metric. The construction of the bilinear potentials follows the method used for the tidal potentials, generalized to $\ell = 1, 2, 3$; details can be found in Refs. [62, 63]. The bilinear gravitoelectric moments $\mathcal{F}_a$, $\mathcal{E}_{ab}$ and $\mathcal{F}_{abc}$ give rise to the odd-parity dipole vector potential

$$\mathcal{F}^d_a := \epsilon_{abc} n^b \mathcal{F}^c;$$  

(2.3.8)
the even-parity quadrupole scalar, vector and tensor potentials

$$\hat{\mathcal{E}}^q := \hat{\mathcal{E}}_{ab} n^a n^b, \quad \hat{\mathcal{E}}_a := \gamma_a \hat{\mathcal{E}}_{bc} n^c, \quad \hat{\mathcal{E}}_{ab} := 2\gamma_a \gamma_b \hat{\mathcal{E}}_{cd} + \gamma_{ab} \hat{\mathcal{E}}^q; \quad (2.3.9)$$

and the odd-parity octupole vector and tensor potentials

$$\mathcal{F}^o := \epsilon_{acd} n^c \mathcal{F}_{bde} n^b n^e, \quad \mathcal{F}_{ab} := (\epsilon_{acd} n^c \mathcal{F}_{de} + \epsilon_{bed} n^d \mathcal{F}_{ae}) n^e. \quad (2.3.10)$$

The bilinear gravitomagnetic moments $\mathcal{K}_{a}$, $\hat{\mathcal{B}}_{ab}$ and $\mathcal{K}_{abc}$ give rise to the even-parity dipole scalar and vector potentials

$$\mathcal{K}^d := \mathcal{K}_a n^a, \quad \mathcal{K}^d_a := \gamma_a \mathcal{K}_b; \quad (2.3.11)$$

the odd-parity quadrupole vector and tensor potentials

$$\mathcal{B}^q := \epsilon_{acd} \mathcal{B}_{bde} n^b n^e, \quad \mathcal{B}_{ab} := (\epsilon_{acd} \mathcal{B}_{de} + \epsilon_{bed} \mathcal{B}_{ae}) n^e; \quad (2.3.12)$$

and the even-parity octupole scalar, vector and tensor potentials

$$\mathcal{K}^o := \mathcal{K}_{abc} n^a n^b n^c, \quad \mathcal{K}^o_a := \gamma_a \mathcal{K}_{bde} n^b n^c, \quad \mathcal{K}^o_{ab} := 2\gamma_a \gamma_b \mathcal{K}_{dec} n^c + \gamma_{ab} \mathcal{K}^o. \quad (2.3.13)$$

The conversion of the bilinear tidal potentials to spherical coordinates is accomplished by multiplication with appropriate factors of the Jacobian matrix $n^A$, as above; e.g.

$$\mathcal{F}^d_a := \mathcal{F}^d_a n^A, \quad \mathcal{B}^q_{AB} := \mathcal{B}^q_{ab} n^A n^B, \quad \mathcal{K}^o_{AB} := \mathcal{K}^o_{ab} n^A n^B, \quad (2.3.14)$$

and so on. The spherical-harmonic decomposition of the potentials is achieved with the STF identities

$$\chi_a n^a = \sum_m \chi^d n^1 Y^1_m, \quad \mathcal{F}^d_a n^a = \sum_m \mathcal{F}^d_m Y^1_m, \quad \mathcal{K}^o_a n^a = \sum_m \mathcal{K}^o_m Y^1_m \quad (2.3.15a)$$

$$\hat{\mathcal{E}}_{ab} n^a n^b = \sum_m \hat{\mathcal{E}}^q_m Y^2_m, \quad \hat{\mathcal{B}}_{ab} n^a n^b = \sum_m \hat{\mathcal{B}}^q_m Y^2_m, \quad (2.3.15b)$$

$$\mathcal{F}_{abc} n^a n^b n^c = \sum_m \mathcal{F}^o_m Y^3_m, \quad \mathcal{K}_{abc} n^a n^b n^c = \sum_m \mathcal{K}^o_m Y^3_m. \quad (2.3.15c)$$

Differentiating these identities in the manner described in Appendix A of Ref. [66], one arrives at the spherical-harmonic decompositions of the vector and tensor potentials. The complete collection of decompositions is

$$\chi^d = \sum_m \chi^d n^1 X^1_A, \quad \mathcal{F}^d = \sum_m \mathcal{F}^d n^1 X^1_A, \quad (2.3.16a)$$

$$\hat{\mathcal{E}}^d = \sum_m \hat{\mathcal{E}}^d n^1 X^1_A, \quad (2.3.16b)$$

14
\[ K^d = \sum_m K^d_{m1}, \quad K^d_A = \sum_m K^d_{m1}Y^1_A, \] (2.3.16c)

\[ \hat{E}^q = \sum_m \hat{E}^q_{m2}, \quad \hat{E}^q_A = \frac{1}{2} \sum_m \hat{E}^q_{m2}Y^2_A, \quad \hat{E}^q_{AB} = \sum_m \hat{E}^q_{m2}Y^2_{AB}, \] (2.3.16d)

\[ \hat{B}^q_A = \frac{1}{2} \sum_m \hat{B}^q_{m2}X^2_A, \quad \hat{B}^q_{AB} = \sum_m \hat{B}^q_{m2}X^2_{AB}, \] (2.3.16e)

\[ F^o_A = \frac{1}{3} \sum_m F^o_{m3}X^3_A, \quad F^o_{AB} = \frac{1}{3} \sum_m F^o_{m3}X^3_{AB}, \] (2.3.16f)

\[ K^o = \sum_m K^o_{m3}, \quad K^o_A = \frac{1}{3} \sum_m K^o_{m3}X^3_A, \quad K^o_{AB} = \frac{1}{3} \sum_m K^o_{m3}X^3_{AB}. \] (2.3.16g)
CHAPTER 3

SPACETIME OUTSIDE A TIDALLY DEFORMED, SLOWLY ROTATING BODY

3.1 EXTERIOR METRIC

The preceding chapter envisioned the tidal environment as a perturbation of the vacuum spacetime near the geodesic $\lambda$ in the absence of the reference body. The tidal environment metric was constructed to reflect the influence of the tidal quadrupole moments $E_{ab}$ and $B_{ab}$. We now place the material body of mass $M$, radius $R$ and dimensionless spin vector $\chi^a$ on $\lambda$, and determine its response to the tidal field\footnote{This work was originally carried out in Paper I with Poisson.}. We suppose that $M \ll b$, which ensures that the body is contained within $\mathcal{N}$, and further implies that it is weakly perturbed by the tides\cite{66}.

The immersion of the reference body in the tidal environment changes the external spacetime in a number of ways. First, to account for the presence of the body, its unperturbed linearized Kerr metric

\[ ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 - 4\chi M^2 r \sin \theta dt d\phi + r^2 d\Omega^2, \tag{3.1.1} \]

must be added to the tidal environment metric of Eq. (2.1.4). Second, the tidal field generated by $E_{ab}$ and $B_{ab}$ disturbs the body, producing a linear tidal response that contributes terms proportional to the tidal potentials to the metric. Third, the body’s spin couples to the tidal moments to produce a bilinear tidal response associated with bilinear-potential terms. To formulate a metric ansatz which accounts for all these contributions, it is helpful to switch perspectives and view the spacetime outside the tidally deformed body as a perturbation $p_{\alpha\beta}$ of the background Schwarzschild metric of an isolated, nonrotating body, rather than a perturbation of the linearized Kerr metric (which is physically the case), or a perturbation of the local Minkowski metric of the original vacuum region (which is formally how the construction is designed). From this practical point of view, Eq. (3.1.1) arises from a dipole perturbation of the...
background Schwarzschild metric
\[ ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2d\Omega^2 \] (3.1.2)
that results in slow, rigid rotation. The rotational perturbation is linear in \( \chi \), and its only nonzero component is
\[ P_{t\phi}^{\text{rotation}} = \frac{2M^2}{r}\chi_{\phi}. \] (3.1.3)

The tidal perturbation \( P_{\alpha\beta}^{\text{tidal}} \) created by \( E_{ab} \) and \( B_{ab} \) is treated as a separate addition to Eq. (3.1.2). It is helpful to decompose \( P_{\alpha\beta}^{\text{tidal}} \) in spherical harmonics; Martel & Poisson [74] have performed the decomposition for generic perturbations of the Schwarzschild metric, and we make use of their results here. The even-parity sector associated with \( Y^{\ell m} \), \( Y^{\ell m}_A \) and \( Y^{\ell m}_{AB} \) has components
\[ P_{tt}^{\text{tidal}} = \sum_{\ell m} h_{tt}^{\ell m} Y^{\ell m}, \] (3.1.4a)
\[ P_{tr}^{\text{tidal}} = \sum_{\ell m} h_{tr}^{\ell m} Y^{\ell m}, \] (3.1.4b)
\[ P_{rr}^{\text{tidal}} = \sum_{\ell m} h_{rr}^{\ell m} Y^{\ell m}, \] (3.1.4c)
\[ P_{tA}^{\text{tidal}} = \sum_{\ell m} j_{t}^{\ell m} Y^{\ell m}_A, \] (3.1.4d)
\[ P_{rA}^{\text{tidal}} = \sum_{\ell m} j_{r}^{\ell m} Y^{\ell m}_A, \] (3.1.4e)
\[ P_{AB}^{\text{tidal}} = r^2 \sum_{\ell m} (K^{\ell m}_A \Omega^{\ell m} + G^{\ell m}_A Y^{\ell m}_{AB}); \] (3.1.4f)

while the nonzero components of the odd-parity sector associated with \( X^{\ell m}_A \) and \( X^{\ell m}_{AB} \) are
\[ P_{tA}^{\text{tidal}} = \sum_{\ell m} h_{t}^{\ell m} X^{\ell m}_A, \] (3.1.5a)
\[ P_{rA}^{\text{tidal}} = \sum_{\ell m} h_{r}^{\ell m} X^{\ell m}_A, \] (3.1.5b)
\[ P_{AB}^{\text{tidal}} = \sum_{\ell m} h_{2}^{\ell m} X^{\ell m}_{AB}. \] (3.1.5c)

Certain degrees of freedom of the perturbation are redundant, and we eliminate this redundancy by fixing the perturbation’s gauge. We impose the Regge-Wheeler gauge conditions of Ref. [75],
\[ j_{t}^{\ell m} = j_{r}^{\ell m} = G_{t}^{\ell m} = 0, \quad h_{2}^{\ell m} = 0 \] (3.1.6)
for \( \ell \geq 2 \), and Eqs. (3.1.4) and (3.1.5) simplify accordingly.

The metric ansatz is populated with the tidal potentials of Sec. 2.2 based on their own spherical-
harmonic decompositions: for example, because \( \mathcal{E}^q \) is related to the scalar harmonics \( Y^{2m}_m \), it appears in the \( tt, tr, rr \) and angular components of the even-parity tidal perturbation; since \( \mathcal{B}^q_{A} \) is related to the vector harmonics \( X^{2m}_{A} \), it appears in the \( tA \) and \( rA \) components of the odd-parity tidal perturbation; and so on. The tidal potentials are purely quadrupolar in nature, so only the \( \ell = 2 \) pieces of Eqs. (3.1.4) and (3.1.5) are relevant. Dimensionless radial functions \( \{ e^q_{tt}, e^q_{tr}, ..., b^q_r \} \) are introduced to encode the spatial dependence of the tidal response. We write

\[
\begin{align*}
 p_{tt}^{\text{tidal}} &= -r^2 e^q_{tt}(r) \mathcal{E}^q \\
 p_{tr}^{\text{tidal}} &= -r^2 e^q_{tr}(r) \mathcal{E}^q \\
 p_{rr}^{\text{tidal}} &= -r^2 e^q_{rr}(r) \mathcal{E}^q \\
 p_{AB}^{\text{tidal}} &= -r^2 e^q_{AB}(r) \Omega_{AB} \mathcal{E}^q
\end{align*}
\] (3.1.7a)

for the nonzero components of the tidal perturbation’s gravitoelectric sector, and

\[
\begin{align*}
 p_{tA}^{\text{bilinear}} &= \frac{2}{3} r^3 b^q_{tA}(r) \mathcal{B}^q_{A}, \\
 p_{rA}^{\text{bilinear}} &= \frac{2}{3} r^3 b^q_{rA}(r) \mathcal{B}^q_{A}
\end{align*}
\] (3.1.8a)

for the gravitomagnetic sector’s nonvanishing components. Appropriate powers of \( r \) have been inserted to give the perturbation the proper dimensions. The radial functions \( \{ e^q_{tt}, e^q_{rr}, b^q_t, e^q \} \) are designed to asymptote to one in the \( r \to \infty \) and \( M \to 0 \) limits, such that the tidal environment metric is recovered far away from the body, and in the test-body limit. In this way, the radial functions encapsulate both the applied tidal field and the tidal response.

The couplings between the body’s spin and the tidal moments produce a bilinear perturbation \( p_{\alpha \beta}^{\text{bilinear}} \). The bilinear tidal response is characterized by the bilinear potentials of Sec. 2.3. They are inserted in the metric in the same manner as the tidal potentials, using their correspondence with spherical harmonics of multipole order \( \ell = 1, 2, 3 \). We continue to express the ansatz in the Regge-Wheeler gauge, with the extension of Campolattaro & Thorne [76] for the dipole terms generated by \( F_a \) and \( K_a \): since neither \( Y^{lm}_{AB} \) and \( G^{lm} \), nor \( X^{lm}_{AB} \) and \( h^{lm}_2 \), are defined for \( \ell = 1 \), we adopt \( K^{1m} = 0 \) and \( h^{1m}_{2} = 0 \) as alternate gauge conditions. The components of the bilinear perturbation that results from the coupling of \( \chi_a \) and \( \mathcal{E}_{ab} \) are

\[
\begin{align*}
 p_{tt}^{\text{bilinear}} &= r^2 e^q_{tt}(r) \mathcal{E}^q \\
 p_{tr}^{\text{bilinear}} &= r^2 e^q_{tr}(r) \mathcal{E}^q \\
 p_{rr}^{\text{bilinear}} &= r^2 e^q_{rr}(r) \mathcal{E}^q \\
 p_{tA}^{\text{bilinear}} &= -r^3 f^d_t(r) \mathcal{F}^d_{A} + r^3 f^o_t(r) \mathcal{F}^o_{A} \\
 p_{rA}^{\text{bilinear}} &= r^3 f^o_r(r) \mathcal{F}^o_{A} \\
 p_{AB}^{\text{bilinear}} &= r^4 e^q_{AB}(r) \Omega_{AB} \mathcal{E}^q
\end{align*}
\] (3.1.9a)

By the same token, the radial functions \( e^q_{tr} \) and \( b^q_r \) tend to zero in the \( r \to \infty \) and \( M \to 0 \) limits.
Here we have introduced new dimensionless radial functions with suitable factors of \( r \). The components arising from the coupling of \( \chi_a \) and \( B_{ab} \) are

\[
p_{bl}^{tt} = r^2 k_d^{t}(r) K^{d} - r^2 k_o^{t}(r) K^{o} \tag{3.1.10a}
\]
\[
p_{bl}^{tr} = r^2 k_d^{t}(r) K^{d} - r^2 k_o^{t}(r) K^{o} \tag{3.1.10b}
\]
\[
p_{bl}^{rr} = r^2 k_d^{r}(r) K^{d} - r^2 k_o^{r}(r) K^{o} \tag{3.1.10c}
\]
\[
p_{bl}^{tA} = -r^3 \hat{b}_q^{t}(r) \hat{B}_A \tag{3.1.10d}
\]
\[
p_{bl}^{rA} = -r^3 \hat{b}_q^{r}(r) \hat{B}_A \tag{3.1.10e}
\]
\[
p_{bl}^{AB} = r^4 k^{o}(r) \Omega_{AB} K^{o}. \tag{3.1.10f}
\]

The radial functions of Eqs. (3.1.9) and (3.1.10) vanish in the \( r \to \infty \) and \( M \to 0 \) limits.

Combining the background metric of Eq. (3.1.1) with the tidal perturbations of Eqs. (3.1.7a) and (3.1.8a), and the bilinear perturbations of Eqs. (3.1.9) and (3.1.10), we obtain a metric ansatz for the spacetime outside a tidally deformed, slowly rotating body:

\[
g_{tt} = -(1 - 2M/r) - r^2 e^{tt} \hat{E}^{tt} + r^2 \hat{e}^{tt} \hat{\hat{E}}^{tt} + r^2 k^{tt} K^{tt} - r^2 k^{tt} K^{o}, \tag{3.1.11a}
\]
\[
g_{tr} = -r^2 e^{tr} \hat{E}^{tr} + r^2 \hat{e}^{tt} \hat{\hat{E}}^{tr} + r^2 k^{tr} K^{tr} - r^2 k^{tt} K^{o}, \tag{3.1.11b}
\]
\[
g_{rr} = (1 - 2M/r)^{-1} - r^2 e^{rr} \hat{E}^{rr} + r^2 \hat{e}^{rr} \hat{\hat{E}}^{rr} + r^2 k^{rr} K^{rr} - r^2 k^{tt} K^{o}, \tag{3.1.11c}
\]
\[
g_{tA} = \frac{2M^2}{r} \chi_A + \frac{2}{3} r^3 \hat{b}_q^{t} \hat{B}_A - r^3 \hat{b}_q^{t} \hat{B}_A - r^3 f_{t} \hat{F}_{A} + r^3 f_{o} \hat{F}_{A}, \tag{3.1.11d}
\]
\[
g_{rA} = \frac{2}{3} r^3 \hat{b}_q^{t} \hat{B}_A - r^3 \hat{b}_q^{r} \hat{B}_A + r^3 f_{o} \hat{F}_{A}, \tag{3.1.11e}
\]
\[
g_{AB} = r^2 \Omega_{AB} - r^4 e^{AB} \Omega_{AB} \hat{E}^{AB} + r^4 \hat{e}^{AB} \Omega_{AB} \hat{\hat{E}}^{AB} + r^4 k^{o} \Omega_{AB} K^{o}. \tag{3.1.11f}
\]

The metric is stationary by virtue of the assumed time independence of the tidal moments, and the hatted potentials vanish when the perturbation is axisymmetric. In the nonrotating limit \( \chi \to 0 \), Eq. (3.1.11) represents the external geometry of a static, tidally deformed body (cf. Refs. [21, 22, 77]). At large distances from the body, the metric is dominated by the tidal field: it asymptotes the tidal environment metric of Eq. (2.1.4). The tidal environment metric is also recovered in the test-body limit \( M \to 0 \).

### 3.2 EXTERIOR FIELD EQUATIONS

The radial functions appearing in the metric of Eq. (3.1.11) are determined by the vacuum field equations

\[
R_{\alpha\beta} = 0. \tag{3.2.1}
\]
We calculate the Ricci tensor $R_{\alpha\beta}$ from the metric ansatz, neglecting time derivatives on account of the assumed stationarity of the tidal moments. The Ricci tensor is then expanded in powers of $\chi, \mathcal{E}_m^q$ and $\mathcal{B}_m^q$. The expansion includes contributions that are linear in $\mathcal{E}_m^q$ or $\mathcal{B}_m^q$, and independent of $\chi$: these are the linearized field equations which govern the tidal perturbation of Eqs. (3.1.7a) and (3.1.8a). The contributions are collected in a spherical-harmonic decomposition and the field equations reduce to a system of coupled differential equations for the radial functions. At this order in the expansion, they decouple into separate $\ell = 2$ gravitoelectric and gravitomagnetic sectors respectively associated with $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$.

The field equations have the schematic form

$$\mathcal{L}^j_k w^k_1 = 0,$$

(3.2.2)

where $\mathcal{L}^j_k$ is a second-order differential operator, and $w^j_1(r)$ is the collection of radial functions that appear in the tidal perturbation of the metric. Integrating Eq. (3.2.2), one obtains a general solution formed from two linearly independent modes – one decaying as $r$ increases, and the other growing. The decaying solution scales like $r^{-(\ell+1)}$ at leading order; it diverges at $r = 2M$ and remains finite as $r \to \infty$. The growing solution scales like $r^\ell$ at leading order; it is regular at $r = 2M$, but diverges as $r \to \infty$. While a solution’s appearance may be altered by a gauge transformation, its identity as a growing or decaying mode is preserved, provided the transformation respects the working assumptions of the perturbation theory.

The growing mode of the tidal perturbation created by $\mathcal{E}_{ab}$ (respectively $\mathcal{B}_{ab}$) represents the external tidal field, and the decaying mode represents the body’s response to this tidal field, measured by the gravitational Love number $K^\text{el}_2$ (respectively $K^\text{mag}_2$), which appears in the solution as an integration constant. The amplitude of the growing mode is fixed by the tidal environment metric of Eq. (2.1.4), to which Eq. (3.1.11) must asymptote in the $r \to \infty$ limit.

The effective gravitational potential,

$$U = \frac{1}{2} (g_{tt} + 1)$$

(3.2.3)

to leading order in $M/r$, is matched to the Newtonian gravitational potential of a tidally deformed, nonrotating body, Eq. (1.1.3). Since there is no Newtonian analogue of the gravitomagnetic field, we adopt the conventional normalization of Binnington & Poisson [22] for $K^\text{mag}_2$.

With these normalizations for the integration constants, the explicit tidal solutions to the exterior field equations are displayed in Table 3.1. The radial functions $c^q_{1r}$ and $b^q_{1r}$, which do not figure in the table, are found to vanish. The specific differential equations which govern the solutions can be deduced from the interior field equations of Sec. 4.3.

The expansion of the Ricci tensor also includes contributions that scale like $\chi \mathcal{E}_m^q$ and $\chi \mathcal{B}_m^q$; higher-order contributions are neglected. The terms proportional to $\chi \mathcal{E}_m^q$ and $\chi \mathcal{B}_m^q$ are collected in a spherical-harmonic decomposition, and the vacuum field equations give rise to a system of coupled differential equations for the remaining radial functions; these equations decouple into gravitoelectric and gravitomagnetic sectors of dipole, quadrupole and octupole order. The field

\[^{3}\text{In principle, there is also an } \ell = 1 \text{ sector associated with the rotational perturbation, but the corresponding field equations are automatically solved by the ansatz of Eq. (3.1.3).}\]

\[^{4}\text{The growing mode of the rotational perturbation is set to zero, because it corresponds to an uninteresting coordinate transformation to a rotating frame.}\]
\[ e_1^q = \left(1 - \frac{1}{x}\right)^2 + \frac{2}{x^2} \left[-30x^3(x-1)^2 \log (1 - 1/x) - \frac{5}{2} x (2x-1)(6x^2 - 6x - 1)\right] K^\text{el}_2 \]
\[ e_1^{q r} = \left(1 - \frac{1}{x}\right)^{-2} e_1^q \]
\[ e_0^q = 1 - \frac{1}{2x} \left[\frac{15}{32} \left[20x^4 - 40x^2 + 140x + 3\right] \log (1 - 1/x) \right] K^\text{mag}_2 + \frac{3}{2x^2 - 49x + 38} K^\text{el}_2 \]
\[ e_0^{q r} = \left(1 - \frac{1}{x}\right)^{-2} e_0^q \]
\[ e_0^q = \frac{2}{x} \left[-15x^3(2x^2 - 1) \log (1 - 1/x) - 5x^2(6x^2 + 3x - 1)\right] (e^q - 1/120) \]

\[ b_1^q = \left(1 - 1/x\right) - \frac{3}{2x^2} \left[20x^4(x-1) \log (1 - 1/x) + \frac{5}{2} x (12x^3 - 6x^2 - 2x - 1)\right] K^\text{mag}_2 \]
\[ e_1^q = \frac{2}{x^2} \left[-30x^5(x-1)^2 \log (1 - 1/x) - \frac{5}{2} x (2x-1)(6x^2 - 6x - 1)\right] (e^q - 1/120) \]
\[ e_1^{q r} = \left(1 - \frac{1}{x}\right)^{-2} e_1^q \]
\[ e_1^q = \frac{2}{x} \left[-15x^3(2x^2 - 1) \log (1 - 1/x) - 5x^2(6x^2 + 3x - 1)\right] (e^q - 1/120) \]

\[ b_0^q = \frac{3}{4x^2} \left[\frac{20}{9} \frac{x^3(2x-1)}{x-1} \log (1 - 1/x) - \frac{10}{9} \frac{x^3(12x^3 + 1)}{x-1}\right] K^\text{mag}_2 + \frac{1}{12} \frac{2x-1}{x^2(x-1)^2} \]

\[ k_1^q = \left(1 - \frac{1}{x}\right)^{-2} \left[\frac{15}{4} \frac{(20x - 9)(x-1)x^4 \log (1 - 1/x)}{K^\text{mag}_2} \right] + \frac{3}{20} \frac{x^2}{x(x-1)^3} \left[240x^5 - 468x^4 + 242x^3 - 16x^2 - x + 2\right] K^\text{mag}_2 + \frac{3}{x(x-1)^3} \left[20x^2 - 49x + 38\right] K^\text{el}_2 \]

\[ k_1^{q r} = \left(1 - \frac{1}{x}\right)^{-2} k_1^q \]

\[ k_0^q = \left(1 - \frac{1}{x}\right)^{-2} \left[\frac{10}{7} \frac{x^6}{x-1} \log (1 - 1/x) \right] K^\text{mag}_2 + \frac{3}{5} \frac{x^4}{x^3(x-1)} \left[3360x^7 - 7670x^6 + 3640x^5 + 268x^4 - 34x^3 - 2x^2 + 3x - 5\right] K^\text{mag}_2 \]

\[ k_0^{q r} = \left(1 - \frac{1}{x}\right)^{-2} k_0^q \]

\[ k_0^q = \left(1 - \frac{1}{x}\right)^{-2} \left[\frac{10}{7} \frac{x^6}{x-1} \log (1 - 1/x) \right] K^\text{mag}_2 + \frac{3}{5} \frac{x^4}{x^3(x-1)} \left[3360x^7 - 7670x^6 + 3640x^5 + 268x^4 - 34x^3 - 2x^2 + 3x - 5\right] K^\text{mag}_2 \]

\[ k_0^{q r} = \left(1 - \frac{1}{x}\right)^{-2} k_0^q \]

\[ k_0^q = \left(1 - \frac{1}{x}\right)^{-2} \left[\frac{10}{7} \frac{x^6}{x-1} \log (1 - 1/x) \right] K^\text{mag}_2 + \frac{3}{5} \frac{x^4}{x^3(x-1)} \left[3360x^7 - 7670x^6 + 3640x^5 + 268x^4 - 34x^3 - 2x^2 + 3x - 5\right] K^\text{mag}_2 \]

\[ k_0^{q r} = \left(1 - \frac{1}{x}\right)^{-2} k_0^q \]

\[ k_0^q = \left(1 - \frac{1}{x}\right)^{-2} \left[\frac{10}{7} \frac{x^6}{x-1} \log (1 - 1/x) \right] K^\text{mag}_2 + \frac{3}{5} \frac{x^4}{x^3(x-1)} \left[3360x^7 - 7670x^6 + 3640x^5 + 268x^4 - 34x^3 - 2x^2 + 3x - 5\right] K^\text{mag}_2 \]

\[ f_1^q = \frac{1}{2} \left[\frac{1}{x^2} \left[-84x^4(10x^3 - 10x^2 + 1) \log (1 - 1/x) - 14x^3(60x^3 - 30x^2 - 10x + 1)\right] K^\text{el}_2 \right] + \frac{1}{12} \frac{2x-1}{x^2} \]

\[ f_1^{q r} = \frac{1}{2} \left[\frac{1}{x^2} \left[-84x^4(10x^3 - 10x^2 + 1) \log (1 - 1/x) - 14x^3(60x^3 - 30x^2 - 10x + 1)\right] K^\text{el}_2 \right] + \frac{1}{12} \frac{2x-1}{x^2} \]

\[ f_1^q = \frac{1}{2} \left[\frac{1}{x^2} \left[-84x^4(10x^3 - 10x^2 + 1) \log (1 - 1/x) - 14x^3(60x^3 - 30x^2 - 10x + 1)\right] K^\text{el}_2 \right] + \frac{1}{12} \frac{2x-1}{x^2} \]

\[ f_1^{q r} = \frac{1}{2} \left[\frac{1}{x^2} \left[-84x^4(10x^3 - 10x^2 + 1) \log (1 - 1/x) - 14x^3(60x^3 - 30x^2 - 10x + 1)\right] K^\text{el}_2 \right] + \frac{1}{12} \frac{2x-1}{x^2} \]
equations governing the bilinear perturbation have the schematic form

\[ \mathcal{L}^j_k \mathbf{w}_2^k = S^j(\mathbf{w}_1^k), \]  

(3.2.4)

where \( \mathbf{w}_2^j \) is the collection of radial functions that appear in the bilinear perturbation, \( S^j \) is a set of source terms constructed from the previously determined radial functions \( \mathbf{w}_1^j \), and \( \mathcal{L}^j_k \) is the same differential operator as in Eq. (3.2.2). The general solution to Eq. (3.2.4),

\[ \mathbf{w}_2^j = \mathbf{w}_2^j[\text{particular}] + \mathbf{w}_2^j[\text{decaying}] + \mathbf{w}_2^j[\text{growing}], \]  

(3.2.5)

is a linear superposition of a particular solution to the system of differential equations, a decaying solution to the homogeneous system \( \mathcal{L}^j_k \mathbf{w}_2^k = 0 \), and a growing solution to the same homogeneous system. The explicit solutions to Eq. (3.2.1) are given in Table 3.1. The radial functions \( \mathbf{k}_d^\text{tr}, \mathbf{k}_o^\text{tr} \) and \( \mathbf{f}_o^\text{r} \) do not figure in the table because they are found to vanish by virtue of the field equations.

The integration constants associated with the decaying solutions measure the strength of the body’s response to the bilinear moments \( \hat{\mathbf{E}}_{ab}, \hat{\mathbf{F}}_{abc}, \hat{\mathbf{B}}_{ab} \) and \( \hat{\mathbf{K}}_{abc} \): they are called rotational-tidal Love numbers, and are denoted by \( \mathbf{E}^q, \mathbf{F}^o, \mathbf{B}^q \) and \( \mathbf{K}^o \), respectively. There are no rotational-tidal Love numbers associated with the bilinear moments \( \mathbf{F}_a \) and \( \mathbf{K}_a \), since the corresponding dipolar responses can be eliminated via a shift to the body’s centre-of-mass frame [18]. The precise normalizations of the rotational-tidal Love numbers are a matter of convention; unlike the case of the gravitoelectric Love number, we cannot appeal to the Newtonian limit, as the body’s spin does not couple to the external tidal field in Newtonian theory. Nonetheless, provided that consistent definitions for the Love numbers are used inside and outside the body, the exterior metric is unaffected by the choices of normalization. In this thesis, we retain the definitions of the rotational-tidal Love numbers introduced in Paper I. This explains the curious factors of \((\mathbf{E}^q - 1/120)\) and \((\mathbf{B}^q - 1/120)\) which appear in Table 3.1: the normalization choices of Paper I were formulated in the lightcone gauge [78], and these shifts are required to ensure that the solutions of Table 3.1 can be recovered from those of Table III of Paper I after a coordinate and gauge transformation of the metric. The amplitude of the growing solutions is set to zero to agree with the tidal environment metric of Eq. (2.1.4) far away from the body; the meaning of the growing solutions is nevertheless explained in Sec. II of Paper I. The particular solutions depend on gauge constants \( c^d \) and \( \gamma^d \) which serve to specify the residual freedom of the \( \ell = 1 \) extension to the Regge-Wheeler gauge [76]; for example, Poisson has shown in Ref. [62] that \( \gamma^d \) is associated with a small precession of the spatial coordinates around the body’s rotation axis.

With the solutions of Table 3.1 for the radial functions, Eq. (3.1.11) provides a complete description of the spacetime outside a slowly rotating body deformed by stationary tides to leading order in the tidal interaction. The metric is universal up to the Love numbers \( K^s_2, K^\text{mag}_2, \mathbf{E}^q, \mathbf{B}^q, \mathbf{F}^o, \) and \( \mathbf{K}^o \), which encode the dependence of the body’s response on its internal structure. This means that the metric for a black hole differs from that of a material body, such as a neutron star, only by the values of its Love numbers. Indeed, we note that Eq. (3.1.11) reduces to the metric of a tidally deformed, slowly rotating black hole, as constructed by Poisson [62], when...
We therefore conclude that a black hole has vanishing gravitational and rotational-tidal Love numbers. This result is consistent with the findings of Refs. [21, 22], which determined that the gravitational Love numbers are zero for nonrotating black holes, and has been confirmed in the rotating case by Ref. [60] (and by Ref. [79]'s nonperturbative calculation). For a material body, the Love numbers depend on the equation of state, and must be computed by matching the exterior metric to the internal solution at the body’s surface. This task involves formulating and solving the field equations inside the body; we undertake this calculation in the next chapter.

\[ K_2^q = K_2^{\text{mag}} = \mathcal{C}^q = \mathcal{B}^q = \mathcal{R}^q = \mathcal{R}^q = 0 \]

\footnote{A direct comparison will necessitate transforming Ref. [62]'s metric from the lightcone gauge [78] to the Regge-Wheeler gauge. The required transformation is identical to the one described in Appendix A, with the added coordinate change \( d\phi[\text{LC}] = d\phi[\text{RW}] + 2(1 - 2M/r)^{-1}(\chi M^2/r^3)dr \).}
CHAPTER 4

SPACETIME INSIDE A TIDALLY DEFORMED, SLOWLY ROTATING BODY

4.1 INTERIOR METRIC

The construction of the exterior metric demonstrated that the deformed spacetime outside a slowly rotating body subject to a quadrupolar tidal field is described by a set of six Love numbers, \( \{K_2^a, K_2^m, \beta^a, \zeta, \eta, \gamma\} \), which depend on the body’s equation of state. To compute them, it is necessary to solve the interior field equations for the metric inside the body, and we undertake the first step in this task by specifying an ansatz for the interior metric in this section.

We focus on the interior region \( r < R \) of the material body which was placed on the wordline \( \lambda \) in Ch. 3. Its dimensionless spin \( \chi \ll 1 \) is related to the rotational angular velocity \( \Omega \) measured by an observer at rest at infinity according to

\[
\Omega = \frac{\chi M^2}{I},
\]

where \( I \) is the body’s moment of inertia. We find it convenient to work in terms of \( \Omega \), rather than \( \chi \), in the interior.

We suppose that the body consists of a perfect fluid of pressure \( p \) and total energy density

\[
\mu = \rho + \epsilon,
\]

the sum of mass density \( \rho \) and internal energy density \( \epsilon \). The fluid’s equation of state is taken to be barotropic, so that \( p = p(\rho) \) and \( \epsilon = \epsilon(\rho) \). To construct the interior metric, we adopt the same practical viewpoint as in Sec. 3.1 and build it up from successive perturbations of the background interior Schwarzschild metric

\[
ds^2 = -e^{2\psi}dt^2 + f^{-1}dr^2 + r^2d\Omega^2
\]

of an isolated, nonrotating body in hydrostatic equilibrium. The functions \( f := 1 - 2m(r)/r \) and

\[\text{This work, as well as the work described in Secs. 4.2 and 4.3 was originally carried out in Paper II with Poisson and in Paper III.}\]
\( \psi(r) \) are determined by the Einstein field equations

\[
\frac{dm}{dr} = 4\pi r^2 \mu, \quad \frac{d\psi}{dr} = \frac{m + 4\pi r^3 p}{r^2 f}.
\] (4.1.4)

The fluid’s pressure \( p \) satisfies the condition of hydrostatic equilibrium

\[
\frac{dp}{dr} = -\frac{(\mu + p)(m + 4\pi r^3 p)}{r^2 f}.
\] (4.1.5)

The interior metric matches on to the exterior Schwarzschild solution at the body’s surface \( r = R \), where \( p = 0 \). In particular, this means that \( m(R) = M \) and \( \psi(R) = \frac{1}{2} \log (1 - 2M/R) \).

The body’s slow, rigid rotation is added as a linear, dipole perturbation of the Schwarzschild metric; its only nonvanishing component is

\[
\mu_{\text{rotation}} = (1 - \omega)r^2 \chi^d_{\phi}.
\] (4.1.6)

Here, and throughout this chapter, \( \chi^d_{\phi} \) is the same odd-parity rotational potential as in Eq. (2.3.3), except that we construct it with \( \Omega \) instead of \( \chi \). It therefore differs from its external version by a factor of \( \chi/\Omega = I/M^2 \). The function \( \omega(r) \) satisfies the field equation

\[
2r^2 f \frac{d^2 \omega}{dr^2} + \left[ 4f - 4\pi r^2 (\mu + p) \right] r \frac{d\omega}{dr} - 16\pi r^2 (\mu + p) \omega = 0
\] (4.1.7)

and matches on to the exterior solution \( \omega_{\text{ext}} = 1 - 2I/r^3 \) at \( r = R \), which ensures that Eq. (4.1.6) agrees with Eq. (3.1.3) \cite{71}.

The quadrupolar tidal field disturbs the background configuration described by Eq. (4.1.3). The linear tidal response to \( E_{ab} \) and \( B_{ab} \) contributes tidal-potential terms to the perturbed metric, while the bilinear tidal response associated with \( F_a, \hat{E}_{ab}, F_{abc}, \mathring{K}_a, \hat{B}_{ab}, K_{abc} \) contributes bilinear-potential terms. We continue to work in the Regge-Wheeler gauge, and consequently the interior metric perturbations take the same form as their external counterparts in Eqs. (3.1.7a), (3.1.8a), (3.1.9a) and (3.1.10a). The radial functions inserted in the interior metric satisfy regularity conditions at \( r = 0 \) and agree with their external versions at \( r = R \); however, we simplify this chapter’s notation by absorbing the prefactors which appear in Eq. (3.1.11) directly into the radial functions themselves. These explicit numerical factors and powers of \( r \) facilitated the asymptotic matching of the external solution to the tidal environment metric, but are of limited use for the interior radial functions, which need not be dimensionless. Hence, we write, for instance,

\[
e^{q}_{tt}\text{[here]} = -r^2 e^{q}_{tt}\text{[ext]}, \quad b^{q}_{r}\text{[here]} = \frac{2}{3} r^3 b^{q}_{r}\text{[ext]}, \quad k^{o}\text{[here]} = r^4 k^{o}\text{[ext]},
\] (4.1.8)

and so on. With this change of notation, the interior metric ansatz is written as

\[
g_{tt} = -e^{2\psi} + e^{q}_{tt}(r) E^{q} + \hat{e}^{q}_{tt}(r) \hat{E}^{q} + k^{d}_{tt}(t,r) K^{d} + k^{o}_{tt}(t,r) K^{o},
\] (4.1.9a)

\[
g_{tr} = e^{q}_{tr}(r) E^{q} + \hat{e}^{q}_{tr}(r) \hat{E}^{q} + k^{d}_{tr}(t,r) K^{d} + k^{o}_{tr}(t,r) K^{o},
\] (4.1.9b)

\footnote{The regularity and matching conditions are understood to apply both to the functions and to their first derivatives.}
\[ g_{rr} = f^{-1} + e_{rr}^q(r)\mathcal{E}_q + \hat{e}_{rr}^q(r)\hat{\mathcal{E}}_q + k_{rr}^d(t, r)\mathcal{K}_q^d + k_{rr}^o(t, r)\mathcal{K}_o, \]  
\[ g_{tA} = (1 - \omega)r^2\mathcal{A}_A^d + \hat{b}_t^d(r, t, r)\hat{\mathcal{A}}_A^d + f_t^d(r)\mathcal{F}_A^d + f_t^o(r)\mathcal{F}_A^o, \]  
\[ g_{rA} = b_t^o(r)\mathcal{B}_A^q + \hat{b}_t^o(t, r)\hat{\mathcal{B}}_A^q + f_r^q(r)\mathcal{F}_A^q, \]  
\[ g_{AB} = r^2\Omega_{AB} + e^q(r)\Omega_{AB}\mathcal{E}_q + \hat{e}^q(r)\hat{\Omega}_{AB}\hat{\mathcal{E}}_q + k^o(t, r)\Omega_{AB}\mathcal{K}_o. \]  

The bilinear potentials appearing in Eq. (4.1.9) are constructed with \( \Omega \) rather than \( \chi \), and they consequently differ from their external versions by a factor of \( \chi/\Omega = I/M^2 \). Because the tidal potentials are independent of \( \chi \), they remain unchanged.

The exterior metric ansatz of Eq. (3.1.11) was taken to be stationary in accordance with the assumed time independence of the tidal moments. While it would be natural to make the same assumption about the interior metric, we find that a completely time-independent internal solution is remarkably incompatible with the Einstein field equations (cf. Refs. [60, 61]). In the following section, we demonstrate that the fluid variables associated with bilinear gravitomagnetic perturbations generically acquire a time dependence as a result of the fluid equations; the rest of the fluid variables remain time independent. We therefore allow the interior radial functions which make up the bilinear gravitomagnetic sector, namely \( \{k_t^d, k_t^o, k_r^d, k_r^o, \hat{b}_t^q, \hat{b}_r^q, k_o \} \), to depend on both \( t \) and \( r \).

### 4.2 Energy-momentum tensor

#### 4.2.1 Background

The radial functions appearing in the interior metric of Eq. (4.1.9) are governed by the Einstein field equations

\[ G_{\alpha\beta} = 8\pi T_{\alpha\beta}. \]  

In this section, we construct the energy-momentum tensor \( T_{\alpha\beta} \) for the matter making up the slowly rotating reference body. Because we take the body to consist of a perfect fluid, the energy-momentum tensor for the unperturbed equilibrium configuration is

\[ T_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + p g_{\alpha\beta}. \]  

We consider the body’s spin to be part of the background configuration in this section. The slow, rigid rotation is described by the velocity vector

\[ u^\alpha = u^t(t^\alpha + \Omega^\phi^\alpha), \]  

where \( t^\alpha \) and \( \phi^\alpha \) are respectively the time-translation and rotational Killing vectors for the spacetime [71]. In the Boyer-Lindquist coordinates \( (t, r, \theta, \phi) \) we employ, they are simply \( t^\alpha = [1, 0, 0, 0] \) and \( \phi^\alpha = [0, 0, 0, 1] \). The time component \( u^t = e^{-\psi} \) of Eq. (4.2.3) is determined by normalizing the velocity vector so that \( g_{\alpha\beta}u^\alpha u^\beta = -1 \). The covariant version of Eq. (4.2.3), \( u_\alpha = g_{\alpha\beta}u^\beta \), has
components
\[ u_t = -e^\psi, \quad u_A = -e^\psi r^2 \omega_A, \quad (4.2.4) \]
expressed in terms of the rotational potential \( \chi_A \). Because of the rotation, the background configuration has nonzero vorticity, encoded by the vorticity tensor
\[ \omega_{\alpha\beta} := \nabla_\alpha (hu_\beta) - \nabla_\beta (hu_\alpha), \quad (4.2.5) \]
where \( h : = (\mu + p)/\rho \) is the fluid’s specific enthalpy. The nonvanishing components of Eq. (4.2.5) are
\[ \omega_{rA} = -re^{-\psi} h \left[ r \frac{d\omega}{dr} - 2\omega \left( r \frac{d\psi}{dr} - 1 \right) \right] \chi_A, \quad \omega_{AB} = -2r^2 e^{-\psi} h \omega \left( D_A \chi_B - D_B \chi_A \right). \quad (4.2.6) \]

The energy-momentum tensor must satisfy the conservation law
\[ \nabla_\alpha T^{\alpha\beta} = 0, \quad (4.2.7) \]
as well as the statement of baryon conservation
\[ \nabla_\alpha (\rho u^\alpha) = 0. \quad (4.2.8) \]
As shown in Ref. [70], the projection of Eq. (4.2.7) in the direction of \( u^\alpha \) produces the first law of thermodynamics
\[ (\mu + p) \nabla_\alpha u^\alpha + u^\alpha \nabla_\alpha \mu = 0, \quad (4.2.9) \]
which gives rise to
\[ d\mu = h d\rho \quad (4.2.10) \]
for a barotropic fluid when Eq. (4.2.8) holds. From Eq. (4.2.10), the definition of \( h \) and a first integral of Eq. (4.1.5), one obtains the explicit expression
\[ h = (1 - 2M/R)^{1/2} e^{-\psi} \quad (4.2.11) \]
for the specific enthalpy.

The projection of Eq. (4.2.7) in the direction orthogonal to \( u^\alpha \) produces the relativistic Euler equations
\[ (\mu + p) u^\beta \nabla_\beta u_\alpha = - \left( \delta_\alpha^\beta + u_\alpha u^\beta \right) \nabla_\beta p. \quad (4.2.12) \]
Using the definitions of the enthalpy and the vorticity tensor to re-express Eq. (4.2.12) as a condition on \( \omega_{\alpha\beta} \), one arrives at the statement of vorticity conservation
\[ \mathcal{L}_u \omega_{\alpha\beta} = 0, \quad (4.2.13) \]
where \( \mathcal{L}_u \) denotes a Lie derivative in the direction of \( u^\alpha \). Eq. (4.2.13) says that \( \omega_{\alpha\beta} \) is conserved.
along the fluid worldlines; this is the differential form of Synge’s circulation theorem \cite{synge1946} (see e.g. Ref. [71] for a derivation). Hence, to satisfy the fluid equations, a fluid with specific enthalpy of the form of Eq. (4.2.11) need only satisfy Eq. (4.2.7). A simple calculation reveals that $\mathcal{L}_u \omega_{\alpha\beta} = 0$ for the unperturbed fluid.

### 4.2.2 Perturbed fluid

The equilibrium configuration described in the preceding section is disturbed by the influence of the tidal and bilinear moments. We give the fluid disturbance a Lagrangian formulation, as summarized in e.g. Sec. 2.2 of Ref. \cite{ref1}. The Lagrangian formulation views the fluid at the microscopic level: under the action of the perturbation, fluid elements are displaced by $\xi^\alpha$, and the Lagrangian change $\Delta Q := Q(x^\alpha + \xi^\alpha) - Q_0(x^\alpha)$ in a scalar fluid variable $Q$ compares a given fluid element before and after the perturbation. This stands in contrast with the Eulerian formulation, which takes the macroscopic perspective: the Eulerian change $\delta Q := Q(x^\alpha) - Q_0(x^\alpha)$ compares $Q$ to its unperturbed value $Q_0$ at the same location in spacetime (thereby comparing different fluid elements). The two types of perturbations are related by $\Delta Q = \delta Q + \mathcal{L}_\xi Q$, which also serves to define the Lagrangian perturbation for tensors.

We continue to assume that the fluid is barotropic, and we suppose that the perturbed fluid’s one-parameter equation of state is the same as the unperturbed fluid’s. The perturbation of the body’s energy-momentum tensor takes the form

$$\delta T_{\alpha\beta} = (\delta \mu + \delta p) u_\alpha u_\beta + 2(\mu + p) \delta u_{(\alpha} u_{\beta)} + \delta p g_{\alpha\beta} + p g_{\alpha\beta},$$

(4.2.14)

and the perturbed fluid must continue to satisfy the conservation equations. Taking the Lagrangian perturbation of Eq. (4.2.13), the condition becomes \cite{ref2}

$$\mathcal{L}_u \Delta \omega_{\alpha\beta} = 0,$$

(4.2.15)

the statement that $\Delta \omega_{\alpha\beta}$ is conserved along the fluid worldlines. To implement this condition, we imagine that despite the assumption of stationarity, the tidal perturbation was switched on adiabatically in the remote past, so that the fluid began in an unperturbed, slowly rotating state. In this initial state $\Delta \omega_{\alpha\beta} = 0$, and the conservation statement implies that $\Delta \omega_{\alpha\beta}$ continues to vanish on each worldline. Conservation of vorticity therefore guarantees that

$$\Delta \omega_{\alpha\beta} = 0$$

(4.2.16)

at any time throughout the fluid. The vorticity conservation condition has far-reaching consequences for the fluid variables, which decouple by parity into separate gravitoelectric and gravitomagnetic sectors.

**Gravitoelectric sector**

As was done for the metric perturbations in Sec. \cite{ref3}, we decompose the fluid variables in scalar, vector and tensor spherical harmonics, and then translate the spherical harmonics into irreducible potentials using the identities of Eqs. (2.2.9) and (2.3.16). We begin our treatment of the fluid
perturbations by giving the disturbances associated with $E_{ab}$, $F_{a}$, $\hat{E}_{ab}$ and $F_{abc}$ a Lagrangian formulation.

The Lagrangian displacement vector $\xi_{\alpha}$ describes how the fluid elements are translated by the perturbation. It is decomposed as

$$\xi_{r} = \xi_{r}^{q}(t, r) E^{q} + \hat{\xi}_{r}^{q}(t, r) \hat{E}^{q}, \tag{4.2.17a}$$
$$\xi_{A} = \xi_{A}^{q}(t, r) E^{q}_{A} + \hat{\xi}_{A}^{q}(t, r) \hat{E}^{q}_{A} + \xi_{o}^{q}(t, r) F^{o}_{A}, \tag{4.2.17b}$$

in terms of the irreducible potentials of Ch. 2; the time component of $\xi_{\alpha}$ is irrelevant for our purposes. The Eulerian perturbation of the velocity vector is expressed as

$$\delta u_{r} = v_{r}^{q}(r) E^{q} + \hat{v}_{r}^{q}(t, r) \hat{E}^{q}, \tag{4.2.18a}$$
$$\delta u_{A} = v_{A}^{q}(r) E^{q}_{A} + v_{d}^{q}(t, r) F^{d}_{A} + \hat{v}_{A}^{q}(t, r) \hat{E}^{q}_{A} + v_{o}^{q}(t, r) F^{o}_{A}; \tag{4.2.18b}$$

$\delta u_{t}$ can be related to the other components by properly normalizing the perturbed velocity vector. The Eulerian perturbation of the pressure is decomposed as

$$\delta p = p^{q}(r) E^{q} + \hat{p}^{q}(t, r) \hat{E}^{q}, \tag{4.2.19}$$

and the perturbations in energy density $\mu$ and specific enthalpy $h$ are given by $\delta \mu = (d\mu/d\rho)\delta \rho$ and $\delta h = h(\mu + p)^{-1}\delta \rho$ on account of the barotropic assumption.

We compute the Lagrangian perturbation of the vorticity tensor from its Eulerian perturbation

$$\delta \omega_{\alpha\beta} = \partial_{\alpha} (\delta h u_{\beta} + h \delta u_{\beta}) - \partial_{\beta} (\delta h u_{\alpha} + h \delta u_{\alpha}), \tag{4.2.20}$$

and obtain a number of assignments for the fluid variables. The $tA$ components of Eq. (4.2.16) produce

$$p^{q} = \frac{1}{2} e^{-2\psi} (\mu + p) \epsilon_{tt}^{q}, \tag{4.2.21a}$$
$$\hat{p}^{q} = \frac{1}{2} e^{-\psi} (\mu + p) (e^{\psi} \epsilon_{tt}^{q} + v^{q}). \tag{4.2.21b}$$

The $rA$ components then yield

$$v_{r}^{q} = \frac{1}{2} \left( \frac{dv_{r}^{q}}{dr} - \frac{m + 4\pi r^{3} p}{r^{2} f} v^{q} \right), \tag{4.2.22}$$

as well as an algebraic equation relating $\hat{v}_{r}^{q}$ to $\hat{v}^{q}$, $\xi^{q}$ and $\xi_{r}^{q}$. However, Eq. (4.2.16) does not provide enough information to specify all four of these fluid variables. Consequently, we leave the determination of $\hat{v}_{r}^{q}$ and $\hat{v}^{q}$ up to Sec. 4.3's analysis of the Einstein field equations. Similarly, the angular components of the vorticity conservation condition fail to produce a definite assignment for $v^{d}$ and $v^{o}$, and the remaining $tr$ component is redundant.

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3In fact, we will show that $\hat{v}_{r}^{q}$ and $\hat{v}^{q}$ are left undetermined by the field equations, and that this corresponds to the freedom to add a $g$-mode to the interior solution.
Gravitomagnetic sector

We switch our focus to the gravitomagnetic sector, and give the disturbances created by $B_{ab}$, $K_{a}$, $\hat{B}_{ab}$ and $K_{abc}$ a Lagrangian formulation. The displacement vector $\xi_{\alpha}$ is decomposed as

$$
\xi_r = \xi^d_r(t, r)K^d + \xi^o_r(t, r)K^o, \tag{4.2.23a}
$$
$$
\xi_A = \xi^q(t, r)B^q_A + \xi^d(t, r)K^d_A + \hat{\xi}^q(t, r)\hat{B}^q_A + \xi^o(t, r)K^o_A, \tag{4.2.23b}
$$

and the Eulerian perturbation of the velocity vector is decomposed as

$$
\delta u_r = v^d_r(t, r)K^d + v^o_r(t, r)K^o, \tag{4.2.24a}
$$
$$
\delta u_A = v^q(r)B^q_A + v^d(t, r)K^d_A + \hat{v}^q(t, r)\hat{B}^q_A + v^o(t, r)K^o_A; \tag{4.2.24b}
$$

the time component of $\xi_{\alpha}$ plays no role in the discussion, and $\delta u_t$ can be related to the other components by properly normalizing the perturbed velocity vector. The Eulerian pressure perturbation is expressed as

$$
\delta p = p^d(t, r)K^d + p^o(t, r)K^o. \tag{4.2.25}
$$

The various expansion coefficients of the Eulerian velocity perturbation are related to the displacement vector via the relation $\delta u_{\alpha} = \Delta u_{\alpha} - L\xi u_{\alpha}$; using the identity

$$
\Delta g_{\alpha\beta} = p_{\alpha\beta} + \nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha},
$$

we calculate

$$
v^q = e^{-\psi}(\partial_t\xi^q + b^q_t). \tag{4.2.27}
$$

We shall not have need for explicit relations among the other components of $\delta u_{\alpha}$ and $\xi_{\alpha}$; nonetheless, they can be found in Sec. III of Paper II.

The perturbed fluid variables of the gravitomagnetic sector are also constrained by Eq. (4.2.16). The angular components of the equation imply that

$$
\hat{v}^q = 0. \tag{4.2.28}
$$

This is the piece of $\delta u_A$ associated with the tidal perturbation of a nonrotating body, and the vorticity constraint implies the existence of internal motions within the fluid. These motions are described by $\delta u_A = -r^{-2}e^{-\psi}b^q(t)\Omega^{AB}\hat{B}^B_{pr}$, and they are gradually established as the tidal field is adiabatically switched on. From Eq. (4.2.27), we find next that $\xi^q = -\hat{b}^q(t)$.

The angular components of Eq. (4.2.16) also produce $\hat{v}^q = -\frac{1}{3}e^{-\psi}\omega\xi^q$, in which we insert our previous result for $\xi^q$. We arrive at

$$
\hat{v}^q = \frac{1}{3}te^{-\psi}\omega(r)(b^q_t(r), \tag{4.2.29}
$$

the striking statement that the response of a slowly rotating body to a stationary tidal field is
necessarily dynamical. The $rA$ components of the vorticity conservation condition give

$$
\rho^d = \frac{3}{5} t e^{-\psi} \left[ \omega \frac{db^d}{dr} + 2 \left( \frac{d\omega}{dr} + \frac{r - 4m - 8\pi r^3 p}{r^2 f} \omega \right) b^d_l \right] + \partial_r \rho^d - \frac{m + 4\pi r^3 p}{r^2 f} \rho^d,
$$

(4.2.30a)

$$
\rho^o = \frac{1}{3} t e^{-\psi} \left[ 2\omega \frac{db^o}{dr} - \left( \frac{d\omega}{dr} + \frac{6r - 14m - 8\pi r^3 p}{r^2 f} \omega \right) b^o_l \right] + \frac{1}{3} \partial_r \rho^o - \frac{m + 4\pi r^3 p}{3r^2 f} \rho^o,
$$

(4.2.30b)

and these also reveal the time dependence of the velocity field. The $tA$ components of Eq. (4.2.16) relate the pressure perturbations to other variables; we have

$$
\rho^d = e^{-\psi}(\mu + p) \left[ \frac{1}{2} e^{-\psi} k^d_{tt} - \frac{3}{5} e^{-\psi}(1 + \omega)b^d_l - \partial_t \rho^d \right],
$$

(4.2.31a)

$$
\rho^o = e^{-\psi}(\mu + p) \left[ \frac{1}{2} e^{-\psi} k^o_{tt} + \frac{1}{3} e^{-\psi}(3 - 2\omega)b^o_l - \frac{1}{3} \partial_t \rho^o \right].
$$

(4.2.31b)

This exhausts the information disclosed by the vorticity conservation statement; the remaining $tr$ component is redundant.

The time dependence exhibited in Eq. (4.2.29) and subsequent relations describes a steady growth. We remark, however, that our calculation is carried out to first order in $\Omega$, and that consequently it cannot distinguish between $\Omega t$ and $\sin \Omega t$. We consider it likely that the time dependence is actually bounded.

### 4.3 INTERIOR FIELD EQUATIONS

#### 4.3.1 Zero frequency modes

In this chapter, we determine the field equations which govern the interior metric and fluid variables. Before going on to examine the equations in detail, we pause to explain the meaning of the freedom that remains in the fluid variables after the preceding section’s analysis. The fluid variables of the gravitomagnetic sector were fully specified by the vorticity conservation condition, but the variables $\{v^d, \dot{v}^d, \dot{v}^o\}$ from the gravitoelectric sector were left undetermined. In this section, we find that the residual freedom in these variables is not eliminated by the Einstein field equations. Rather, as made clear by Ref. [80], it is associated with the freedom to add zero-frequency $r$- and $g$-modes to the interior solution.

The $g$-modes are polar (even-parity), stationary fluid disturbances which characterize the perturbation of a static, spherically symmetric material body [75]. To investigate how they manifest themselves in the context of this work, we set $\Omega = 0$, switch off the external tidal field, and focus on stationary perturbations in this discussion. We also suppress the multipole labels (i.e. $d, q$ and $o$) to indicate that the discussion is valid for all multipole orders $\ell$. A polar perturbation is described by the metric variables $\{k_{tt}, k_{tr}, k_{rr}, k\}$ and the fluid variables $\{v_r, v, p\}$. By virtue of the field equations, the variables decouple into the groups $\{k_{tt}, k_{rr}, k, p\}$ and $\{k_{tr}, v_r, v\}$. The first group of variables vanishes for a homogeneous perturbation (one not driven by an external tidal field). The
second group, however, admits an infinite number of solutions when the fluid is barotropic; each one is characterized by a freely specifiable $v_r$. These solutions define the class of zero-frequency $g$-modes. A free $g$-mode can be eliminated by making the assignment $v_r = e^{-\psi} k_{tr}$, which sets the corresponding component of the (contravariant) Eulerian velocity perturbation to zero \[80\].

The $r$-modes are axial (odd-parity), stationary disturbances of the fluid in a perturbed static, spherically symmetric material body. An axial perturbation is described by the metric variables \{$f_t, f_r, v\$}, where we have again omitted the multipole labels. In this case, the field equations admit another infinite set of solutions; each one is characterized by a freely specifiable $v$ and a vanishing $f_r$. These solutions define the class of zero-frequency $r$-modes, which are not restricted to barotropes. A free $r$-mode can be removed by setting $v = e^{-\psi} f_t$, which eliminates its associated (contravariant) Eulerian velocity perturbation \[80\].

To simplify the solution to the field equations, we choose to discard the free $r$- and $g$-modes when they appear. Nonetheless, the freedom to add zero-frequency modes to the solution remains, and the $r$- and $g$-modes can be restored at will.

### 4.3.2 Field equations: Tidal perturbation

**Gravitoelectric sector**

Our examination of the field equations governing the body’s tidal response begins with the substitution of the metric of Eq. (4.1.9) and the fluid variables of Sec. 4.2.2 into the Einstein field equations, which are then expanded to first order in $\Omega$, $E^q_m$ and $B^q_m$, and decomposed in (scalar, vector and tensor) spherical harmonics. The field equations decouple according to multipole order, parity and spin, and in this subsection we examine the contributions that are linear in $E^q_m$, and independent of $\Omega$: these constitute the $\ell = 2$ gravitoelectric sector of the linear tidal response, which is associated with $E_{tt}$. This sector is characterized by the metric variables \{$e^q_{tt}, e^q_{tr}, e^q_{rr}, e^q_{tt}\$} and the fluid variables \{$v^q_r, v^q, p^q\$}. The variables decouple into the groups \{$e^q_{tt}, e^q_{tt}, e^q_{tr}, e^q\$} and \{$e^q_{tr}, v^q, v^q\$} by virtue of the field equations.

The pressure perturbation is eliminated with Eq. (4.2.21a). The angular components of the field equations, and a combination of the $rA$ and $rr$ components, relate $e^q_{tr}$ and $e^q$ algebraically to $e^q_{tt}$. The remaining first-group variable satisfies the homogeneous differential equation

$$r^2 f \frac{d^2 e^q_{tt}}{dr^2} - 2 \left[ \frac{3m}{r} + 1 + 2\pi r^2 (\mu + 3p) \right] r \frac{de^q_{tt}}{dr} - 2 \left[ 3 - 2\pi r^2 (\mu + p) \left( \frac{d\mu}{dp} \right) \right] e^q_{tt} = 0 \quad (4.3.1)$$

supplied by the $tt$ component of the field equations.

The $tr$ component of the field equations gives rise to

$$\left[ 8\pi r^2 (\mu + p) - 3 \right] e^q_{tr} - 8\pi r^2 (\mu + p)e^q v^q_r = 0. \quad (4.3.2)$$

Since $e^q_{tr}$ depends on $v^q_r$, Eq. (4.2.22) relates $v^q$ to $v^q_r$, and $v^q_r$ is unconstrained by the field equations, it is clear that the second group of variables, \{$e^q_{tr}, v^q, v^q\$}, represents a $g$-mode. The $g$-mode is removed by setting $v^q_r = e^{-\psi} e^q_{tr}$. Eq. (4.3.2) then produces the assignment $e^q_{tr} = 0$, and Eq. (4.2.22)
in turn sets $v^q = 0$.  

The other radial functions are given by

$$e^q_{rr} = f^{-1} e^{-2\psi} e^q_{tt}$$

and

$$e^q = r^2 e^{-2\psi} \left\{ \frac{1}{2} \left( \frac{m}{r} + 4\pi r^2 p \right) r \frac{de^q_{tt}}{dr} + \left[ 1 + \frac{m}{r} + 4\pi r^2 p - 2\pi r^2 (\mu + 3p) \right] e^q_{tt} \right\}.$$  \hspace{1cm} (4.3.4)

The interior solutions must be matched with the exterior solutions of Table 3.1 at $r = R$, after taking into account the change of notation described in Eq. (4.1.8). In addition, the radial functions must satisfy regularity conditions at $r = 0$. The matching conditions on $e^q_{tt}$ determine it fully, both inside and outside the body, thereby fixing the value of the gravitational Love number $K_{el}^2$ that appears in the exterior solution.

**Gravitomagnetic sector**

We turn now to the contributions that are linear in $B^q_m$, and independent of $\Omega$, which correspond to the $\ell = 2$ gravitomagnetic sector of the linear tidal response. This sector is associated with $B_{ab}$, and is described by the metric and fluid variables $\{b^q_t, b^q_r, v^q\}$. The velocity perturbation is eliminated with Eq. (4.2.28). The $rA$ components of the field equations imply that $b^q_r = 0$, and the $tA$ components give rise to a homogeneous differential equation for $b^q_t$,

$$r^2 f \frac{d^2 b^q_t}{dr^2} - 4\pi r^3 (\mu + p) \frac{db^q_t}{dr} - 2[3 - \frac{2m}{r} - 4\pi r^2 (\mu + p)] b^q_t = 0.$$  \hspace{1cm} (4.3.5)

The interior solution for $b^q_t$ must agree with its external counterpart at $r = R$, subject to the change of notation described in Eq. (4.1.8), and it must also satisfy regularity conditions at $r = 0$. The matching conditions completely determine $b^q_t$ inside and outside the body, including the gravitational Love number $K_{2}^{mag}$ that appears in the exterior metric.

**4.3.3 Field equations: Bilinear perturbation**

**Gravitoelectric sector: $\ell = 2$**

In this section, we examine the contributions to the field equations which are linear in both $\Omega$ and $E^q_m$ or $B^q_m$; the corresponding equations govern the body’s bilinear tidal response, and they decouple according to their spherical-harmonic decompositions into dipole, quadrupole and octupole sectors of even or odd parity. We begin by studying the $\ell = 2$ gravitoelectric sector of the field equations, characterized by quadrupole contributions proportional to $\Omega E^q_m$. This sector is associated with $\tilde{E}_{ab}$.
and is described by the metric variables \( \{ \hat{e}_tt^q, \hat{e}_tr^q, \hat{e}_{rr}^q, \hat{e}_r^q \} \) and the fluid variables \( \{ \hat{\nu}^q, \hat{\rho}^q \} \). The variables decouple into the groups \( \{ \hat{e}_tt^q, \hat{e}_r^q, \hat{e}_r^q, \hat{p}^q \} \) and \( \{ \hat{e}_{tr}^q, \hat{e}_r^q, \hat{\nu}^q \} \).

The pressure perturbation \( \hat{p}^q \) is eliminated with Eq. (4.2.4b). The angular components of the field equations, together with the combined \( rA \) and \( rr \) components, further eliminate \( \hat{e}_{tr}^q \) and \( \hat{e}_r^q \). The only remaining first-group variable, \( \hat{e}_{tt}^q \), satisfies a homogeneous differential equation supplied by the \( tt \) component of the field equations. This is the type of equation which governs a stationary, homogeneous perturbation of a static, spherically symmetric body; therefore, the first-group variables all vanish in accordance with Sec. 4.3.1's discussion.

The \( tr \) component of the field equations gives rise to

\[
\hat{e}_{tr}^q = \frac{e^\psi}{8\pi r^2(\mu + p) - 3} \left[ 8\pi r^2(\mu + p)\hat{\nu}_r^q + \frac{1}{2} r^2 e^{-3\psi}(1 - \omega) \frac{de_{tt}^q}{dr} + \frac{1}{2} r e^{-3\psi} \left( r \frac{d\omega}{dr} + 2\omega - 2 \right) \hat{e}_{tt}^q \right],
\]

an algebraic equation for \( \hat{e}_{tr}^q \), while the \( tA \) components produce

\[
-8\pi r^2(\mu + p)e^\psi \hat{\nu}_r^q = r^2 f \frac{de_{tt}^q}{dr} + \left[ 2m - 4\pi r^3(\mu - p) \right] \hat{e}_{tr}^q + \frac{1}{3} r^3 e^{-2\psi} \left( \frac{3m}{r} - r + 4\pi r^2 p \right) (1 - \omega) \frac{de_{tt}^q}{dr} + \frac{2}{3} r^2 e^{-2\psi} \left\{ (1 + \frac{m}{r}) (1 - \omega) + \pi r^2(\mu + p) \left[ 1 + \frac{d\mu}{dp} + \left( 3 + \frac{d\mu}{dp} \right) \omega \right] \right\} \hat{e}_{tt}^q. \tag{4.3.6}
\]

The field equations do not fully determine the second-group variables \( \{ \hat{e}_{tr}^q, \hat{\nu}_r^q, \hat{\nu}^q \} \), and the residual freedom is interpreted as the freedom to add a \( g \)-mode to the solution. To simplify the description, we remove the \( g \)-mode by setting \( \hat{\nu}_r^q = e^{-\psi} \hat{e}_{tr}^q \). Eq. (4.3.6) then reduces to

\[
\hat{e}_{tr}^q = -\frac{1}{6} r e^{-2\psi} \left[ (1 - \omega) r \frac{de_{tt}^q}{dr} + \left( r \frac{d\omega}{dr} + 2\omega - 2 \right) \hat{e}_{tt}^q \right]. \tag{4.3.8}
\]

The interior metric variables must be matched with the exterior solutions listed in Table 3.1 at \( r = R \); we remind the reader of the changes of notation described in Eq. (4.1.8) and immediately following Eq. (4.1.9). It is straightforward to verify that the expression given for \( \hat{e}_{tt}^q \) in Eq. (4.3.6) automatically agrees with its external counterpart at the body’s surface regardless of the choice of \( \hat{\nu}_r^q \), since the equation of state approaches polytropic form near the surface \( \mu \) and \( p \) vanish at \( r = R \) like \( \mu \propto (1 - r/R)^n \) and \( p \propto (1 - r/R)^{n+1} \), with \( n > 0 \). The matching of \( \hat{e}_{tt}^q = 0 \) to its external version determines the rotational-tidal Love number \( E^q \) which appears in the exterior metric. The matching conditions produce the assignment \( E^q = 1/120 \) independently of the material body’s equation of state.

**Gravitomagnetic sector: \( \ell = 2 \)**

We now examine the field equations governing the \( \ell = 2 \) gravitomagnetic sector of the bilinear tidal response, corresponding to quadrupole contributions linear in \( \Omega B_m^q \). This sector, associated
This result is universal for material bodies in the absence of internal \( B \) in the exterior solution: the assignment \( \hat{b}_{r_0}^q \). The solution to this equation represents an \( r \)-mode, and to simplify the solution we choose to eliminate this degree of freedom by setting \( \hat{b}_{r_0}^q = 0 \).

We are left with

\[
\hat{b}_t^q = t\hat{b}_{t_1}^q(r), \quad \hat{b}_r^q = \hat{b}_{r_0}^q(r),
\]

and the \( t.A \) components of the field equations give rise to

\[
r^2 f \frac{d^2 \hat{b}_t^q}{dr^2} - 4\pi r^3 (\mu + p) \frac{d\hat{b}_t^q}{dr} - 2 \left[ 3 - \frac{2m}{r} - 4\pi r^2 (\mu + p) \right] \hat{b}_t^q - \frac{16\pi}{3} r^2 (\mu + p) \omega \hat{b}_1^q = 0,
\]

a differential equation that determines \( \hat{b}_{t_1}^q(r) \). The \( r.A \) components return

\[
\hat{b}_r^q = \frac{1}{4} e^{-2\omega} \left[ r^2 \frac{d\hat{b}_t^q}{dr} - 2r \hat{b}_t^q - r^2 (1 - \omega) \frac{d\hat{b}_t^q}{dr} - \frac{1}{3} r \left( r \frac{d\omega}{dr} + 6\omega - 6 \right) \hat{b}_t^q \right],
\]

an algebraic equation for \( \hat{b}_{r_0}^q \).

The interior metric variables must be matched with the exterior solutions at \( r = R \), subject to the change of notation described in Sec. 4.1. This implies that \( \hat{b}_{t_1}^q \) must satisfy the boundary conditions

\[
\hat{b}_{t_1}^q = 0 = \frac{d\hat{b}_{t_1}^q}{dr}
\]

at \( r = R \). One can verify that \( \hat{b}_{t_0}^q \), as given in Eq. (4.3.11), automatically agrees with its external counterpart at the body’s surface. In addition, these functions must satisfy regularity conditions at \( r = 0 \).

The boundary conditions of Eq. (4.3.12) determine the rotational-tidal Love number \( \mathcal{B}^q \) appearing in the exterior solution: the assignment \( \mathcal{B}^q = 1/120 \) is dictated by the matching conditions. This result is universal for material bodies in the absence of internal \( r \)-modes. The addition of an \( r \)-mode characterized by \( \{ \hat{b}_{t_0}^q(r), \hat{b}_{r_0}^q(r) = 0, \hat{v}_0^q(r) \} \) would impact the value of \( \mathcal{B}^q \) through the change in \( \hat{b}_t^q := t\hat{b}_{t_1}^q + \hat{b}_{t_0}^q \) and its first derivative at \( r = R \).

**Gravitoelectric sector: \( \ell = 1 \)**

In this subsection, we turn our attention to the dipole contributions linear in \( \Omega \mathcal{E}_m^a \) which represent the \( \ell = 1 \) sector of the bilinear tidal response. The metric and fluid variables that characterize this \( \mathcal{F}_a \)-generated sector are \( \{ f_t^d, v^a \} \). The metric variable \( f_r^d \) does not figure in the ansatz because of the gauge choice \( f_r^d = 0 \) made in Sec. 3.1. The \( t.A \) components of the field equations give rise to the inhomogeneous differential equation

\[
\frac{d^2 f_t^d}{dr^2} + 2r \frac{df_t^d}{dr} = 0.
\]
\[ 0 = r^2 f d^2 f^d_t \left| \begin{array}{c} dr^2 \end{array} \right. - 4\pi r^3 (\mu + p) \frac{df^d_t}{dr} + \left( \frac{4m}{r} - 2 + 8\pi r^2 (\mu + p) \right) f^d_t \]

\[ - 16\pi r^2 (\mu + p) e^\psi v^d + \frac{2}{5} r^3 f e^{-2\psi} (1 - \omega) \frac{de^\omega}{dr} \]

\[ + \frac{1}{5} r^2 e^{-2\psi} \left[ 6(1 - \omega) - 4\pi r^2 (\mu + p) \left( 1 + \frac{d\mu}{dp} \right) \omega - 4\pi r^2 (\mu + p) \left( 3 + \frac{d\mu}{dp} \right) \right] e^\omega \]  \hspace{1cm} (4.3.13)

for \( f^d_t \). The fluid variable \( v^d \) is left undetermined, and the corresponding residual freedom in the variables \( \{ f^d_t, f^d_r = 0, v^d \} \) represents an \( r \)-mode. We choose to eliminate the \( r \)-mode to simplify the solution by setting \( v^d = e^{-\psi} f^d_t \).

The matching conditions on the interior and exterior solutions require \( f^d_t \) to agree at \( r = R \) with the value appearing in Table 3.1, after taking into account Sec. 4.1’s changes of notation. In addition, it must satisfy regularity conditions at \( r = 0 \). These boundary conditions fully determine \( f^d_t \) inside and outside the body, including the residual gauge constant \( \gamma^d \) which appears in the exterior solution. The precise value of \( \gamma^d \) depends on the choice made for \( v^d \).

**Gravitomagnetic sector: \( \ell = 1 \)**

The \( \ell = 1 \) gravitomagnetic sector of the bilinear tidal response, associated with \( K_\alpha \) and consisting of dipole contributions proportional to \( \Omega B^a_m \), involves the metric variables \( \{ k^t_t, k^t_r, k^r_r \} \) and the fluid variables \( \{ v^d_t, v^d_r, p^d \} \), which decouple into the groups \( \{ k^d_t, k^d_r, p^d \} \) and \( \{ k^d_t, v^d_r, v^d \} \). The variables \( v^d_t \) and \( p^d \) are eliminated with Eqs. (4.2.30a) and (4.2.31a), respectively, and we find that the \( tA \) and \( rA \) components of the field equations further eliminate \( v^d \) and \( k^d_r \). We are left with \( k^d_t \) from the first group of variables, and \( k^d_r \) from the second. We next assume the explicit forms \( k^d_t(t, r) := k^d_t(t, R) \) and \( k^d_r(t, r) := k^d_r(t, R) + tk^d_r(t, R) \), which ensures that all variables from the first group are time independent, while those from the second group are linear in time. The \( tr \) component of the field equations then implies that \( k^d_{tr0} \) satisfies a homogeneous differential equation. The solution to this equation represents a \( g \)-mode, and we simplify the solution by setting \( k^d_{tr0} = 0 \).

We are left with

\[ k^d_t(t, r) = k^d_{t0}(r), \quad k^d_r(t, r) = tk^d_{r1}(r), \]  \hspace{1cm} (4.3.14)

and these functions are determined by the \( tr \) and \( rr \) components of the field equations, respectively. We have

\[ r^2 f \left| \frac{d^2 k^d_{r1}}{dr^2} \right. + \left( \frac{3(m - 4\pi r^3 \mu) + (m + 4\pi r^3 p) \frac{d\mu}{dp}}{dr} \right) \frac{dk^d_{r1}}{dr} \]

\[ - \frac{2}{r^2 f} \left( 1 - 10\pi r^2 (\mu + p) + 16\pi^2 r^4 p^2 \right) r^2 + 4\pi r^3 (5\mu + 7p)m - 3m^2 - (m + 4\pi r^3 p) \frac{d\mu}{dp} \} k^d_{r1} \]

\[ - \frac{48\pi}{5} r^2 (\mu + p) \frac{db^d_t}{dr} - \frac{96\pi}{5} (\mu + p) \left( r^2 \frac{d\omega}{dr} + \frac{r - 4m - 8\pi r^3 p}{f} \omega \right) b^d_t = 0 \]  \hspace{1cm} (4.3.15)
\begin{equation}
0 = r \frac{dk_{tt0}^d}{dr} + \frac{2}{r} \frac{m - 2\pi r^3(\mu + p)}{r f(m + 4\pi r^3 p)} \frac{r^2}{k_{tt0}^d} - \frac{r^2}{2(m + 4\pi r^3 p)} \frac{dk_{tr1}^d}{dr} \\
+ \frac{[1 + 2\pi r^2(1 + 4\pi r^2 p)(\mu - p)] r^2 - [5 + 2\pi r^2(\mu + p)] r m + 5 m^2}{f(m + 4\pi r^3 p)} k_{tr1}^d \\
- \frac{3 r}{10} \frac{r(m + 4\pi r^3 p) d\omega}{m + 4\pi r^3 p} - 4 \omega + 4 \frac{db_i^q}{dr} \\
- \frac{3}{5(m + 4\pi r^3 p)} \frac{r d\omega}{r f} - 4 \frac{m + 2\pi r^3(\mu + p)}{r f} \omega + 4 \frac{m - 2\pi r^3(\mu + p)}{r f} \frac{db_i^q}{dr}.
\end{equation}

(4.3.16)

The remaining variables are given by

\begin{equation}
e^{2\psi \omega^d r} = - \frac{r^2}{r - m + 4\pi r^3 p} \frac{dk_{tt0}^d}{dr} + \frac{1}{f} k_{tt0}^d + \frac{r^2}{r - m + 4\pi r^3 p} k_{tr1}^d + \frac{6(1 - \omega)}{5(r - m + 4\pi r^3 p)} \frac{db_i^q}{dr} \\
- \frac{6(1 - \omega)}{5 f} b_i^q,
\end{equation}

(4.3.17a)

\begin{equation}
p^d = e^{-2\psi} \left[ \frac{f}{16\pi} \frac{dk_{tr1}^d}{dr} + \frac{m - 2\pi r^3(\mu - p)}{8\pi r^2} k_{tr1}^d + \frac{1}{2} (\mu + p) k_{tt0}^d - \frac{3}{5} (\mu + p)(1 + \omega) b_i^q \right].
\end{equation}

(4.3.17b)

and

\begin{equation}
e^d = - \frac{t}{8\pi r^2(\mu + p)} \left[ e^{-\psi} [1 - 8\pi r^2(\mu + p)] \right] k_{tr1}^d.
\end{equation}

(4.3.18a)

\begin{equation}
e^d = - \frac{t}{16\pi r^2(\mu + p)} \left\{ r^2 f \frac{dk_{tr1}^d}{dr} + 2 [m - 2\pi r^3(\mu - p)] k_{tr1}^d \right\}.
\end{equation}

(4.3.18b)

The interior solutions must be matched with the exterior solutions of Table 3.1 at \( r = R \) (subject to the aforementioned notational changes), and this reveals that \( k_{tr1}^d \) must satisfy the boundary conditions

\begin{equation}
k_{tr1}^d = 0 = \frac{dk_{tr1}^d}{dr}
\end{equation}

at \( r = R \). In addition, all the functions must satisfy regularity conditions at \( r = 0 \). We note that the matching conditions on \( k_{tt}^d \) set the value of the residual gauge constant \( e^d \) appearing in Table 3.1. The matching is unaffected by the addition of a \( g \)-mode \( \{ k_{tr0}^d(r), v_{r0}^d(r), v_{t0}^d(r) \} \) to the solution, since only the time derivative of \( k_{tr}^d := t k_{tr1}^d + k_{tr0}^d \) appears in Eq. (4.3.16).
Gravitoelectric sector: $\ell = 3$

The $\ell = 3$ gravitoelectric sector of the bilinear tidal response, associated with $F_{abc}$ and consisting of octupole contributions proportional to $\Omega \mathcal{E}_m^3$, is characterized by the variables $\{f_t^o, f_r^o, v^o\}$. The $rA$ components of the field equations indicate that $f_r^o = 0$. The metric variable $f_t^o$ is governed by the inhomogeneous differential equation

\[
0 = r^2 f_t f_t^o \frac{d^2 f_t^o}{dr^2} - 4\pi r^3 (\mu + p) \frac{df_t^o}{dr} + 4 \left[ \frac{m}{r} - 3 + 2\pi r^2 (\mu + p) \right] f_t^o \\
- 16\pi r^2 (\mu + p)e^\psi v^o + r^3 e^{-2\psi}(1 - \omega) \left( \frac{9m}{r} - 2 + 20\pi r^2 p \right) \frac{df_t^o}{dr} \\
+ 2r^2 e^{-2\psi} \left[ \left( \frac{5m}{r} + 2 \right)(1 - \omega) + 2\pi r^2(\mu + p) \left( 6 + \frac{d\mu}{dp} \right) \omega + 2\pi r^2(\mu + p) \left( \frac{d\mu}{dp} - 2 \right) \right] e_t^o
\]

which results from the $tA$ components of the field equations. The fluid variable $v^o$ is not specified by the field equations, and the associated freedom in the variables $\{f_t^o, f_r^o = 0, v^o\}$ represents an $r$-mode. We choose again, for simplicity, to eliminate this freedom in the solution; the assignment $v^o = e^\psi f_t^o$ has the desired effect.

The matching conditions on the interior and exterior solutions demand that $f_t^o$ agree at $r = R$ with the value listed in Table 3.1, subject to the changes of notation mentioned above. It must moreover satisfy regularity conditions at $r = 0$. These boundary conditions fully specify $f_t^o$ inside and outside the body, thereby determining the rotational-tidal Love number $\mathcal{F}_t$ which appears in the exterior solution. We note that the value of $\mathcal{F}_t$ depends on the choice made for $v^o$; it is sensitive to the presence of internal $r$-modes.

Gravitomagnetic sector: $\ell = 3$

Finally, we examine the $\ell = 3$ gravitomagnetic sector associated with the tidal response to $K_{abc}$, which contributes octupole terms linear in $\Omega B_{m}^2$ to the field equations. This sector is described in terms of the metric variables $\{k_t^o, k_r^o, k_r^o, k^o\}$ and the fluid variables $\{v^o, v^o, p^o\}$. By virtue of the Einstein field equations, they decouple into the groups $\{k_t^o, k_r^o, k_r^o, p^o\}$ and $\{k_t^o, v^o, v^o\}$. The variables $v^o$ and $p^o$ are eliminated with Eqs. (4.2.30) and (4.2.31), respectively, and we find that the $tA$ and angular components of the field equations further eliminate $v^o$ and $k_r^o$. We are left with $k_t^o$ and $k^o$ from the first group of variables, and $k_r^o$ from the second. We then assume the explicit forms $k_t^o(t, r) := k_{t0}^o(r), k_r^o(t, r) := k_{r0}^o(r)$ and $k_r^o(t, r) := k_{r0}^o(r) + tk_{r1}^o(r)$. The $tr$ component of the field equations implies that $k_{t0}^o$ satisfies a homogeneous differential equation, and its solution represents a $g$-mode that we allow ourselves to discard.

We are left with

\[
k_{t}^o(t, r) = k_{t0}^o(r), \quad k_{r}^o(t, r) = k_{r0}^o(r), \quad k_{r1}^o(t, r) = tk_{r1}^o(r), \quad (4.3.21)
\]

and the differential equations satisfied by these functions are obtained from the $tt, rr$ and $rA$ components of the field equations, respectively. We find

38
\begin{align*}
0 &= r^2 f \frac{d^2 k_{\nu_0}^o}{dr^2} + 2 \left[ 1 - \frac{3m}{r} - 2\pi r^2(\mu + 3p) \right] \frac{dk_{\nu_0}^o}{dr} + 4 \left[ \frac{\pi r^2(\mu + p)(3 + \frac{d\mu}{dp}) - 3}{r} \right] k_{\nu_0}^o \\
&+ \frac{1}{2} r^2 f \left( \frac{d\mu}{dp} - 1 \right) \frac{dk_{\nu_1}^o}{dr} + \left\{ 11 + \frac{d\mu}{dp} \right\} m + 2\pi r^3 \left[ (\mu + 7p) - (\mu - p) \frac{d\mu}{dp} - 4r \right] k_{\nu_1}^o \\
&+ S_1 \frac{db_i^q}{dr} + S_0 b_i^q, \tag{4.3.22}
\end{align*}

where

\begin{align*}
S_1 &= -\frac{2}{3} \left\{ \frac{5m^2}{r^2} + 3 - 16\pi^2 r^4 p^2 + 4\pi^2 r^2 \mu - \frac{m}{r} \left[ 9 + 8\pi r^2(\mu + p) \right] \right\} r \frac{d\omega}{dr} \\
&- \frac{4}{3} \left\{ 3 - \frac{m}{r} \left[ 9 - 4\pi r^2(\mu + p) \right] - 4\pi r^2 p \left[ 3 - 4\pi r^2(\mu + p) \right] \right\} \omega \\
&+ 4 \left( 1 - \frac{3m}{r} - 4\pi r^2 p \right), \tag{4.3.23a}
\end{align*}

\begin{align*}
S_0 &= \frac{2}{3} \left\{ \frac{10m^2}{r^2} + 4\pi r^2 \left[ (3 - 8\pi r^2 p) p + 2\mu \right] - \frac{m}{r} \left[ 3 + 16\pi r^2(\mu + p) \right] \right\} r \frac{d\omega}{dr} \\
&+ \frac{4}{3} \left\{ \frac{m}{r} \left[ 6 + 8\pi r^2(\mu + p) \right] + \left[ 9 - 4\pi r^2(\mu + p) \right] \left( 6 + \frac{d\mu}{dp} - 8\pi r^2 p \right) \right\} \omega \\
&- 4 \left\{ \frac{2m}{r} - 2\pi r^2(\mu + p) \left( 1 + \frac{d\mu}{dp} \right) - 3 \right\}, \tag{4.3.23b}
\end{align*}

as well as

\begin{align*}
&\ r^2 f \frac{d^2 k_{\nu_1}^o}{dr^2} + \left[ 3(m - 4\pi r^3 \mu) + (m + 4\pi r^3 p) \frac{d\mu}{dp} \right] \frac{dk_{\nu_1}^o}{dr} \\
&- \frac{2}{r^2 f} \left\{ 2 \left[ 3 - 5\pi r^2(\mu + p) + 8\pi^2 r^4 p^2 \right] r^2 - 2 \left[ 5 - 2\pi r^2(5\mu + 7p) \right] \right\} \frac{m}{r} \left[ 3m^2 - (m + 4\pi r^3 p)^2 \frac{d\mu}{dp} \right] k_{\nu_1}^o \\
&- \frac{32\pi}{3} \frac{2\pi r^2(\mu + p)\omega}{dr} \frac{db_i^q}{dr} + \frac{16\pi}{3} (\mu + p) \left[ r^2 \frac{d\omega}{dr} + \frac{23r - 7m - 4\pi r^3 p}{f} \omega \right] b_i^q = 0 \tag{4.3.24}
\end{align*}

and

\begin{align*}
&\ r \frac{dk_{\nu_0}^o}{dr} - \frac{5r + 2m + 8\pi r^3 p}{m + 4\pi r^3 p} k_{\nu_0}^o - \frac{e^{2\psi} r^5 f}{m + 4\pi r^3 p} \frac{dk_{\nu_1}^o}{dr} + \frac{e^{-2\psi} r^3 \left[ r - 3m + 2\pi r^3 (\mu - p) \right]}{m + 4\pi r^3 p} k_{\nu_1}^o \\
&+ \frac{e^{-2\psi} r^2 [5r + 2m - 4\pi r^3 (\mu + p)]}{m + 4\pi r^3 p} k_{\nu_0}^o + \frac{e^{-2\psi} r^4 f (3r + 2m + 8\pi r^3 p)}{6(m + 4\pi r^3 p)} \frac{db_i^q}{dr} \frac{d\omega}{dr} \\
&- \frac{2}{3} \frac{e^{-2\psi} r^2 \left[ rf (r + m + 4\pi r^3 p) \right]}{m + 4\pi r^3 p} \frac{d\omega}{dr} + \frac{[15r + 6m - 8\pi r^3 (\mu + p)] \omega}{dr} \\
&- 3 \left[ 5r + 2m - 4\pi r^3 (\mu + p) \right] b_i^q = 0. \tag{4.3.25}
\end{align*}
The remaining variables are given by

\[ k^0_{rr} = e^{-2\psi} \left\{ \frac{1}{f} k^0_{tr0} - \frac{r^2}{3} \frac{d\omega}{dr} \frac{db^3}{dr} + 2 \left[ \frac{d\omega}{dr} + \frac{3(1 - \omega)}{f} \right] b^3 \right\}, \]  
\[ p^0 = e^{-2\psi} \left[ \frac{f}{16\pi} \frac{dk^0_{tr}}{dr} + \frac{m - 2\pi r^3 (\mu - p)}{8\pi r^2} k^0_{tr1} + \frac{1}{2} (\mu + p) k^0_{tr0} + \frac{1}{3} (\mu + p)(3 - 2\omega)b^3 \right], \]  

and

\[ v^0 = -t e^{-\psi} \left\{ 3 - 4\pi r^2 (\mu + p) \right\} k^0_{tr1}, \]  
\[ v^o = -t \frac{3e^{-\psi}}{16\pi r^2 (\mu + p)} \left\{ r^2 f \frac{dk^0_{tr1}}{dr} + 2 \left[ m - 2\pi r^3 (\mu - p) \right] k^0_{tr1} \right\}. \]  

As before, the interior solutions must be matched with the exterior solution of Table 3.1 (subject to the aforementioned notational changes), and this reveals that \( k^0_{tr1} \) must satisfy the boundary conditions

\[ k^0_{tr1} = 0 = \frac{dk^0_{tr1}}{dr} \]  

at \( r = R \). In addition, the radial functions must satisfy regularity conditions at \( r = 0 \). The boundary conditions on \( k^0_{tr} \) fully determine it inside and outside the body, including the rotational-tidal Love number \( \mathcal{K}^0 \) which appears in the exterior solution. The presence of a \( g \)-mode within the body does not affect the value of \( \mathcal{K}^0 \), since the addition of terms \( \{ k^0_{tr0}(r), v^0(r), v^0(r) \} \) to the solution does not impact Eq. (4.3.22), in which only the time derivative of \( k^0_{tr1} + k^0_{tr0} \) appears.

By virtue of the assignments dictated for \( \mathcal{E}^q \) and \( \mathcal{B}^q \) by the matching conditions, in addition to the consequences of the exterior field equations examined in Sec. 3.2, the metric outside the tidally deformed slowly rotating material body simplifies a great deal relative to the original ansatz, Eq. (3.1.11). Reverting to the notation of Ch. 3, the remaining nonzero components of the perturbed metric are

\[ g_{tt} = -(1 - 2M/r) - r^2 \mathcal{E}_{tt}^q + r^2 k^d_{tt} \mathcal{K}^d - r^2 k^o_{tt} \mathcal{K}^o, \]  
\[ g_{tr} = r^2 \mathcal{E}_{tr}^q, \]  
\[ g_{rr} = (1 - 2M/r)^{-1} - r^2 \mathcal{E}_{rr}^q + r^2 k^d_{rr} \mathcal{K}^d - r^2 k^o_{rr} \mathcal{K}^o, \]  
\[ g_{tA} = \frac{2M^2}{r} k^d_{tA} + \frac{2}{3} r^3 b^3_{iA} \mathcal{B}^q_i - r^3 f^d_{tA} \mathcal{F}^d + r^3 f^o_{tA} \mathcal{F}^o, \]  
\[ g_{rA} = -r^3 b^3_{tA} \mathcal{B}^q_A, \]  
\[ g_{AB} = r^2 \Omega_{AB} - r^4 \mathcal{E}_{AB} \mathcal{E}^q + r^4 k^o_{AB} \mathcal{K}^o. \]  

The radial functions appearing here are listed in Table 3.1 those that do not appear in this metric vanish outside the body. Thus, the external solution is completely determined by the numerical values of the Love numbers \( K^e_2, K^m_2, \mathcal{K}^q \) and \( \mathcal{K}^o \).
### 4.4 Discussion: Static Fluid State

In Sec. 4.2, we demonstrated that the fluid equations for a barotrope naturally give rise to a conservation law $\Delta \omega_{\alpha\beta} = 0$ for the vorticity tensor when the tidal perturbation is switched on adiabatically, provided the fluid’s equation of state is unchanged by the perturbation. The perturbation’s adiabatic growth mirrors the gradual increase in the amplitude of the tidal field sourced by a companion in a binary system as the orbital separation shrinks over the radiation-reaction timescale. Stationary tides in an astrophysical setting are thus expected to conserve vorticity.

The vorticity conservation condition is a key physics input that gives rise to stationary, irrotational fluid motions inside nonrotating neutron stars, as well as dynamical fluid motions in rotating neutron stars. Despite the fact that the irrotational currents have been discussed by Shapiro [81] and Favata [20] (the dynamical currents, in contrast, are a new discovery), most authors have overlooked the vorticity conservation condition in their studies of the tidal deformation of static [21, 22] and rotating [60] material bodies. Instead, they make direct assumptions about the fluid velocity perturbation $\delta u^\alpha$, which lead them to conclusions that differ markedly from ours.

The gravitational Love numbers computed in Sec. 5.2 of this thesis, and those computed in earlier work by Damour & Nagar [21] and Binnington & Poisson [22], are calculated under the assumption that the tidal field varies so slowly that it never takes the body out of hydrostatic equilibrium. This is a perfectly valid approximation for circumstances such as the orbital evolution of a compact binary well before merger, and it is manifested in the assumed time independence of the tidal moments. However, the hydrostatic equilibrium considered in the earlier work is a strict one that forbids all fluid motions within the body: the fluid is assumed to be strictly static, in the sense that $\delta u^\alpha = 0$.

Traditionally, authors imposed the static state on the fluid, and then proceeded to solve the relativistic Euler equations for the other fluid variables. At zeroth order in spin, the angular components of Eq. (4.2.12) return the condition

$$e^{-\psi}(\mu + p)\partial_t \left( \delta u^A + e^{-\psi}p_{tA} \right) = 0 \quad (4.4.1)$$

in the gravitomagnetic sector [24], which is satisfied for the static fluid only when the tidal field is strictly time independent (recall that $p_{tA} = b^A_{t}(r)B^3_{tA}$ here). As soon as the tidal field has a time dependence – even one that is arbitrarily slow – the static fluid state violates the Euler equations. In contrast, the irrotational fluid state which obeys $\Delta \omega_{\alpha\beta} = 0$ automatically satisfies Eq. (4.4.1) whether the tidal field is stationary or not; conservation of vorticity assigns $\delta u^A = -e^{-\psi}p_{tA}$ (cf. Eq. (4.2.28) and Sec. IV of [24]). The irrotational state is therefore a more realistic configuration for the fluid than the static state.

The different assignments the static and irrotational states dictate for $\delta u^A$ impact the field equation for $b^A_t$ – the sign of the last term is changed, relative to Eq. (4.3.5), for the static fluid. This, in turn, has important consequences for the calculation of the gravitomagnetic Love number $K^\text{mag}_2$. As we will show in Sec. 5.2, $K^\text{mag}_2$ is positive for barotropes in the static fluid state, but negative for ones in the irrotational state.

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5A more complete treatment of the fluid state would account for the fact that astrophysical neutron stars are believed to have a composition gradient that is not maintained by perturbations [82]. This feature could potentially alter the analysis; in particular, it endows the $g$-modes with finite frequencies.
The assumption of strict hydrostatic equilibrium has no bearing on the gravitoelectric sector of the perturbation, since the condition $\Delta \omega_{\alpha\beta} = 0$ automatically implies that the even-parity pieces of $\delta u^\alpha$ vanish – this is true both at zeroth and at first order in spin. On the other hand, as we argue below, the static fluid state is generically incompatible with the field equations governing bilinear gravitomagnetic perturbations of a rotating material body.

Pani, Gualtieri and Ferrari studied the tidal deformation of a rotating material body to second order in its spin in Ref. [60], and saw no trace of the dynamics which characterized the internal metric and fluid variables in Sec. 4.3. There are two reasons for this discrepancy. First and foremost, these authors place the fluid in the static state. The condition $\delta u^\alpha = 0$ sets $v^q = e^{-\psi} b^q_t$ and $\hat{v}^q = e^{-\psi} \hat{b}^q_t$, and the conservation equations $\nabla_\alpha T^\alpha\beta = 0$ imply that $\hat{b}^q_t$ must be of the form $\hat{b}^q_t(t, r) = \hat{b}^q_{t0}(r) + tb^q_t(r)$. The field equations produce a homogeneous differential equation for $\hat{b}^q_{t0}(r)$, which is identified as a free $r$-mode and consequently discarded. The matching with the exterior solution then creates a contradiction: the interior expression for $\hat{b}^q_t$ is necessarily time dependent and does not vanish at $r = R$, but the exterior expression is necessarily stationary. We conclude that the static fluid is an unphysical configuration in the context of a tidally deformed, slowly rotating body. We note that there is no contradiction in the dipole or octupole sectors of the bilinear gravitomagnetic response, and that the contradiction in the quadrupole sector is lifted when the tidal field is axisymmetric; in this case, $\hat{B}^q_A = 0$ and the variables $\{\hat{b}^q_t, \hat{b}^q_r, \hat{v}^q\}$ are not defined. This is the second reason the dynamical response was not noticed by Pani, Gualtieri and Ferrari: these authors restricted their attention to axisymmetric tidal fields.
Chapter 5

Tidal response

5.1 Equation of state

In this chapter, we integrate the field equations derived previously to calculate the Love numbers which complete the exterior metric of Eq. (4.3.29). We also calculate the perturbed velocity field inside the material body to get a picture of the fluid motions induced by the tidal field. The field equations depend on the background fluid variables $\mu$ and $p$, and to determine these functions we must choose a specific barotropic equation of state $p = p(\rho)$, $\epsilon = \epsilon(\rho)$ for the body and solve the structure equations, Eqs. (4.1.4) and (4.1.5).

We are interested in modelling neutron stars, whose equation of state is poorly constrained in the relevant supranuclear density range. A variety of candidate equations of state have been proposed based on nuclear physics calculations (e.g. APR \[83\], FPS \[84\] using variational methods; SLy \[85\], MS1 \[86\] using the relativistic mean-field approximation; BCSK20 \[87\] using a hybrid method; etc.), but to investigate the neutron star tidal response in a model-agnostic way, we shall focus on the polytropic equation of state

$$p = K \rho^{1+1/n}. \tag{5.1.1}$$

Polytropes neatly parameterize the space of barotropic equations of state; the polytropic index $n > 0$ controls the stiffness of the models (polytropes of a given mass have larger radii for smaller values of $n$), and $K$ is a constant which sets the compactness $M/R$ of the polytrope for a given central mass density $\rho_c$. In practice, we specify the polytropic model by choosing $n$ and $b := p_c/\rho_c$, the ratio of central pressure to central density; substituting $K = b/\rho_c^{1/n}$, the scale $\rho_c$ drops out of all the equations for the Love numbers and the fluid variables. Tooper \[88\] has shown that a polytrope of the form of Eq. (5.1.1) has internal energy density $\epsilon = np$, such that

$$\mu = \rho + np. \tag{5.1.2}$$

Further, Ref. \[88\] demonstrates that there exists a maximum value of $b$ for each $n$, say $b_{\text{max}}$, beyond which the polytrope becomes unstable against radial oscillations. The point $b_{\text{max}}$ corresponds to
the maximum of $M$ for a given $n$. We consider only stable polytropes with $b < b_{\text{max}}$ in this work.\footnote{A different consideration sets $b_{\text{max}}$ for sufficiently stiff equations of state: causality demands that the sound speed $c_s = (d\mu/dp)^{-1/2}$ of the fluid be less than unity. This is automatically satisfied for $n \geq 1$, but requires $b \leq n/(1-n^2)$ for $n < 1$. This requirement is more stringent than the stability condition for $n \lesssim 0.54$.}

The observation of a 2 $M_\odot$ neutron star \cite{1,2} effectively rules out the polytropic models softer than $n \approx 1$ as suitable proxies for the nuclear equation of state, since they cannot support a sufficiently massive star. Consequently, we focus on polytropes with $n \leq 1$ and $0.13 \leq M/R \leq 0.21$ in our study of the tidal response; the compactness range is limited to values for which all these polytropes are stable (the lower bound corresponds to a 1.4 $M_\odot$ star with a radius of 16 km). We make an exception, however, when we calculate the Love numbers: we also explore somewhat softer polytropes to better observe the dependence on the equation of state’s stiffness.

In this section, we calculate the equation-of-state-dependent Love numbers $K_{el}^2$, $K_{\text{mag}}^2$, $\mathcal{S}^0$ and $\mathcal{S}_0^0$ which appear in the exterior metric of Eq. (4.3.29).\footnote{This work was originally carried out in Ref. \cite{77} and Paper III.} The quadrupolar rotational-tidal Love numbers were found to have a fixed universal value of $C^4 = 1/120 = \mathcal{S}^0$ in Ch. 4.

We begin the computation by focusing on the gravitoelectric sector of the tidal perturbation, characterized by the Love number $K_2^e$. We specialize to the polytropic equation of state of Eq. (5.1.1) and integrate Eq. (4.3.1) for $\epsilon_{el}^q$, the integration proceeds from $r = 0$ to $r = R$. A local analysis of the differential equation near $r = 0$ reveals that the regularity conditions imply $r d\epsilon_{el}^q/dr = 2\epsilon_{el}^q$ at the centre of the body. Matching $\epsilon_{el}^q$ to the external version of Table 3.1 at $r = R$ determines $K_2^el$. A practical implementation of this procedure can be formulated in terms of the logarithmic derivative $\eta := r (d\epsilon_{el}^q/dr)/\epsilon_{el}^q$. This formulation is outlined in Sec. VIII of Ref. \cite{23}, and we make use of it for our numerical integrations.

The results of the integration of Eq. (4.3.1) are displayed in Fig. 5.1. The gravitoelectric Love numbers obtained from the matching procedure are found to scale like $(R/M)^5$; accordingly, we introduce the scale-free version $k_2^el := (2M/R)^5K_2^el$, which remains finite in the zero-compactness limit. The Love numbers $k_2^el$ are plotted against compactness $M/R$ for different polytropes in Fig. 5.2. They are observed to decrease monotonically in magnitude with increasing $M/R$. More-
The radial function \( e_{tt}^n \) for various polytropes. The surficial value of this numerical solution to Eq. (4.3.1) determines the gravitoelectric Love number \( K_{el}^2 \).

Over, stiffer equations of state produce larger values of \( K_{el}^2 \) overall. This is a natural consequence of the fact that stiffer equations of state generate more uniform density profiles with weaker internal gravity: it is easier to raise a mass quadrupole moment \( Q_{ab} \sim \int \delta \rho x_a x_b \, d^3x \) when the mass density is evenly distributed, rather than concentrated near the body’s centre of mass. We point out that the results shown in Fig. 5.2 agree with the calculations of Refs. [19, 21, 22].

### 5.2.2 Gravitomagnetic Love number

We now turn our attention to the gravitomagnetic sector of the field equations, which we specialize to the equation of state of Eq. (5.1.1). We integrate Eq. (4.3.5) from \( r = 0 \) to \( r = R \) to determine \( b_t^1 \), and match it with its external counterpart to compute \( K_{mag}^2 \). The regularity conditions on \( b_t^1 \) imply, via a local analysis of the differential equation, that \( r db_t^1/dr = 3 b_t^1 \) at \( r = 0 \). The differential equation can be cast into scale-free form to facilitate numerical integration; this is done in Sec. V of Ref. [24] by introducing the logarithmic derivative \( \kappa := r(db_t^1/dr)/b_t^1 \), and it is this formulation we employ for our numerical integrations.

We plot the numerical solution for \( b_t^1 \) in Fig. 5.3. The gravitomagnetic Love numbers calculated from the matching procedure are observed to scale with \( (R/M)^4 \); we therefore introduce a scale-free version \( K_{mag}^2 := (2M/R)^4 K_{mag}^2 \). The results for \( K_{mag}^2 \), plotted against compactness for different polytropes in Fig. 5.4, also decrease monotonically in magnitude with increasing compactness, and are smaller for softer equations of state. The intuition from the gravitoelectric sector holds here: it is more difficult to induce a current quadrupole when the body’s mass density is centrally
Figure 5.2: Scale-free gravitoelectric Love number $k_2^{\text{el}} := (2M/R)^5 K_2^{\text{el}}$ as a function of compactness $M/R$ for polytropes of index $n$. The Love numbers are computed up to the maximum compactness supported by the given equation of state. The circled data points at $M/R = 0$ represent the Newtonian Love number $k_2$ for each polytrope.
concentrated. The values we obtain for $k_2^{\text{mag}}$ correspond to the negative branches in the figure, and they are consistent with the results of Ref. [24], which used the correct irrotational state for the fluid. Our results differ from earlier work (e.g. Refs. [19, 21, 22]) that made the physically unjustified assumption of strict hydrostatic equilibrium, which gives rise to the positive branches in Fig. 5.4.

5.2.3 Rotational-tidal Love number $\tilde{\mathcal{F}}^0$

In this subsection, we calculate the rotational-tidal Love number $\tilde{\mathcal{F}}^0$ by integrating the gravitoelectric field equations of octupole order. Specializing to the polytropic equation of state of Eq. (5.1.1), we integrate Eq. (4.3.20) from $r = 0$ to $r = R$. At $r = 0$, $r df_t^o/dr = 4f_t^o$ by a local analysis of the differential equation, and at $r = R$, $f_t^o$ matches its external counterpart from Table 3.1. Because the boundary conditions at $r = 0$ and $r = R$ are each known only up to a scale, we use a shooting method to integrate the differential equation, which also involves the functions $\omega$ and $c_{tt}^o$. We plot the numerical solution for $f_t^o$ in Fig. 5.5.

The matching conditions at the body’s surface determine the rotational-tidal Love number $\tilde{\mathcal{F}}^0$. We find that it remains finite in the limit of zero compactness when scaled with a factor of $(M/R)^5$. Accordingly, we define $\mathcal{F}^0 := -(2M/R)^5\tilde{\mathcal{F}}^0$ as the scale-free version of the gravitoelectric rotational-tidal Love number, and plot it as a function of the compactness for various polytropes in Fig. 5.6. We note that the numerical results we display refer to a material body that is free of $r$-
Figure 5.4: Scale-free gravitomagnetic Love number $k_{2}^{\text{mag}} := (2M/R)^4 K_{2}^{\text{mag}}$ as a function of compactness $M/R$ for polytropes of index $n$. The Love numbers are computed up to the maximum compactness supported by the given equation of state. The positive branches refer to polytropes in the static fluid state; the negative branches correspond to the irrotational state. The legend of Fig. 5.2 identifies the curves shown here.
modes; they would be modified if such a mode were added to the solution. The gravitoelectric tidal-
rotational Love number is observed to decrease in magnitude with increasing compactness $M/R$; however, it changes sign from positive to negative along a sequence of increasing compactness for sufficiently stiff equations of state. The sign of the Love number reflects whether the tide has a stretching or compressing effect on the body. The existence of a compressive component in relativistic tides was first pointed out by Shapiro \[81\], and has been discussed in Ref. \[20\]. The scaling of $\tilde{\mathcal{S}}^0$ agrees with the post-Newtonian prediction of Ref. \[63\].

5.2.4 Rotational-tidal Love number $\mathcal{S}^0$

Finally, we integrate the gravitomagnetic field equations of octupole order, which determine the rotational-tidal Love number $\mathcal{S}^0$. The Love number is computed by matching $k_{tt}^0$, governed by Eq. (4.3.22), to its external version from Table 3.1 at $r = R$. The boundary condition at $r = 0$ for integration of Eq. (4.3.22) is $r d k_{tt}^0 / dr = 3 k_{tt}^0$ via a local analysis of the differential equation. We must again use a shooting method, since the boundary conditions on $k_{tt}^0$ are known only up to a scale. The differential equation for $k_{tt}^0$ involves the functions $\omega$, $b_1^0$ and $k_{tr,l}^0$, the last of which is determined by solving Eq. (4.3.24). The numerical solutions we obtain for $k_{tr,1}^0$ and $k_{tt}^0$ are plotted in Figs. 5.7 and 5.8 respectively.

The matching conditions on $k_{tt}^0$ at $r = R$ fix the value of the rotational-tidal Love number $\mathcal{S}^0$. We find that this Love number has the same scaling as $\tilde{\mathcal{S}}^0$, and we plot the scale-free version...
Figure 5.6: Scale-free gravitoelectric rotational-tidal Love number $f_o := -(2M/R)^{5/3}g_{\alpha}$ as a function of compactness $M/R$ for polytropes of index $n$. The Love numbers are computed up to the maximum compactness supported by the given equation of state. The legend of Fig. 5.2 identifies the curves shown here.
The radial function $k_{tr1}^0$ for various polytropes. This numerical solution to Eq. (4.3.24) is implicated in the dynamical tidal response, and it enters in the differential equation that governs $k_{tt}^0$. The legend of Fig. 5.1 identifies the curves shown here.

$k^0 := -(2M/R)^5 R^0$ versus compactness for various polytropes in Fig. 5.9. The scale-free nature of $k^0$ is ensured by the fact that it remains finite in the $M/R \to 0$ limit. The Love numbers are observed to decrease monotonically in magnitude with increasing $M/R$.

For a polytrope of index $n = 1$, the rotational-tidal Love number has the value of $k^0 = 9.0493 \times 10^{-2}$ in the zero-compactness limit. This figure agrees to one part in $10^5$ with the result of Ref. [89], which calculated the same quantity in a post-Newtonian approximation. The small discrepancy reflects the accuracy of our numerical integrations near $M/R = 0$.

5.2.5 Discussion: Pani, Gualtieri and Ferrari

The calculation of the rotational-tidal Love numbers was pursued in parallel by Pani, Gualtieri and Ferrari in Ref. [60]. We shall briefly clarify the differences between their work and ours. Just as this thesis depends on a series of papers (Papers I-III), which first identified the rotational-tidal Love numbers and then computed them, Ref. [60]’s computation of the Love numbers relies on the formalism of a prior paper by Pani et al., Ref. [65], in which the external geometry of a tidally deformed, rotating star is calculated to second order in its spin. Because the formalism of Ref. [65] is adapted to axisymmetric settings, Ref. [60] lacks the analogues of our Love numbers $\mathcal{E}_q^0$ and $\mathcal{B}_q^0$; however, since Ref. [65] treats the response to the tidal field’s higher multipole moments, Ref. [60]

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\(^3\)This discussion originally appeared in Paper III.
has additional Love numbers relative to this thesis. Nevertheless, both Ref. [60] and this work compute the octupolar rotational-tidal Love numbers which measure the response to spin-coupled quadrupolar tidal fields. By comparing the external metric of Ref. [60] to the one presented in Sec. 3.2, it is possible to establish the mapping between the definitions of the Love numbers in each formalism. We find that the Love numbers $\delta \tilde{\lambda}_M^{(32)}$ and $\delta \tilde{\lambda}_E^{(32)}$ of Ref. [60] can be expressed as a linear combination of our gravitational and rotational-tidal Love numbers; specifically,

$$\delta \tilde{\lambda}_M^{(32)} = -\frac{32}{\sqrt{5\pi}} \left( 5K_{2}^\text{eq} - 3\delta^o \right), \quad \delta \tilde{\lambda}_E^{(32)} \propto -144 \sqrt{\frac{7}{5}} \delta^o. \quad (5.2.1)$$

The mapping has been established modulo a prefactor in the latter case since $\delta \tilde{\lambda}_E^{(32)}$ was defined only up to an overall scale in Ref. [60].

Pani, Gualtieri and Ferrari compute rotational-tidal Love numbers for polytropes of index $n = 1$ and tabulated neutron star equations of state in Ref. [60]. Since we restrict our attention to polytropic equations of state in this dissertation, our results coincide only in the former case. Even so, the fact that we make different physical assumptions about the material body means that we cannot expect our results to be in quantitative agreement. First and foremost, Pani, Gualtieri and Ferrari place the material body in the static fluid state, which artificially prevents internal currents from developing through gravitomagnetic induction. As shown in Paper II, and discussed in Sec. 4.4, the static state is compatible with the Einstein field equations only in axisymmetry; the irrotational state employed in this thesis is valid in generic settings. The numerical value of the

Figure 5.8: The radial function $k_{tt}^o$ for various polytropes. The surficial value of this numerical solution to Eq. (4.3.22) determines the gravitomagnetic rotational-tidal Love number $\delta^o$. The legend of Fig. [5.1] identifies the curves shown here.
Figure 5.9: Scale-free gravitomagnetic rotational-tidal Love number $k_o := -(2M/R)^5 \mathcal{R}_o$ as a function of compactness $M/R$ for polytropes of index $n$. The Love numbers are computed up to the maximum compactness supported by the given equation of state. The circled data point at $M/R = 0$ represents the post-Newtonian result of Ref. [89] for an $n = 1$ polytrope. The legend of Fig. 5.2 identifies the curves shown here.
Love numbers depends on the choice of fluid state (cf. Ref. [24]), and our results will therefore
differ in this respect.

Second, the polytropic model adopted in Ref. [60] is different from the one employed here.
Pani, Gualtieri and Ferrari use the equation of state

\[ p = K \mu^{1+1/n}, \] (5.2.2)

which describes an “energy polytrope” whose pressure \( p \) is related to the total energy density \( \mu \). In
contrast, our chosen polytropic equation of state, Eq. (5.1.1), describes a “mass polytrope” whose
pressure is related to the mass density \( \rho \), and whose total energy density is \( \mu = \rho + np \). The mass-
polytrope equation of state is more closely tied to the thermodynamics of a degenerate Fermi gas,
the simplest idealization of neutron star matter [25]. The difference between the two equations of
state becomes more pronounced for larger values of the compactness.

These significant differences in formulation prevent us from making a precise comparison of
numerical results with Pani, Gualtieri and Ferrari in this thesis. However, we do note that when
subjected to the mapping prescribed in Eq. (5.2.1), our results are similar in magnitude to those
of Pani, Gualtieri and Ferrari, and display the same qualitative behaviour as a function of the
compactness as can be seen in Figs. 5 and 9 of Ref. [60]. Although the rotational-tidal Love
numbers may not change as drastically as \( K_{mag}^2 \) when going from the static to the irrotational fluid
state, the restrictions placed on the fluid have a dramatic effect on the internal currents generated
by the gravitomagnetic part of the tidal field, as discussed below.

5.3 TIDAL CURRENTS

5.3.1 Discussion: Dynamical response

The developments of Sec. 4.2 and Sec. 4.3 revealed that a slowly rotating body undergoes a
dynamical response when subjected to a stationary tidal field: a number of internal metric and
fluid variables are time dependent. The field equations of Sec. 4.3 also confirmed that the exterior
geometry bears no trace of the internal dynamics: the time-dependent metric variables are sup-
ported inside the body only, and they vanish smoothly at \( r = R \) (this was verified explicitly in
Paper II). The exterior metric is thus perfectly stationary, despite the time-dependent interior.

All the time-dependent terms in the metric and fluid variables require the existence of a grav-
itomagnetic field \( B_{ab} \); the gravitoelectric field \( E_{ab} \) does not provoke a dynamical response, and all
associated quantities are stationary. The time dependence revealed by the fluid and field equations
describes a steady growth. We show in Sec. 5.3.3 that the time dependence is in fact bounded
when \( B_{ab} \) is identified with the gravitomagnetic field sourced by a companion in a compact binary
system.

The dynamical response of a slowly rotating body to a stationary tidal field is a startling
outcome. To better understand its origin, we relate two intuitive explanations for the phenomenon
which have been articulated in the literature. The first is due to Poisson & Douçot [89], and it

\footnote{This fact is a consequence of the linearization in \( \chi \); a calculation taken to higher order in the spin would reveal
the gravitational waves produced by the time-dependent interior.}
is formulated in post-Newtonian theory: it is therefore approximately true in general relativity. Nonetheless, it captures the essence of the phenomenon.

The reference body of angular velocity $\Omega$ is immersed in an external gravitomagnetic field $B$. As a result of the post-Newtonian equations of motion (see e.g. Ref. [18]), the body experiences a Lorentz-like force density

$$ f = \rho v \times B, \quad (5.3.1) $$

where $v = \Omega \times x$ is the body’s rotational velocity, and $x$ is the position relative to its centre of mass. In contrast to the situation in Newtonian or general-relativistic gravitoelectric tides, this force is not equilibrated by pressure gradients within the fluid (cf. Eqs. (4.2.19) and (4.2.25)). Instead, the action of the gravitomagnetic force establishes a velocity perturbation $\delta v$ proportional to the time integral of the tidal field via the equation of motion

$$ \rho \dot{\delta v} = f \quad (5.3.2) $$

at first order in the perturbation, where $\dot{\cdot} := d/dt$. When $B$ is idealized as time-independent, the velocity field grows linearly in time, exactly as observed in Sec. 4.2. In the more realistic case of a time-varying $B$, $\delta v$ is modulated by the changes in the tidal environment [89].

The unbalanced Lorentz-like force argument also provides an intuitive picture of the irrotational currents induced inside a nonrotating body by an external gravitomagnetic field. In this case, because the unperturbed body is static, $v = 0$. This implies that $f = 0$, and Eq. (5.3.2) then gives rise to stationary fluid motions with $\delta \dot{v} = 0$. These motions are necessarily irrotational, because the vorticity vector $\omega := \nabla \times v$ vanishes for the unperturbed configuration, and the circulation theorem [72] guarantees that $\omega$ is conserved throughout the evolution of the fluid.

The second explanation for the origin of the dynamical response was presented in Paper II. It is essentially equivalent to the first, though it does not rely on a post-Newtonian approximation, and it is framed in terms of oscillation modes rather than forces. The argument, as formulated here, is neither rigorous nor precise, but it accounts for the essential features of the effect.5

The dynamics of the tidally perturbed fluid that makes up the material body are described in terms of the Lagrangian displacement $\xi$, which maps the position of a perturbed fluid element to its original, unperturbed position. The displacement field satisfies a complicated partial differential equation which can be solved by decomposing $\xi$ in a complete basis of normal modes $z_\lambda$, with $\lambda$ denoting a mode label. Writing $\xi(t, x) = \sum_\lambda a_\lambda(t) z_\lambda(x)$, the fluid equations produce a mode equation of the form

$$ \ddot{a}_\lambda + \omega_\lambda^2 a_\lambda = f_\lambda, \quad (5.3.3) $$

where $\omega_\lambda$ is the mode’s natural frequency and $f_\lambda = \int F \cdot z_\lambda d^3x$ is the forcing function, obtained by evaluating an overlap integral between the external perturbation $F$ and each mode function. The mode equation is simply the familiar equation that governs a simple harmonic oscillator driven by an external force $f_\lambda$.

In the analogy with the tidal deformation problem, $f_\lambda$ represents the time-independent tidal forces, and the expected stationary solution to the mode equation is obtained by balancing the

---

5A more detailed version of the argument was developed by Poisson & Douçot in the post-Newtonian regime in Ref. [89].
external force with the oscillator’s restoring force: \( a_\lambda = \omega_\lambda^{-2} f_\lambda \). However, when the mode spectrum includes zero-frequency modes, there is no restoring force to balance out the external force, and the solution cannot be stationary. Choosing an unperturbed state at \( t = 0 \), one obtains instead \( a_\lambda = \frac{1}{2} f_\lambda t^2 \). The displacement field grows quadratically with time, and it gives rise to a velocity field proportional to \( t \). This is precisely the situation observed in Sec. 4.2. We recall, however, that the linear time dependence is likely an artifact of the linearization in \( \Omega \) we implement in our calculations – our analysis cannot distinguish between \( \Omega t \) and \( \sin \Omega t \) growth. We consider it probable that the time dependence is actually bounded.

The existence of zero-frequency modes in relativistic fluids is well documented \[80\], and includes the \( r \)- and \( g \)-modes we discussed in Sec. 4.3.1. These two types of modes, which describe a steady velocity field inside the body, are implicated in the tidal response of a slowly rotating body. In the absence of rotation, the overlap integral between the external tidal field and each zero-mode function vanishes, and the response is stationary. In the presence of rotation, however, the forcing function no longer vanishes, and the tidal response is dynamical. The computations of Paper II indicate that the overlap integral vanishes whenever \( B_{ab} = 0 \); this explains why the dynamical response requires a gravitomagnetic tidal field.

The picture of the dynamical response delineated above is subject to one important caveat, which we now state. It is well-known that rotating relativistic stars, in contrast to static ones, have no pure \( r \)- or \( g \)-modes \[90\]. Instead, they have a spectrum of rotational modes of finite frequency and indefinite parity \[80, 91\]. The connection between the rotational modes and the dynamical tidal currents is not totally clear; one expects nonetheless that a more complete treatment of the fluid would incorporate the finite mode frequencies in Eq. (5.3.3). As we shall not attempt a full mode decomposition of the tidal perturbations in this dissertation, we restrict our analysis to the zero-frequency modes of spherical stars. At any rate, the rotational mode frequencies are proportional to \( \Omega \) and are therefore small in slowly rotating stars \[71\].

### 5.3.2 Perturbed velocity field

The preceding section’s discussion interpreted the time-dependent fluid perturbations revealed in Sec. 4.2 as dynamical tidal currents induced by the external gravitomagnetic field. In this section, we calculate the dynamical part of the perturbed velocity field by integrating the relevant field equations for polytropes. In doing so, a clear picture of the time-dependent velocity field induced within the body emerges. The computations of this section will then be used to calculate the tidal currents that arise in a neutron star binary in Sec. 5.3.3.

We begin with the \( \ell = 2 \) gravitomagnetic sector of the bilinear response; we remind the reader that the gravitoelectric sector of the problem was found to be completely stationary. The time-dependent fluid variable \( \hat{v}^d \) is given by Eq. (4.2.29) in terms of \( \hat{b}^d_{11} \), which was calculated in the preceding section. The metric variable \( \hat{b}^d_{11} \) is also dynamical, and to understand its contribution to the response we integrate Eq. (4.3.10). At \( r = 0 \), a local analysis of the differential equation returns \( r dB^d_{11}/dr = 3 \hat{b}^d_{11} \), and at \( r = R \), \( \hat{b}^d_{11} \) matches its external version from Table 3.1. The numerical results for \( \hat{v}^d \) and \( \hat{b}^d_{11} \) are plotted as a function of radius in Figs. 5.10 and 5.11.

Turning to the dipole sector of the bilinear gravitomagnetic response, we note that the time-dependent fluid variables \( \hat{v}^d \) and \( \hat{v}^d \) of Eqs. (4.3.18a) and (4.3.18b) depend on the radial function \( k^d_{tr1} \), governed by Eq. (4.3.15). We integrate Eq. (4.3.15) from \( r = 0 \) to \( r = R \); the regularity
Figure 5.10: Radial profile of the fluid variable $\hat{v}$ for several polytropes. The velocity perturbation is calculated from Eq. (4.2.29), and it is implicated in the dynamical tidal response. The legend of Fig. 5.1 identifies the curves shown here.

Figure 5.11: The radial function $\hat{b}$ for various polytropes. This numerical solution to Eq. (4.3.10) is implicated in the dynamical response. The legend of Fig. 5.1 identifies the curves shown here.
conditions on $k^d_{tr1}$ imply that $rdk^d_{tr1}/dr = 2k^d_{tr1}$ at $r = 0$, and the matching conditions imply that $k^d_{tr1}$ and its first derivative vanish at $r = R$. The numerical results we obtain for $v^d_r$, $v^d$ and $k^d_{tr1}$ are plotted against radius in Figs. 5.12, 5.13 and 5.14.

Finally, using the previously determined solution to Eq. (4.3.24) for the dynamical metric variable $k^o_{tr1}$, we compute $v^o_r$ and $v^o$ from Eqs. (4.3.27a) and (4.3.27b). These are the dynamical fluid variables which make up the octupole sector of the bilinear gravitomagnetic response. We plot the radial profiles of $v^o_r$ and $v^o$ in Figs. 5.15 and 5.16.

We remark that the angular components of the velocity perturbation (namely $v^d$, $\hat{v}^q$ and $v^o$) vanish at the centre of the body, and reach their maximum magnitude at the surface. On the other hand, the radial components $v^d_r$ and $v^o_r$ vanish at $r = R$.

5.3.3 Tidal currents in a binary

The external gravitomagnetic tidal field induces time-dependent velocity perturbations, despite the assumed time independence of $B_{ab}$. As argued in Paper II, and shown convincingly by the post-Newtonian analysis of Ref. [89], the dynamical velocity perturbations represent time-varying internal currents which are driven by the zero-frequency modes of the fluid when the gravitomagnetic tidal field is allowed to couple to the body’s spin. In this section, we calculate the amplitude
Figure 5.13: Radial profile of the fluid variable $v^d$ for several polytropes. This fluid variable is calculated from Eq. (4.3.18b), and it is implicated in the dynamical tidal response. The legend of Fig. 5.1 identifies the curves shown here.

We suppose that the reference body is a neutron star in a binary system with a companion of mass $M'$ moving on a circular orbit of radius $b$ and orbital angular frequency $\Omega_{\text{orb}} = [(M + M')/b^3]^{1/2}$. We adopt a Cartesian coordinate system oriented so that the orbital plane lies in the $x$-$y$ plane. In these coordinates, the gravitomagnetic tidal quadrupole moment sourced by the companion’s orbital motion has nonvanishing components

$$B_{13} = -\frac{3M'\Omega_{\text{orb}}}{b^2}\cos\Omega_{\text{orb}}t, \quad B_{23} = -\frac{3M'\Omega_{\text{orb}}}{b^2}\sin\Omega_{\text{orb}}t.$$  \hfill (5.3.4)

So far in this thesis, we have assumed that the orbital timescale $T \sim 1/\Omega_{\text{orb}}$ is much longer than the internal hydrodynamical timescale $T \sim (R^3/M)^{1/2}$ for the body’s tidal response, so that the tidal moments are effectively stationary. The assumption of strict stationarity is responsible for the linear time dependence of $\hat{v}^q$ which follows from Eqs. (4.2.28) and (4.2.29). We now relax this assumption to better model the binary setting, and allow the tidal moments to retain the sinusoidal time dependence exhibited in Eq. (5.3.4).

The effect of taking $B^q_m(t) = B^q_m(t)$ can be understood by examining Eq. (4.2.28), in which we reinsert the implicit factors of the tidal potential $B^q_A$:

$$e^{-\psi} [\partial_t (\xi^q B^q_A) + b^q B^q_A] = 0.$$  \hfill (5.3.5)

---

6This work was originally carried out in Paper III.
Previously, for stationary $B^q_m$, this statement produced a linear time dependence $\xi^q B^q_A = -t b^q_A B^q_A$. In contrast, for time-dependent $B^q_m$, it yields a bounded time dependence

$$
\xi^q B^q_A = -b^q_A \int_0^t B^q_A(t')dt' = -b^q_A \sum_m \int_0^t B^q_m(t')dt' X^2_m.
$$

(5.3.6)

The net effect of the tidal moments’ sinusoidal time dependence is therefore to take

$$
t B^q_m \rightarrow \int_0^t B^q_m(t')dt' = \frac{1}{\Omega_{\text{orb}}} \int_0^\Phi B^q_m(\Phi')d\Phi',
$$

(5.3.7)

where $\Phi := \Omega_{\text{orb}} t$ is the orbital phase.\(^7\) Because the spherical-harmonic coefficients $K^d_m$, $\tilde{B}^q_m$ and $K^o_m$ are related to $B^q_m$ (see Table 3.1), the time dependence further implies

$$
t K^d_m \rightarrow \frac{1}{\Omega_{\text{orb}}} \int_0^\Phi K^d_m(\Phi')d\Phi', \quad t \tilde{B}^q_m \rightarrow \frac{1}{\Omega_{\text{orb}}} \int_0^\Phi \tilde{B}^q_m(\Phi')d\Phi', \quad t K^o_m \rightarrow \frac{1}{\Omega_{\text{orb}}} \int_0^\Phi K^o_m(\Phi')d\Phi'.
$$

(5.3.8)

These transformations modify the time-dependent metric and fluid variables $\{k^d_{tr}, \tilde{b}^q_{tr}, k^o_{tr}\}$ and

---

\(^7\)In keeping with the caveat stated at the end of Sec. 5.3.1, one might expect a treatment incorporating the finite rotational-mode frequencies $\omega_\lambda \sim \Omega$ to find that the dynamical response varies over the rotational timescale $1/\Omega$, rather than the orbital timescale $1/\Omega_{\text{orb}}$, when $\Omega_{\text{orb}} \ll \Omega$. 
\{v^d_r, v^o_r, v^d, \dot{v}^q, v^o\}, but leave time-independent ones like \(b^q_t\) unchanged.

Implementing these changes in the results of Sec. 4.2.2 and Sec. 4.3, the Eulerian velocity perturbation \(\delta u^\alpha\) can be calculated explicitly for the binary neutron star. The contravariant expression \(\delta u^\alpha\) is related to the covariant components of \(\delta u_\alpha\) given in Eq. (4.2.24) as follows:

\[
\begin{align*}
\delta u^r &= f \int_0^\Phi \left[ \left( \frac{v^d_r}{t\Omega_{\text{orb}}} - e^{-\psi} \frac{k^d_{tr1}}{\Omega_{\text{orb}}} \right) \sum_m \mathcal{K}^d_m(\Phi')Y^{1m} + \left( \frac{v^o_r}{t\Omega_{\text{orb}}} - e^{-\psi} \frac{k^o_{tr1}}{\Omega_{\text{orb}}} \right) \sum_m \mathcal{K}^o_m(\Phi')Y^{3m} \right] d\Phi', \\
\delta u^A &= r^{-2}e^{-\psi}A^B \left[ e^\psi \delta u_B - p_{tB} + p_{BC}^{CD}\chi^d_{BD} \right] - \frac{1}{2}e^{-3\psi}(1 - \omega)p_{tt}\Omega^{AB}\chi^d_{BA}, \tag{5.3.9b}
\end{align*}
\]

Expressed in terms of the tidal potentials and the variables of this subsection, one finds that

\[
\begin{align*}
\delta u^r &= f \int_0^\Phi \left[ \left( \frac{v^d_r}{t\Omega_{\text{orb}}} - e^{-\psi} \frac{k^d_{tr1}}{\Omega_{\text{orb}}} \right) \sum_m \mathcal{K}^d_m(\Phi')Y^{1m} + \left( \frac{v^o_r}{t\Omega_{\text{orb}}} - e^{-\psi} \frac{k^o_{tr1}}{\Omega_{\text{orb}}} \right) \sum_m \mathcal{K}^o_m(\Phi')Y^{3m} \right] d\Phi', \\
\delta u^A &= r^{-2}\Omega^{AB} \int_0^\Phi \left[ \frac{v^d}{t\Omega_{\text{orb}}} \sum_m \mathcal{K}^d_m(\Phi')Y^{1m} + \left( \frac{\dot{v}^q}{t\Omega_{\text{orb}}} - e^{-\psi} \frac{\hat{b}^q_{t1}}{\Omega_{\text{orb}}} \right) \sum_m \frac{1}{2}B^q_m(\Phi')X^{2m} \right. \\
&\quad + \left. \frac{v^o}{t\Omega_{\text{orb}}} \sum_m \frac{1}{3}K^o_m(\Phi')Y^{3m} \right] d\Phi' - r^{-2}e^{-\psi}b^q_t\Omega^{AB}B^{q}_{\text{B}}, \tag{5.3.10b}
\end{align*}
\]
Figure 5.16: Radial profile of the fluid variable $v^o$ for several polytropes. This velocity perturbation is calculated from Eq. (4.3.27b), and it is implicated in the dynamical tidal response. The legend of Fig. 5.1 identifies the curves shown here.

Eq. (5.3.10) provides a complete description of the currents induced in a neutron star by the gravitomagnetic tidal field of a binary companion to first order in the tidal deformation and the neutron star’s spin. The entire radial component $\delta u^r$ is first order in spin, while the angular components $\delta u^A$ consist of a piece $\delta u^A_{\text{tidal}} := -r^{-2}e^{-\psi}b^\theta_1 \Omega^{AB} B_{B}^\theta$, which is zeroth order in spin, and a first-order piece $\delta u^A_{\text{bilinear}}$. The former piece is an irrotational tidal current directly induced by the gravitomagnetic quadrupole moment $\mathcal{B}_{\text{ab}}$; its time dependence is merely parametric, in the sense that it simply reflects the modulation of $\mathcal{B}_{\text{ab}}$ with the orbital phase $\Phi$. The latter piece is a genuinely dynamical tidal current induced by the spin-coupled, bilinear gravitomagnetic moments. The integrals of $\mathcal{K}_m^d$, $\mathcal{B}_m^d$, and $\mathcal{K}_m^o$ produce a sinusoidal variation of the amplitude that is superposed on the time dependence inherited from Eq. (5.3.4). We show the fluid currents described by Eq. (5.3.10) on an equatorial slice $(r, \theta = \pi/2, \phi)$ of the neutron star in Fig. 5.17.

For concreteness, we now proceed to calculate the amplitude of the tidal currents at the neutron star’s equator $(r = R, \theta = \pi/2, \phi)$ by evaluating the tangential velocity perturbation $\delta v^A := R \delta u^A$. The radial component of the velocity perturbation automatically vanishes at the surface because the functions $k_{\text{tr}1}^d$ and $k_{\text{tr}1}^o$, as well as their first derivatives, are zero at $r = R$. The function $b^\theta_1$ also vanishes at $r = R$, and only the $\phi$ component of $\delta v^A$ is nonzero at the equator. Thus, on the basis of Eq. (5.3.10), we compute

$$\delta v^\phi_{\text{tidal}} = -\frac{3M'R^2\Omega_{\text{orb}}}{b^2}e^{-\psi}b^\theta_1 \frac{b^q}{R^3} \cos (\phi - \Phi)$$

at zeroth order in spin, and
Figure 5.17: Equatorial projection of the internal fluid currents defined by Eq. (5.3.10). The velocity fields are evaluated at orbital phase $\Phi = \pi/2$. The binary companion’s position and direction of motion are indicated by the dot and arrow above the disk. Left: the stationary piece $\delta v_{\text{tidal}} = \delta u_{\text{tidal}}^r \hat{r} + r \delta u_{\text{tidal}}^\theta \hat{\theta} + r \delta u_{\text{tidal}}^\phi \hat{\phi}$. Right: the dynamical piece $\delta v_{\text{bilinear}} = \delta u_{\text{bilinear}}^r \hat{r} + r \delta u_{\text{bilinear}}^\theta \hat{\theta} + r \delta u_{\text{bilinear}}^\phi \hat{\phi}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M/R$</th>
<th>$\sigma$</th>
<th>$\delta v$ (km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.13</td>
<td>1.117</td>
<td>2.234</td>
</tr>
<tr>
<td></td>
<td>0.17</td>
<td>0.998</td>
<td>1.997</td>
</tr>
<tr>
<td></td>
<td>0.21</td>
<td>0.871</td>
<td>1.743</td>
</tr>
<tr>
<td>0.75</td>
<td>0.13</td>
<td>1.233</td>
<td>2.466</td>
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<tr>
<td></td>
<td>0.17</td>
<td>1.096</td>
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<tr>
<td></td>
<td>0.21</td>
<td>0.952</td>
<td>1.903</td>
</tr>
<tr>
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<td>0.13</td>
<td>1.327</td>
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<tr>
<td></td>
<td>0.17</td>
<td>1.174</td>
<td>2.348</td>
</tr>
<tr>
<td></td>
<td>0.21</td>
<td>1.009</td>
<td>2.018</td>
</tr>
</tbody>
</table>

Table 5.1: The dimensionless, equation-of-state-dependent parameter $\sigma$ from Eq. (5.3.13) for polytropic models of index $n$ and compactness $M/R$, for use in the estimate Eq. (5.3.14). The adoption of the fiducial values of Eq. (5.3.15) for the binary system produces the values of the equatorial velocity perturbation $\delta v$ listed here.
\[ \delta v_{\text{bilinear}} = -\frac{M'R^2\Omega}{b^2} \sigma \sin \left( \Phi/2 \right) \sin \left( \phi - \Phi/2 \right) \] (5.3.12)

at first order. We have defined the dimensionless quantity

\[ \sigma := -6 \left( \frac{v^d}{tR^3} + \frac{\dot{q}q}{tR^3} - \frac{2}{15} \frac{v^q}{tR^3} \right) \] (5.3.13)

to encode all the equation-of-state dependence of \( \delta v_{\text{bilinear}} \). As shown in Table 5.1, \( \sigma \) is roughly of order unity for the polytropic models studied in this paper. Restoring factors of \( G \) and \( c \), the amplitude \( \delta v \) of Eq. (5.3.12) can be written as

\[ \delta v = \sigma \frac{(2\pi)^{7/3}G^{1/3}}{c^2} \frac{M'}{(M + M')^{2/3}} \frac{R^2}{P} f^{4/3} \] (5.3.14)

in terms of the neutron star’s rotational period \( P := 2\pi/\Omega \) and the orbital frequency \( f := \Omega_{\text{orb}}/2\pi \). Evaluated with the parameters of a typical equal-mass neutron star binary system during inspiral, the dynamical tidal currents have amplitude

\[ \delta v = 2\sigma \left( \frac{M'}{1.4 M_\odot} \right) \left( \frac{2.8 M_\odot}{M + M'} \right)^{2/3} \left( \frac{R}{12 \text{ km}} \right)^2 \left( \frac{100 \text{ ms}}{P} \right) \left( \frac{f}{100 \text{ Hz}} \right)^{4/3} \text{ km/s} \] (5.3.15)

at the equator. This figure differs only by a factor of \( \sigma \) from the post-Newtonian order-of-magnitude estimate of Poisson & Douçot [89].
CHAPTER 6

CONCLUSIONS AND FUTURE DIRECTIONS

6.1 Conclusions

The study of tides on rotating neutron stars is an important undertaking in light of the prospect of constraining the nuclear equation of state through gravitational-wave observations of coalescing compact binaries. In this dissertation, we have laid out the general-relativistic theory of tidal deformations of spinning bodies with this motivation in mind. We focused on the case of a slowly rotating material body disturbed by weak, slowly varying tides, such as those sourced by a binary companion at the large orbital separations which characterize the early inspiral stage of a binary’s orbital evolution. The weakness of the tides permitted us to work to first order in a perturbative expansion in the reference body’s deformation, and to leading (quadrupole) order in the tidal interaction – we idealized the tidal field as a pure quadrupole. The slow variation of the tidal field allowed us to further idealize it as stationary, and to work in an adiabatic approximation whereby the perturbation never takes the body out of hydrostatic equilibrium. This set of approximations constitutes the regime of stationary tides in which our results provide an accurate description of the tidal deformation; this regime breaks down, for instance, in the late stage of the inspiral, when the orbital frequency approaches resonance with the natural frequencies of the oscillation modes of the neutron star, and dynamical tides dominate the body’s response.

The slow-rotation approximation was motivated by astrophysical observations which indicate that neutron stars in binaries rotate at only a few percent of the speed of light, and it enabled us to approximate the equilibrium shape of the body as spherical. Linearization in the spin parameter also made the coupling between the body’s spin angular momentum and the tidal moments analytically tractable. Studies of the deformation of a rotating body have been performed to second order in spin by Refs. [60, 65] and to higher orders by Refs. [92, 93], but they are limited to axisymmetric situations. In contrast, our approach is valid in generic settings, despite its restriction to slow rotation.

We also made a number of assumptions about the composition of the material body. We neglected dissipation and took the matter to consist of a perfect fluid. In order to model a neutron star, which has negligible temperature, we adopted a one-parameter zero-temperature equation of state for the fluid. Under these conditions, the fluid equations give rise to a conservation law for the body’s vorticity tensor. Conservation of vorticity is a key physics input that was overlooked in earlier studies of tidally deformed nonrotating bodies (e.g. Refs. [21, 22]), and in more recent
studies of rotating bodies (e.g. Refs. [60, 65]).

In Papers I-III, we reached a number of important conclusions about the tidal response of a slowly rotating barotrope in the regime of stationary tides. These results were presented in detail in Ch. 2-5 of this thesis, and we summarize the main achievements of our research program here.

In Ch. 3 (originally in Paper I with Poisson), we identified as rotational-tidal Love numbers a class of integration constants associated with decaying terms in the metric outside a tidally deformed, slowly rotating body. The rotational-tidal Love numbers measure the body’s quadrupole and octupole responses to couplings between its spin and the applied quadrupolar tidal field. We found in Ch. 4 (originally in Paper III) that the two quadrupole rotational-tidal Love numbers are surprisingly independent of the material body’s equation of state: they have a fixed, universal value that causes the metric perturbations with which they are associated to vanish outside the body. The integrations of Ch. 5 revealed that the octupole rotational-tidal Love numbers scale like $(R/M)^{5}$, as predicted by the post-Newtonian calculations of Paper I and Ref. [89]. Using the polytropic relation $p \propto \rho^{1+1/n}$ as a stand-in for the unknown neutron star equation of state, we computed the octupole rotational-tidal Love numbers for polytropes and found that they generally decrease as a function of increasing compactness of the model, and do likewise as a function of decreasing stiffness of the equation of state. The value of $\tilde{K}_o$ computed for an $n = 1$ polytrope by Ref. [89] in a post-Newtonian approximation was recovered in the zero-compactness limit of our results.

Although this thesis focused on the case of a tidally deformed material body, the exterior metric constructed in Ch. 3 (originally in Paper I with Poisson) was found to describe a perturbed black hole (cf. Ref. [62]) when the gravitational and rotational-tidal Love numbers are set to zero. This finding, corroborated by the work of Refs. [60, 79], demonstrates that the rotational-tidal Love numbers of black holes vanish, just like their gravitational counterparts [22], and implies that the tidal deformations of material bodies and black holes can be treated in a unified framework.

In Sec. 4.4 (originally in Ref. [24] with Poisson), we explained that the vorticity conservation condition gives rise to stationary, irrotational fluid currents inside a nonrotating body. These currents are automatically generated by the gravitomagnetic part of the tidal field; they were first discussed by Shapiro [81] and Favata [20]. The irrotational currents impact the calculation of the gravitomagnetic Love number $K_2^{\text{mag}}$: while it is positive for a body in strict hydrostatic equilibrium, it is negative for the irrotational state.

The vorticity conservation condition revealed that gravitomagnetic induction has an even more significant impact on the fluid in a rotating body. Remarkable dynamical tidal currents, characterized by time-dependent internal metric and fluid variables, were discovered in Ch. 4 (originally in Paper II with Poisson). Despite the dynamics inside the body, the exterior metric remains perfectly stationary. In Sec. 5.3.3, we explained that Pani, Gualtieri and Ferrari [60] did not observe the dynamical fluid motions in a similar analysis of theirs because of the artificially restrictive assumption of strict hydrostatic equilibrium they placed on the fluid. We estimated the amplitude of the tidal currents in a typical neutron star binary in Ch. 5 (originally Paper III), and found it to be on the order of a kilometer per second. This figure depends only weakly on the equation of state, and is in close agreement with the post-Newtonian estimate of Poisson & Douçot [89].
6.2 Future Directions

The work presented in this thesis is by no means an exhaustive treatment of the tidal deformation problem in general relativity, and many interesting questions remain open for future investigation. Firstly, Pani, Gualtieri and Ferrari claim in Ref. [60] that the most important spin corrections to the tidal gravitational-wave phase of Eq. (1.1.1) come not from the octupole rotational-tidal Love numbers $F^o$ and $K^o$ we calculated in Sec. 5.2, but rather from quadrupole rotational-tidal Love numbers that arise from the coupling of the body’s spin to the octupole moments of the tidal field. These quadrupole Love numbers, denoted $\delta\lambda_E^{(23)}$ and $\delta\lambda_M^{(23)}$ in Ref. [60], are distinct from our Love numbers $E^q$ and $B^q$, which result from spin-coupled tidal quadrupole moments. The authors of Ref. [60] claim that $\delta\lambda_E^{(23)}$ produces a correction of $\sim 13\%$ to the induced quadrupole moment of a tidally deformed, slowly rotating neutron star. However, this calculation was performed for a fluid body in the unphysical static state, and it would be interesting to see if the estimate changes when the quadrupole rotational-tidal Love numbers are calculated for the irrotational fluid state. For astrophysical relevance, it would be desirable to carry out the computation with true candidate neutron star equations of state, rather than polytropes, and the calculation could therefore serve as an opportunity to recompute all the rotational-tidal Love numbers with realistic, tabulated equations of state – at present, these have only been used in the context of the static fluid state.

Secondly, the large-amplitude dynamical currents which were revealed in Sec. 5.3.3 to arise generically in coalescing binary systems of rotating neutron stars may themselves contribute to the tidal phasing of the binary’s gravitational waves. For instance, the piece $\delta^q_0$ of the perturbed velocity field produces purely nonaxisymmetric, time-varying quadrupole currents, and should consequently generate gravitational radiation. These gravitational waves are likely too weak to be directly detectable, but the current quadrupole induced in the body could impact the orbital dynamics of the inspiral and thereby leave an imprint on the gravitational-wave signal from the binary as a whole. To investigate this possibility, it would be interesting to conduct the type of analysis described by Essick, Vitale & Weinberg, who discovered a potentially significant effect of tides on inspiral waveforms due to nonlinear $p$- and $g$-mode couplings [94].

Thirdly, it seems plausible that the time-dependent metric and fluid variables discovered in Secs. 4.2 and 4.3 which constitute the dynamical response discussed in Sec. 5.3.1 may be related to the rotational modes of relativistic stars (also known as inertial modes) described by Lockitch, Andersson & Friedman [80, 91]. The rotational modes generalize the well-known $r$-modes of Newtonian gravity, and this connection raises a tantalizing possibility: like the $r$-modes [90], the rotational modes are susceptible to the Chandrasekhar-Friedman-Schutz instability [95, 96], and this suggests that an external tidal field could naturally create rotational modes that may be driven unstable by the emission of gravitational radiation. The driving of the modes would occur on a shorter timescale than the timescale for viscous damping, and it would persist as long as the binary companions orbit one another. On the other hand, one might expect the growth of the modes to be limited by viscosity, since the viscous damping timescale is longer than the radiation-reaction timescale. In any case, the tidal excitation of rotational modes could conceivably be a significant factor in the tidal interaction of neutron stars.

Finally, for nonrotating bodies there exists a complimentary description of the tidal deformation in terms of the distortion of a body’s shape, rather than its perturbed exterior spacetime.
geometry [97]. This description characterizes the deformation of the body’s surface in terms of the perturbation of its intrinsic curvature; quadrupolar deformations are measured by a *surficial Love number* $h_2$, which is related to the gravitoelectric Love number $K_{2}^{el}$ in a simple way for perfect fluids [23]. At present, no analogue of $h_2$ exists to measure the deformation of a rotating body’s surface under the influence of spin-coupled tidal quadrupole moments; the formulation of a general-relativistic theory of surficial deformations for rotating compact objects could lead to a more intuitive understanding of some aspects of the tidal interaction.

Beyond these specific avenues for further inquiry, there are plenty of opportunities to relax some of the assumptions made regarding the tidal deformation problem in this thesis. Dispensing with the linearization in spin would allow the tidal response of rapidly rotating bodies to be modelled perturbatively; considering deformations quadratic in the tidal moments would capture more strong-field aspects of the tidal interaction; and allowing the tidal field to vary over a shorter timescale would provide a window into the regime of dynamical tides. In this way, more features of the tidal interaction of coalescing compact binaries may come to be understood from an analytic perspective, providing theoretical insight to complement numerical simulations and astrophysical observations.
APPENDIX A
DERIVATION OF THE TIDAL ENVIRONMENT METRIC

The metric of the tidal environment, Eq. (2.1.4), is derived from Eq. (3.4) of Ref. [66]. In this equation, the tidal environment metric is expressed in lightcone coordinates \((v, r, \theta, \phi)\), and it contains a variety of irreducible potentials which are irrelevant for our purposes: octupole and higher multipole tidal potentials, time derivatives of the tidal potentials, and potentials that are quadratic in the tidal moments. We discard all but the leading-order quadrupole tidal potentials, leaving

\[
\begin{align*}
g_{vv} &= -1 - r^2 \mathcal{E}^q, \\
g_{vr} &= 1, \\
g_{rr} &= 0, \\
g_{vA} &= -\frac{2}{3} r^3 (\mathcal{E}_A^q - B_A^q), \\
g_{rA} &= -\frac{2}{3} r^3 (\mathcal{E}_A^q - B_A^q), \\
g_{AB} &= r^2 \Omega_{AB} - \frac{1}{3} r^4 (\mathcal{E}_{AB}^q - B_{AB}^q).
\end{align*}
\]

We then convert Eq. (A.0.1) to our preferred Minkowski-space spherical coordinates \((t, r, \theta, \phi)\) with the transformation

\[
v = t + r
\]

which gives \(v\) its meaning as an advanced time coordinate. The resulting metric can be viewed as a perturbation \(p_{\alpha\beta}\) of the Minkowski metric, with components

\[
\begin{align*}
p_{tt} &= -r^2 \mathcal{E}^q, \\
p_{tr} &= -r^2 \mathcal{E}^q, \\
p_{rr} &= -r^2 \mathcal{E}^q,
\end{align*}
\]
\[ p_{tA} = -\frac{2}{3} r^3 (E_A^q - B_A^q), \tag{A.0.3d} \]
\[ p_{rA} = -\frac{2}{3} r^3 (E_A^q - B_A^q), \tag{A.0.3e} \]
\[ p_{AB} = -\frac{1}{3} r^4 (E_{AB}^q - B_{AB}^q). \tag{A.0.3f} \]

While written in the desired coordinates, Eq. (A.0.3) cannot be used with the metrics of Ch. 3 and Ch. 4 because it is expressed in a different gauge. Performing a spherical-harmonic decomposition of the perturbation in the same manner as Eqs. (3.1.4) and (3.1.5), we obtain

\[ h^{2m}_{tt} = -r^2 E^q_m, \tag{A.0.4a} \]
\[ h^{2m}_{tr} = -r^2 E^q_m, \tag{A.0.4b} \]
\[ h^{2m}_{rr} = -r^2 E^q_m, \tag{A.0.4c} \]
\[ j^{2m}_t = -\frac{1}{3} r^3 E^q_m, \tag{A.0.4d} \]
\[ j^{2m}_r = -\frac{1}{3} r^3 E^q_m, \tag{A.0.4e} \]
\[ h^{2m}_t = \frac{1}{3} r^3 B^q_m, \tag{A.0.4f} \]
\[ h^{2m}_r = \frac{1}{3} r^3 B^q_m, \tag{A.0.4g} \]
\[ K^{2m} = 0, \tag{A.0.4h} \]
\[ G^{2m} = -\frac{1}{3} r^2 E^q_m, \tag{A.0.4i} \]
\[ h^{2m}_2 = \frac{1}{3} r^4 B^q_m. \tag{A.0.4j} \]

for the spherical-harmonic coefficients of the perturbation. Under a gauge transformation generated by the vector field \( \Xi_\alpha \), which is expanded as

\[ \Xi_t = \sum_{\ell m} \xi_{\ell m} Y^{\ell m}, \tag{A.0.5a} \]
\[ \Xi_r = \sum_{\ell m} \xi_{\ell m} Y^{\ell m}, \tag{A.0.5b} \]
\[ \Xi_A = \sum_{\ell m} (\xi_{\ell m} Y^{\ell m}_A + \xi_{\ell m} X^{\ell m}_A), \tag{A.0.5c} \]

the spherical-harmonic coefficients of the perturbation change according to (see e.g. Ref. [74])

\[ h^{2m}_{tt} \to h^{2m'}_{tt} = h^{2m}_{tt} - 2 \partial_t \xi^{2m}_{tt}, \tag{A.0.6a} \]
\[ h^{2m}_{tr} \to h^{2m'}_{tr} = h^{2m}_{tr} - \partial_t \xi^{2m}_{tr} - \partial_r \xi^{2m}_{rt}, \tag{A.0.6b} \]
\[ h^{2m}_{rr} \to h^{2m'}_{rr} = h^{2m}_{rr} - 2 \partial_r \xi^{2m}_{rr}, \tag{A.0.6c} \]
\[ j_t^{2m} \rightarrow j_t^{2m'} = j_t^{2m} - \xi_t^{2m} - \partial_t \xi_t^{2m}, \quad (A.0.6d) \]
\[ j_r^{2m} \rightarrow j_r^{2m'} = j_r^{2m} - \xi_r^{2m} - \partial_r \xi_r^{2m} + \frac{2}{r} \xi_r^{2m}, \quad (A.0.6e) \]
\[ h_t^{2m} \rightarrow h_t^{2m'} = h_t^{2m} - \partial_t \xi_t^{2m}, \quad (A.0.6f) \]
\[ h_r^{2m} \rightarrow h_r^{2m'} = h_r^{2m} - \partial_r \xi_r^{2m} + \frac{2}{r} \xi_r^{2m}, \quad (A.0.6g) \]
\[ K^{2m} \rightarrow K^{2m'} = K^{2m} + \frac{6}{r^2} \xi_2^{2m} - \frac{2}{r} \xi_r^{2m}, \quad (A.0.6h) \]
\[ G^{2m} \rightarrow G^{2m'} = G^{2m} - \frac{2}{r^2} \xi_2^{2m}, \quad (A.0.6i) \]
\[ h_2^{2m} \rightarrow h_2^{2m'} = h_2^{2m} - 2 \xi_2^{2m}. \quad (A.0.6j) \]

We now apply the gauge conditions
\[ j_t^{2m} = j_r^{2m} = G^{2m} = 0, \quad h_2^{2m} = 0 \quad (A.0.7) \]

to bring the perturbation into the Regge-Wheeler gauge we use throughout this thesis. A simple computation reveals that the gauge vector \( \Xi_\alpha \) is determined by
\[ \xi_t^{2m} = -\frac{1}{3} r^3 \mathcal{E}_m, \quad (A.0.8a) \]
\[ \xi_r^{2m} = 0, \quad (A.0.8b) \]
\[ \xi_2^{2m} = -\frac{1}{6} r^4 \mathcal{E}_m, \quad (A.0.8c) \]
\[ \xi_2^{2m} = \frac{1}{6} r^4 B_m^q \quad (A.0.8d) \]

when the tidal moments are time independent. The remaining spherical-harmonic coefficients of the perturbation are thus
\[ h_{tt}^{2m} = -r^2 \mathcal{E}_m, \quad (A.0.9a) \]
\[ h_{tr}^{2m} = 0, \quad (A.0.9b) \]
\[ h_{rr}^{2m} = -r^2 \mathcal{E}_m, \quad (A.0.9c) \]
\[ h_t^{2m} = \frac{1}{3} r^3 B_m^q, \quad (A.0.9d) \]
\[ h_r^{2m} = 0, \quad (A.0.9e) \]
\[ K^{2m} = -r^2 \mathcal{E}_m. \quad (A.0.9f) \]

in the Regge-Wheeler gauge. When added to the Minkowski metric in \((t, r, \theta, \phi)\) coordinates, the perturbations corresponding to Eq. \((A.0.9)\) give rise to the tidal environment metric of Eq. \((2.1.14)\).
APPENDIX B

ROTATIONAL-TIDAL LOVE NUMBERS FOR POLYTROPS

In this appendix, we present tables of the rotational-tidal Love numbers $f_0$ and $k_0$ for polytropes. We consider values of the parameter $b$ up to the critical value $b_{\text{max}}$ at which instability to radial oscillations sets in. Based on the accuracy of our zero-compactness value for $k_0$ compared to the post-Newtonian result of Ref. [89], we display figures to five significant digits of accuracy. Each table is described in the caption that accompanies it.

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<tr>
<th>$b$</th>
<th>$M/R$</th>
<th>$f_0$</th>
</tr>
</thead>
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<td>0.00086</td>
<td>$-6.9998 \times 10^{12}$</td>
</tr>
<tr>
<td>0.01042</td>
<td>0.02048</td>
<td>$-7.4281 \times 10^{5}$</td>
</tr>
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<td>$-2.2367 \times 10^{3}$</td>
</tr>
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<tr>
<td>0.57042</td>
<td>0.29577</td>
<td>$4.9018 \times 10^{-2}$</td>
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</table>

Table B.1: Rotational-tidal Love number $f_0$ for $n = 0.5$ polytropes.
<table>
<thead>
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<th>$M/R$</th>
<th>$\tilde{\mathcal{F}}^o$</th>
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<td>0.42781</td>
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Table B.2: Rotational-tidal Love number $\tilde{\mathcal{F}}^o$ for $n = 0.75$ polytropes.

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<td>0.00021</td>
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Table B.3: Rotational-tidal Love number $\tilde{\mathcal{F}}^o$ for $n = 1$ polytropes.

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Table B.4: Rotational-tidal Love number $\tilde{\mathcal{F}}^o$ for $n = 1.5$ polytropes.
Table B.5: Rotational-tidal Love number $\tilde{\mathfrak{F}}^o$ for $n = 2$ polytropes.

<table>
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Table B.6: Rotational-tidal Love number $\tilde{\mathfrak{F}}^o$ for $n = 2.5$ polytropes.

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Table B.7: Rotational-tidal Love number $\tilde{\mathfrak{F}}^o$ for $n = 0.5$ polytropes.

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<td>0.00781</td>
<td>0.01535</td>
<td>$-3.7158 \times 10^6$</td>
</tr>
<tr>
<td>0.02531</td>
<td>0.04568</td>
<td>$-1.2775 \times 10^4$</td>
</tr>
<tr>
<td>0.05281</td>
<td>0.08437</td>
<td>$-4.3679 \times 10^2$</td>
</tr>
<tr>
<td>0.09031</td>
<td>0.12448</td>
<td>$-4.3479 \times 10^1$</td>
</tr>
<tr>
<td>0.13781</td>
<td>0.16137</td>
<td>$-8.0483$</td>
</tr>
<tr>
<td>0.19531</td>
<td>0.19288</td>
<td>$-2.2215$</td>
</tr>
<tr>
<td>0.26281</td>
<td>0.21856</td>
<td>$-8.0548 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.34031</td>
<td>0.23885</td>
<td>$-3.5449 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.42781</td>
<td>0.25454</td>
<td>$-1.7990 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Table B.8: Rotational-tidal Love number $R^o$ for $n = 0.75$ polytropes.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$M/R$</th>
<th>$R^o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00021</td>
<td>0.00041</td>
<td>$-2.3750 \times 10^{14}$</td>
</tr>
<tr>
<td>0.00516</td>
<td>0.01002</td>
<td>$-2.6051 \times 10^7$</td>
</tr>
<tr>
<td>0.01671</td>
<td>0.03053</td>
<td>$-8.5314 \times 10^4$</td>
</tr>
<tr>
<td>0.03486</td>
<td>0.05817</td>
<td>$-2.7293 \times 10^3$</td>
</tr>
<tr>
<td>0.05961</td>
<td>0.08879</td>
<td>$-2.5226 \times 10^2$</td>
</tr>
<tr>
<td>0.09096</td>
<td>0.11901</td>
<td>$-4.3450 \times 10^1$</td>
</tr>
<tr>
<td>0.12891</td>
<td>0.14664</td>
<td>$-1.1273 \times 10^1$</td>
</tr>
<tr>
<td>0.17346</td>
<td>0.17059</td>
<td>$-3.9024$</td>
</tr>
<tr>
<td>0.22461</td>
<td>0.19058</td>
<td>$-1.6711$</td>
</tr>
<tr>
<td>0.28236</td>
<td>0.20676</td>
<td>$-8.4218 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Table B.9: Rotational-tidal Love number $R^o$ for $n = 1$ polytropes.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$M/R$</th>
<th>$R^o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>0.00019</td>
<td>$-1.2005 \times 10^{16}$</td>
</tr>
<tr>
<td>0.0025</td>
<td>0.00457</td>
<td>$-1.2739 \times 10^9$</td>
</tr>
<tr>
<td>0.0081</td>
<td>0.01428</td>
<td>$-3.8759 \times 10^6$</td>
</tr>
<tr>
<td>0.0169</td>
<td>0.02824</td>
<td>$-1.1154 \times 10^5$</td>
</tr>
<tr>
<td>0.0289</td>
<td>0.04502</td>
<td>$-9.0781 \times 10^3$</td>
</tr>
<tr>
<td>0.0441</td>
<td>0.06318</td>
<td>$-1.3637 \times 10^3$</td>
</tr>
<tr>
<td>0.0625</td>
<td>0.08147</td>
<td>$-3.0898 \times 10^2$</td>
</tr>
<tr>
<td>0.0841</td>
<td>0.09890</td>
<td>$-9.4355 \times 10^1$</td>
</tr>
<tr>
<td>0.1089</td>
<td>0.11482</td>
<td>$-3.6228 \times 10^1$</td>
</tr>
<tr>
<td>0.1369</td>
<td>0.12886</td>
<td>$-1.6697 \times 10^1$</td>
</tr>
</tbody>
</table>

Table B.10: Rotational-tidal Love number $R^o$ for $n = 1.5$ polytropes.
<table>
<thead>
<tr>
<th>(b)</th>
<th>(M/R)</th>
<th>(K^o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00005</td>
<td>0.00008</td>
<td>(-2.7663 \times 10^{17})</td>
</tr>
<tr>
<td>0.00125</td>
<td>0.00206</td>
<td>(-2.9109 \times 10^{10})</td>
</tr>
<tr>
<td>0.00405</td>
<td>0.00653</td>
<td>(-8.6842 \times 10^{7})</td>
</tr>
<tr>
<td>0.00845</td>
<td>0.01318</td>
<td>(-2.4232 \times 10^{6})</td>
</tr>
<tr>
<td>0.01445</td>
<td>0.02159</td>
<td>(-1.8911 \times 10^{5})</td>
</tr>
<tr>
<td>0.02205</td>
<td>0.03124</td>
<td>(-2.6941 \times 10^{4})</td>
</tr>
<tr>
<td>0.03125</td>
<td>0.04162</td>
<td>(-5.7264 \times 10^{3})</td>
</tr>
<tr>
<td>0.04205</td>
<td>0.05224</td>
<td>(-1.6236 \times 10^{3})</td>
</tr>
<tr>
<td>0.05445</td>
<td>0.06268</td>
<td>(-5.7308 \times 10^{2})</td>
</tr>
<tr>
<td>0.06845</td>
<td>0.07260</td>
<td>(-2.4056 \times 10^{2})</td>
</tr>
</tbody>
</table>

Table B.11: Rotational-tidal Love number \(K^o\) for \(n = 2\) polytropes.

<table>
<thead>
<tr>
<th>(b)</th>
<th>(M/R)</th>
<th>(K^o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00002</td>
<td>0.00003</td>
<td>(-4.5792 \times 10^{19})</td>
</tr>
<tr>
<td>0.00047</td>
<td>0.00067</td>
<td>(-4.7511 \times 10^{12})</td>
</tr>
<tr>
<td>0.00152</td>
<td>0.00214</td>
<td>(-1.3720 \times 10^{10})</td>
</tr>
<tr>
<td>0.00317</td>
<td>0.00440</td>
<td>(-3.6404 \times 10^{8})</td>
</tr>
<tr>
<td>0.00542</td>
<td>0.00738</td>
<td>(-2.6567 \times 10^{7})</td>
</tr>
<tr>
<td>0.00827</td>
<td>0.01098</td>
<td>(-3.4854 \times 10^{6})</td>
</tr>
<tr>
<td>0.01172</td>
<td>0.01510</td>
<td>(-6.7278 \times 10^{5})</td>
</tr>
<tr>
<td>0.01577</td>
<td>0.01962</td>
<td>(-1.7109 \times 10^{5})</td>
</tr>
<tr>
<td>0.02042</td>
<td>0.02443</td>
<td>(-5.3575 \times 10^{4})</td>
</tr>
<tr>
<td>0.02567</td>
<td>0.02941</td>
<td>(-1.9760 \times 10^{4})</td>
</tr>
</tbody>
</table>

Table B.12: Rotational-tidal Love number \(K^o\) for \(n = 2.5\) polytropes.
REFERENCES


