

SIGN PATTERN MATRICES AND SEMIRINGS

by

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A Thesis

Presented to

The University of Guelph

In partial fulfilment of requirements

for the degree of

Master of Science

in

Mathematics

Guelph, Ontario, Canada

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ABSTRACT

SIGN PATTERN MATRICES AND SEMIRINGS

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2011

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Sign pattern theory examines what can be said about a matrix if one knows the signs of all or some of its entries but not the exact values. Since all we know is the sign of each entry, we can write these sign patterns as matrices whose entries come from the set $\{+1, -1, 0, \#\}$, where $\#$ is used for an unknown sign. Semirings satisfy all properties of rings with unity except the existence of additive inverses. The set $\{+1, -1, 0, \#\}$ can be viewed as a commutative semiring in natural way. In the thesis, we give a semiring version of the Cayley-Dickson construction which allows one to construct the sign pattern semiring from the Boolean semiring. We use tools from Boolean matrices to study sign nonsingular (SNS) matrices. We also investigate different notions of rank of matrices over semirings. For these rank functions we simplify proofs of classical inequalities for the sum and the product of matrices using the semiring versions of the Cauchy-Binet and Laplace theorems. For matrices over the sign pattern semiring, the minimum rank of the sign pattern is compared

with the other versions of the rank. We also characterize irreducible powerful sign pattern matrices and investigate the period and base of an SNS matrix.

To My Grandparents, Chand Rani and Chaman Lal Mohindru,

For instilling in me the desire to learn

Acknowledgements

First and foremost I want to thank my supervisor Dr. Rajesh Pereira. It has been an honor to be his M.Sc. student. I appreciate all his contributions of time, ideas, and findings that make my M.Sc. experience productive and stimulating. I am extremely grateful to him for his guidance, encouragement and patience throughout the course of my research.

My thanks also to my Co-advisor, Dr. Dan Ashlock and the additional members of my examining committee consisting of Dr. Monica Cojocaru (Chair) and Dr. Pal Fischer. I would also like to thank the Department of Mathematics & Statistics at the University of Guelph for their help and support during my stay at the University of Guelph as a graduate student.

Many thanks to Professor Arvind Kaur Saini and Professor Shri Ram Vaid for inspiring and motivating me to pursue my studies in mathematics. I wish to thank my entire family for providing a loving environment for me. Mr. Vijay Kumar (Raj), Payal Dhingra, Parveen K. Dhingra, Parshant Mohindru, Akash Lal Mohindru, Kavita Mohindru, Nishima, Ashima, Amar, Divya and Gaurav were particularly supportive. I wish to thank my friend Miss Anandvir Kaur Saini, Ph.D. scholar, Department of Geography, Panjab University, Chandigarh, (India), for helping me get through the difficult times, and for all the emotional support, entertainment, and caring she provided.

Lastly, and most importantly, I wish to thank my parents, Mrs. Vijay Rani Mohindru and Mr. Parshotam Lal Mohindru, for teaching me from a very young age that there is nothing greater in this world than learning.

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Chapter 1

Introduction

In this thesis, we explore the connections between matrix sign pattern theory and the theory of semirings. Sign pattern theory examines what can be said about a matrix if one knows the signs of all or some of its entries but not the exact values. Since all we know is the sign of each entry, we can write these sign patterns as matrices whose entries come from the set $\{+1, -1, 0, \#\}$, where $\#$ is used for an unknown sign.

Semirings satisfy all properties of rings with unity except the existence of additive inverses. Semirings constitute a fairly natural generalization of rings, with broad applications in the mathematical foundations of computer science. The set $\{+1, -1, 0, \#\}$ can be viewed as a commutative semiring in natural way. This fact allows us to use the theory of matrices over semirings to study matrix sign pattern theory. In particular we study the relation between the sign pattern matrices and the Boolean matrices, and formulate a semiring version of the Cayley-Dickson construction. This allows Boolean matrices to be

used to study matrix sign pattern theory. We compare the different notions of rank of the matrices over the sign pattern semiring. We also study the sequence of powers of certain matrices over the sign pattern semiring.

In chapter 2, we discuss the notations we use and introduce the theory of matrices over semirings. We define a special class of semirings called S -squared semiring, denoted as S^2 . This S^2 construction allows one to construct the sign pattern semiring from the Boolean semiring.

In chapter 3, we talk about the sign pattern matrices and construct a $2n$ by $2n$ Boolean matrix $B(A)$ from an n by n sign pattern matrix A . We show that there is a one to one homomorphism from the matrices over the sign pattern semiring to the matrices over the Boolean semiring. This allows Boolean matrices to be used to study matrix sign pattern theory. Some relations between the bideterminant of a sign pattern and the corresponding Boolean matrix are also described.

In chapter 4, we investigate different notions of rank of matrices over semirings. For matrices over the sign pattern semiring, we compare the minimum rank of the sign pattern with the other versions of the rank. We simplify proofs of some inequalities on the rank of the sum and the product of two matrices using the semiring versions of the Cauchy-Binet and Laplace theorems.

In chapter 5, we discuss the powers of certain matrices over the sign pattern semiring. For example, we look at the bounds for the period and the base of a sign non-singular non-negative irreducible pattern. We also extend some results about the base and the period of the Boolean matrices to the base and the period of the sign pattern matrices.

Chapter 2

Background

2.1 Notation

2.1.1 Abstract Algebra

We begin by reviewing some of the fundamental structures of abstract algebra.

Definition 2.1.1. (*Ring*) [23] A non empty set S together with the operation $(+)$ and (\cdot) is called a ring if for all $a, b, c \in S$, the following laws hold:

1. **Laws of Addition:**

(a) *Closure Law:* $a + b \in S$.

(b) *Associative Law:* $a + (b + c) = (a + b) + c$.

(c) *Commutative Law:* $a + b = b + a$.

(d) *Identity Law:* There exists an element $\mathbf{0} \in S$ such that $a + \mathbf{0} = a = \mathbf{0} + a$, for all $a \in S$. Here $\mathbf{0}$ is called the additive identity.

(e) *Inverse Law:* For each $a \in S$ there exists an element $-a \in S$ such that $a + (-a) = 0 = (-a) + a$. Here $-a$ is called the additive inverse.

2. **Laws of multiplication:**

(a) *Closure Law:* $a \cdot b \in S$.

(b) *Associative Law:* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

3. **Distributive Law:**

(a) $a \cdot (b + c) = ab + ac$

(b) $(a + b) \cdot c = ac + bc$

Definition 2.1.2. (Commutative Ring) [23] A ring $(S, +, \cdot)$ is called a commutative ring if the following property holds for all $a, b \in S$:

$$a \cdot b = b \cdot a$$

Definition 2.1.3. (Unitary Ring) [23] A ring $(S, +, \cdot)$ is called a unitary ring or a ring with unity if there exists an element $\mathbf{1} \in S$ such that for each $a \in S$

$$a \cdot \mathbf{1} = a = \mathbf{1} \cdot a$$

Definition 2.1.4. (Monoid) [23] A non empty set S together with the given operation \cdot and an element e is called a monoid if it satisfies the following laws:

1. *Closure law:* For all a, b in S , $a \cdot b$ is also in S .

2. *Associative law:* For all a, b, c in S , $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

3. *Identity law:* There exists an element e in S , such that for all elements a in S ,
- $$e \cdot a = a \cdot e = a .$$

We now introduce the concept of a semiring. Semirings are a fairly natural generalization of rings. Semirings satisfy all properties of unitary rings except the existence of additive inverses. H. S. Vandiver introduced the concept of semiring in [35], in connection with the axiomatization of the arithmetic of the natural numbers. Over the years, semirings have been studied by various researchers in an attempt to broaden techniques coming from semigroup theory or ring theory.

Definition 2.1.5. (*Semiring*) [20] A semiring is a set S together with two operations \oplus and \otimes and two distinguished elements $\mathbf{0}$, $\mathbf{1}$ in S with $\mathbf{0} \neq \mathbf{1}$, such that

1. $(S, \oplus, \mathbf{0})$ is a commutative monoid,
2. $(S, \otimes, \mathbf{1})$ is a monoid,
3. \otimes is both left and right distributive over \oplus ,
4. Whose additive identity $\mathbf{0}$ satisfies the property $r \otimes \mathbf{0} = \mathbf{0} \otimes r = \mathbf{0}$, for all $r \in S$.

In the other words semirings are commutative unital rings without the requirement that each element has additive inverse. If $(S, \otimes, \mathbf{1})$ is a commutative monoid then S is called a *commutative semiring*.

Definition 2.1.6. (Antinegative Semiring) [20] A semiring is said to be antinegative or zerosumfree if the only element with an additive inverse is the additive identity 0.

Definition 2.1.7. (Zero Divisor) [20] Let S be a commutative semiring. A non-zero element $x \in S$ is called a zero divisor if there exists a non-zero $y \in S$ such that $x \otimes y = 0$.

Examples of semirings:

- The set of all *non-negative numbers* forms a semiring under the usual addition and multiplication [20].
- $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$ forms a semiring with $a \oplus b = \max\{a,b\}$ and $a \otimes b = a + b$. This is called *the max-plus semiring*. Note that in this case $\mathbf{0} = -\infty$ and $\mathbf{1} = 0$ [29].
- A totally ordered set S with greatest element and least element forms a semiring with $a \oplus b = \max\{a,b\}$ and $a \otimes b = \min\{a,b\}$. This is called *chain semiring*. The chain semiring with two elements $\{0,1\}$ is called the *Boolean semiring* and it is denoted by β [20].
- The set $\{+1, -1, 0, \#\}$ forms a semiring, where the operations of addition and multiplication are defined as follows:

\oplus	0	+1	-1	#
0	0	+1	-1	#
+1	+1	+1	#	#
-1	-1	#	-1	#
#	#	#	#	#

\otimes	0	+1	-1	#
0	0	0	0	0
+1	0	+1	-1	#
-1	0	-1	+1	#
#	0	#	#	#

This is called the *sign pattern semiring* [4]. Here 0 is the additive identity and +1 is the multiplicative identity. Later we will discuss the sign pattern semiring in detail.

All these semirings are commutative, antinegative and have no zero divisors.

Much of the standard terminology about mapping also applies to semirings.

Definition 2.1.8. (*Semiring Homomorphism*) [5] *Let R and S are be two semirings.*

A semiring homomorphism is a function $F : R \rightarrow S$, such that

$$F(\mathbf{0}) = \mathbf{0} \text{ and } F(\mathbf{1}) = \mathbf{1}$$

$$F(a \oplus b) = F(a) \oplus F(b) \text{ for all } a \text{ and } b \text{ in } R.$$

$$F(a \otimes b) = F(a) \otimes F(b) \text{ for all } a \text{ and } b \text{ in } R.$$

Definition 2.1.9. (*Semiring Isomorphism*) [5] *Let R and S are be two semirings. A*

semiring homomorphism $F : R \rightarrow S$, is called a semiring isomorphism if the homomorphism

F is bijective.

Definition 2.1.10. (*Semiring Monomorphism*) [5] *Let R and S are be two semirings.*

A semiring homomorphism $F : R \rightarrow S$ is called a semiring monomorphism if the homomor-

phism F is injective.

We can consider matrices over semirings. Let $M_{m,n}(S)$ denotes the set of all m by n matrices over S and $M_n(S)$ denotes the set of all n by n matrices over S . Addition and multiplication of these matrices can be defined in the usual way. Let A be an m by n matrix. Then the element a_{ij} is called the (i,j) -entry of A . The (i,j) -entry of A is sometimes denoted by A_{ij} . Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be m by n matrices over a semiring (S, \oplus, \otimes)

and let $C = [c_{ij}]$ be an n by p matrix over the same semiring. Then $A + B = [a_{ij} \oplus b_{ij}]$ and $AC = [\bigoplus_{k=1}^n a_{ik} \otimes c_{kj}]$. The set of n by n matrices over a semiring is itself a semiring. For any $k \in S$, the matrix $kA = [k \otimes a_{ij}]$.

2.1.2 Linear Algebra

Definition 2.1.11. (Standard determinant expression) [32] *If $A = [a_{ij}]$ be a n by n matrix over commutative ring, then the standard determinant expression of A is:*

$$\det(A) = \bigoplus_{\sigma \in S_n} \text{Sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

where S_n is the symmetric group of order n and $\text{Sign}(\sigma) = +$ if σ is even permutation and $\text{Sign}(\sigma) = -$ if σ is odd permutation. Here $\text{Sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$ is called a term of the determinant.

Since we do not have subtraction in a semiring, we can not write the determinant of a matrix over a semiring in this form. We split the determinant into two parts, the positive determinant and the negative determinant.

Definition 2.1.12. (Positive and Negative Determinant) [32] *Let A be an n by n matrix over a commutative semiring S , then we define the positive and the negative determinant as:*

$$\begin{aligned} \det^+(A) &= \bigoplus_{\sigma \in A_n} \bigotimes_{i=1}^n a_{i\sigma(i)} \\ \det^-(A) &= \bigoplus_{\sigma \in S_n \setminus A_n} \bigotimes_{i=1}^n a_{i\sigma(i)} \end{aligned}$$

Where A_n is the alternating group of order n i.e, the set of all even permutations of order n and $S_n \setminus A_n$ is the set of all odd permutations of order n .

As such we note that the determinant of a matrix A over a ring takes the form:

$$\det(A) = \det^+(A) - \det^-(A)$$

In the semiring theory the bideterminant is an analog of the determinant.

Definition 2.1.13. (*Bideterminant*) [19] Let A be an n by n matrix over a commutative semiring. The bideterminant of A is $(\det^+(A), \det^-(A))$.

Definition 2.1.14. (*Permanent of a Matrix*) [32] Let $A = [a_{ij}]$ be an n by n matrix over a semiring, then the permanent of A is :

$$\text{per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

The sequence $(a_{1\sigma(1)}, a_{2\sigma(2)}, \dots, a_{n\sigma(n)})$ is called the diagonal of A , and the product $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$ is called the *diagonal product* of A . Thus the permanent of A is the sum of all diagonal products of A .

In terms of positive and negative determinant, the permanent of $A = \det^+(A) \oplus \det^-(A)$.

Definition 2.1.15. (*Balanced Determinant*) [31] Let A be an n by n matrix over a commutative semiring S . A is said to have the balanced determinant if $\det^+(A) = \det^-(A)$.

Definition 2.1.16. (*Semimodule*) [2] A semimodule, M , over a semiring S is an abelian monoid under addition which has a neutral element, 0 , and is equipped with a law

$$S \times M \rightarrow M$$

$$(s, m) \rightarrow s \otimes m$$

called scalar multiplication such that for all m and m' in M and $r, s \in S$

1. $(s \otimes r) \otimes m = s \otimes (r \otimes m)$,
2. $(s \oplus r) \otimes m = (s \otimes m) \oplus (r \otimes m)$,
3. $s \otimes (m \oplus m') = (s \otimes m) \oplus (s \otimes m')$,
4. $1 \otimes m = m$,
5. $s \otimes 0 = 0 = 0 \otimes m$.

Definition 2.1.17. (*Linear Combination*) [2] An element m in a semimodule M over S is called a linear combination of elements from a certain subset $P \subseteq M$ if there exists $k \geq 0$, $s_1, \dots, s_k \in S$, $m_1, \dots, m_k \in P$ such that $m = \sum_{i=1}^k s_i \otimes m_i$ with the convention that an empty sum is equal to 0 . In this case $\sum_{i=1}^k s_i \otimes m_i$ is called a linear combination of the elements m_1, \dots, m_k from P with coefficients s_1, \dots, s_k in S .

Definition 2.1.18. (*Linear Span*) [2] The linear span of a family or set P of elements of a semimodule M over a semiring S is the set of all linear combinations of elements from

P with coefficients from S . The linear span of P is denoted by $\langle P \rangle$. We say that the family P generates or spans M if $\langle P \rangle = M$, and that P envelopes a subset $V \subseteq M$ in M if $V \subseteq \langle P \rangle$.

Definition 2.1.19. (*Linearly Dependent in Gondran-Minoux Sense*) [2] A family m_1, \dots, m_k of elements of a semimodule M over a semiring S is linearly dependent in the Gondran-Minoux sense if there exist two subsets $I, J \subseteq K = \{1, \dots, k\}$, $I \cap J = \emptyset$, $I \cup J = K$, and the scalars $\alpha_1, \dots, \alpha_k \in S$, not all equal to 0, such that $\sum_{i \in I} \alpha_i \otimes m_i = \sum_{j \in J} \alpha_j \otimes m_j$. Otherwise it is called linearly independent in the Gondran-Minoux sense.

Definition 2.1.20. (*Weakly Linearly Dependent*) [2] A family P of elements of a semimodule M over a semiring S is weakly dependent if there is an element in P that can be expressed as a linear combination of other elements of P . Otherwise it is called weakly linearly independent.

Later we will prove that in the case of semirings, the linear dependence in the Gondran-Minoux sense is different from the linear dependence in the weak sense.

2.2 Boolean Matrices

A Boolean matrix is a matrix whose entries lie in the Boolean semiring. Boolean matrices are sometimes called relation matrices, Boolean relation matrices, binary relation matrices, binary Boolean matrices, or (0,1) Boolean matrices. The set of all m by n Boolean

matrices is denoted as β_{mn} . The set of all n by n Boolean matrices is denoted as β_n . Let B be an m by n Boolean matrix. The i^{th} row of B is the sequence $b_{i1}, b_{i2}, \dots, b_{in}$, and j^{th} column of B is the sequence $b_{1j}, b_{2j}, \dots, b_{mj}$. Rows of B are the Boolean vectors of length n and columns of B are the Boolean vectors of length m .

Remark 2.2.1. *The set of all 2 by 2 matrices over β , β_2 , forms a semiring under usual addition and multiplication. The four element subset*

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

of β_2 is itself a semiring under usual addition and multiplication of the matrices.

2.3 S-Squared Semirings

The Cayley-Dickson construction gives an infinite sequence of normed division algebras, doubling in dimension each time. This construction explains why each normed division algebra from the sequence $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, fits neatly inside the next.

The complex number $a + bi$ can be thought of as a pair (a, b) of real numbers. Addition is done component-wise, and multiplication is defined as follows:

$$(a, b)(c, d) = (ac - db, ad + cb)$$

We can also define the conjugate of a complex number by

$$(a, b)^* = (a, -b).$$

Now that we have the complex numbers \mathbb{C} , we can define the quaternions, \mathbb{H} , in a similar way. A quaternion can be thought of as a pair of complex numbers. Addition is done component-wise, and multiplication is defined as follows:

$$(a, b)(c, d) = (ac - db^*, a^*d + cb)$$

This is just like our formula for multiplication of complex numbers, but with a couple of conjugates thrown in. If we included them in the previous formula nothing would change, since the conjugate of a real number is just itself. We can also define the conjugate of a quaternion by

$$(a, b)^* = (a^*, -b).$$

Now we can define an octonion to be a pair of quaternions. We add and multiply them using the same formulas that worked for the quaternions. This trick for getting new algebras from old is called the Cayley-Dickson construction, [[12], p. 237 and [13]].

We use a similar construction called S-squared construction. This construction is from [29], in which it was applied to the max-plus semiring. In our thesis we explore applications of this construction to the Boolean semiring.

If (S, \oplus, \otimes) is a commutative semiring then $S^2 = \{(a, b) | a, b \in S\}$ is also a commutative semiring with addition and multiplication defined as follows: For all a, b, c and $d \in S$,

$$(a, b) \oplus (c, d) = (a \oplus c, b \oplus d)$$

$$(a, b) \otimes (c, d) = ((a \otimes c) \oplus (b \otimes d), (b \otimes c) \oplus (a \otimes d))$$

We can see that all the properties of a semiring are satisfied with $(0,0)$ being the additive identity and $(1,0)$ being the multiplicative identity.

Remark 2.3.1. - Let $\beta = \{0, 1\}$ be a Boolean semiring. Then $\beta^2 = \{(0,0), (1,0), (0,1), (1,1)\}$ is also a semiring with the addition and the multiplication defined above for S^2 . Moreover it is isomorphic to the sign pattern semiring as $(0,0) = 0$, $(1,0) = +1$, $(0,1) = -1$, $(1,1) = \#$.

We know that if $\mathbf{1}$ is the unity of S then $(\mathbf{1},\mathbf{0})$ is the unity of S^2 . Since $\mathbf{1}$ is the unity in S , so $\mathbf{1} \cdot s = s$ for all $s \in S$. Now consider $(\mathbf{1},\mathbf{0}) \otimes (a,b) = (a,b)$, and this is true for all $(a,b) \in S^2$. This implies that $(\mathbf{1},\mathbf{0})$ is the unity of S^2 and it is unique. Similarly if $\mathbf{0}$ is the additive identity of S then $(\mathbf{0},\mathbf{0})$ is the additive identity of S^2 .

Theorem 2.3.1. - Let S be a commutative semiring. If S is antinegative and has no zero divisors then S^2 is also antinegative and has no zero divisors.

Proof. - Since S is antinegative, only $\mathbf{0}$ has an additive inverse. Let us suppose that $(a_1, b_1) \in S^2$ has an additive inverse, so there exists $(a_2, b_2) \in S^2$, such that $(a_1, b_1) \oplus (a_2, b_2) = (\mathbf{0},\mathbf{0})$. This implies that $a_1 \oplus a_2 = \mathbf{0}$ and $b_1 \oplus b_2 = \mathbf{0}$. Since S is antinegative so $a_1 = a_2 = b_1 = b_2 = \mathbf{0}$. Hence $(a_1, b_1) = (\mathbf{0},\mathbf{0})$. Thus only the additive identity has an additive inverse in S^2 which means S^2 is an antinegative semiring. Now suppose that $(a_1, b_1) \in S^2$ is a zero divisor, so there exists a nonzero $(a_2, b_2) \in S^2$, such that $(a_1, b_1) \otimes (a_2, b_2) = (\mathbf{0},\mathbf{0})$. This implies that $((a_1 \otimes a_2) \oplus (b_1 \otimes b_2)) = 0$ and $((a_1 \otimes b_2) \oplus (a_2 \otimes b_1)) = 0$. Since S is

antinegative so $a_1 \otimes a_2 = 0$ and $b_1 \otimes b_2 = 0$. Also S has no zero divisors so either $a_1 = 0$ or $a_2 = 0$ and either $b_1 = 0$ or $b_2 = 0$, combining this with $((a_1 \otimes b_2) \oplus (a_2 \otimes b_1)) = 0$, we get either $(a_1, b_1) = (0,0)$ or $(a_2, b_2) = (0,0)$. Hence S^2 has no zero divisors. ■

2.3.1 Determinant of a Matrix over S^2

The bideterminant of a matrix over S can be viewed as an element of S^2 . Now we will define the determinant of a matrix over S -squared semiring.

Let $A \in M_n(S^2)$. The bideterminant of A , $\text{bidet}(A) = (\det^+(A), \det^-(A)) \in (S^2)^2$. Then determinant of A , denoted by $\det_2(A) = ((1,0) \otimes \det^+(A)) \oplus ((0,1) \otimes \det^-(A))$.

Suppose that

$$\text{bidet}(A) = ((a_1, b_1), (a_2, b_2)), \text{ where } (a_1, b_1) \text{ and } (a_2, b_2) \in S^2.$$

then

$$\begin{aligned} \det_2(A) &= ((1,0) \otimes (a_1, b_1)) \oplus ((0,1) \otimes (a_2, b_2)) \\ &= (a_1 \oplus b_2, b_1 \oplus a_2) \in S^2 \end{aligned}$$

Note that if $A = [a_{ij}] \in M_n(S)$ we can view $\tilde{A} \in M_n(S^2)$ by $[(a_{ij}, 0)]$. Clearly $\det_2(\tilde{A}) = \text{bidet}(A)$.

We will discuss the bideterminant and the relation between the Boolean semiring and the sign pattern semiring as the applications of S^2 semiring.

Chapter 3

The Relationship Between Sign Pattern Matrices and Boolean Matrices

3.1 Sign Pattern Matrices

In qualitative and combinatorial matrix theory, we study properties of a matrix based combinatorial information, such as the signs of entries in the matrix. A matrix whose entries are from the set $\{+1, -1, 0\}$ is called a *sign pattern matrix* (or sign pattern, or pattern). Let $A = [a_{ij}]$ is a real matrix. Then sign pattern (portrait) of A is obtained from A , by replacing each entry by its signs [21]. The sign pattern of A is denoted by $Sg(A) = [sg(a_{ij})]$, where

$$\text{sg}(a_{ij}) = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ +1 & \text{if } a_{ij} > 0 \\ -1 & \text{if } a_{ij} < 0. \end{cases}$$

Thus in a sign pattern matrix all we know is the sign of each entry. We do not know the exact values of the entries. We denote the set of all n by n sign pattern matrices by Q_n . Sometimes we may not know the signs of certain entries, so a new symbol, $\#$, has been introduced to denote such entries. The addition and the multiplication involving the symbol $\#$ are defined as follows:

$$\begin{aligned} (+1) + (-1) &= \# , & \# + (\pm 1) &= \# , & \# + \# &= \# \\ (\pm 1) \cdot \# &= \# , & \# \cdot \# &= \# , & \# \cdot 0 &= 0. \end{aligned}$$

Definition 3.1.1. (*Generalized Sign Pattern matrix*) [25] *The generalized sign pattern matrices are the matrices over the set $\{+1, -1, 0, \#\}$, where $\#$ corresponds to an unknown sign.*

The addition and the multiplication of generalized sign pattern matrices are defined in the usual way, so the sum and the product of generalized sign pattern matrices are still generalized sign pattern matrices.

The set $\{+1, -1, 0, \#\}$ can be viewed as a semiring. If $S = \{+1, -1, 0, \#\}$, then (S, \oplus, \otimes) is a commutative semiring with identity, where the operations of addition and multiplication are defined as follows:

\oplus	0	+1	-1	#
0	0	+1	-1	#
+1	+1	+1	#	#
-1	-1	#	-1	#
#	#	#	#	#

\otimes	0	+1	-1	#
0	0	0	0	0
+1	0	+1	-1	#
-1	0	-1	+1	#
#	0	#	#	#

Clearly all the properties of a semiring are satisfied where 0 is the additive identity and 1 is multiplicative identity. Here 1 and -1 are the units of S [4].

Definition 3.1.2. (*Qualitative Class*) [11] *Let A be a real matrix. The qualitative class of A is $Q(A)$, the set of all real matrices with the same sign pattern as A .*

Definition 3.1.3. (*Zero Non-zero Pattern*) [11] *The zero non-zero pattern of a real matrix A is $(0, 1)$ matrix obtained from A by replacing each non zero entry by 1.*

3.1.1 SNS Matrices

In this section we will study the *sign-nonsingular matrices*, abbreviated SNS matrices. We will use the determinant in order to obtain characterizations of SNS matrices.

Definition 3.1.4. (*L-Matrix*) [11] *An L-matrix matrix is an m by n $(0, +1, -1)$ -matrix A such that every m by n real matrix with the same sign pattern as A has linearly independent rows.*

Definition 3.1.5. (*SNS Matrix*) [11] *A square L-matrix is called a sign-nonsingular*

(SNS) matrix.

Definition 3.1.6. (Signed Determinant) [11] A square sign pattern matrix is said to have a signed determinant if the determinant of all real matrices in $Q(A)$ have the same sign.

Lemma 3.1.1. [11] Let $A = [a_{ij}] \in Q_n$. Then A has a signed determinant if and only if one of the following holds:-

1. Every term in the standard determinant expression of A is zero.
2. There is a non zero term in standard determinant expression of A and every such term has same sign.

Theorem 3.1.1. [11] - Let $A = [a_{ij}]$ be a matrix of order n . Then the following are equivalent:

1. A is SNS matrix.
2. There is a non zero term in the standard determinant expression of A and every nonzero term has same sign.

Remark 3.1.1. 1. A necessary condition for a sign pattern matrix to be an SNS matrix is that its row and columns can be permuted such that the entries in the main diagonal of the resulting matrix are all nonzero.

2. A is an SNS matrix if and only if the bideterminant of A is one of the following six elements

$$(\pm 1, 0) \quad (0, \pm 1) \quad (\pm 1, \mp 1).$$

3. An n by n sign pattern Matrix A is called SNS matrix if for all real matrices $B \in Q(A)$, $\det(B) \neq 0$.

Using the determinant of a matrix over S^2 semiring defined in section 2.3.1, we conclude the following result. We will use the fact that if β is the Boolean semiring then β^2 is the sign pattern semiring.

Proposition 3.1.1. - *Let A be an n by n sign pattern matrix. A is an SNS matrix if and only if $\det_2(A)$ is a unit.*

Proof. - This follows from remark 3.1.1 # 2.

■

Now we will explore the connection between the sign pattern matrices and the Boolean matrices .

3.1.2 Matrix Interpretation of S^2

S^2 is isomorphic to 2 by 2 circulant matrices over S . i.e.

$$(a,b) \cong \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \forall (a,b) \in S^2.$$

Thus the 2 by 2 circulant matrices over S form a semiring with $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as an additive

identity and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as a multiplicative identity.

Note that the sign pattern semiring is isomorphic to 2 by 2 Boolean circulant matrices. Where the isomorphism is

$$\begin{array}{ll} 0 \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ -1 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \# \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

This allows us to relate the Boolean matrices to the sign pattern matrices.

3.2 Construction of a 2n by 2n Boolean matrix from a given n by n sign pattern matrix

Let A be a given n by n sign pattern matrix.

1. Define a Boolean matrix A^+ whose $(ij)^{th}$ entry is 1 if $(ij)^{th}$ entry of A is $+1$ or $\#$

and 0 otherwise.

2. Define another Boolean matrix A^- whose $(ij)^{th}$ entry is 1 if $(ij)^{th}$ entry of A is -1 or # and 0 otherwise.

Now $B(A)$, the Boolean matrix associated with an n by n sign pattern matrix A , is the $2n$ by $2n$ Boolean matrix whose upper left and lower right n by n blocks are both A^+ and whose upper right and lower left n by n blocks are both A^- , i.e,

$$B(A) = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}$$

Remark 3.2.1. - *By the construction of $B(A)$, we see that entries of $B(A)$ depend upon the entries of A . We label the n rows and the n columns of A by $1, 2, \dots, n$ and the $2n$ rows and the $2n$ columns of $B(A)$ by $1^+, 2^+, \dots, n^+, 1^-, 2^-, \dots, n^-$. Let a_{ij} be the $(ij)^{th}$ entry of A and b_{i^*, j^*} be the $(i^*, j^*)^{th}$ entry of $B(A)$, where $*$ is + or -. Then we have the following relations:*

$a_{i,j}$	b_{i^+, j^+}	b_{i^+, j^-}	b_{i^-, j^+}	b_{i^-, j^-}
+1	1	0	0	1
-1	0	1	1	0
0	0	0	0	0
#	1	1	1	1

Proposition 3.2.1. - *Let A be an n by n sign pattern matrix and $B(A)$ be the corresponding Boolean matrix. The rows of A are linearly dependent if and only if $B(A)$ has linearly dependent rows.*

Proof. - Let

$$A = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix} \Rightarrow B(A) = \begin{pmatrix} V_{1,+} \\ V_{2,+} \\ \vdots \\ V_{n,+} \\ V_{1,-} \\ V_{2,-} \\ \vdots \\ V_{n,-} \end{pmatrix}$$

where V_i , for $i = 1, 2, \dots, n$ be the rows of A and $V_{i,*}$ for all $i = 1, 2, \dots, n$ and $* = +, -$, be the rows of the corresponding Boolean matrix $B(A)$.

Suppose there exists a number r , $1 \leq r \leq n$, such that

$$V_r = \bigoplus_{k=1}^t \lambda_k V_{S_k}, \quad (3.1)$$

where $S_k \neq r$, $\lambda_k = +1, -1$ and $1 \leq t \leq n$. Then by the construction of $B(A)$ we get:

$$V_{r,+} = \bigoplus_{k=1}^t V_{S_k, \lambda_k}, \quad (3.2)$$

Thus rows of A are linearly dependent implies that rows of $B(A)$ are linearly dependent.

Now suppose that rows of $B(A)$ are linearly dependent. This implies that

$$V_{r,+} = \bigoplus_{k=1}^t V_{S_k, \lambda_k}, \quad (3.3)$$

Where $S_k \neq r$, since A is not a generalized sign pattern matrix, and $\lambda_k = +1, -1$ and $1 \leq t \leq n$. Also note that both $V_{S_i,+}$ and $V_{S_i,-}$ can not belong to the right hand side because

if so then

$$V_r = \left(\bigoplus_{k=1, k \neq i}^t \lambda_k V_{S_k} \right) \oplus +V_{S_i} \oplus -V_{S_i} \quad (3.4)$$

This means that some of the entries of V_r are #, which is a contradiction because A is a sign pattern matrix (not generalized sign pattern matrix). Hence we get

$$V_r = \bigoplus_{k=1}^t \lambda_k V_{S_k}, \quad (3.5)$$

Where $S_k \neq r$ and $\lambda_k = +1, -1$ and $1 \leq t \leq n$. Thus rows of A are linearly dependent if and only if $B(A)$ has linearly dependent rows. ■

Remark 3.2.2. - *The proposition 3.2.1 is not true for generalized sign pattern matrices.*

We will show this by an example.

Example 3.2.1. - Let $A = \begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix}$, then $B(A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Clearly the rows of $B(A)$ are linearly dependent but the rows of A are not linearly dependent.

3.3 Semiring Isomorphism and Monomorphism

In this section we will prove there is a one to one semiring homomorphism from the matrices over the sign pattern semiring to the matrices over the Boolean semiring.

Let $M_n(S)$ and $M_n(S^2)$ be the set of all n by n matrices over the semirings S and S^2 . We know that $M_n(S)$ and $M_n(S^2)$ form semirings. If $M_n(S)$ forms a semiring then $(M_n(S))^2$ will also form a semiring (using the S-squared Construction). Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices in $M_n(S)$, where a_{ij} is the ij^{th} entry of A and b_{ij} is the ij^{th} entry of B , then $(A,B) = ([a_{ij}], [b_{ij}]) \in (M_n(S))^2$ and $[A,B] = [(a_{ij}, b_{ij})] \in M_n(S^2)$. Clearly $[(0,0)]$ is the additive identity and $\text{diag}\{(1,0), (1,0), \dots, (1,0)\}$ is the multiplicative identity of $M_n(S^2)$, and $([0],[0])$ is the additive identity and $(\text{diag}\{1, 1, \dots, 1\}, \text{diag}\{0,0, \dots, 0\})$ is the multiplicative identity of $(M_n(S))^2$.

Theorem 3.3.1. - *There is a semiring isomorphism between $M_n(S^2)$ and $(M_n(S))^2$.*

Proof. - Let $F: M_n(S^2) \rightarrow (M_n(S))^2$, such that $F([(a_{ij}, b_{ij})]) = ([a_{ij}], [b_{ij}])$, where $[(a_{ij}, b_{ij})] \in M_n(S^2)$, and $([a_{ij}], [b_{ij}]) \in (M_n(S))^2$.

Clearly F takes the additive identity and the multiplicative identity of $M_n(S^2)$ to the additive identity and the multiplicative identity of $(M_n(S))^2$. Now we will show the following conditions:

1. $F(A_1 \oplus A_2) = F(A_1) \oplus F(A_2)$ for all A_1 and $A_2 \in M_n(S^2)$.
2. $F(A_1 \otimes A_2) = F(A_1) \otimes F(A_2)$ for all A_1 and $A_2 \in M_n(S^2)$.
3. F is injective.
4. F is surjective.

Proof of 1.- Let $A_1 = [(a_{ij}, b_{ij})]$ and $A_2 = [(a'_{ij}, b'_{ij})] \in M_n(S^2)$. Then $A_1 \oplus A_2 = [(a_{ij} \oplus a'_{ij}, b_{ij} \oplus b'_{ij})]$.

$$\begin{aligned}
F(A_1 \oplus A_2) &= F([(a_{ij} \oplus a'_{ij}, b_{ij} \oplus b'_{ij})]) \\
&= ([a_{ij} \oplus a'_{ij}], [b_{ij} \oplus b'_{ij}]), \\
&= ([a_{ij}], [b_{ij}]) \oplus ([a'_{ij}], [b'_{ij}]), \\
&= F([(a_{ij}, b_{ij})]) \oplus F([(a'_{ij}, b'_{ij})]), \\
&= F(A_1) \oplus F(A_2).
\end{aligned}$$

Proof of 2.- Let $A_1 = [(a_{ij}, b_{ij})]$ and $A_2 = [(a'_{ij}, b'_{ij})] \in M_n(S^2)$. Then the ij^{th} entry of $A_1 \otimes A_2$ is $= \bigoplus_{k=1}^n (a_{ik}, b_{ik}) \otimes (a'_{kj}, b'_{kj})$.

$$= \bigoplus_{k=1}^n ((a_{ik} \otimes a'_{kj}) \oplus (b_{ik} \otimes b'_{kj}), (b_{ik} \otimes a'_{kj}) \oplus (a_{ik} \otimes b'_{kj})).$$

$$\begin{aligned}
\text{Now } F(A_1 \otimes A_2) &= F([\bigoplus_{k=1}^n ((a_{ik} \otimes a'_{kj}) \oplus (b_{ik} \otimes b'_{kj}), (b_{ik} \otimes a'_{kj}) \oplus (a_{ik} \otimes b'_{kj}))]), \\
&= \bigoplus_{k=1}^n F([((a_{ik} \otimes a'_{kj}) \oplus (b_{ik} \otimes b'_{kj}), (b_{ik} \otimes a'_{kj}) \oplus (a_{ik} \otimes b'_{kj}))]), \\
&= \bigoplus_{k=1}^n ([((a_{ik} \otimes a'_{kj}) \oplus (b_{ik} \otimes b'_{kj})], [(b_{ik} \otimes a'_{kj}) \oplus (a_{ik} \otimes b'_{kj})])), \\
&= (\bigoplus_{k=1}^n [((a_{ik} \otimes a'_{kj}) \oplus (b_{ik} \otimes b'_{kj})], \bigoplus_{k=1}^n [(b_{ik} \otimes a'_{kj}) \oplus (a_{ik} \otimes b'_{kj})]), \\
&= ([a_{ij}] \otimes [a'_{ij}] \oplus ([b_{ij}] \otimes [b'_{ij}]), ([b_{ij}] \otimes [a'_{ij}] \oplus ([a_{ij}] \otimes [b'_{ij}])), \\
&= ([a_{ij}], [b_{ij}]) \otimes ([a'_{ij}], [b'_{ij}]), \\
&= F(A_1) \otimes F(A_2).
\end{aligned}$$

Proof of 3.- To prove that F is injective, we will show that if $F(A_1) = F(A_2)$, then $A_1 = A_2$ for all $A_1, A_2 \in M_n(S^2)$.

Let $A_1 = [(a_{ij}, b_{ij})]$ and $A_2 = [(a'_{ij}, b'_{ij})] \in M_n(S^2)$ and $F(A_1) = F(A_2)$. This im-

plies that $([a_{ij}], [b_{ij}]) = ([a'_{ij}], [b'_{ij}])$. From this we get, $[a_{ij}] = [a'_{ij}]$ and $[b_{ij}] = [b'_{ij}]$. Thus $a_{ij} = a'_{ij}$ for all i and j and $b_{ij} = b'_{ij}$ for all i and j . This implies that $(a_{ij}, b_{ij}) = (a'_{ij}, b'_{ij})$ for all i and j . Thus $[(a_{ij}, b_{ij})] = [(a'_{ij}, b'_{ij})]$. Hence $A_1 = A_2$.

Proof of 4.- To prove that F is surjective.

Let $([a_{ij}], [b_{ij}]) \in (M_n(S))^2$, where a_{ij} and $b_{ij} \in S$ for all i and j . This implies that $(a_{ij}, b_{ij}) \in S^2$. Thus $[(a_{ij}, b_{ij})] \in M_n(S^2)$, such that $F([(a_{ij}, b_{ij})]) = ([a_{ij}], [b_{ij}])$. Thus for every matrix (say A) in $(M_n(S))^2$ we get a matrix (say A') in $M_n(S^2)$ such that $F(A') = A$. This implies that F is surjective.

Hence we have proved that there is a semiring isomorphism between $M_n(S^2)$ and $(M_n(S))^2$.

■

We will use theorem 3.3.1 to connect the matrices over the Boolean semiring and the matrices over the sign pattern semiring. Now we will prove that there is a one to one semiring homomorphism from the n by n matrices over the sign pattern semiring to the $2n$ by $2n$ matrices over the Boolean semiring.

Let S be a semiring then $M_{2n}(S)$, the set of all $2n$ by $2n$ matrices over the semiring S and $M_n(S^2)$, the set of all n by n matrices over the semiring S^2 . We know that $M_{2n}(S)$ and $M_n(S^2)$ form semirings under the usual addition and multiplication of the matrices. Clearly $[0]$ is the additive identity and I_{2n} is the multiplicative identity of $M_{2n}(S)$, and $([0], [0])$ is the additive identity and $(\text{diag}\{1, 1, \dots, 1\}, \text{diag}\{0, 0, \dots, 0\})$ is the multiplicative identity of

$(M_n(S))^2$. The following theorem shows the relation between $M_{2n}(S)$ and $M_n(S^2)$.

Theorem 3.3.2. - *There is a semiring monomorphism from $(M_n(S))^2$ to $M_{2n}(S)$.*

Proof. - Let $A = [a_{ij}] \in M_n(S)$ and $B = [b_{ij}] \in M_n(S)$. Then $(A,B) = ([a_{ij}], [b_{ij}]) \in (M_n(S))^2$, where a_{ij} is the ij^{th} entry of A and b_{ij} is the ij^{th} entry of B .

$$\text{Let } F: (M_n(S))^2 \rightarrow M_{2n}(S), \text{ such that } F((A,B)) = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \text{ where } (A,B) \in (M_n(S))^2 \text{ and } \begin{bmatrix} A & B \\ B & A \end{bmatrix} \in M_{2n}(S).$$

Clearly F takes the additive identity and the multiplicative identity of $(M_n(S))^2$ to the additive identity and the multiplicative identity of $M_{2n}(S)$. Now we will show the following conditions:

1. $F((A,B) \oplus (C,D)) = F((A,B)) \oplus F((C,D))$ for all (A,B) and $(C,D) \in (M_n(S))^2$.
2. $F((A,B) \otimes (C,D)) = F((A,B)) \otimes F((C,D))$ for all (A,B) and $(C,D) \in (M_n(S))^2$.
3. F is injective.

Proof of 1.- Let $(A,B) = ([a_{ij}], [b_{ij}])$ and $(C,D) = ([c_{ij}], [d_{ij}]) \in (M_n(S))^2$. Then $(A,B) \oplus (C,D) = (A \oplus C), (B \oplus D) = (([a_{ij}] \oplus [c_{ij}]), ([b_{ij}] \oplus [d_{ij}]))$.

$$\begin{aligned} F((A,B) \oplus (C,D)) &= F((A \oplus C), (B \oplus D)) = \begin{bmatrix} A \oplus C & B \oplus D \\ B \oplus D & A \oplus C \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ B & A \end{bmatrix} \oplus \begin{bmatrix} C & D \\ D & C \end{bmatrix} \end{aligned}$$

$$= F((A,B)) \oplus F((C,D)).$$

Proof of 2.- Let $(A,B) = ([a_{ij}], [b_{ij}])$ and $(C,D) = ([c_{ij}], [d_{ij}]) \in (M_n(S))^2$. Then $(A,B) \otimes (C,D) = (A \otimes C) \oplus (B \otimes D), (B \otimes C) \oplus (A \otimes D)$.

$$\begin{aligned} F((A,B) \otimes (C,D)) &= F((A \otimes C) \oplus (B \otimes D), (B \otimes C) \oplus (A \otimes D)) \\ &= \begin{bmatrix} (A \otimes C) \oplus (B \otimes D) & (B \otimes C) \oplus (A \otimes D) \\ (B \otimes C) \oplus (A \otimes D) & (A \otimes C) \oplus (B \otimes D) \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ B & A \end{bmatrix} \otimes \begin{bmatrix} C & D \\ D & C \end{bmatrix} \\ &= F((A,B)) \otimes F((C,D)). \end{aligned}$$

Proof of 3.- To prove that F is injective, we will show that if $F((A,B)) = F((C,D))$, then $(A,B) = (C,D)$ for all $(A,B), (C,D) \in (M_n(S))^2$.

Let $(A,B) = ([a_{ij}], [b_{ij}])$ and $(C,D) = ([c_{ij}], [d_{ij}]) \in (M_n(S))^2$ and $F((A,B)) = F((C,D))$. This implies that $\begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} C & D \\ D & C \end{bmatrix}$
From this we get, $A = C$ and $B = D$. Thus $(A,B) = (C,D)$.

Hence we have proved that there is a semiring monomorphism from $(M_n(S))^2$ to $M_{2n}(S)$. ■

Remark 3.3.1. - Now we have $M_n(S^2) \xrightarrow{\text{isomorphism}} (M_n(S))^2 \xrightarrow{\text{monomorphism}} M_{2n}(S)$. This implies that there is a monomorphism from $M_n(S^2)$ to $M_{2n}(S)$. In particular, if we suppose

that S is the Boolean semiring, then we know that S^2 will be the sign pattern semiring. Thus we conclude that there is a one to one semiring homomorphism from the matrices over the sign pattern semiring to the matrices over the Boolean semiring.

3.4 Diagonal Permutations

The set of all permutations on the set $\{1, 2, \dots, n\}$, denoted by S_n , is called the symmetric group of order n . Let A be an n by n matrix. If $\sigma \in S_n$ then $(a_{1\sigma(1)}, a_{2\sigma(2)}, \dots, a_{n\sigma(n)})$ is called the diagonal of the matrix A .

Now we will relate all the diagonals of an n by n sign pattern matrix A to some of the diagonals of the corresponding $2n$ by $2n$ Boolean matrix $B(A)$. We will use this to find the relation between the permanent of A and the bideterminant of $B(A)$ in the next section. Let A be an n by n sign pattern matrix and $\sigma \in S_n$ be a permutation on A then $\tau \in S_{2n}$ is a permutation on the corresponding Boolean matrix $B(A)$, where S_{2n} is the set of all permutations on the set $\{1^+, 2^+, \dots, n^+, 1^-, 2^-, \dots, n^-\}$.

Given a permutation $\sigma \in S_n$ we can construct a permutation $\tau \in S_{2n}$ as follows:

- If $a_{i\sigma(i)} = +1$ then;

$$i^+ \rightarrow \tau(i^+) \quad \text{such that } a_{i^+, \tau(i^+)} = 1,$$

$$\text{and } i^- \rightarrow \tau(i^-) \quad \text{such that } a_{i^-, \tau(i^-)} = 1,$$

- If $a_{i\sigma(i)} = -1$ then;

$$i^+ \rightarrow \tau(i^-) \quad \text{such that } a_{i^+, \tau(i^-)} = 1,$$

$$\text{and } i^- \rightarrow \tau(i^+) \quad \text{such that } a_{i^-, \tau(i^+)} = 1,$$

- If $a_{i\sigma(i)} = 0$ then $a_{i^-, \tau(i^-)} = a_{i^+, \tau(i^+)} = a_{i^-, \tau(i^+)} = a_{i^+, \tau(i^-)} = 0$.
- If $a_{i\sigma(i)} = \#$ then $a_{i^-, \tau(i^-)} = a_{i^+, \tau(i^+)} = a_{i^-, \tau(i^+)} = a_{i^+, \tau(i^-)} = 1$.

The permutation $\sigma \in S_n$ can be written as the product of cycles of even and odd length. If σ has a cycle $(i_1, \sigma(i_1), \sigma^2(i_1), \dots, \sigma^{k-1}(i_1))$ of length k , then $\sigma^k(i_1) = i_1$.

Lemma 3.4.1. - Let A be an n by n sign pattern matrix and $\sigma \in S_n$ be a permutation on A . If odd numbers of $a'_{i, \sigma(i)}$ in the sign pattern matrix A are negative then σ corresponds to an odd permutation $\tau \in S_{2n}$.

Proof. - Let σ have a cycle $(i_1, \sigma(i_1), \sigma^2(i_1), \dots, \sigma^{k-1}(i_1))$ of length k and odd numbers of $a'_{i, \sigma(i)}$ correspond to this cycle are negative. Then this cycle of length k can be extended to a cycle of length $2k$ of the form,

$$(i_1^+, \tau(i_1^-), \tau^2(i_1^-), \dots, \tau^{k-1}(i_1^+), i_1^-, \tau(i_1^+), \tau^2(i_1^+), \dots, \tau^{k-1}(i_1^-))$$

in S_{2n} . The sign of this cycle is equal to $(-1)^{2k-1} = -1$. i.e, this is an odd cycle. ■

Lemma 3.4.2. - Let A be an n by n sign pattern matrix and $\sigma \in S_n$ be a permutation on A of length k . If even numbers of $a'_{i,\sigma(i)}$ in the sign pattern matrix A are negative then σ corresponds to an even permutation $\tau \in S_{2n}$.

Proof. - Let σ have a cycle $(i_1, \sigma(i_1), \sigma^2(i_1), \dots, \sigma^{k-1}(i_1))$ of length k and even numbers of $a'_{i,\sigma(i)}$ correspond to this cycle are negative. Then this cycle of length k can be extended to a cycle consisting of two cycles of length k of the form,

$$M_1 = (i^+, \tau(i^-), \tau^2(i^-), \dots, \tau^{k-1}(i^+))$$

$$\text{and } M_2 = (i^-, \tau(i^+), \tau^2(i^+), \dots, \tau^{k-1}(i^-))$$

in S_{2n} . The sign of this cycle is equal to $(-1)^{k-1} (-1)^{k-1} = 1$. i.e, this is an even cycle. ■

Lemma 3.4.3. - Let A be an n by n sign pattern matrix and $\sigma \in S_n$ be a permutation on A . If $\{\sigma \in S_n \mid \prod_{i=1}^n a_{i\sigma(i)} = -1\}$, where $a_{i\sigma(i)} \in A$, then there exists an odd permutation τ on $B(A)$ such that $\prod_{i=1}^{2n} b_{i,\tau(i)} = 1$.

Proof. - Given that $\prod_{i=1}^n a_{i\sigma(i)} = -1$, this implies that there are an odd number of cycles in σ which have an odd number of negative entries in the sign pattern matrix A . All other cycles (suppose l cycles) have even number of negative entries. Thus by lemma 3.4.1 and 3.4.2, we get odd number of cycles in $\tau \in S_{2n}$ have negative signs and all other cycles (l cycles) have positive signs. Clearly $\text{sign}(\tau) = (-1)^{\text{odd}} \cdot (1)^l = -1$. Thus τ is an odd permutation of order $2n$, also $\prod_{i=1}^{2n} b_{i,\tau(i)} = 1$. ■

Lemma 3.4.4. - Let A be an n by n sign pattern matrix and $\sigma \in S_n$ be a permutation on A . If $\{\sigma \in S_n \mid \prod_{i=1}^n a_{i\sigma(i)} = +1\}$, where $a_{i\sigma(i)} \in A$, then there exists an even permutation τ on $B(A)$ such that $\prod_{i=1}^{2n} b_{i,\tau(i)} = 1$.

Proof. - Given that $\prod_{i=1}^n a_{i\sigma(i)} = +1$, this implies that there are an even number of cycles in σ which have an odd number of negative entries in the sign pattern matrix A . All other cycles (suppose l cycles) have even number of negative entries. Thus by lemma 3.4.1 and 3.4.2, we get even number of cycles in $\tau \in S_{2n}$ have negative signs and all other cycles (l cycles) have positive signs. Clearly $\text{sign}(\tau) = (-1)^{\text{even}} \cdot (1)^l = 1$. Thus τ is an even permutation of order $2n$, also $\prod_{i=1}^{2n} b_{i,\tau(i)} = 1$. ■

3.5 Bideterminant of a Boolean Matrix and SNS Matrix

This section explores the connection between the bideterminant of a sign pattern matrix A and the bideterminant of its corresponding Boolean matrix $B(A)$. We relate the determinantal rank of $B(A)$ with the sign nonsingularity of A .

To prove our next result, we make use of König-Frobenius theorem.

Theorem 3.5.1. (König-Frobenius) [17] *Let A be an n by n matrix. A necessary and sufficient condition for every diagonal of A to contain a zero entry is that A contains an k by m zero submatrix where $k+m \geq n+1$.*

Theorem 3.5.2. - *Let A be a sign pattern matrix and $B(A)$ is the corresponding Boolean matrix. Then the permanent of A and the bideterminant of $B(A)$ has the following relation:*

1. If $\text{per}(A) = 0$ then $\text{bidet}(B(A)) = (0,0)$.
2. If $\text{per}(A) = +1$ then $\det^+(B(A)) = 1$.
3. If $\text{per}(A) = -1$ then $\det^-(B(A)) = 1$.
4. If $\text{per}(A) = \#$ then $\text{bidet}(B(A)) = (1,1)$.

Proof. of 1.- Let $\text{per}(A) = 0$. This implies that $\text{bidet}(A) = (0,0)$. Thus $\prod_{i=1}^n a_{i\sigma(i)} = 0$, for all $\sigma \in S_n$. i.e. every diagonal of A contains a zero entry. So by König-Frobenius theorem there exists a k by m zero submatrix of A , where $k+m \geq n+1$. Thus $B(A)$ has $2k$ by $2m$ zero submatrix, where $2k + 2m \geq 2n + 2 > 2n + 1$. Hence $\text{bidet}(B(A)) = (0,0)$, (using König-Frobenius theorem).

Proof of 2.- Let $\text{per}(A) = +1$. This implies that there exists a permutation $\sigma \in S_n$ such that $\prod_{i=1}^n a_{i\sigma(i)} = +1$. By lemma 3.4.4 there exists an even permutation $\tau \in S_{2n}$ such that $\prod_{i=1}^{2n} b_{i,\tau(i)} = 1$. Hence $\det^+(B(A)) = 1$.

Proof of 3.- Let $\text{per}(A) = -1$. This implies that there exists a permutation $\sigma \in S_n$ such that $\prod_{i=1}^n a_{i\sigma(i)} = -1$. By lemma 3.4.3 there exists an odd permutation $\tau \in S_{2n}$ such that $\prod_{i=1}^{2n} b_{i,\tau(i)} = 1$. Hence $\det^-(B(A)) = 1$.

Proof of 4.- Let $\text{per}(A) = \#$. i.e. the bideterminant of A is either $(+1,-1)$ or $(-1,+1)$. This implies that there exists two permutations σ_1 and $\sigma_2 \in S_n$ such that

$$\prod_{i=1}^n a_{i\sigma_1(i)} = +1 \quad \text{and} \quad \prod_{i=1}^n a_{i\sigma_2(i)} = -1$$

So by lemma 3.4.3 and 3.4.4 there exists an even and an odd permutation τ_1 and $\tau_2 \in S_{2n}$ such that

$$\prod_{i=1}^{2n} b_{i,\tau_1(i)} = 1 \quad \text{and} \quad \prod_{i=1}^{2n} b_{i,\tau_2(i)} = 1.$$

Thus $\text{bidet}(B(A)) = (1,1)$.

■

Remark 3.5.1. - *In the above result we have if the $\text{per}(A) = +1$, then $\det^+(B(A)) = 1$.*

Here we can not say anything about $\det^-(B(A))$. It depends upon the bideterminant of A.

In the next result we will show that if $\text{bidet}(A) = (+1,+1)$ then $\det^-(B(A)) = 1$.

Remark 3.5.2. - *Let A be a sign pattern matrix. If $\det^+(A) = +1$, then rows and columns of A can be permuted in such a way that all main diagonal entries of the resulting matrix are +1, and the bideterminant of the resulting matrix is same as the bideterminant of A.*

Theorem 3.5.3. - *Let A be a sign pattern matrix and B(A) is the corresponding Boolean matrix. If $\text{bidet}(A) = (+1,+1)$ then $\det^-(B(A)) = 1$.*

Proof. - Given that the $\text{bidet}(A) = (+1,+1)$. By remark 3.5.2, we can assume that the main diagonal of A has all positive entries. So the corresponding Boolean matrix B(A)

has all the main diagonal entries equal to 1. Also $\det^-(A) = +1$. This implies that there exists an odd permutation $\sigma \in S_n$ such that $\prod_{i=1}^n a_{i\sigma(i)} = +1$. This implies that there are an even number of cycles in σ which have an odd number of negative entries in the sign pattern matrix A. All other cycles (suppose l cycles) have even numbers of negative entries. By using lemma 3.4.1 and lemma 3.4.2 we get, the corresponding permutation $\tau \in S_{2n}$ has even numbers of cycles of even lengths and l cycles consisting of two cycles of same length. Also by lemma 3.4.4, $\prod_{i=1}^{2n} b_{i,\tau(i)} = 1$. Now construct a permutation $\tau_1 \in S_{2n}$ consists of one of the cycles of length $2k$ from τ and fixes all other entries (such a permutation exists as τ contains cycles of even length and all the main diagonal entries of B(A) are 1). Clearly τ_1 is an odd permutation of order $2n$ and $\prod_{i=1}^{2n} b_{i,\tau_1(i)} = 1$. Thus $\det^-(B(A)) = 1$. ■

Example 3.5.1. - Let $A = \begin{bmatrix} +1 & -1 & 0 \\ -1 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}$ be a 3 by 3 sign pattern matrix.

Clearly the bidet(A) = (+1,+1). The Boolean matrix corresponding to this sign pattern matrix A is

$$B(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly $\sigma = (24) \in S_6$ is an odd permutation, and the entries in $B(A)$ correspond to the σ are all equals to 1. This implies that $\det^-(B(A)) = 1$.

Similarly we can show that if $\text{bidet}(A) = (-1,-1)$ then $\det^+(B(A)) = 1$.

Definition 3.5.1. (*Determinantal Rank*) [31] Let A be an m by n matrix over the semiring S . The determinantal rank of A , denoted by $rk_{\det}(A)$, is the biggest positive integer k such that there exists a k by k submatrix A' of A with $\det^+(A') \neq \det^-(A')$.

For a Boolean matrix B the bideterminant is one of the following

$$(0,0) \quad (1,0) \quad (0,1) \quad (1,1)$$

and the determinantal rank of a Boolean matrix B is the largest natural number k for which there exists a k by k submatrix of B having bideterminant equal to $(1,0)$ or $(0,1)$. A Boolean matrix is said to have *full determinantal rank* if its bideterminant is either $(1,0)$ or $(0,1)$.

Definition 3.5.2. (*Zero Pattern*) [11] Let A be a sign pattern matrix. The zero pattern of A , denoted by $|A|$, is a $(0, +1)$ -pattern obtained by replacing -1 by $+1$.

Clearly $|A|$ is a sign nonnegative matrix.

Lemma 3.5.1. - Let A be a nonnegative sign pattern matrix. A is an SNS matrix if and only if the bideterminant of A is either $(+1,0)$ or $(0,+1)$.

Proof. - One side is obvious from the definition of a sign nonsingular matrix, so we will prove the other side. Let A be a sign nonnegative matrix. This implies that A has no negative entry. So the both $\det^+(A)$ and $\det^-(A)$ have two possibilities, either $+1$ or 0 . Therefore either

$$\det^+(A) = \bigoplus_{\sigma \in A_n} \bigotimes_{i=1}^n a_{i\sigma(i)} \text{ is positive.}$$

or

$$\det^-(A) = \bigoplus_{\sigma \in S_n \setminus A_n} \bigotimes_{i=1}^n a_{i\sigma(i)} \text{ is positive.}$$

because if both are positive then the bideterminant of $A = (+1, +1)$. Which is a contradiction because A is SNS. Hence the bideterminant of A is either $(+1, 0)$ or $(0, +1)$.

■

Now we are ready to prove the following theorem.

Theorem 3.5.4. - *Let A be a sign pattern matrix. $|A|$ is a sign nonnegative matrix obtained from A by replacing all negative signs by positive signs and $B(A)$ is the Boolean matrix corresponding to the sign pattern matrix A . Then the following are equivalent:*

1. $B(A)$ has full determinantal rank.
2. Both A and $|A|$ are SNS matrices.
3. A has a non-zero signed determinant and a non-zero signed permanent.

Proof. - $(1 \Leftrightarrow 2)$ Given that A is a sign pattern matrix. Let A_1 be a real matrix such that $A_1 \in Q(A)$, the qualitative class of A . Construct two real matrices A_1^+ and A_1^- from

A_1 as follows: Let $[A_1]_{ij}$ is the ij^{th} entry of A_1 then,

$$[A_1^+]_{ij}, \text{ the } ij^{th} \text{ entry of } A_1^+ = \begin{cases} [A_1]_{ij}, & \text{if } [A_1]_{ij} > 0 \\ 0, & \text{otherwise.} \end{cases} \quad \text{and}$$

$$[A_1^-]_{ij}, \text{ the } ij^{th} \text{ entry of } A_1^- = \begin{cases} -[A_1]_{ij}, & \text{if } [A_1]_{ij} < 0 \\ 0, & \text{otherwise.} \end{cases}$$

Let $B(A_1)$ is a real matrix defined as follows:

$$B(A_1) = \begin{bmatrix} A_1^+ & A_1^- \\ A_1^- & A_1^+ \end{bmatrix}$$

and $P = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix}$ is a unitary matrix with $P^{-1} = P$.

$$\begin{aligned} P^{-1}B(A_1)P &= \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} A_1^+ & A_1^- \\ A_1^- & A_1^+ \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix} \\ &= \begin{bmatrix} A_1^+ + A_1^- & 0 \\ 0 & A_1^+ - A_1^- \end{bmatrix} \\ &= \begin{bmatrix} |A_1| & 0 \\ 0 & A_1 \end{bmatrix}, \quad \text{where } |A_1| \text{ is the non negative real matrix obtained from } A_1 \text{ by} \\ &\text{replacing each entry with its absolute value. Clearly } |A_1| \in Q(|A|). \end{aligned}$$

This implies that $\det(B(A_1)) = \det(P^{-1}B(A_1)P) = \det(|A_1|)\det(A_1)$. Thus $\det(B(A_1)) \neq 0$ if and only if $\det(|A_1|) \neq 0$ and $\det(A_1) \neq 0$, and this is true for all real matrices $\in Q(A)$. Hence $B(A)$ has full determinantal rank if and only if both $|A|$ and A are SNS matrices.

(2 \Rightarrow 3): Firstly suppose that both A and $|A|$ are SNS matrices. A is an SNS matrix so clearly A has a non-zero signed determinant and the bideterminant of A has the following choices:

$$(\pm 1, 0), (0, \pm 1), (-1, +1) \text{ and } (+1, -1).$$

Also $|A|$ is an SNS matrix. This implies that the bideterminant of A can not be $(-1, +1)$ or $(+1, -1)$, (because if so then the bideterminant of $|A| = (+1, +1)$. Thus $|A|$ is not SNS, which is a contradiction). Thus the bideterminant of A has only the following choices:

$$(\pm 1, 0) \text{ and } (0, \pm 1).$$

Hence A has a non-zero signed permanent. Thus if both A and $|A|$ are SNS matrices then A has a non-zero signed determinant and a non-zero signed permanent.

(3 \Rightarrow 2):- Now suppose that A has a non-zero signed determinant and a non-zero signed permanent, we will show that both A and $|A|$ are SNS matrices. A has a non-zero signed determinant, so clearly by the definition of sign nonsingular matrices A is an SNS matrix. Also given that A has a non-zero signed permanent. This implies that the bideterminant of A has only the following choices:

$$(\pm 1, 0) \text{ and } (0, \pm 1).$$

(Because if $\text{bidet}(A) = (-1, +1)$ or $(+1, -1)$, then $\text{per}(A) = \#$ which is a contradiction to the fact that A has a non-zero signed permanent.) Thus the bideterminant of $|A|$ is $(+1, 0)$ or $(0, +1)$. Hence $|A|$ is an SNS matrix (using lemma 3.5.1). Thus if A has a non-zero signed

determinant and a non-zero signed permanent then both A and $|A|$ are SNS matrices. ■

Remark 3.5.3. - *From the above theorem we get, if $B(A)$ has full determinantal rank then A is SNS. The converse may not be true. In the other words if A is an SNS matrix then the corresponding Boolean matrix $B(A)$ may not have full determinantal rank. We will show this by an example.*

Example 3.5.2. - Let $A = \begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix}$ be a 2 by 2 sign pattern matrix.

Clearly $\text{bidet}(A) = (+1, -1)$, this implies that A is an SNS matrix. The Boolean matrix corresponding to this sign pattern matrix A is

$$B(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

All the main diagonal entries of $B(A)$ are 1. This implies that the $\det^+(B(A)) = 1$. Also $\sigma = (1432) \in S_4$ is an odd permutation, and the entries in $B(A)$ correspond to the σ are all equals to 1. This implies that $\det^-(B(A)) = 1$. Thus the $\text{bidet}(B(A)) = (1, 1)$, i.e, $B(A)$ does not have full determinantal rank. Thus if A is an SNS matrix then the corresponding Boolean matrix $B(A)$ may not have full determinantal rank.

Chapter 4

Different Notions of the Rank of Matrices Over Semiring

In this chapter we investigate different notions of the rank of matrices over semirings. For these rank functions we simplify proofs of classical inequalities for the sum and the product of matrices using the semiring versions of the Cauchy-Binet and Laplace theorems. For matrices over the sign pattern semiring, the minimum rank of the sign pattern is compared with the other versions of the rank.

4.1 Some Basic Results

There are many different yet equivalent ways of defining the rank of a matrix over a field. Many of these definitions may be generalized to matrices over semirings. With matrices over semirings, these different definitions are no longer equivalent and they yield

different rank functions. We will introduce the rank functions used in our thesis, these and many others are discussed extensively in [2, 6, 8].

Definition 4.1.1. (*Basis*)- A collection of weakly linearly independent vectors is said to be a basis of a semimodule M over a semiring S , if its linear span is M . The dimension of M is a minimal number of vectors in any basis of M .

Definition 4.1.2. (*Row Rank*)- Let S be a semiring. The row rank of a matrix $A \in M_{mn}(S)$, denoted by $r(A)$, is the dimension of the linear span of the rows of A .

Definition 4.1.3. (*Column Rank*)- Let S be a semiring. The column rank of a matrix $A \in M_{mn}(S)$, denoted by $c(A)$, is the dimension of the linear span of the columns of A .

Definition 4.1.4. (*Maximal Row Rank*)- Let S be a semiring. The maximal row rank of a matrix $A \in M_{mn}(S)$ in weak or Gondran-Minoux sense, denoted respectively by $mr_w(A)$ or $mr_{GM}(A)$, is the maximum number k such that A contains k weakly or Gondran-Minoux, linearly independent rows.

Similarly we define maximum column rank and denoted by $mc_w(A)$ or $mc_{GM}(A)$.

Remark 4.1.1. [2] Note that for matrices over general semirings, we have $r(A) \leq mr_w(A)$.

Definition 4.1.5. (Factor Rank)- Let S be a semiring. The factor rank $f(A)$ of a non zero matrix $A \in M_{mn}(S)$ is the smallest integer k such that $A = BC$ for some matrices $B \in M_{mk}(S)$ and $C \in M_{kn}(S)$.

Definition 4.1.6. (Term rank)- Let S be a semiring. The term rank of a matrix $A \in M_{mn}(S)$, denoted by $term(A)$, is defined as the minimum number of rows needed to include all nonzero elements of A .

Unlike the other types of rank, the term rank is not a straightforward generalization of one of the equivalent definitions of the rank of a matrix over a field. For matrices over fields, the factor rank, the row rank, the column rank, the maximal row rank in weak or Gondran-Minoux sense and the maximal column rank in weak or Gondran-Minoux sense are all equal while term rank may be different from these. For the matrices over semirings, all of these ranks may differ from one other.

Remark 4.1.2. -

1. It is proved in [6] that the inequality

$$f(A) \leq term(A)$$

holds for matrices with entries in the arbitrary semiring.

2. $f(A) \leq \min\{r(A), c(A)\}$, Lemma 2.3 of [8].

Now we will define the minimum rank and the maximum rank of a sign pattern matrix.

Definition 4.1.7. (*Minimum Rank*) [3] *Let A be a sign pattern matrix. The minimum rank of A , denoted by $mr(A)$, is defined as*

$$mr(A) = \min_{B \in Q(A)} \{ \text{rank of } B \}$$

Definition 4.1.8. (*Maximum Rank*) [3] *Let A be a sign pattern matrix. The maximum rank of A , denoted by $MR(A)$, is defined as*

$$MR(A) = \max_{B \in Q(A)} \{ \text{rank of } B \}$$

The minimum rank of a sign pattern is not only of interest theoretically, it is also of practical value. For instance [14] is devoted to the question of constructing real m by n matrices of low rank under the constraint that each entry is nonzero and has a given sign. This problem arises from a topic in neural networks or, more specifically, multilayer perceptrons. In this application, the rank of a real matrix can be interpreted as the number of elements in a hidden layer, which motivates a search for low rank solutions.

Remark 4.1.3. - *Both the minimum rank and the maximum rank of an n by n SNS matrix A are equals to n . Since A is SNS, so every real matrix $B \in Q(A)$, determinant of $B \neq 0$. This implies that for all $B \in Q(A)$, rows of B are linearly independent. This implies that $\min_{B \in Q(A)} \{ \text{rank of } B \} = n = \max_{B \in Q(A)} \{ \text{rank of } B \}$.*

Proposition 4.1.1. [3] *Let A be an m by n sign pattern matrix. Then $MR(A)$ is the maximum number of nonzero entries of A no two of which lie on the same row or column.*

We will refer to either a row or a column of the matrix as a line of the matrix. Now we will mention the famous fundamental minimax theorem of König (1936). This theorem deals exclusively with the properties of $(0,1)$ - matrix that remain invariant under arbitrary permutation of the lines of the matrix.

Theorem 4.1.1. [10]- *Let A be a $(0,1)$ -matrix of size m by n . The minimal number of lines in A that cover all of the 1's in A is equal to the maximum number of 1's in A with no two of the 1's on a line.*

Remark 4.1.4. - *From the above theorem we conclude that the maximum rank of the sign pattern matrix is the term rank of the matrix over sign pattern semiring, i.e, $MR(A) = term(A)$.*

4.2 Gondran-Minoux linear dependence

In fact linear dependence in Gondran-Minoux sense of the rows a matrix over a sign pattern semiring is same as the linear dependence of its zero non-zero pattern in Gondran-Minoux sense. Let β be a Boolean semiring. Consider a mapping $\phi: \beta^2 \rightarrow \beta$, such that $\phi(0) = 0$ and $\phi(+1) = \phi(-1) = \phi(\#) = 1$. Clearly this is a semiring homomorphism.

Let A be a matrix over sign pattern semiring, then $\phi(A)$ is the zero non-zero pattern of A . Claim: $mr_{GM}(A) = mr_{GM}\phi(A)$. Suppose that k rows of A , $\{V_{r_1}, V_{r_2}, \dots, V_{r_k}\}$ are linearly dependent in Gondran-Minoux sense. This implies that there exist two subsets I and $J \subseteq K = \{1, 2, \dots, k\}$, where $I \cap J = \emptyset$, $I \cup J = K$, such that

$$\sum_{i \in I} \lambda_i V_{r_i} = \sum_{j \in J} \lambda_j V_{r_j},$$

where $\lambda_i \neq 0$ and $\lambda_j \neq 0 \in S$ for all i and j . Since the mapping is well defined, so $\phi(\sum_{i \in I} \lambda_i V_{r_i}) = \phi(\sum_{j \in J} \lambda_j V_{r_j})$. This implies that $\sum_{i \in I} \phi(\lambda_i) \phi(V_{r_i}) = \sum_{j \in J} \phi(\lambda_j) \phi(V_{r_j})$. Thus we get $\sum_{i \in I} \phi(V_{r_i}) = \sum_{j \in J} \phi(V_{r_j})$, where $\phi(\lambda_i)$ and $\phi(\lambda_j) = 1$ because all λ_i and λ_j are nonzero. This implies that rows of $\phi(A)$ are linearly dependent in Gondran-Minoux sense. Now suppose that rows of $\phi(A)$ are linearly dependent in Gondran-Minoux sense. This implies that $\sum_{i \in I} \phi(V_{r_i}) = \sum_{j \in J} \phi(V_{r_j})$. Then clearly $\sum_{i \in I} \# V_{r_i} = \sum_{j \in J} \# V_{r_j}$. Thus rows of A are also linearly dependent in Gondran-Minoux sense. Hence $mr_{GM}(A) = mr_{GM}\phi(A)$.

Remark 4.2.1. - *If there is no zero entry in the matrix A over the sign pattern semiring then $mr_{GM}(A) = 1$.*

Remark 4.2.2. - *If the rows of a sign pattern matrix are independent in the Gondran-Minoux sense then they are also independent in weak sense. But the converse may not be true, as it is shown in the following example.*

Example 4.2.1. - Let $A = \begin{bmatrix} 0 & +1 & +1 & +1 \\ +1 & 0 & +1 & +1 \\ +1 & +1 & 0 & +1 \\ +1 & +1 & +1 & 0 \end{bmatrix}$

Clearly rows of A are weakly linearly independent but not linearly independent in the Gondran-Minoux sense.

4.3 Rank Inequalities

In this section we compare different rank functions for the matrices over the sign pattern semiring. For these rank functions we investigate the semiring versions of classical inequalities for the sum and product of matrices. Numerous examples are given to illustrate the behaviour of rank functions.

Let $M_{mn}(S)$ denotes the set of m by n sign pattern matrices. Throughout we assume that $m \leq n$. Let A and $C \in M_{nn}(S)$, then $A + C$ exists (i.e, each entry is distinct from $\#$) if $a_{ij}c_{ij} \neq -1$ for all i and j in $\{1, 2, \dots, n\}$. The product AC exists (i.e, each entry is distinct from $\#$) if no two terms in the sum

$$\sum_{k=1}^n a_{ik}c_{ki}$$

are oppositely signed, for all i and j in $\{1, 2, \dots, n\}$.

There are classical inequalities for the rank function ρ of sums and products of matrices over fields: The *rank-sum inequality*:

$$| \rho(A) - \rho(B) | \leq \rho(A + B) \leq \rho(A) + \rho(B);$$

The *Sylvester's laws*:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min \{ \rho(A) , \rho(B) \};$$

Where A and B are real or complex matrices. These inequalities may or may not hold when S is not a field. Now we will compare these inequalities for the minimal rank of the matrices over the sign pattern semiring.

In [6] Beasley and Guterman showed that $\rho(A + B) \leq \rho(A) + \rho(B)$ and $\rho(AB) \leq \min \{ \rho(A) , \rho(B) \}$ holds and $\rho(A + B) \geq | \rho(A) - \rho(B) |$ and $\rho(AB) \geq \rho(A) + \rho(B) - n$ does not hold when ρ is a factor rank and term rank. We will use his examples to show similar results for the minimal rank and the ideal rank.

The matrix I_n is the n by n identity sign pattern matrix with all diagonal entries equal to +1, $J_{m,n}$ is the m by n matrix with all entries equal to +1, $O_{m,n}$ is the m by n zero matrix. We omit the subscripts when the size of the matrix is obvious from the context, and write I , J , and O , respectively. The matrix $E_{i,j}$, called a *cell*, denotes the matrix whose (i, j) entry is +1 or -1 and all other entries are zero. Let U_k denote the k by k matrix with +1s above and on the main diagonal, L_k denote the k by k strictly lower triangular matrix of +1s. The block diagonal matrix of the form $\begin{bmatrix} A & O \\ O & B \end{bmatrix}$, is denoted by $A \oplus B$. Note that in this sense \oplus is not commutative. A diagonal sign pattern matrix is denoted by D and D is called a *signature sign pattern* if each of its diagonal entries is either +1 or -1.

4.3.1 The Minimum Rank

Let A be a sign pattern matrix. The minimum rank of A , denoted by $\text{mr}(A)$, is defined as

$$\text{mr}(A) = \min_{B \in Q(A)} \{ \text{rank of } B \}$$

Now we will give an example to show that the inequality $\text{mr}(A + B) \geq | \text{mr}(A) - \text{mr}(B) |$ does not hold for the sign pattern matrices.

Let us consider $A = K_7 =$

$$\begin{bmatrix} 0 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & 0 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & 0 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & 0 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 & 0 & +1 & +1 \\ +1 & +1 & +1 & +1 & +1 & 0 & +1 \\ +1 & +1 & +1 & +1 & +1 & +1 & 0 \end{bmatrix}$$

and $B = I_7$, the identity sign pattern matrix, [6]. Then $\text{mr}(A + B) = \text{mr}(J_7) = 1$, but $\text{mr}(K_7) \leq 5$ because

$$C = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 2 & 1 & 0 \end{bmatrix} \in Q(A)$$

and rank of $C = 5$. This implies that $\text{mr}(A) \leq 5$, also $\text{mr}(B) = 7$.

$$\text{Thus } |\text{mr}(A) - \text{mr}(B)| = \text{mr}(B) - \text{mr}(A) \geq 7 - 5 = 2 > 1 = \text{mr}(A + B).$$

Proposition 4.3.1. - *Let $A, B \in M_{mn}(S)$. Then*

$$1. \text{mr}(A + B) \leq \text{mr}(A) + \text{mr}(B);$$

$$2. \text{mr}(A + B) \geq \begin{cases} \text{mr}(A) & \text{if } B = O \\ \text{mr}(B) & \text{if } A = O \\ 1 & \text{if } A \neq O, B \neq O \text{ and } O \notin Q(A + B). \end{cases}$$

These bounds are exact in the sense the upper bound and the lower bound are both the best possible for matrices over the sign pattern semiring.

Proof. 1. - Let $\text{mr}(A) = k$ and $\text{mr}(B) = l$. This implies that there exists a real matrix $\tilde{A} \in Q(A)$ such that rank of \tilde{A} (say $\rho(\tilde{A})$) = k , and a real matrix $\tilde{B} \in Q(B)$ such that rank of \tilde{B} (say $\rho(\tilde{B})$) = l . So by the rank-sum inequality for the real matrices we get: $\rho(\tilde{A} + \tilde{B}) \leq$

$\rho(\tilde{A}) + \rho(\tilde{B})$. Also $\tilde{A} + \tilde{B} \in Q(A + B)$, this implies that $\text{mr}(A + B) \leq \rho(\tilde{A} + \tilde{B}) \leq \rho(\tilde{A}) + \rho(\tilde{B})$. Thus $\text{mr}(A + B) \leq \rho(\tilde{A}) + \rho(\tilde{B}) = \text{mr}(A) + \text{mr}(B)$. Hence we get $\text{mr}(A + B) \leq \text{mr}(A) + \text{mr}(B)$. To prove that this bound is exact and the best possible, for each pair (r,s) , $0 \leq r, s \leq n$ consider the matrices $A_r = D_r \oplus O_{n-r}$ and $B_s = O_{n-s} \oplus D_s$ in the case $m = n$.

2.- Since $A + B = O$ if and only if both $A = O$ and $B = O$. Thus $\text{mr}(A + B) \geq 1$ unless $A = B = O$ and $O \notin Q(A + B)$, and clearly if $A = O$ then $\text{mr}(A + B) = \text{mr}(B)$ and if $B = O$ then $\text{mr}(A + B) = \text{mr}(A)$. To show that the equality holds let $A = B = E_{1,1}$. To prove that this bound is best possible, consider the following family of matrices, [6]: for each pair (r,s) , $0 \leq r, s \leq m$ consider the matrices

$$A_r = \begin{bmatrix} U_r & J_{r,n-r} \\ J_{m-r,r} & J_{m-r,n-r} \end{bmatrix}$$

and

$$B_s = \begin{bmatrix} J_{s,n-s} & L_s + I_s \\ J_{m-s,n-s} & J_{m-s,s} \end{bmatrix}$$

Then $\text{mr}(A_r) = r$, $\text{mr}(B_s) = s$ and $\text{mr}(A_r + B_s) = 1$, since $A_r + B_s = J_{m,n}$.

■

Now we will give an example to show that the inequality $\text{mr}(AB) \geq \text{mr}(A) + \text{mr}(B) - n$ need not hold for the sign pattern matrices A and B .

$$\text{Let us consider } A = \begin{bmatrix} +1 & 0 & 0 & \cdot & 0 \\ +1 & +1 & 0 & \cdot & 0 \\ +1 & 0 & +1 & \cdot & 0 \\ +1 & 0 & 0 & \cdot & +1 \end{bmatrix}$$

and $B = A^t$, [6]. Then $\text{mr}(A) = \text{mr}(B) = n$ and $\text{mr}(AB) = 1 \leq \text{mr}(A) + \text{mr}(B) - n = n$, since $AB = J$, where J is a square matrix with all entries equals to $+1$.

Proposition 4.3.2. - *Let $A \in M_{m,n}(S)$ and $B \in M_{n,r}(S)$. Then*

$$\text{mr}(AB) \leq \min \{ \text{mr}(A), \text{mr}(B) \};$$

Proof. - Let $\text{mr}(A) = k$ and $\text{mr}(B) = l$. This implies that there exists a real matrix $\tilde{A} \in Q(A)$ such that rank of \tilde{A} (say $\rho(\tilde{A})$) = k , and a real matrix $\tilde{B} \in Q(B)$ such that rank of \tilde{B} (say $\rho(\tilde{B})$) = l . So by the Sylvester's law for real matrices we get: $\rho(\tilde{A}\tilde{B}) \leq \min \{ \rho(\tilde{A}), \rho(\tilde{B}) \}$. Also $\tilde{A}\tilde{B} \in Q(AB)$, this implies that $\text{mr}(AB) \leq \rho(\tilde{A}\tilde{B}) \leq \min \{ \rho(\tilde{A}), \rho(\tilde{B}) \}$. From this we get $\text{mr}(AB) \leq \min \{ \rho(\tilde{A}), \rho(\tilde{B}) \} \leq \min \{ \text{mr}(A), \text{mr}(B) \}$. Thus $\text{mr}(AB) \leq \min \{ \text{mr}(A), \text{mr}(B) \}$. ■

4.3.2 Rank Relations

In this section we compare the minimum rank of a sign pattern matrix with the other ranks. we also define the rank functions of matrices over commutative semirings in

terms of minors and compare them using the semiring versions of the Cauchy-Binet and Laplace theorems.

Proposition 4.3.3. - *Let $A \in M_{m,n}(S)$, then $mr(A) \leq f(A)$.*

Proof. - If $f(A) = \min\{m,n\}$ then we are done. Now suppose that $f(A) = r < \min\{m,n\}$, we have to show that there exists at least one $\tilde{A} \in Q(A)$ such that $\text{rank of } \tilde{A} < r + 1$. Since $f(A) = r$, this implies that there exist matrices $B \in M_{m,r}(S)$ and $C \in M_{r,n}(S)$ such that $A = BC$. Now construct two real matrices by replacing $+1$ with 1 , -1 with -1 and 0 and $\#$ with 0 in B and C , and name these matrices \tilde{B} and \tilde{C} . Clearly $\tilde{B} \in Q(B)$ and $\tilde{C} \in Q(C)$, and $\tilde{B}\tilde{C} \in Q(A)$. The dimension of \tilde{B} is m by r ($r < m$), this implies that the rank of $\tilde{B} < r + 1$. Similarly the dimension of \tilde{C} is r by n ($r < n$), this implies that the rank of $\tilde{C} < r + 1$. Now the rank of $\tilde{B}\tilde{C} \leq \min \{\text{rank of } \tilde{B}, \text{rank of } \tilde{C}\}$, using Sylvester's laws for real matrices. Thus we get, $\text{rank of } \tilde{B}\tilde{C} < r+1$. Thus there exists a real matrix $\tilde{A} = \tilde{B}\tilde{C} \in Q(A)$ such that $\text{rank of } \tilde{A} < r + 1$. Hence $mr(A) \leq r$. ■

Now we will define the Laplace expansion of the determinant of a matrix over commutative semiring, and the ideals of a commutative semiring.

We adopt the following notations. For any $m, n \in \mathbb{N}$ with $m \leq n$, we define $Q_{m,n}$ to be the set of m -tuples of integers satisfying $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_{m-1} < \alpha_m \leq n$. If $\alpha, \beta \in Q_{m,n}$, then $A[\alpha|\beta]$ is the m by m submatrix of A whose (i,j) th entry is the (α_i, β_j) entry of A . If $m = 0$ and hence α and β are empty sets, then we define $\text{bidet}(A[\alpha|\beta]) = (1,0)$. For any $\alpha \in Q_{m,n}$, we define $s(\alpha) = \sum_{k=1}^m \alpha_k$.

We begin with stating the Laplace expression of the determinant of a matrix over commutative semiring S^2 given in [32].

Proposition 4.3.4. (Laplace Expression)- Let $A \in M_n(S^2)$, $0 \leq m \leq n$ and $\alpha \in Q_{m,n}$, then

$$\det_2(A) = \bigoplus_{\beta \in Q_{m,n}} \left((0, 1)^{s(\alpha)+s(\beta)} \otimes (\det_2(A[\alpha|\beta]) \otimes \det_2(A[\alpha^c|\beta^c])) \right).$$

Proposition 4.3.5. - Let $A, B \in M_n(S^2)$, then

$$\det_2(A + B) = \bigoplus_{m=0}^n \bigoplus_{\alpha, \beta \in Q_{m,n}} \left((0, 1)^{s(\alpha)+s(\beta)} \otimes (\det_2(A[\alpha|\beta]) \otimes \det_2(B[\alpha^c|\beta^c])) \right).$$

Proposition 4.3.6. (Cauchy-Binet Theorem for commutative semirings S^2) [32]

Let $A \in M_{m,n}(S^2)$, $B \in M_{n,p}(S^2)$, $k \leq \min \{m, n, p\}$. Suppose $I \in Q_{k,m}$ and $J \in Q_{k,p}$; then

$$\det_2(AB[I|J]) = \left(\bigoplus_{K \in Q_{k,n}} \det_2(A[I|K]) \otimes \det_2(B[K|J]) \right) \oplus (d_{I,J}, d_{I,J}), \text{ where } d_{I,J}$$

is an element of S .

We have the semiring monomorphism $\phi: S \rightarrow S^2$, such that $\phi(x) = (x, 0)$. We can extend this monomorphism to the matrices as $\phi: M_n(S) \rightarrow M_n(S^2)$ such that $\phi([a_{ij}]) = [(a_{ij}, 0)]$, where $a_{ij} \in S$ for all i and j . Now let $A \in M_n(S)$ then $\phi(A) \in M_n(S^2)$. Clearly $\det_2(\phi(A)) = \text{bidet}(A)$. Thus the bideterminant results follow by considering matrices of the form $[(a_{ij}, 0)]$. We have equivalent expressions for the bideterminant of a matrix over the commutative semiring S .

Proposition 4.3.7. (Laplace Expression)- Let $A \in M_n(S)$, $0 \leq m \leq n$ and $\alpha \in Q_{m,n}$, then

$$\text{bidet}(A) = \bigoplus_{\beta \in Q_{m,n}} \left((0, 1)^{s(\alpha)+s(\beta)} \otimes (\text{bidet}(A[\alpha|\beta]) \otimes \text{bidet}(A[\alpha^c|\beta^c])) \right).$$

Proposition 4.3.8. - Let $A, B \in M_n(S)$, then

$$\text{bidet}(A + B) = \bigoplus_{m=0}^n \bigoplus_{\alpha, \beta \in Q_{m,n}} \left((0, 1)^{s(\alpha)+s(\beta)} \otimes (\text{bidet}(A[\alpha|\beta]) \otimes \text{bidet}(B[\alpha^c|\beta^c])) \right).$$

Proposition 4.3.9. (Cauchy-Binet Theorem for commutative semirings) [32] Let $A \in M_{m,n}(S)$, $B \in M_{n,p}(S)$, $k \leq \min \{m, n, p\}$. Suppose $I \in Q_{k,m}$ and $J \in Q_{k,p}$; then

$$\text{bidet}(AB[I|J]) = \left(\bigoplus_{K \in Q_{k,n}} \text{bidet}(A[I|K]) \otimes \text{bidet}(B[K|J]) \right) \oplus (d_{I,J}, d_{I,J}),$$

where $d_{I,J}$ is an element of S .

Definition 4.3.1. (Ideals) [1] Let S be a commutative semiring. A non empty subset I of S is called an ideal if the following two properties hold:

1. $a \oplus b \in I$, for all $a, b \in I$
2. $s \otimes a \in I$, for all $a \in I$ and $s \in S$.

In [9] the determinantal ideals are defined for commutative rings. Here we are defining the determinantal ideals for commutative semirings in a similar way. New rank functions such as ideal rank and determinantal rank are introduced, by using the determinantal ideals, for the matrices over commutative semirings.

Definition 4.3.2. (Determinantal Ideal)- Let S^2 be a commutative semiring and $A \in M_{m,n}(S^2)$. The determinantal ideal I_t of S^2 is the ideal generated by the set of t by t minors

of A . The determinantal ideal I_t is defined to be $\{0\}$ for all $t > \min \{m, n\}$.

Note that I_t consists of all the elements which can be written as $\bigoplus s_{\alpha,\beta} \det_2(A[\alpha,\beta])$, where α ranges over $Q_{t,m}$, β ranges over $Q_{t,n}$ and $s_{\alpha,\beta}$ ranges over S^2 for all α, β .

Remark 4.3.1. - By Laplace expansion each k by k minor can be written as the linear combination of k minors of order $(k-1)$ by $(k-1)$. This implies that $I_1 \supseteq I_2 \supseteq \dots \supseteq I_t \supseteq I_{t+1} \supseteq \dots$.

Definition 4.3.3. (Ideal Rank)- Let S^2 be a commutative semiring and $A \in M_{m,n}(S^2)$. The ideal rank of A , denoted by $rk_{Id}(A)$, is the largest positive integer t such that $I_t = S^2$.

Definition 4.3.4. (Diagonal Ideal)- Let S^2 be a commutative semiring. The diagonal ideal, Δ , of S^2 is the set $\{ (s,s) : s \in S \}$.

The following characterization of the determinantal rank of a matrix over the commutative semiring S^2 is now immediate.

Proposition 4.3.10. - Let S^2 be a commutative semiring and $A \in M_{m,n}(S^2)$. The determinantal rank of A , denoted by $rk_{det}(A)$, is the largest positive integer t such that I_t is not contained in the diagonal ideal.

Proposition 4.3.11. - Let S^2 be a commutative semiring and $A \in M_{m,n}(S^2)$. Then $rk_{det}(A) \geq rk_{Id}(A)$.

Proof. - Let $rk_{det}(A) = k$, this implies that k is the largest integer such that I_k is not contained in the diagonal ideal and I_{k+1} is contained in the diagonal ideal. Thus $I_{k+1} \neq S^2$, this implies that $rk_{Id}(A) < k+1$. Hence $rk_{det}(A) \geq rk_{Id}(A)$. ■

Proposition 4.3.12. - Let S^2 be a commutative semiring and $A \in M_{m,n}(S^2)$. Then $rk_{det}(A) \leq f(A)$.

Proof. - If $f(A) = \min\{m,n\}$ then we are done. Now suppose that $f(A) = r < \min\{m,n\}$, we have to show that I_{r+1} is contained in the diagonal ideal. Since $f(A) = r$, this implies that there exist matrices $B \in M_{m,r}(S^2)$ and $C \in M_{r,n}(S^2)$ such that $A = BC$. Let $\hat{B} \in M_{m,r+1}(S^2)$ be the matrix obtained by adding a zero column to the right of B and $\hat{C} \in M_{r+1,n}(S^2)$ be the matrix obtained by adding a zero row to the bottom of C . Clearly $A = \hat{B}\hat{C}$. Now compute the minors of $A (= \hat{B}\hat{C})$ of order $(r+1) \times (r+1)$, using Cauchy Binet theorem, i.e, $det_2(\hat{B}\hat{C}[I|J]) = (\bigoplus_{K \in Q_{r+1,r+1}} det_2(\hat{B}[I|K]) \otimes det_2(\hat{C}[K|J])) \oplus (d_{I,J}, d_{I,J})$, where $I \in Q_{r+1,m}$ and $J \in Q_{r+1,n}$. Clearly the first part of the right hand summand is zero. Hence $(r+1) \times (r+1)$ minor of $A = (d_{I,J}, d_{I,J})$, where $I \in Q_{r+1,m}$ and $J \in Q_{r+1,n}$, and this is true for all minors of order $(r+1) \times (r+1)$. Thus I_{r+1} is contained in the diagonal ideal. This implies that $rk_{det}(A) \leq r$. ■

4.3.3 Sublocal Semirings

In this section we will characterize sublocal semirings. The sublocal semirings are used in the section 4.3.4 to prove the rank sum and Sylvester's inequality for the ideal rank of matrices over a sublocal semiring.

Definition 4.3.5. (*k*-ideal) [1] *An ideal I of a commutative semiring S is called a k -ideal if for any $y \in S$ with $x, x+y \in I$ we have $y \in I$.*

Definition 4.3.6. (*Maximal (resp. k-Maximal) ideal*) [1] *A proper ideal I of a commutative semiring S is called a maximal (resp. k -maximal) ideal if there exists no other proper ideal (resp. k -ideal) J such that $I \subset J$.*

Remark 4.3.2. - *If S is a semiring and $A \in M_{m,n}(S)$, such that $rk_{Id}(A) = k$. Then for all $i > k$, I_i is contained in a maximal ideal of S*

Definition 4.3.7. (*Local Semiring*) [1] *Let S be a commutative semiring. We say that S is a local semiring if S has only one k -maximal ideal.*

Now we will define sublocal semirings using maximal ideals instead of k -maximal ideals. This is useful as some semirings do not have proper k -ideals. For example the sign pattern semiring has only one proper ideal $\{0, \#\}$ and this is not a k -ideal.

Definition 4.3.8. (Sublocal Semiring)- Let S be a commutative semiring. We say that S is a sublocal semiring if S has only one maximal ideal.

Both the local and sublocal semirings are the semiring version of the local ring.

Examples of sublocal semirings:

- The sign pattern semiring is a sublocal semiring having only one maximal ideal $I = \{0, \# \}$.
- The set of all natural numbers, $N = \{0, 1, 2, \dots\}$, forms a sublocal semiring having only one maximal ideal $I = \{N / \{1\} \}$.
- All chain semirings S are sublocal semirings with $S/\{1\}$ as a unique maximal ideal.
- All semifields are sublocal semirings as zero ideal is the maximal ideal.

Lemma 4.3.1. [1] Let S be a semiring. An ideal $Sa = S$ if and only if a is a unit of S .

Proof. - Let a be a unit in S . This implies that an ideal of S which contains a is equal to S . Since $a \in Sa$, so $Sa = S$. Now suppose that $Sa = S$. This implies that there exists a $s_1 \in S$ such that $s_1a = \mathbf{1}$, since S is a semiring so it has $\mathbf{1}$. Hence a is a unit in S . ■

Lemma 4.3.2. [1] Let S be a semiring and let $a \in S$. Then a is a unit of S if and only if a lies outside each maximal ideal of S .

Proof. - Let a be a unit of S and let $a \in M$ for some maximal ideal M of S . Then we should have $Sa \subseteq M \subset S$, but by lemma 4.3.1, $Sa = S$. Thus a can not be contained in M for some maximal ideal M of S . Conversely, if a is not a unit of S , then there does not exist any $s_1 \in S$ such that $s_1a = \mathbf{1}$. Thus $\mathbf{1} \notin Sa$ yields that Sa is a proper ideal of S . Hence $Sa \subseteq M$ for some maximal ideal M of S . ■

In [1] the following result is proved for local semiring. We prove the analog for sublocal semirings.

Theorem 4.3.1. - *Let S be a semiring. Then S is a sublocal semiring if and only if the set of all non units forms an ideal.*

Proof. - Firstly suppose that S is a sublocal semiring. This means that S has only one maximal ideal. Let M be the maximal ideal of S . By lemma 4.3.2, M is precisely the set of all non units of S . Conversely, suppose that the set of all non units of S forms an ideal I of S (so $I \neq S$ since $\mathbf{1}$ is a unit of S). Since S is non trivial, it has at least one maximal ideal (say J). By lemma 4.3.2, J consists of non units of S , and $J \subseteq I \subset S$. Thus $J = I$ since J is the maximal ideal and this is true for all maximal ideals of S . Hence I is a unique maximal ideal of S . ■

Corollary 4.3.1. - *Let S be a sublocal semiring. For any two elements a_1, a_2 in S if $a_1 \oplus a_2 \in U(S)$, where $U(S)$ denotes the set of units of S , then either $a_1 \in U(S)$ or $a_2 \in U(S)$.*

Theorem 4.3.2. - *Let S be a commutative antinegative semiring with no zero divisors. If x is a unit in S then $U(S^2) = \{(x,0): x \in U(S)\} \cup \{(0,x): x \in U(S)\}$.*

Proof. - Let x be the unit in S . This implies that there exists a nonzero element a in S such that $x \otimes a = 1$. Now clearly $(x,0)$ and $(0,x) \in S^2$ and $(a,0)$ and $(0,a)$ are nonzero elements in S^2 . Consider $(x,0) \otimes (a,0) = (x \otimes a, 0) = (1,0)$ and $(0,x) \otimes (0,a) = (x \otimes a, 0) = (1,0)$. This implies that $(x,0)$ and $(0,x)$ are units of S^2 . Now we have to prove that these are the only units for S^2 . Suppose $(x,y) \in S^2$ is a unit in S^2 . This implies that there exists a nonzero element $(a,b) \in S^2$ such that $(x,y) \otimes (a,b) = (1,0)$. Thus $((a \otimes x) \oplus (b \otimes y), (b \otimes x) \oplus (a \otimes y)) = (1,0)$. This implies that $(a \otimes x) \oplus (b \otimes y) = 1$ and $(b \otimes x) \oplus (a \otimes y) = 0$. Since $(b \otimes x) \oplus (a \otimes y) = 0$ and S is an antinegative semiring, so $b \otimes x = 0$ and $a \otimes y = 0$. This implies that either $b = 0$ or $x = 0$ (note that both b and x can not be zero because $(a \otimes x) \oplus (b \otimes y) = 1$) and either $a = 0$ or $y = 0$ (here also both a and y can not be zero because $(a \otimes x) \oplus (b \otimes y) = 1$). Since (x,y) and (a,b) are nonzero elements of S^2 so the units of S^2 , (x,y) , have only two choices which are $(x,0)$ and $(0,y)$. Put $(x,y) = (x,0)$ in $(a \otimes x) \oplus (b \otimes y) = 1$, we get $a \otimes x = 1$, this means that x is a unit of S . put $(x,y) = (0,y)$ in $(a \otimes x) \oplus (b \otimes y) = 1$, we get $b \otimes y = 1$, this means that y is a unit of S . Thus all the units in S^2 are of the type $(x,0)$ and $(0,x)$ where x is a unit in S . ■

Theorem 4.3.3. - *If S is a sublocal antinegative semiring with no zero divisors then S^2 is also a sublocal semiring.*

Proof. - It is enough to prove that the set of all non units in S^2 forms an ideal of S^2 . Let $M = \{ (a,b): \text{ where } (a,b) \text{ is not a unit of } S^2 \}$ be the set of all non units of S^2 . Let (a_1, b_1)

and $(a_2, b_2) \in M$ such that $(a_1, b_1) \oplus (a_2, b_2) = (x, 0)$ (or $(0, x)$), where x is a unit in S . This implies that $a_1 \oplus a_2 = x$ and $b_1 \oplus b_2 = 0$. Since S is antinegative so $b_1 = 0$ and $b_2 = 0$, also $a_1 \oplus a_2 = x$, where x is a unit in S so (using corollary 4.3.1) either a_1 is a unit or a_2 is a unit. Thus either $(a_1, b_1) = (a_1, 0)$, where a_1 is a unit in S or $(a_2, b_2) = (a_2, 0)$, where a_2 is a unit in S . This implies that either (a_1, b_1) is a unit or (a_2, b_2) is a unit in S^2 . Which is a contradiction to the fact that both (a_1, b_1) and $(a_2, b_2) \in M$. Thus the sum of non units in S^2 is a non unit. A similar argument works if $(a_1, b_1) \oplus (a_2, b_2) = (0, x)$, where x is a unit in S . Now suppose that for $(a, b) \in M$ and $(s_1, s_2) \in S^2$ we have $(a, b) \otimes (s_1, s_2) = (x, 0)$, where x is a unit in S . This implies that $(a \otimes s_1) \oplus (b \otimes s_2) = x$ and $(a \otimes s_2) \oplus (b \otimes s_1) = 0$. Since S is antinegative so $a \otimes s_2 = 0$ and $b \otimes s_1 = 0$, also S has no zero divisor so either $a = 0$ or $s_2 = 0$ and either $b = 0$ and $s_1 = 0$. Clearly (a, b) or (s_1, s_2) can not be $(0, 0)$ since $(a \otimes s_1) \oplus (b \otimes s_2) = x$, so we have $(a, b) = (0, b)$ or $(a, 0)$. Further x is a unit in S and $(a \otimes s_1) \oplus (b \otimes s_2) = x$, so either $a \otimes s_1$ is a unit in S or $b \otimes s_2$ is a unit in S . This implies that either a is a unit in S or b is a unit in S . We get $(a, b) = (0, b)$ or $(a, 0)$ is a unit in S^2 , which is a contradiction to the fact that $(a, b) \in M$. Thus $(a, b) \otimes (s_1, s_2) \in M$ for all $(a, b) \in M$ and $(s_1, s_2) \in S^2$. Hence the set of all non units in S^2 forms an ideal of S^2 . This implies that S^2 is a sublocal semiring. A similar argument works if $(a, b) \otimes (s_1, s_2) = (0, x)$, where x is a unit in S .

■

4.3.4 The Ideal Rank

Now we will prove the rank sum and Sylvester's inequality for the ideal rank of matrices over a sublocal semiring. For rank sum inequality we will use the Laplace expres-

sion of the determinant of the sum of matrices over sublocal commutative semiring. To prove the Sylvester's inequality we will use the Cauchy-Binet theorem for semirings.

Proposition 4.3.13. - *Let S^2 be a sublocal semiring and A and $B \in M_{m,n}(S^2)$, then $rk_{Id}(A + B) \leq \min \{ rk_{Id}(A) + rk_{Id}(B), m, n \}$.*

Proof. - If $rk_{Id}(A) + rk_{Id}(B) \geq \min \{m, n\}$, then we are done. Suppose $rk_{Id}(A) + rk_{Id}(B) < k \leq \min \{m, n\}$. Now consider the determinantal ideal I_k of S^2 generated by the k by k minors of $A + B$. Computing k by k minors of $A + B$ (using proposition 4.3.5), we get every element of the right hand side summand contains a minor of A of dimension $> rk_{Id}(A)$ or a minor of B of dimension $> rk_{Id}(B)$. Thus every k by k minor of $A + B$ lies in the maximal ideal of S^2 . This implies that I_k is contained in the maximal ideal of S^2 , i.e, $I_k \neq S^2$. This implies that $rk_{Id}(A + B) < k$. Hence $rk_{Id}(A + B) \leq rk_{Id}(A) + rk_{Id}(B)$. ■

Remark 4.3.3. - *The inequality $rk_{Id}(A + B) \geq | rk_{Id}(A) - rk_{Id}(B) |$ does not hold for the matrices over sublocal semirings S^2 . Here we will give an example of sign pattern matrices.*

Let us consider $A = K_7$ and $B = I_7$, the identity sign pattern matrix. Then $rk_{Id}(A + B) = rk_{Id}(J_7) = 1$, but $rk_{Id}(K_7) = 3$ and $rk_{Id}(B) = 7$. Thus $| rk_{Id}(A) - rk_{Id}(B) | = rk_{Id}(B) - rk_{Id}(A) = 7 - 3 = 4 > 1 = rk_{Id}(A + B)$.

Proposition 4.3.14. - *Let $A \in M_{m,n}(S^2)$ and $B \in M_{n,p}(S^2)$, where S^2 is a sublocal commutative semiring. Then $rk_{Id}(AB) \leq \min \{ rk_{Id}(A), rk_{Id}(B) \}$.*

Proof. - If $\min \{rk_{Id}(A), rk_{Id}(B)\} = k = \min \{m, p\}$, then we are done. Suppose $\min \{rk_{Id}(A), rk_{Id}(B)\} < k \leq \min \{m, p\}$. Now consider the determinantal ideal I_k of S^2 generated by the k by k minors of AB . Compute k by k minors of AB (using proposition 4.3.6), we get every element of the right hand side summand contains a k by k minor of A and a k by k minor of B . Also we have $rk_{Id}(A) < k$. This implies that every k by k minor of AB lies in the maximal ideal of S^2 , This implies that I_k is contained in the maximal ideal of S^2 , i.e, $I_k \neq S^2$. Hence $rk_{Id}(AB) < k$. ■

Remark 4.3.4. - *The inequality $rk_{Id}(AB) \geq rk_{Id}(A) + rk_{Id}(B) - n$ does not hold for the matrices over sublocal semirings S^2 . We will give an example of sign pattern matrices.*

$$\text{Let us consider } A = \begin{bmatrix} +1 & 0 & 0 & \cdots & 0 \\ +1 & +1 & 0 & \cdots & 0 \\ +1 & 0 & +1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ +1 & 0 & 0 & \cdots & +1 \end{bmatrix}$$

and $B = A^t$, [6]. Then $rk_{Id}(A) = rk_{Id}(B) = n$ and $rk_{Id}(AB) = 1 \leq rk_{Id}(A) + rk_{Id}(B) - n = n$, since $AB = J$.

Now we will study the relationship between the SNS rank of a sign pattern matrix A and the SNS submatrix of A . We will use this study to compare the SNS rank of a sign pattern matrix A with the minimum rank of A .

We know that if an ideal of a semiring contains a unit then the ideal is equal to the semiring.

Proposition 4.3.15. - *Let S^2 be a sublocal semiring and $A \in M_{m,n}(S^2)$. For any positive t , $I_t = S^2$ if and only if at least one of the t by t minors of A is a unit of S^2 .*

Proof. - We know that the ideal I_t is the linear combination of t by t minors of A . Clearly if at least one of the t by t minors of A is a unit of S^2 , then $I_t = S^2$. Conversely, suppose no t by t minor of A is a unit of S^2 . Then all t by t minors of A lie in the maximal ideal of S^2 . This implies that I_t is contained in the maximal ideal of S^2 . Thus $I_t \neq S^2$. ■

Remark 4.3.5. - *We know that the sign pattern semiring is a sublocal semiring of the type S^2 , where S is equal to β . Thus the proposition 4.3.15 holds for the sign pattern semiring.*

Lemma 4.3.3. - *If S is a sign pattern semiring and $A \in M_{m,n}(S)$. For any positive k , $I_k = S$ if and only if at least one k by k submatrix of A is SNS.*

Proof. - Using remark 4.3.5, for any k , $I_k = S$ if and only if at least one of the k by k minors of A is a unit of S . We also know that a sign pattern matrix is SNS if and only if the \det_2 of the matrix is a unit of S . Thus $I_k = S$ if and only if at least one of the k by k submatrix

of A is SNS. ■

Lemma 4.3.4. - *If S is a sign pattern semiring and $A \in M_{m,n}(S)$, then $rk_{Id}(A)$ is the biggest positive integer k such that there exists a k by k SNS submatrix of A .*

Proof. - We know that $rk_{Id}(A)$ is the biggest positive integer k such that $I_k = S$. So by lemma 4.3.3, $rk_{Id}(A)$ is the biggest positive integer k such that there exists at least one k by k SNS submatrix of A . ■

Proposition 4.3.16. - *Let A be a sign pattern matrix then $rk_{Id}(A) \leq mr(A)$.*

Proof. - Let $rk_{Id}(A) = k$. Using lemma 4.3.4, k is the biggest integer such that there exists a k by k , SNS submatrix B of A . This implies that any real matrix in the sign pattern class of A will have an invertible k by k submatrix. Thus $mr(A) \geq k = rk_{Id}(A)$. ■

This inequality can be strict as the next example shows.

Example 4.3.1. [22] *Let*

$$H = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix} \quad \text{and} \quad H' = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \end{bmatrix}$$

$$rk_{Id}(H) = rk_{Id}(H') = 2 \quad \text{and} \quad mr(H) = mr(H') = 3.$$

4.3.5 Relation Between the ranks of A and B(A)

It is clear from the definitions of ideal rank and the determinantal rank that if B is a submatrix of A then $rk_{Id}(B) \leq rk_{Id}(A)$ and $rk_{det}(B) \leq rk_{det}(A)$.

Proposition 4.3.17. - *Let A be a sign pattern matrix of order n and B(A) is the corresponding Boolean matrix, then $2mr(A) \leq r(B(A))$.*

Proof. - Let $mr(A) = r$. This implies that r rows of A are linearly independent in Gondran-Minoux sense. This implies that there exists an r by n submatrix (say F) of A which is an L-matrix. Using proposition 3.2.1 we get, row rank of $B(F) = 2r$. Since $B(F) \subseteq B(A)$, so $r(B(F)) \leq r(B(A))$. Thus we get $r(B(A)) \geq 2r = 2mr(A)$. ■

Corollary 4.3.2. - *Let A be a sign pattern matrix and B(A) be the corresponding Boolean matrix, then we have the following inequality:*

$$2rk_{Id}(A) \leq 2mr(A) \leq r(B(A))$$

Proof. - Combining proposition 4.3.17 and 4.3.16, we get the required inequality. ■

4.4 Ideals of S-Squared Semiring

We know that if S is a semiring then S^2 is also a semiring. In this section we will discuss the ideals of S-squared semirings.

Examples of Ideals of S^2 :

1. Let I be an ideal of S then $I^2 = \{(a,b) : a, b \in I\}$ is an ideal of S^2 .
2. In S^2 the diagonal ideal, $\Delta = \{(a,a) : a \in S\}$ always exists.
3. If I is an ideal of S , $I^2 \cap \Delta = \{(a,a) : a \in I\}$ is also an ideal of S^2 contained in I^2 and the diagonal ideal.

Definition 4.4.1. (*Principal Ideal*) [9] *Let S be a commutative semiring. An ideal I of S is called a principal ideal if it is generated by a single element, that is, $I = Sx$ for some $x \in I$.*

Definition 4.4.2. (*Principal Ideal Semiring (PIS)*) [9] *A semiring is called a principal ideal semiring if every ideal I of S is a principal ideal.*

Theorem 4.4.1. - *Let S be a commutative semiring. If I is a principal ideal of S then I^2 is a principal ideal of S^2 .*

Proof. - Let the principal ideal I of S is generated by $p \in I$. Then $I^2 = \{(a,b) : a, b \in I\}$. Since I is generated by p so we can assume that $a = p \otimes x$ and $b = p \otimes y$, where x and $y \in S$. This implies that $I^2 = \{(p \otimes x, p \otimes y) : p \otimes x \text{ and } p \otimes y \in I\}$, and $(p \otimes x, p \otimes y) = (p,0) \otimes (x,y)$. Thus we get $I^2 = \{(p,0) \otimes (x,y) : p \in I \text{ and } x \text{ and } y \in S\}$. Hence I^2 is a principal ideal generated by $(p,0)$. ■

Corollary 4.4.1. - *In the semiring S^2 the ideal generated by $(p,0)$ is same as the ideal generated by $(0,p)$.*

Proof. - We know that $(p,0) = (0,1) \otimes (0,p)$ and $(0,1)$ is a unit in S^2 . Thus the ideal generated by $(p,0)$ is the same as the ideal generated by $(0,p)$. ■

Corollary 4.4.2. - *Let the ideal I of a commutative semiring S be generated by p then the ideal $I^2 \cap \Delta$ of S^2 is generated by (p,p) .*

Proof. - By theorem 4.4.1, if I is generated by p then I^2 is generated by $(p,0)$. Also $I^2 \cap \Delta = \{(p,0) \otimes (a,a) : a \in I\}$. We know that $(p,0) \otimes (a,a)$, where $a \in I = (p,p) \otimes (a,b)$, where $a, b \in I$. Thus we get $I^2 \cap \Delta = \{(p,p) \otimes (a,b), \text{ where } a, b \in I\}$. ■

4.5 Subsemirings With Unity

If one omits the requirement that semirings have a unity element then ideals become subsemirings. Ideals may or may not have their own multiplicative identity (distinct from the identity of the semiring). In this section we will characterize semirings which have ideals with their own unity.

Definition 4.5.1. (Idempotent Element) [34] *An element x of a semiring is called an idempotent element, or simply an idempotent if $x^2 = x$.*

Lemma 4.5.1. - *Let S be a semiring with unity. Proper ideals of S generated by idempotent elements of S are subsemirings with unity.*

Proof. - Let p be an idempotent element of S , then the ideal generated by p , denoted by $(p) = \{a \otimes p : a \in S\}$. We know that (p) is subsemiring of S . Now we will show that (p) has its own multiplicative identity different from the identity of S . Clearly $p \in (p)$, since S is a semiring with unity. Now $(a \otimes p) \otimes p = a \otimes p$. This implies that p is the multiplicative identity of the ideal (p) . Thus (p) is the subsemiring with unity. ■

Definition 4.5.2. (*Regular Semiring*) [34] *A commutative semiring is called regular if for each $x \in S$ there exists $y \in S$ such that $x = x^2y$.*

The regular rings has been characterized in several ways [30]. There are also many similar characterizations of what can be called regular semigroup. The following proposition is proved essentially like these in terms of semirings.

Proposition 4.5.1. [34] *Let S be a semiring. S is a regular semiring if and only if every principal ideal is generated by an idempotent.*

Corollary 4.5.1. - *If S is a regular semiring with unity, then every principal ideal is a subsemiring with its own unity.*

Proof. - Suppose that S is a regular semiring, then by proposition 4.5.1 every principal ideal of S is generated by idempotent elements of S . Hence principal ideals are subsemirings with unity, using lemma 4.5.1. ■

4.6 Group Semirings

The group semiring is a generalization of S-squared semiring. The S^2 construction is a group semiring where $G = Z_2$. Now we will define the group semiring for general group G and verify that the properties of the S^2 construction hold when G is an arbitrary finite Abelian group.

Let S be a semiring and G be an Abelian group of order n with identity e , then the formal sum

$$\left\{ \sum_{g \in G} s_g \cdot g \right\}, \text{ where } s_g \in S,$$

denoted by $S[G]$, is a group semiring with addition and multiplication defined as follows:

$$\sum_{g \in G} s_g \cdot g \oplus \sum_{g \in G} r_g \cdot g = \sum_{g \in G} (s_g \oplus r_g) \cdot g.$$

$$\sum_{g \in G} s_g \cdot g \otimes \sum_{g \in G} r_g \cdot g = \sum_{g \in G} t_g \cdot g, \text{ where } t_g = \bigoplus_{hk=g} (s_h \otimes r_k).$$

Clearly all the properties of a semiring are satisfied where $\sum_{g \in G} 0 \cdot g = 0$ is the additive identity and $\sum_{g \in G} s_g \cdot g$, where the coefficient corresponding to the identity of the group is equal to 1 and $s_g = 0$ for every other element $g \in G$, is the multiplicative identity. i.e., $\sum_{g \in G} s_g \cdot g = 1 \cdot e$. We will denote the multiplicative identity of the group semiring $S[G]$ by $\mathbf{1}$.

Theorem 4.6.1. - *Let S be a semiring. If S is an antinegative semiring with no zero divisors then $S[G]$ is also an antinegative semiring with no zero divisors.*

Proof. - Let $\sum_{g \in G} s_g \cdot g$ and $\sum_{g \in G} r_g \cdot g \in S[G]$ such that

$$\sum_{g \in G} s_g \cdot g \oplus \sum_{g \in G} r_g \cdot g = \sum_{g \in G} 0 \cdot g.$$

This implies that $\sum_{g \in G} (s_g \oplus r_g) \cdot g = \sum_{g \in G} 0 \cdot g$. Thus $s_g \oplus r_g = 0$, for all $g \in G$. Since S is antinegative so $s_g = 0$ and $r_g = 0$ for all $g \in G$. Thus $\sum_{g \in G} s_g \cdot g = \sum_{g \in G} 0 \cdot g$ and $\sum_{g \in G} r_g \cdot g = \sum_{g \in G} 0 \cdot g$, this implies that only the additive identity has additive inverse in $S[G]$. Hence $S[G]$ is an antinegative semiring. Now suppose that $\sum_{g \in G} s_g \cdot g$ and $\sum_{g \in G} r_g \cdot g \in S[G]$ such that

$$\sum_{g \in G} s_g \cdot g \otimes \sum_{g \in G} r_g \cdot g = \sum_{g \in G} 0 \cdot g.$$

This implies that $\sum_{g \in G} t_g \cdot g = \sum_{g \in G} 0 \cdot g$. Thus $t_g = 0$, for all $g \in G$. Thus $\bigoplus_{hk=g} (s_h \otimes r_k) = 0$, for all $g \in G$. Since S is an antinegative semiring, so $s_h \otimes r_k = 0$ for all h and k such that $hk = g$ and this is true for all $g \in G$. Also S has no zero divisors so for all h and k , one of the s_h and r_k is equal to zero at a time.

We claim that either $s_h = 0$ or $r_k = 0$ for all h and $g \in G$.

Suppose neither $s_h = 0$ or $r_k = 0$ for all h and $g \in G$. This implies that there exists at least one $s_{h_i} \neq 0$ and $r_{k_j} \neq 0$. Thus there exists a $g \in G$, where $g = h_i \cdot k_j$, such that $t_g \neq 0$. Which is a contradiction. Hence either $s_h = 0$ or $r_k = 0$ for all h and $g \in G$. Thus either $\sum_{g \in G} s_g \cdot g = 0$ or $\sum_{g \in G} r_g \cdot g = 0$. ■

Theorem 4.6.2. - Let $S[G]$ be a group semiring. The set of all units in $S[G]$, denoted as $U(S[G])$, is $\{s_g \cdot g\}$, where $g \neq 0$ and s_g is a unit in S .

Proof. - Let s_{g_i} be a unit in S then there exists a nonzero element r_{g_j} in S such that $s_{g_i} \otimes r_{g_j} = 1$. Clearly $\sum_{g \in G} s_g \cdot g = s_{g_i} \cdot g_i \in S[G]$, where s_{g_i} is a unit of S and $\sum_{g \in G} r_g \cdot g = r_{g_j} \cdot g_j$,

where $g_i \cdot g_j = e$, is a nonzero element of $S[G]$ since r_{g_j} is a nonzero element of S . Now $\sum_{g \in G} s_g \cdot g \otimes \sum_{g \in G} r_g \cdot g = s_{g_i} \otimes r_{g_j} \cdot g_i \cdot g_j = 1 \cdot e$. This implies that $\sum_{g \in G} s_g \cdot g = s_{g_i} \cdot g_i$, where s_{g_i} is a unit of S , is a unit in $S[G]$. Now we will prove that these are the only units in $S[G]$. Let $\sum_{g \in G} s_g \cdot g$ is a unit in $S[G]$, this implies that there exists a nonzero element $\sum_{g \in G} r_g \cdot g \in S[G]$ such that $\sum_{g \in G} s_g \cdot g \otimes \sum_{g \in G} r_g \cdot g = \mathbf{1}$. This implies that $\sum_{g \in G} t_g \cdot g = 1 \cdot e$. This implies that the t_g corresponding to the identity of G is equal to 1 and $t_g = 0$ for every other elements $g \in G$. Suppose $t_{g_i} = 1$ and $t_{g_j} = 0$ for all $j \neq i$. Since $t_{g_j} = 0$, so $s_h \otimes r_k = 0$ for all $hk = g_j$, using the antinegativity of S . This implies that either $s_h = 0$ or $r_k = 0$ for all $hk = g_j$, where $j \neq i$, using that S has no zero divisors. Also we have $t_{g_i} = 1$, so

$$\bigoplus_{hk=g_i} (s_h \otimes r_k) = 1 \quad (4.1)$$

We claim that $\sum_{g \in G} s_g \cdot g = s_h \cdot h + \sum_{g \in G, g \neq h} 0 \cdot g$, where $h \in G$. Suppose at least two s_g 's are nonzero, s_h and $s_{h'}$, from 4.1 we get at least one r_g is nonzero for $g \in G$. Suppose $r_j \neq 0$ and $hj = g_i$, then we get $s_{h'} \otimes r_j \neq 0$ and $h'j \neq g_i$ which is a contradiction to the fact that either $s_h = 0$ or $r_k = 0$ for all $hk = g_j$, where $j \neq i$. Thus our claim is proved. Combining (2) and claim we get $s_h \otimes r_k = 1$ for h and $k \in G$ such that $hk = g_i$. This implies that s_h is a unit in S . ■

Theorem 4.6.3. - *Let $S[G]$ be a group semiring. If I is an ideal of the semiring S then $I[G]$ is an ideal of $S[G]$.*

Proof. - Let $\sum_{g \in G} s_g \cdot g$ and $\sum_{g \in G} r_g \cdot g$ be any two elements of $I[G]$, where s_g and $r_g \in I$.

Then

$$\sum_{g \in G} s_g \cdot g \oplus \sum_{g \in G} r_g \cdot g = \sum_{g \in G} (s_g \oplus r_g) \cdot g.$$

Since I is an ideal and both s_g and $r_g \in I$, this implies that $s_g \oplus r_g \in I$. Thus $\sum_{g \in G} (s_g \oplus r_g) \cdot g \in I[G]$. Now let $\sum_{g \in G} s_g \cdot g \in I[G]$ and $\sum_{g \in G} r_g \cdot g \in S[G]$. Then

$$\sum_{g \in G} s_g \cdot g \otimes \sum_{g \in G} r_g \cdot g = \sum_{g \in G} t_g \cdot g, \text{ where } t_g = \bigoplus_{hk=g} (s_h \otimes r_k).$$

Since I is an ideal and $s_g \in I$ for all g and $r_k \in S$ for all g , this implies that $\bigoplus_{hk=g} (s_h \otimes r_k) \in I$. Thus $\sum_{g \in G} t_g \cdot g \in I[G]$. Hence $I[G]$ is an ideal of $S[G]$. ■

Theorem 4.6.4. - *If S is a sublocal antinegative semiring with no zero divisors then $S[G]$ is also a sublocal antinegative semiring with no zero divisors.*

Proof. - It is enough to prove that the set of all non units in $S[G]$ forms an ideal of $S[G]$.

Let M be the set of all non units of $S[G]$. Suppose $\sum_{g \in G} s_g \cdot g$ and $\sum_{g \in G} r_g \cdot g$ be any two elements of M with the property that

$$\sum_{g \in G} s_g \cdot g \oplus \sum_{g \in G} r_g \cdot g = c \cdot h, \quad (4.2)$$

where $c \in S$ and $h \in G$. We claim that c is not a unit in S . From 4.2 we get that only the $g = h$ entry in $\sum_{g \in G} s_g \cdot g$ and $\sum_{g \in G} r_g \cdot g$ is non zero. Since $\sum_{g \in G} s_g \cdot g$ and $\sum_{g \in G} r_g \cdot g$ are non units in $S[G]$, so s_h is not a unit and r_h is not a unit in S . This implies that $s_h \oplus r_h$ is not a unit in S . Thus c is not a unit in S . Hence the sum of non units in $S[G]$ is a non unit.

Now suppose that for $\sum_{g \in G} s_g \cdot g \in M$ and $\sum_{g \in G} r_g \cdot g \in S[G]$. Suppose

$$\sum_{g \in G} s_g \cdot g \otimes \sum_{g \in G} r_g \cdot g = c \cdot h, \quad (4.3)$$

where $c \in S$ and $h \in G$. This implies that $\sum_{hk=h} t_h \cdot h = c \cdot h$, where $c \in S$ and $t_h = \bigoplus_{jk=h} (s_j \otimes r_k)$.

We claim that c is not a unit in S . From 4.3 we get that only one entry in $\sum_{g \in G} s_g \cdot g$ and $\sum_{g \in G} r_g \cdot g$ is non zero. This implies that $t_h = \bigoplus_{j,k=h} (s_j \otimes r_k) = s_j \otimes r_k = c$. Since $\sum_{g \in G} s_g \cdot g$ is a non unit in $S[G]$, so s_j is not a unit in S where $j \in G$. Also $s_j \otimes r_k = c$ and the set of all non units in S forms an ideal, this implies that c is not a unit in S . Hence the set of all non units in $S[G]$ forms an ideal of $S[G]$. This implies that $S[G]$ is a sublocal semiring. ■

4.7 Determinantal Identities

Now, because we have no minus signs in semirings, we cannot define cofactors of a given square matrix A . To get around this, we define the (i, j) -minor matrix, denoted by A_{ij} or $(A)_{ij}$, to be the submatrix obtained by deleting the i^{th} row and the j^{th} column from A . We define the positive and negative minors $|A_{ij}|^+$ and $|A_{ij}|^-$, which for convenience, we abbreviate to A_{ij}^+ and A_{ij}^- , respectively. We will use the results proved in [32] to relate SNS matrices with their adjoints.

Definition 4.7.1. (The Positive and Negative Adjoint) [32] Let $A \in S_n$, then the positive and negative adjoint of A are defined as:

$$adj^+(A) = \begin{bmatrix} A_{11}^+ & A_{21}^- & A_{31}^+ & \cdots \\ A_{12}^- & A_{22}^+ & A_{32}^- & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

$$\text{adj}^-(A) = \begin{bmatrix} A_{11}^- & A_{21}^+ & A_{31}^- & \cdots \\ A_{12}^+ & A_{22}^- & A_{32}^+ & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

We will write \mathbf{A}^+ for $\text{adj}^+(A)$ and \mathbf{A}^- for $\text{adj}^-(A)$. We also note that $[\mathbf{A}^+]_{ij} = A_{ij}^{(-)^{i+j}}$, and $[\mathbf{A}^-]_{ij} = A_{ij}^{(-)^{i+j+1}}$.

Definition 4.7.2. (*Super-pattern*)- Let A be a sign pattern matrix. The super-pattern of A is a sign pattern matrix obtained by replacing some (possibly none) of the 0 entries in A with $+1$ or -1 .

Lemma 4.7.1. [32](*Cofactor expansion*)- If $A \in M_n(S)$, then the bideterminant of A , $(\det^+(A), \det^-(A))$ is defined as follows, if we expand by i^{th} row:

$$\det^+(A) = [A \cdot \text{adj}^+(A)]_{ii} = \sum_{j=1}^n a_{ij} A_{ij}^{(-)^{i+j}}$$

and

$$\det^-(A) = [A \cdot \text{adj}^-(A)]_{ii} = \sum_{j=1}^n a_{ij} A_{ij}^{(-)^{i+j+1}}$$

Corollary 4.7.1. - Let $A \in M_n(S)$. A is an SNS matrix if the determinant of $A \neq 0$ and A is a common super-pattern of both $+\text{adj}^+(A)$ and $-\text{adj}^-(A)$ or $-\text{adj}^+(A)$ and $+\text{adj}^-(A)$.

Proof. - Since the determinant of $A \neq 0$, this implies that either $\det^+(A)$ or $\det^-(A)$ or both are nonzero. Further A is a common super-pattern of both $+\text{adj}^+(A)$ and $-\text{adj}^-(A)$ or $-\text{adj}^+(A)$ and $+\text{adj}^-(A)$ then using the above lemma, $\det^+(A)$ and $\det^-(A)$ have opposite

signs or only one of them is zero. Hence A is an SNS matrix. ■

But the converse of the above corollary is not true.

Example 4.7.1. *Let*

$$A = \begin{bmatrix} +1 & +1 & 0 \\ 0 & +1 & +1 \\ +1 & 0 & +1 \end{bmatrix}$$

Then

$$\text{adj}^+(A) = \begin{bmatrix} +1 & 0 & +1 \\ +1 & +1 & 0 \\ 0 & +1 & +1 \end{bmatrix}$$

Clearly A is an SNS matrix, but A is not a super-pattern of either $+\text{adj}^+(A)$ or $-\text{adj}^+(A)$.

Chapter 5

On the Powers of Matrices over semirings

This chapter describes the relationship between sign pattern matrices and signed directed graphs. We characterize the signed directed graph of a sign pattern matrix A when A^2 is entrywise nonnegative. Different ranks of the powers of a sign pattern matrix are studied and several results are obtained. The bounds for the period and the base of a sign nonsingular nonnegative irreducible pattern are studied.

5.1 Basic Definitions and Notations

We begin by representing the sign pattern matrix as a signed directed graph.

Definition 5.1.1. (*Signed Directed Graph*) [25] Let $A = [a_{ij}] \in Q_n$, then $D(A)$, the signed directed graph of A is the digraph with vertex set $\{ 1, 2, 3, \dots, n \}$ and edge set $E = \{$

$(i, j) \mid a_{ij} \neq 0 \}$, where each edge is labeled + or - as follows:

If $a_{ij} = +1$, then there is a positive directed edge from i to j .

If $a_{ij} = -1$, then there is a negative directed edge from i to j .

If $a_{ij} = 0$, then there is no directed edge from i to j .

We introduce some graph theoretical concepts.

Definition 5.1.2. (Walk) [28] A sequence of vertices of the form i_0, i_1, \dots, i_k is called a walk W in $D(A)$ from i_0 to i_k if $(i_0, i_1), (i_1, i_2), \dots, (i_{k-1}, i_k)$ are all edges of $D(A)$.

A walk W of a sign pattern matrix A is a formal product of the form $W = a_{i_0 i_1} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k}$, where entries involved are nonzero. The length of W , denoted by $l(W)$, is k . The above walk W can also be represented as $W: (i_0, i_1, i_2, \dots, i_k)$. We say that W is positive [negative], and we write $\text{sign}(W) = +$ [$\text{sign}(W) = -$], if W contains an even [odd] numbers of negative edges.

Definition 5.1.3. (Closed Walk) [28] A walk $W: (i_0, i_1, i_2, \dots, i_k)$ is called a closed walk if $i_0 = i_k$.

Definition 5.1.4. (Cycle) [28] A closed walk $\gamma: (i_0, i_1, i_2, \dots, i_{k-1}, i_0)$ is called a cycle if $i_0, i_1, i_2, \dots, i_{k-1}$ are all distinct. Sometimes it is referred to as a simple cycle.

A cycle with length k is called a **k-cycle**.

Definition 5.1.5. (*Strongly Connected Directed Graph*) [28] *A directed graph is called strongly connected if for each pair of vertices i, j there is a walk from i to j .*

Definition 5.1.6. (*Cyclically Nonnegative*) [25] *$A \in Q_n$ is called cyclically nonnegative if every simple cycle of A is positive.*

Now we are ready to consider the powers of a pattern $A \in Q_n$. For any positive integer k , A^k is defined, that is, exists as a pattern, if the (i,j) entry of A^k is unambiguously defined for all $1 \leq i, j \leq n$, and write $A^k \in Q_n$. If we denote the (i,j) entry of A^k by $(A^k)_{ij}$, then it is clear that $(A^k)_{ij}$ is unambiguously defined if and only if no two walks of length k from i to j have opposite signs. If A^k exists as a pattern, then $(A^k)_{ij} = \sum \text{sign}(W)$, where W runs over the set of walks of A of length k from i to j , henceforth indicated by $W(i \rightarrow j; k)$. Of course the sum is zero if there is no such walk.

Definition 5.1.7. (*Powerful Sign pattern*) [25] *The pattern $A \in Q_n$ is said to be powerful if all the powers A, A^2, A^3, \dots , are defined.*

Notice that $A \in Q_n$ powerful means that for any $k \geq 2$ and $B_i \in Q(A)$, $i = 1, 2, \dots, k$, the signs of the entries of $B_1 B_2 \dots B_k$ are independent of the choice of B_i^s . In fact, we must have $\text{sign}(B_1 B_2 \dots B_k) = A^k$.

In the generalized sign pattern matrix the powers of any square pattern always exists.

5.2 Patterns with nonpositive squares

In this section we will give a generic characterization of sign patterns whose squares contain only nonpositive entries. Of course, this implies that the squared pattern does not contain any qualitative ambiguous entries. We will also describe the Boolean matrices corresponding to the sign patterns whose squares contain only nonpositive entries.

Lemma 5.2.1. - Let $A \in Q_n$. $A^2 \leq 0$ if and only if $(B(A))^2 = \begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix}$, where L is a square matrix obtained from A^2 by replacing all -1s by 1s.

Proof. - We know that there is one to one semiring homomorphism from the matrices over sign pattern semiring to the matrices over Boolean semiring. Thus if A is a sign pattern matrix then $B(A^2) = (B(A))^2$. From this we conclude that $A^2 \leq 0$, if and only if $B^2(A) = \begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix}$, where L is a square matrix obtained from A^2 by replacing all the minus one entries with ones. ■

Corollary 5.2.1. - Let $A \in Q_n$. $A^2 \leq 0$ if and only if there are no paths of length two in either $D(A^+)$ or $D(A^-)$.

Proof. - Let A is an n by n sign pattern matrix then the corresponding Boolean matrix

$$B(A) = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}$$

and

$$B^2(A) = \begin{bmatrix} A^+A^+ + A^-A^- & A^+A^- + A^-A^+ \\ A^-A^+ + A^+A^- & A^+A^+ + A^-A^- \end{bmatrix}$$

$$\text{By lemma 5.2.1, } A^2 \leq 0 \text{ if and only if } B^2(A) = \begin{bmatrix} A^+A^+ + A^-A^- & A^+A^- + A^-A^+ \\ A^-A^+ + A^+A^- & A^+A^+ + A^-A^- \end{bmatrix}$$

$$= \begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix},$$

where L is a square matrix obtained from A^2 by replacing all -1s by 1s. From here we get $A^2 \leq 0$ if and only if $(A^+)^2 = 0$ and $(A^-)^2 = 0$. This implies that $A^2 \leq 0$ if and only if there are no paths of length two in $D(A^+)$ and $D(A^-)$.

■

The following lemma has been proved in [16], but we are using the semiring version of the Cayley-Dickson construction to prove this lemma.

Lemma 5.2.2. - *Let $A \in Q_n$. Then $A^2 \leq 0$ if and only if every 2-paths in $D(A)$ is negative.*

$$\text{Proof. - By lemma 5.2.1, } A^2 \leq 0 \text{ if and only if } (B(A))^2 = \begin{bmatrix} A^+A^+ + A^-A^- & A^+A^- + A^-A^+ \\ A^-A^+ + A^+A^- & A^+A^+ + A^-A^- \end{bmatrix}$$

$$= \begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix},$$

where L is a square matrix obtained from A^2 by replacing all -1s by 1s. From here we get $A^2 \leq 0$ if and only if $(A^+A^-)_{ij} = \sum_{k=1}^n a_{ik}^+ a_{kj}^- = 1$ (or 0) and $(A^-A^+)_{ij} = 1$ (or 0), for all i and j . This implies that for every path of length two in A one edge belongs

to $D(A^+)$ and one belongs to $D(A^-)$. Hence $A^2 \leq 0$ if and only if every 2-paths in $D(A)$ is negative. ■

Corollary 5.2.2. [16] *If $A \in Q_n$, and $A^2 \leq 0$, then A has no odd cycles.*

Proof. - From lemma 5.2.2, we know that $A^2 \leq 0$ implies that all 2-paths in $D(A)$ are negative. If $a_{ii} \neq 0$, then $a_{ii}a_{ii}$ would be a positive 2-path in $D(A)$. Hence A has no cycle of length 1. Now suppose $\gamma = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1} \neq 0$ is an odd cycle in A . Replacing A with $-A$, if necessary, we may assume that $a_{i_1 i_2} = +$. Then $i_1 \xrightarrow{+} i_2 \rightarrow i_3$ is a negative 2-path only if $a_{i_2 i_3} = -$. Similarly, $i_2 \xrightarrow{-} i_3 \rightarrow i_4$ is a negative 2-path only if $a_{i_3 i_4} = +$. Continuing in this way, we see that the terms in γ alternate in sign, that is, $\gamma = a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_4} \dots a_{i_{k-2} i_{k-1}} a_{i_{k-1} i_k} a_{i_k i_1} = (+)(-)(+)\dots(+)(-)(+)$. However, then $i_k \xrightarrow{+} i_1 \xrightarrow{+} i_2$ is a positive 2-path in $D(A)$, a contradiction. Thus A has no odd cycle. ■

5.3 Structure of Powers

Let $A \in Q_n$ be powerful pattern. Consider the sequence of the powers of A . Since each power belongs to Q_n and $|Q_n| = 3^{n^2}$, there must be repetitions in the sequence. If $A^l = A^p$ as the sign patterns so does $A^{l+k} = A^{p+k}$ for all $k > 0$.

A nonnegative sign pattern matrix is a Boolean matrix, since its entries come from $\{0, +1\}$, which forms a subsemiring of the sign pattern semiring which is isomorphic to the Boolean semiring.

Remark 5.3.1. - *Nonnegative sign pattern matrices are always powerful, since all the powers A, A^2, A^3, \dots , are defined and nonnegative.*

Definition 5.3.1. (Convergent Boolean Matrix) [24] *Let A be a Boolean matrix. A is said to be convergent (in its powers) if there exists $m \in \mathbf{N}$ such that $A^m = A^{m+1}$. It is said that A converges to the matrix A^m .*

Definition 5.3.2. (Oscillatory or Periodic Matrix) [24] *A Boolean matrix is said to be oscillatory or periodic (in its powers) if there exists $m, p \in \mathbf{N}$ with $p \geq 1$, such that $A^m = A^{m+p}$.*

Lemma 5.3.1. [33] *A Boolean matrix A is either convergent or oscillatory.*

Definition 5.3.3. (Idempotent Matrix) [26] *A Boolean matrix A is said to be idempotent if $A^2 = A$.*

Lemma 5.3.2. [24] *A convergent Boolean matrix converges to an idempotent matrix.*

Proof. - Immediate from definition. ■

Definition 5.3.4. (Universal Matrix) [33] *A Boolean matrix $A = [a_{ij}]$ is said to be a*

universal matrix if $a_{ij} = 1$ for all i, j . The universal matrix is denoted by J .

Definition 5.3.5. (*Primitive Matrix*) [24] A Boolean matrix A is said to be a primitive matrix if some power of A is the universal matrix.

Definition 5.3.6. (*Nilpotent Matrix*) [33] A Boolean matrix A is said to be a nilpotent matrix if there exists $m \in \mathbf{N}$ such that $A^m = 0$.

Remark 5.3.2. [24] Convergent matrices fall into 3 classes: (1) The primitive matrices which converge to the universal matrix. (2) The Nilpotent matrices which converge to the zero matrix. (3) The Boolean matrices which converge to idempotent matrices other than the zero matrix or the universal matrix.

Definition 5.3.7. (*Permutation Matrix*) [24] A permutation matrix is a square matrix whose entries are zeros and ones, where the entry 1 occurs precisely once in each row and in each column.

A matrix X is said to be permutation similar to a matrix Y if there exists a permutation matrix P such that $X = P^T Y P$.

Now we will characterize irreducible sign pattern matrices and the corresponding

Boolean matrices.

Definition 5.3.8. (Irreducible Matrix) [27] Let A be an n by n matrix, where $n \geq 2$. A is called irreducible (indecomposable) if there does not exist any permutation matrix P such that

$$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where B and D are nonempty and square submatrices.

Otherwise, A is *reducible (decomposable)*. It is well known that A is irreducible if and only if $D(A)$ is strongly connected. Clearly, A is reducible if and only if, there exist a permutation $(i_1, i_2, \dots, i_s, j_{s+1}, j_{s+2}, \dots, j_n)$ of $(1, 2, \dots, n)$ such that the submatrix $A[i_1, i_2, \dots, i_s | j_{s+1}, j_{s+2}, \dots, j_n] = 0$.

Lemma 5.3.3. - Let A be a sign pattern matrix and $B(A)$ is the corresponding Boolean matrix. If $B(A)$ is indecomposable then A is indecomposable. The converse may not true.

Proof. - Firstly suppose that A is decomposable, this implies that $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, where B and D are nonempty and square submatrices. The corresponding Boolean matrix $B(A) =$

$$\begin{bmatrix} B^+ & C^+ & B^- & C^- \\ 0 & D^+ & 0 & D^- \\ B^- & C^- & B^+ & C^+ \\ 0 & D^- & 0 & D^+ \end{bmatrix},$$

Clearly $B(A)$ is permutationally similar to
$$\begin{bmatrix} B^+ & B^- & C^+ & C^- \\ B^- & B^+ & C^- & C^+ \\ 0 & 0 & D^+ & D^- \\ 0 & 0 & D^- & D^+ \end{bmatrix},$$

Where $\begin{bmatrix} B^+ & B^- \\ B^- & B^+ \end{bmatrix}$ and $\begin{bmatrix} D^+ & D^- \\ D^- & D^+ \end{bmatrix}$ are nonempty and square submatrices as B

and D are nonempty and square submatrices. This implies that $B(A)$ is decomposable.

Hence a sign pattern matrix is indecomposable if the corresponding Boolean matrix is

indecomposable.

Now we will give an example to show that the converse may not be true.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $B(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Clearly A is indecomposable but $B(A)$ is decomposable. ■

Theorem 5.3.1. - For an indecomposable Boolean matrix A there exists a positive integer x such that $a_{ii}^x = 1$ for some i .

Proof. - Let $i \neq j$. Since A is indecomposable there exist two connecting walks S_1 and S_2 from i to j and from j to i . Suppose $l(S_1) = m$ and $l(S_2) = l$, this implies that $a_{ij}^m = 1$ and $a_{ji}^l = 1$. Then $S_1 + S_2$ be a cycle from i to i of length $m + l$.

So $a_{ii}^{m+l} = \sum_{k=1}^n a_{ik}^m a_{ki}^l = a_{ij}^m a_{ji}^l + \sum_{k=1, k \neq j}^n a_{ik}^m a_{ki}^l = 1 + \sum_{k=1, k \neq j}^n a_{ik}^m a_{ki}^l = 1$. ■

Remark 5.3.3. - From lemma 5.3.3 and theorem 5.3.1 we get, for an indecomposable sign pattern matrix A whose corresponding Boolean matrix $B(A)$ is also indecomposable, there exists a positive integer x such that $a_{ii}^x = +1$ or $\#$ for some i . If A is powerful then there exists a positive integer x such that $a_{ii}^x = +1$ for some i .

Now we will prove that the above remark is true for all indecomposable, powerful sign pattern matrices.

Theorem 5.3.2. - For an indecomposable, powerful sign pattern matrix A there exists a positive integer x such that $a_{ii}^x = +1$ for some i .

Proof. - Let $i \neq j$. Since A is indecomposable then there exist two connecting sequences S_1 and S_2 from i to j and from j to i . Suppose $l(S_1) = m$ and $l(S_2) = l$. Then $S_1 + S_2$ be a cycle from i to i of length $m + l$.

$$\text{So } a_{ii}^{m+l} = \sum_{k=1}^n a_{ik}^m a_{ki}^l = a_{ij}^m a_{ji}^l + \sum_{k=1, k \neq j}^n a_{ik}^m a_{ki}^l.$$

Case 1:- If a_{ij}^m and a_{ji}^l have same signs, then $a_{ij}^m a_{ji}^l = +1$. Since A is powerful, so $\sum_{k=1, k \neq j}^n a_{ik}^m a_{ki}^l = 0$ or $+1$. Hence $a_{ii}^{m+l} = +1$.

Case 1:- If a_{ij}^m and a_{ji}^l have different signs, then $a_{ij}^m a_{ji}^l = -1$. Since A is powerful, so $a_{ii}^{m+l} = -1$ and

$$\begin{aligned} (a_{ii}^{m+l})^2 &= a_{ii}^{m+l} a_{ii}^{m+l} = \sum_{k=1}^n a_{ik}^{m+l} a_{ki}^{m+l} \\ &= a_{ii}^{m+l} a_{ii}^{m+l} + \sum_{k=1, k \neq i}^n a_{ik}^{m+l} a_{ki}^{m+l} \\ &= (+1) + \sum_{k=1, k \neq i}^n a_{ik}^{m+l} a_{ki}^{m+l}. \end{aligned}$$

because A is powerful so $\sum_{k=1, k \neq i}^n a_{ik}^{m+l} a_{ki}^{m+l}$ is either $+1$ or 0 . This implies that $a_{ii}^{2(m+l)} =$

+1. Thus there exists a positive integer x such that $a_{ii}^x = +1$ for some i .

■

5.4 Base and Period of Sign Pattern Matrices

In this section we study the sequence of powers of certain matrices over the sign pattern semiring. We find the minimal rank of the powers of irreducible nonnegative sign pattern matrices and compare it with other rank functions. We also derive bounds for the period and the base of an SNS non-negative irreducible pattern. We will begin with some basic results.

Definition 5.4.1. (*Period of Oscillation*) [25] *The period of oscillation of a sign pattern matrix A , is denoted by $p(A)$, is the least positive integer p such that $A^k = A^{k+p}$ for some integer k .*

Definition 5.4.2. (*Base*) [25] *The base of a sign pattern matrix A , is denoted by $l(A)$, is the least positive integer k such that $A^k = A^{k+p}$ for some integer p .*

Lemma 5.4.1. - [[25],lemma 1.2]- *Let $A \in Q_n$ be a powerful with base l and period p . Then for integers $m, k > 0$, $A^m = A^{m+k}$ if and only if $m > l$ and $p \mid k$.*

Definition 5.4.3. (*Index of imprimitivity*) [7] *Let $A = [a_{ij}]$ be a matrix over an antinegative semiring with no zero divisors. The index of imprimitivity of A , denoted by $h(A)$,*

is the greatest common divisor of $\{x: x > 0, a_{ii}^x \neq 0 \text{ for some } i\}$.

It is well known that for non negative matrices, [[18], p.80] h equals the number of characteristic roots of A having maximum modulus. If $h = 1$, then A is called *primitive*.

Theorem 5.4.1. -[[27], Theorem 3.1]- *Let A be an irreducible matrix with index of imprimitivity $h \geq 2$. Then A is permutation similar to a matrix of the form*

$$\begin{bmatrix} 0 & A_1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & A_{h-1} \\ A_h & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (5.1)$$

where the zero matrices on the diagonal are square. This is called the *block cyclic form* of A . When $h = 1$, the block cyclic form of A is A itself with no partition.

Associated with any pattern $A \in Q_n$, we have a nonnegative pattern $|A|$, obtained from A by replacing every negative one entry with a positive one. Now we will consider the period and the base of a nonnegative pattern.

Theorem 5.4.2. -[[25], theorem 2.1]- *Let $A \in Q_n$ be nonnegative and irreducible with index h . Then the period of A is h , i.e, $p(A) = h(A)$.*

Theorem 5.4.3. -[[25], theorem 2.2]- *Let A be an $n \times n$ ($n \geq 2$) nonnegative irreducible pattern with block cyclic form with index of imprimitivity h . Then the base of A is the smallest positive integer k such that for all i ($1 \leq i \leq h$), $A_i A_{i+1} \dots A_{i+k-1}$ is (entrywise) positive, where the indices are mod h .*

Corollary 5.4.1. - *Let A be a nonnegative irreducible pattern with index of imprimitivity h . Then $A^m = A^{m+k}$ if and only if $h \mid k$ and A^m has h positive blocks.*

Proof. - Let l be the base of A . Consider that $A^m = A^{m+k}$, then clearly by lemma 5.4.1 and theorem 5.4.2 we have $h \mid k$ and $m \geq l$. From theorem 5.4.3, A^l has h positive blocks, this implies that A^m has h positive blocks for all $m \geq l$. Conversely if A^m has h positive blocks then $A^m = A^{m+h} = A^{m+2h} = \dots$, also $h \mid k$ so we have $A^m = A^{m+k}$.

■

The following theorem has been proved by Gantmacher [[18], Theorem 9, p 81], and by Dulmage and Mendelsohn [[15], Theorem 6, p.179] using graph theoretical ideas.

Theorem 5.4.4. - *Let A be an $n \times n$ irreducible, nonnegative matrix with index of imprimitivity h . Let m be a positive integer. If m and h are relatively prime, then A^m is irreducible with the index of imprimitivity h .*

We use this to prove the following theorem.

Theorem 5.4.5. - Let A be an n by n ($n \geq 2$) nonnegative irreducible pattern with index of imprimitivity h . If $A^l = A^{l+h}$ then $\text{mr}(A^l) = h$ and the converse holds if $\text{g.c.d}(l, h) = 1$.

Proof. - We may assume that A is in block cyclic form

$$\begin{bmatrix} 0 & A_1 & & & & \\ & 0 & A_2 & & & \\ & & \ddots & \ddots & & \\ & & & & 0 & A_{h-1} \\ A_h & & & & & 0 \end{bmatrix}$$

Suppose that $A^l = A^{l+h}$, then using corollary 5.4.1 we have A^l has h entrywise positive blocks, say A'_1, A'_2, \dots, A'_h . This implies that the minimal rank of A'_i , for $i = 1, 2, \dots, h$, is equal to one also we see that

$$\begin{aligned} \text{mr}(A^l) &= \text{mr}(A'_1) + \text{mr}(A'_2) + \dots + \text{mr}(A'_h). \\ &= 1 + 1 + \dots + 1 = h \end{aligned}$$

Conversely let $\text{mr}(A^l) = h$ and $\text{g.c.d}(l, h) = 1$. Since A is irreducible with index of imprimitivity h and $\text{g.c.d}(l, h) = 1$, so using theorem 5.4.4, A^l is also irreducible with index of imprimitivity h . This implies that A^l has h nonzero blocks in the block cyclic form (say A'_1, A'_2, \dots, A'_h). Each A'_i has no zero row or column, since A^l is irreducible. This implies that $\text{mr}(A'_i) \geq 1$, for all $i = 1, 2, \dots, h$, also we see that

$$\text{mr}(A^l) = \text{mr}(A'_1) + \text{mr}(A'_2) + \dots + \text{mr}(A'_h) = h$$

$$\text{Thus } \text{mr}(A'_1) = \text{mr}(A'_2) = \dots = \text{mr}(A'_h) = 1.$$

Choose that column of B which has $+1$ in first row and 0 in second row (it is possible because row 1 and row 2 are linearly independent in the Gondran-Minoux sense), and place this column as the first column of B' .

Now choose that column of B which has $+1$ in the second row (it is possible because row 2 is not equals to zero), and place this column as the second column of B' .

Other $h - 1$ columns of B' are the columns of B such that i^{th} column of B' contains first nonzero entry of i^{th} row of B , where $i = 3, 4, \dots, h+1$. Clearly B' is a square matrix with Gondran-Minoux linearly independent rows. This implies that B' is an $(h+1) \times (h+1)$, SNS submatrix of A^k , so $rk_{Id}(A^k) \geq h+1$. Hence $rk_{Id}(A^k) > rk_{Id}(A^l)$.

■

Theorem 5.4.8. - *Let A be an $n \times n$ ($n \geq 2$) nonnegative irreducible pattern. If A is SNS then the base of A is one if and only if A is a permutation matrix.*

Proof. - Let $A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$. This implies that all the blocks of A are positive. So

Using theorem 5.4.3, we get base of A is 1.

We may assume that A is in the block cyclic form. Since A is an SNS matrix, this implies that all the rows of A are linearly independent in the Gondran-Minoux sense. Thus all the rows in each block of A are linearly independent in the Gondran-Minoux sense. This implies that all the blocks of A are not completely positive unless each block is of order 1×1 and no block is completely zero as A is an SNS matrix. Thus the base of $A = 1$ implies

that A is a permutation matrix. ■

Now we will give an example to show that if A is a reducible nonnegative SNS matrix then base of A can be 1 even if A is not a permutation matrix.

Example 5.4.1. - Let $A = \begin{bmatrix} +1 & +1 & \cdots & +1 & +1 \\ 0 & +1 & \cdots & +1 & +1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & +1 & +1 \\ 0 & 0 & \cdots & 0 & +1 \end{bmatrix}$, Clearly A is a reducible nonnegative

SNS matrix and $A^2 = A$. This implies that base of A is 1.

Corollary 5.4.2. - Let A be a nonnegative irreducible pattern. If A is SNS then period of $A \leq n$ and equality holds if and only if A is a permutation matrix.

Proof. - Let l be the base and p be the period of A , this implies that $A^l = A^{l+p}$. Using theorem 5.4.5 we get $\text{mr}(A^l) = p$. This implies that $rk_{Id}(A^l) = p$, where $p = \text{index of imprimitivity } h$ of A , by theorem 5.4.6. Since $1 \leq l$ and $\text{g.c.d}(1,p) = 1$, so using theorem 5.4.7, we get $rk_{Id}(A) \geq rk_{Id}(A^l) = p$. Thus $rk_{Id}(A) \geq p$ and equality holds if and only if $l = 1$. We are also given that A is an SNS matrix, this implies that $rk_{Id}(A) = n$. Thus we get $n = rk_{Id}(A) \geq p$ and equality holds if and only if $l = 1$. Hence the period of $A \leq n$ and equality holds if and only if A is a permutation matrix. ■

5.5 Base of a Boolean Matrix

In [24] it is proved that for an n by n primitive Boolean matrix B , base of $B \leq (n - 1)^2 + 1$. Now we will find a smaller upper bound for the base of a special class of Boolean matrices. For this we will use the results from matrix sign pattern theory.

In section 3.1 we have defined that if A be a real matrix then $Sg(A)$ is the sign pattern of A . The following proposition characterize the sign pattern of the powers of a real matrix.

Proposition 5.5.1. [4] *If the entry (i,j) of the matrix $(Sg(A))^k$ is determined (i.e, distinct from $\#$), then it coincides with the entry (i,j) of the matrix $Sg(A^k)$.*

As it follows from proposition 5.5.1, the matrix $(Sg(A))^k$ contains no information on the sign portrait of A^k if and only if all the entries of $(Sg(A))^k$ are equal to $\#$. If, for a real matrix A , there is an index k such that $(Sg(A))^k$ has all entries equal to $\#$, then the smallest of such indices is denoted by $\#(A)$. If no such k exists we set $\#(A) = \infty$.

Theorem 5.5.1. [4] *Let S be a sign pattern semiring and $A \in M_n(S)$. If $\#(A) < \infty$, then $\#(A) \leq 2n^2 - 3n + 2$.*

Using theorem 5.4.1, we get that every nonnegative irreducible imprimitive matrix is permutation similar to a block cyclic form. Taking into account the structure of an imprimitive matrix $A \in M_n(S)$, One can see that any of its power has zero entries, whence

$\#(A) = \infty$. However it can happen that some power of the matrix A contains only the elements 0 and $\#$. The smallest of such powers will be denoted by $\#'(A)$, and if no such power exists, then we set $\#'(A) = \infty$.

Theorem 5.5.2. [4] *Let a matrix $A \in M_n(S)$ be irreducible and imprimitive. If $\#'(A) < \infty$, then $\#'(A) < 2n^2 - 3n + 2$.*

Now we will give an upper bound for the base of a special kind of Boolean matrices which correspond to the sign pattern matrices.

Theorem 5.5.3. - Let $B = \begin{bmatrix} C & D \\ D & C \end{bmatrix}$ be an $2n$ by $2n$ Boolean matrix, where C and D are square matrices of order n . If B is a primitive matrix then base of $B \leq 2n^2 - 3n + 2$.

Proof. - Given that $B = \begin{bmatrix} C & D \\ D & C \end{bmatrix}$ be a Boolean matrix, where C and D are square matrices. Clearly B corresponds to a sign pattern matrix (say A), i.e, $B = B(A)$, where A is a sign pattern matrix. Suppose that A^k has all entries equal to $\#$ (since $B(A) = B$ is a primitive matrix so such a k exists). Using theorem 5.5.1, we get $k \leq 2n^2 - 3n + 2$ also this implies that $(B(A))^k$ has all entries equal to 1, Since there is one to one homomorphism from the matrices over sign pattern semiring to the matrices over Boolean semiring. Hence base of $B(A) \leq 2n^2 - 3n + 2$.

■

Thus by using results from matrix sign pattern theory, we have proved that for an $2n$ by $2n$ primitive Boolean matrix B of type $\begin{bmatrix} C & D \\ D & C \end{bmatrix}$, where C and D are square matrices, the base of B is less than or equal to $2n^2 - 3n + 2$. This upper bound is smaller than the upper bound for the base of general Boolean matrix of order $2n$, which is $(2n - 1)^2 + 1$. Thus the semiring version of the Cayley-Dickson construction allows matrix sign pattern theory to be used to study Boolean matrices and vice-versa.

5.6 Conclusion

We end by summarizing the novel results of the thesis.

We explored the S^2 construction introduced by [29], where it was used to study the max-plus semiring. We used this construction for the first time to study the sign pattern semiring. We showed that S^2 semiring inherits certain important properties such as antinegativity from S .

In chapter 3, we constructed a $2n$ by $2n$ Boolean matrix from an n by n sign pattern matrix using S -squared construction. we showed that there is a one to one semiring homomorphism from the matrices over the sign pattern semiring to the matrices over the Boolean semiring. We gave a relation between the sign non-singularity of a sign pattern matrix and the determinantal rank of the corresponding Boolean matrix.

In chapter 4, we introduced the concept of a sublocal semiring and the ideal rank of the matrices over a sublocal semiring. We showed that the usual upper bound for the rank of the sum and the product of matrices holds when the rank is the minimal rank and

the ideal rank. We also compared the ideal rank and the minimal rank with the other notions of rank. We showed that the square matrices of full ideal rank are exactly the SNS matrices. We also proved that the general group semiring $S[G]$, for any abelian group G , inherits the same properties from S that S^2 does.

In chapter 5, we found some new results about the ranks of the powers of certain irreducible nonnegative sign pattern matrices.

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