Essays on Optimal Portfolio and Resource Allocation

by

Sergiy Pysarenko

A Thesis
presented to
The University of Guelph

In partial fulfilment of requirements
for the degree of
Doctor of Philosophy
in
Economics

Guelph, Ontario, Canada

ABSTRACT

ESSAYS ON OPTIMAL PORTFOLIO AND RESOURCE ALLOCATION

Sergiy Pysarenko
University of Guelph, 2018

Advisors:
Vitali Alexeev, Kurt Annen, Francis Tapon

This paper proposes a micro-foundation for Contest-Success Functions (CSF) in a principal/agents setting. We characterize the principal's utility function so that the Ratio Form CSF results as the unique optimal allocation rule. The literature assumes a particular CSF without considering that the principal has her own preferences and that the optimal allocation, after having observed efforts, may differ from the allocation stipulated by the CSF. In this case, the CSF is not the principal's best response strategy; thus, the contest is not strongly credible. To ensure strong credibility, we consider sufficient conditions for a non-monotonic utility function, as well as for a larger family of monotonic utility functions compared to the literature.

This paper proposes a novel approach of portfolio allocation. The fundamental indexing (FI) and Markowitz mean-variance optimization (MVO) approaches are complementary but have been considered separately in the portfolio choice literature. Using data on S&P 500 constituents, we evaluate a portfolio construction technique that utilizes the benefits of both approaches. The out-of-sample results of the blended portfolios attest to their superior performance compared to common benchmarks, and to portfolios constructed solely based on the FI or MVO methods. In pursuit of the optimal blend between the MVO and FI,

we find that the ratio of market capitalization to GDP, being a leading indicator for an overpriced market, demonstrates remarkably advantageous properties.

This paper proposes a novel approach of portfolio allocation. The fundamental indexing (FI) and Markowitz mean-variance optimization (MVO) approaches are complementary but have been considered separately in the portfolio choice literature. Using data on S&P 500 constituents, we evaluate a portfolio construction technique that utilizes the benefits of both approaches. The out-of-sample results of the blended portfolios attest to their superior performance compared to common benchmarks, and to portfolios constructed solely based on the FI or MVO methods. In pursuit of the optimal blend between the MVO and FI, we find that the ratio of market capitalization to GDP, being a leading indicator for an overpriced market, demonstrates remarkably advantageous properties.

To Mykhailo, Nataliia, Maryna, and Iphigenie.

ACKNOWLEDGEMENTS

I would like to thank the love of my life, Iphigenie, for simply going through this not so simple journey with me. I would like to thank my parents, Mykhailo and Nataliia, for raising me in love and in pursuit of knowledge. And thank you, my dear sister Dr. Maryna Pysarenko for helping set my path into Economics.

I thank Professors Yuri Yevdokimov, Weiqiu Yu, Pavlo Zhytnyy, Serguei Maliar and Dr. Russ Irwin for inspiring me to continue my education and advising me in my pre-doctorate research journey.

The first chapter would not be possible without the help and advice of Dr. Kurt Annen. Thank you for the great amount of time and help you have provided me over these years. My academic writing, attention to details and teaching style would not be the same without this help.

My second and third chapter would not be possible without the help and advice of Professor Francis Tapon and Dr. Vitali Alexeev. You rejuvenated my interest in Finance, inspired my curiosity in the subject, taught me Matlab and other useful skills and provided ample assistance to me. My professional goals, focus on a larger picture, effectiveness in research and optimism would not be the same without your help.

I owe special thanks to Professors James Atsu Amegashie, Rene Kirkegaard, Alex Maynard, and Thanasis Stengos for their advice, detailed comments during my presentation, time and effort to read and re-read the thesis; and for their great classes. I also equally enjoyed classes that I took from Professors Bram Cadsby, Laurent Cellarier, Kris Inwood, Stephen Kosempel, Henry Thille, Ilias Tsiakas, and Asha Sadanand. Thank you all for your insight.

Also, during my academic program I met very fine people. Some of them became my very close friends and also helped to shape my interests and research. Thank you Brandon Sommer, Dr. Mustafa Koroglu, and Dr. Alexander Maslov for all our numerous discussions and most importantly for your friendship.

Table of Contents

List of Tables

List of Figures

Chapter 1

Micro-foundations of the Ratio Form Contest Success Function as a Credible Resource Allocation Rule

1.1 Introduction

This paper considers the Ratio Form Contest Success Functions (RF CSFs) as a result of a principal's (contest administrator's) maximization problem. We provide sufficient conditions for an allocation rule to be strongly credible (a situation when all parties involved are confident in the allocation rule)[1]. Also, we relate the RF CSF to two types (monotonic and non-monotonic) of a principal's utility function.

[1] We define strong credibility properly in the Model section.

Contest success functions (CSFs) have been widely used in industrial organization, development economics, political science, theories of conflict resolution, etc. They are applied to situations when the choice of relative effort levels across agents affect the allocation of a valuable resource such as when workers compete for a promotion, lobbyists compete for influence in governments, aid recipient countries compete for foreign aid etc.

Similarly to Corchon and Dahm (2010) we define a CSF as:

Definition 1 (CSF) *A contest success function associates, to each vector of efforts e, a lottery specifying for each agent a probability p_i of getting an indivisible object or share p_i of a divisible object. That is, $p_i = p_i(e)$ is such that, for each contestant $i \in N :=$ $1, ..., n,\ p_i(e) \geq 0$, and $\sum_{i=1}^{n} p_i(e) = 1$.*

By far the most widespread CSF is the RF CSF, and its special case - the Tullock CSF. Similarly to the literature (Tullock, 1980; Jia, 2008) we define them as follows:

Definition 2 (RF CSF) *The Ratio Form Contest Success Function is:*

$$p_i = \frac{f_i(e_i)}{\sum_{j=1}^{n} f_j(e_j)}, \tag{1.1}$$

where p_i is the probability of winning the contest or the share of a resource allocated to a contestant i as a result of the contest; $e_i \geq 0$ is the effort exerted by the contestant i, $f_i()$ is defined as strictly increasing efficiency function of the effort.

It is a limitation of our paper to concentrate only on strictly increasing functions $f_i()$.

Definition 3 (Tullock CSF) *The Tullock Contest Success Function is:*

$$p_i = \begin{cases} \frac{e_i^r}{\sum_{j=1}^{n} e_j^r} & \text{if } \exists e_i > 0 \\ \frac{1}{n} & \text{if } e_i = 0 \; \forall i, \end{cases} \tag{1.2}$$

where p_i is the probability of winning the contest or the share of a resource allocated to a contestant i as a result of the contest; $e_i \geq 0$ is the effort exerted by the contestant i, r is the (randomness) parameter.

Most of the literature, assumes a CSF without considering that the principal (i.e. the employer, the government or the donor of foreign aid in the previous three examples), who is allocating the resource among agents, has her own objective function. Given the principal's preferences, the optimal allocation may differ from the assumed CSF once contestants' efforts have been exerted and observed. In this case, the contest is not credible and rational agents may change their effort choice. Thus, the CSF, that is credible, is nothing else than the allocation rule that maximizes a payoff function of the principal given the agents' effort levels. The purpose of this research is to identify payoff functions that produce the RF CSF as the unique optimal allocation rule. This analysis is useful as it will give insights into the circumstances under which one can reasonably expect the RF CSF to occur. In addition, this analysis reveals a possible link between properties of a principal's preferences and parameter values in CSFs, thus providing a micro-foundation for the RF CSF.

This research contributes to the literature on the micro foundations of contest success functions. More specifically, we focus on the RF CSF, in a situation with complete information. It was shown that certain utility functions of a principal generate CSFs. We provide sufficient conditions for the principal's utility function to generate the RF CSF. For an insightful review of micro foundations, see Hao et al. (2013). We deal with a microeconomic foundation, under complete information, when the principal is not necessarily able to commit to the announced allocation rule.

In the literature related to a set-up with complete information, the principal's utility function includes utility functions of contestants and their effort levels. The RF CSF maximizes a principal's utility ex ante. However, there was not enough attention paid to credibility. Being a last mover, she can deviate from the announced RF CSF to maximize her ex post utility.

The closest paper to our study is Corchon and Dahm (2011), where they are concerned about the ex post utility of a social planner that maximizes welfare. They consider what they claim to be a special case of Kahneman and Tversky (1979) value function for regular prospects:

$$W(p,e) = \sum_{i=1}^{n} p_i^{\alpha} f_i(e_i)^{1-\alpha} \quad , \; 0 < \alpha < 1.$$

They prove that the unique solution to this maximization problem is the RF CSF defined in (1.1). We argue that prospect theory cannot justify the CSFs.

More importantly, we consider two types of utility functions (non-monotonic and monotonic) and describe how the RF CSF arises as the result of the principal's maximization problem. First, assuming that the principal's utility function is additively separable in p_i, we back engineer possible properties of the utility function from the RF CSF. This approach generates a (non-monotonic) utility function with a bliss point. Second, we provide a different interpretation and study properties of the monotonic utility functions similar to Corchon and Dahm (2011).

Our study provides a possible microeconomic theoretical justification of the Fundamental Indexation investment approach that is a new trend in portfolio manager practitioners. In their seminal paper Arnott et al. (2005) propose Fundamental Indexation which allocates a portfolio according to equation (1.2), where $r = 1$ and e_i stands for a firm's accounting metrics (it can be book value, free cash flows, dividends paid, number of employees, revenues, etc).

Overall, our primary concern is a situation, where a principal, who is not necessarily a social planner can credibly commit to a contest. For example, when considering a bribe-taking bureaucrat, if she cares only about bribes, then she will not be able to strongly credibly commit to the RF CSF. Finally, we show different ways that stimulate a bureaucrat to strongly credibly commit to contest success functions.

The rest of the chapter will include a literature review (Section 2), the model (Section 3), a discussion (Section 4), and conclusions (Section 5).

1.2 Literature Review

Contest success functions (CSFs) have been widely used in the literature since the seminal paper by Tullock (1980). In his paper, Gordon Tullock proposes a contest success function to capture the probabilities of lobbying success in a rent-seeking game. This function is suggested because it has several desirable properties, like the probability of success should be increasing with one's effort and decreasing with the efforts of other players, and the total sum of probabilities should be equal to one. However, this function does not possess deeper microfoundations that could explain why a principal or a bureaucrat would find this CSF as their optimal allocation rule. Also, it does not explain what determines the specific form of the CSF, given that the aforementioned properties could be satisfied by quite a large family of functions. Thereafter, the literature on CSF microfoundations started evolving.

There are two major forms of CSFs: first, a CSF that assigns a contest outcome from the difference between the efforts of each contestant; and second, a CSF that assigns a contest outcome from the ratio of efforts (e.g. Equation 1.1 is an all-pay auction, when $r = \inf$). The focus of this paper is on the ratio form case defined in (1.1). However, some of our results apply to both types of CSFs. We discuss this below.

Microfoundations of contest success functions can be derived either directly from assumptions (axioms) about the CSF or from microeconomic set-ups.

1.2.1 Axiomatic foundations

This part of the literature starts from axioms (or properties) of the CSF that are desirable. Then, it considers how these properties affect functional forms. For example, Skaperdas (1996) proposes that a resource allocation should be affected only by the efforts of contestants and not anybody else - the Independence of Irrelevant Alternatives Property. Another widely used property of allocations is the homogeneity of degree zero in efforts - factoring efforts of all contestants would not change the allocation:

$$P(\lambda E) = P(E),$$

where E is a vector of efforts, and P is an allocation vector.

Many CSFs are symmetric - switching efforts switches resource allocation the same way. However, symmetry is not always required. In Hirshleifer and Osborne (2001), for example, one party is more advantaged. For a CSF function, this asymmetry can be modelled by allowing different coefficients multiplied by otherwise identical effort efficiency functions:

$$p_i = \frac{f_i(e_i)}{f_i(e_i) + f_j(e_j)} = \frac{a_i f(e_i)}{a_i f(e_i) + a_j f(e_j)}.$$

Similarly to this stream of literature, we also start with axioms (or properties); however, our axioms do not relate to the CSF - as our main focus is not on the RF CSF - but on the principal's preferences. We take a CSF as given and describe sufficient conditions for a principal optimizing problem such that the given CSF results in a unique equilibrium.

1.2.2 Foundations based on a microeconomic set-up

Foundations that are based on a microeconomic set-up can fall into one of two categories: a set-up with incomplete information or a set-up with complete information.

Set-up with incomplete information

Microeconomic foundations based on incomplete information rely on a stochastic process, hidden information, or a hidden action assumption.

Stochastic process. The distinctive feature of stochastic microfoundations of the CSF is that there should be some random process or noise in the model. This noise does not allow for a contestant to guarantee the victory, because effort levels are unobservable. Thus, effort levels influence only a probability of getting a resource. For example, Jia (2008) shows that a performance function

$$h(e_i, \theta_i) = e_i \theta_i,$$

where θ_i is noise, distributed according to the Inverse Exponential Distribution $IEXP(\alpha, k)$, $\alpha > 0, k > 0$, leads to a contest of the form:

$$p_i(e_i, e_{-i}) = \frac{e_i^k}{\sum_{j=1}^{n} e_j^k}.$$

The similarity between approach proposed by Jia (2008) and our approach is that both of them rely on the principal's utility function. The fundamental difference between stochastic foundations and our paper is that in a stochastic based set-up a CSF is a result of a random process while we focus on the opposite case, when the CSF is a result of the deterministic set-up. However, given that the RF CSF is interpreted more often as probabilities than shares, it is the limitation of our approach to focus on the deterministic set-up, where the RF CSF is interpreted as shares.

Hidden information. An example of a hidden information approach is Corchon and Dahm (2010). They show that a CSF is rationalizable by a pair of payoff functions fulfilling the single crossing condition. They employ a hidden information model in which a principal may favour contestants differently, while contestants do not know the preferences of the principal. The preferences of the principal over contestants are captured by a uniformly distributed parameter θ, where a higher θ means that the second contestant is more advantaged than the first contestant:

$$U_1(\theta, e_1).$$

The principal allocates the entire resource to player 1 if:

$$U_1(\theta, e_1) > U_2(\theta, e_2).$$

In this model contestants have incomplete information about the principal's preferences. Corchon and Dahm (2010) show that a form of the CSF depends on how a parameter θ

enters the principal's utility function. The RF CSF is obtained if: $U_1(\theta, e_1) = (1-\theta)f_1(e_1)$ and $U_2(\theta, e_2) = \theta f_2(e_2)$. Since, a resource is fully allocated to one player, p_i can represent only the expected probability of getting the resource and not shares:

$$p_1 = Prob(U_1(\theta, e_1) > U_2(\theta, e_2)) = Prob((1-\theta)f_1(e_1) > \theta f_2(e_2)) =$$

$$= Prob(\frac{f_1(e_1)}{f_1(e_1) + f_2(e_2)} > \theta) = \frac{f_1(e_1)}{f_1(e_1) + f_2(e_2)}.$$

This model runs into problems when the number of contestants is higher than two. Thus, a claim that our model can be a microfoundation for a wide range of contests will require it to be generalizable to more than two players. Corchon and Dahm (2010) require that the utility function of the principal remains unknown to contestants. The advantage of our approach is that the other approach is vulnerable to the revelation of information. If contestants learn the true utility function of the principal, the nature of the Corchon and Dahm (2010) contest would change to an all-pay auction. In turn, the principal may choose to communicate strategically her preferences. Our approach, however, is robust to the revelation of the information.

Hidden action Another type of microfoundations is based on a hidden action. For example, Skaperdas and Vaidya (2012) consider a court (principal) that observes evidence presented by a prosecutor (contestant 1) and evidence presented by a defendant (contestant 2). The observed evidence $F(e_i)$ is a function of an unobserved evidence collection effort

e_i (hidden action) given that the defendant may be guilty or innocent (hidden information). The effort cannot be inferred from the evidence, because it is unclear what the true state of the defendant is: guilt or innocence. However, it is more costly to collect misleading evidence. Skaperdas and Vaidya (2012) consider a situation related to a litigation, when the principal employs a Bayesian rule to update her beliefs:

$$\pi^* = \frac{\pi L^g}{(1-\pi) + \pi L^g},$$

where, π is the principal's original belief and L^g, which represents "the force of the presented evidence in establishing guilt". L^g is defined as follows[2]:

$$L^g(F(e_1), F(e_2)) = \frac{F(e_1)}{F(e_2)},$$

where $F(e_i)$ is a function of evidence creation. More evidence of guilt

$$F(e_1) > F(e_2)$$

increases the belief of the principal that the true state is "guilty". Combining the last three formulas yields:

$$\pi^*(F(e_1), F(e_2)) = \frac{\pi F(e_1)}{(1-\pi)F(e_1) + \pi F(e_2)}.$$

The principal has two common characteristics with the setup we are studying. First, the

[2] In this definition we combined the simplifications which Skaperdas and Vaidya (2012) used in their proof and their original definition.

principal (a judge) decides to allocate her resource (belief in innocence or guilt) according to the RF CSF; second, the resource being distributed is divisible - the belief is not binary. Even though the litigation model requires a prior belief to get a Bayesian rule derivation, if the initial belief is 0.5, and evidence creation functions are linear, then the litigation model generates the Tullock CSF. However, this set-up is problematic when the number of players is more than two. Another problem is that it may have some credibility issues. Evidence creation $F(e_i)$ relies on a contestant's optimization problem, which in turn also depends on the CSF and other parameters (a valuation of the prize, initial endowment, costs, etc). In other words, both the CSF and $F(e_i)$ are endogenous to each other.

Set-up with complete information

In contrast to discussed literature so far, our derivations produce the RF CSF in the situation of complete information: when all preferences and actions are known to all players, and there is no stochastic process. However, our derivations do not rely on complete information: the resource is distributed according to the RF CSF even if the principal does not observe utility functions or types of contestants. Instead, our results rely on the principal's preferences and her ability to observe the result of contestants' efforts - $f_i(e_i)$. Thus, we ask what preferences can the principal have, so that she can commit to the CSF?

The closest paper to our research is Corchon and Dahm (2011). They investigate CSFs that maximize welfare, where a central planner cannot commit. They find that the utilitarian generalized constant elasticity of substitution utility function of the planner with risk neutral contestants is maximized by the difference contest function. A type of the difference function depends on the parameter of the elasticity of substitution. More specifically, the planner is maximizing:

$$W(p,e) = \begin{cases} \left(\sum_{i=1}^{n} u_i(p_i,e)\right)^{\frac{1}{1-r}} & \text{if } r \neq 1 \\ \sum_{i=1}^{n} ln(u_i(p_i,e)) & \text{if } r = 1 \end{cases},$$

where, p is a vector of probabilities or shares, and e is a vector of exerted efforts. The contestants' utility functions are

$$U_i(p_i,e) = p_i(e)V_i - e_i,$$

where V_i is contestant i's valuation of the prize. Then, for $r \neq 1$,

$$p_i = \frac{1 - \sum_{j=1}^{n} \frac{e_j}{V_j}}{\sum_{j=1}^{n} \left(\frac{e_j}{V_j}\right)^{\frac{1-r}{r}}} + \frac{e_i}{V_i}.$$

And for $r = 1$,

$$p_i = \frac{1 - \sum_{j=1}^{n} \frac{e_j}{V_j}}{n} + \frac{e_i}{V_i}.$$

In other words, a planner who maximizes her utility will set a contest rule that depends on the difference in the ratio of individual efforts to valuations of the prize.

More importantly, they show that expected utility theory cannot generate CSFs (other than the all-pay auction), as a utility function of the form

$$W(p,e) = \sum_{i=1}^{n} p_i f_i(e), \tag{1.3}$$

where p is a vector of probabilities and e is a vector of efforts, produces a unique corner solution, which implies an all-pay auction. Hence, to produce more CSFs it is beneficial to consider some utility theory other than the expected utility theory. Then they consider the utility function, "... which corresponds to a special case of the class postulated by (Kahneman and Tversky (1979) , p. 276) for regular prospects, namely

$$W(p,e) = \sum_{i=1}^{n} p_i^{\alpha} f_i(e_i)^{1-\alpha} \quad , \ 0 < \alpha < 1...''$$

(Corchon and Dahm, 2011), and they prove that the unique solution to this maximization problem is the RF CSF defined in (1.1).

Being an important reference point to our research, this finding is limited to the assumed utility function of the social welfare maximizer. More importantly, despite their link to prospect theory, it is not clear where risk comes from in the previously mentioned utility function: the principal controls and knows the resource allocation (discussed in more details below). Instead, we show that the specified utility function cannot be a regular prospect. We also provide an alternative theoretical justification for this form and for the broader family of utility functions. In addition, we suggest a non-monotonic utility function that can justify the RF CSF.

For the set-up with complete information it remains unclear what sufficient conditions are for the principal's utility function to generate the RF CSF as the optimal allocation rule. Thus, our research's aim is to provide additional sufficient conditions of the principal's utility function, that will generate the RF CSF. In Section 1.4, the similarities and differences of our findings compared to those in the literature will be discussed in more details, but first, let's consider our model.

1.3 Model

In this section we propose a game set-up and two ways of RF CSF derivation based on the principal's preferences: for monotonic and non-monotonic utility functions.

1.3.1 Game set-up.

As in Tullock (1980, p. 98), we assume that contestants are engaged in a rent-seeking activity. Tullock, for instance, discusses the example of lobbyists in Washington trying to raise the price of milk, or individual lobbyists spending "time cultivating congressmen and government officials." Tullock focuses on the CSF displayed in (1.2). However, here we consider the more general form of the Tullock CSF - RF CSF displayed in (1.1).

As indicated earlier, our focus, however, is not on the CSF itself but on the preferences of the principal or the congressmen or the government official in Tullock's example. More specifically, we ask what we can say about a principal's preferences so that the RF CSF emerges as the unique optimal allocation rule.

The contest we analyze can be described by a simple game G, which is a game in extensive form with complete information.

Definition 4 (Principal based contest game) *We will consider the following principal-based contest game G: n agents simultaneously choose efforts $e_i \geq 0$ first, where e_i denotes the effort level of agent i. After perfectly observing efforts $e = (e_1, \ldots, e_n)$, the principal chooses an allocation $p = (p_1, \ldots, p_n) \in P$ next, where p_i denotes the share of a resource (normalized to 1) allocated to agent i and $P = \{\mathbb{R}^n_+ | \sum_{i=1}^n p_i = 1\}$. Preferences of the principal can be described by the utility function, U.*

Notice that preferences of contestants are not defined as their definition is not relevant for our purpose. In this game, a strategy of the agent i simply consists of $e_i \geq 0$. A strategy of the principal is more complicated as the strategy needs to define the allocation p for every possible history of the game, $e \in E = \prod_{i=1}^n \mathbb{R}_+$. In other words, the strategy is a function $p(e) : E \to P$. Clearly, the RF CSF defined in (1.1) is one example of such a strategy. We will not solve the game completely as our main focus is on the best-response of the principal given efforts e. Recall the timing of the game is such that contestants choose efforts first and then the principal chooses an allocation p second. Thus, we do not analyze the optimization problem of the agent. Our main focus is on defining the principal's best-response function.

The following two definitions will be useful:

Definition 5 (Weak credibility) *For a strategy $p^*(e)$ to be weakly credible, it should be a principal's best-response in G given e.*

Since the principal is the last mover, weak credibility requires:

$$U(p^*|e) \geq U(p|e)\ \forall p \neq p^* \in P,$$

where U denotes the principal's utility function.

A large range of preferences or utility functions is consistent with weak credibility. For example, when a principal as a bribe taker does not care about how a resource is distributed, then she is indifferent between any allocation p, including the CSF displayed in (1.1). However, she also *could* distribute the resource in a different way. Thus, any allocation rule can be weakly implemented.

Multiple possible outcomes may create a coordination problem between the principal and contestants as the principal's indifference implies multiple Subgame Perfect Equilibria (SPE). One SPE could exist, when contestants choose effort levels e believing that the principal will allocate a resource according to the Tullock CSF (or some other form of the RF CSF), and the principal chooses to do so. Another SPE could exist, when contestants believe that the resource is distributed randomly. In this case, contestants under some reasonable assumptions about their preferences (such as efforts are costly) will exert no effort, and the principal distributes the resource randomly. However, in our example, the bureaucrat prefers the CSF as much as the random or any other allocation rule of the resource.

Thus, the CSF is not necessarily anticipated by fully-informed contestants, and they may not pay up bribes. This coordination problem is resolved when the principal *strictly* prefers to allocate the resource in a certain way. That is why we introduce the distinction between weak and strong credibility:

Definition 6 (Strong credibility) *For a strategy $p^*(e)$ to be strongly credible, it should be the principal's unique best-response in G given e*

In contrast to weak credibility, strong credibility requires:

$$U(p^*|e) > U(p|e)\ \forall p \neq p^*.$$

It is straightforward to show that strong credibility implies weak credibility but not the other way around.

Strong credibility is an important requirement for us to be certain that the RF CSF is obtained in equilibrium. We want to be able to describe environments in which we can be confident that a RF CSF or the Tullock contest emerges. If the strong credibility requirement is not met, then we cannot be confident of obtaining the RF CSF. If confidence of obtaining the RF CSF is not in place, the resource allocation rule may take other forms. If the resource allocation rule takes another form, then the strategic interaction of players changes. If the strategic interaction of players changes, the predictions of the Tullock CSF do not emerge in equilibrium. A specific allocation of a resource cannot be uniquely expected.

For example, let us assume that a bureaucrat has a bias for one of the contestants, then this contestant will receive the full resource once bribes are paid. In this case, the Tullock CSF is not a credible allocation rule. Now consider a bureaucrat who is only interested in maximizing the total amount of bribes. Because she cares only about the total amount of bribes, once bribes are paid, the bureaucrat is indifferent between any allocation of the resource she controls. In this situation, any CSF is weakly credible. Weakly credible RF CSF does not provide confidence in obtaining the RF CSF as discussed before: Weak credibility raises coordination problems.

In the following, we are characterizing environments in which allocation rules are strongly credible. We believe this is a reasonable and important requirement.

Proposition 1 shows that a resource allocation should enter into the principal's utility for her to follow the CSF. Otherwise, the CSF is just one of infinite ways to distribute the resource (given that the number of contestants is greater than 1, and the resource is divisible). Note, that Proposition 1 is general enough to be valid for any CSF, not only RF CSFs.

Proposition 1 *A necessary condition of the principal's utility function to produce a strongly credible strategy (an allocation rule) p: The resource allocation p must have an effect on the principal's utility function. For a special case of a differentiable utility function this condition can be written as:*

$$\exists p \in P : D_p U(p|e) \neq \mathbf{0}.$$

Proof: Let $U_{\text{principal}} = U(p|e)$, where e is a vector of contestants' efforts affecting the principal's utility.

1. Let us consider a credible allocation rule

$$\exists p : U(p|e) > U(\tilde{p}|e) \quad \forall \tilde{p} \neq p.$$

2. Assume that p does not have an effect on the principal's utility function:

$$U(p|e) = U(\tilde{p}|e) \quad \forall \tilde{p} \neq p.$$

3. We see a contradiction:

$$[\exists p : U(p|e) > U(\tilde{p}|e)\ \forall \tilde{p} \neq p] \perp [U(p|e) = U(\tilde{p}|e)\ \forall \tilde{p} \neq p].$$

Hence, our assumption in (2.) is wrong, and for an allocation rule to be strongly credible, $p(e)$ must have an effect on the principal's utility function:

$$\exists p : U(p|e) > U(\tilde{p}|e) \quad \forall \tilde{p} \neq p.$$

□

This proposition seems innocent, but its implications are not. It leads to the following interesting observation:

Observation: When a bureaucrat cares only about bribes, no CSF is guaranteed. The explanation relies on Proposition 1: for a bureaucrat who cares *only* about bribes, any resource allocation is weakly credible and none of them is unique. Tullock (1980) provides an example of lobbyists investing money in Washington to benefit from preferential policies. We show that if the bureaucrat does not care about resource allocation, she will not be able to have a strongly credible allocation rule. Lobbyists will not be sure which allocation rule will be applied even if they pay bribes. Thus, paying bribes does not guarantee success. Of course, there is an equilibrium, where contestants believe that the bureaucrat distributes the resource according to the CSF, and she happens to do so. But such an equilibrium is one out of an infinite number of other ways to allocate a resource.

Note that Proposition 1 holds for any CSF, including RF CSFs. From now on, we will focus on RF CSFs. In RF CSFs, effort levels of contestants affect their share or probability of success, which leads us to an additional requirement regarding the principal's utility function.

Proposition 2 *A necessary condition for a principal's utility function to produce a RF CSF as a strongly credible strategy (allocation-rule) p: As contestants' efforts (e), change, the effect of resource shares (p) on the principal's utility (U) changes. In a special case of differentiable principal's utility function $U(e, p)$ it is:*

$$\exists p \in P : D^2_{ep}U(e, p) \neq 0_{n\times n}.$$

Proof: Let $U(e,p)$ be the principal's utility, where e is a vector of contestants' efforts and p defines resource shares. Assume that p is affecting the principal's utility function (Proposition 1). Let e and $\tilde{e}$ be two vectors of exerted efforts:

$$\tilde{e}, e \in R^n_+.$$

1. Assume a change in e does not affect [3] the effect of resource shares (p) on the principal's utility (U)

2. Then, e cannot affect the optimal $p(e)$:

$$p' = \arg\max U(e,p) = \arg\max U(\tilde{e},p) \quad \forall \tilde{e} \neq e.$$

3. Since p' maximizes U for all e, p' cannot be a strongly credible RF CSF, because according to Definition 2, RF CSF requires efficiency function strictly increasing in efforts.

Hence, the Assumption 1, that e does not affect optimal p is wrong, and the proposition that efforts levels must have an effect on the optimal resource allocation p is proven.

$\square$

To illustrate the difference between Proposition 1 and Proposition 2 let us consider the following game:

[3] Note, that $f_i(e_i)$ is strictly increasing in e_i, according to Definition 2.

Example: The efforts of contestants are such that $f_i(e_i) = 1 \; \forall e_i$.

The utility function of the principal is:

$$U = \begin{cases} 1, & \text{if } p = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}), \\ 0, & \text{otherwise.} \end{cases}$$

This situation satisfies Proposition 1, as the principal's utility function depends on the resource allocation, and $p_i = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$ is a strongly credible allocation rule. However, Proposition 2 is not applicable to this situation as effort levels do not change resource allocation.

We will present two approaches that rationalize two types of principal's utility functions (monotonic and non-monotonic) so that the RF CSF emerges as a strongly credible allocation rule. We will consider the monotonic utility function (the only approach that is similar to what was used in the literature before us). We will start with it. Then we will consider a non-monotonic utility function, which we back-engineer with an approach based on integration.

1.3.2 Monotonic utility function

In this subsection we consider an approach which is similar to what has been done in the literature before, however we generalize the functional forms provided in the literature and we provide a better motivation for it.

Derivation

Corchon and Dahm (2011) use prospect theory (Kahneman and Tversky, 1979) to derive the RF CSF as the optimal allocation rule of a social planner. According to Kahneman and Tversky (1979) prospects (which relate to risky outcomes) can be regular and irregular.

Regular prospects require that at least one of these three conditions is met: $p_1 + p_2 < 1$ or $e_1 \geq 0 \geq e_2$, or $e_1 \leq 0 \leq e_2$. Irregular prospects do not meet any of these conditions. For example, a probability allocation with only positive valued outcomes is an irregular prospect.

The value of a regular prospect in our notation can be written as:

$$V = g(e_1) * h(p_1) + g(e_2) * h(p_2), \tag{1.4}$$

with the requirement that either $p_1 + p_2 < 1$ and/or one of the outcomes e_1 or e_2 smaller or equal than zero, and where g is a utility function measuring the value of certain outcomes. In contrast to this, the RF CSF describes a situation where all efforts are nonnegative and the sum of probabilities is equal to one. Thus, this situation does not meet the requirements for a regular prospect. As a result, a regular prospect cannot be used to motivate the RF CSF.

Since the RF CSF describes a situation where efforts are nonnegative and the sum of probabilities is one, the RF CSF relates to an irregular prospect. According to Kahneman and Tversky (1979) the value of an irregular prospect is:

$$V = g(e_2) + h(p_1)(g(e_1) - g(e_2)), \tag{1.5}$$

where $e_1 \geq e_2$. Differentiating (1.5) with respect to p_1 and setting equal to zero yields:

$$h'(p_1)[g(e_1) - g(e_2)] = 0.$$

Solving for p_1 yields

$$p_1 = h'^{-1}(0),$$

which does not resemble the RF CSF. Thus, irregular prospects cannot be used to motivate the RF CSF.

We conclude that the RF CSF cannot be justified by using prospect theory[4]. However, despite this lack of a connection between the prospect theory and the RF CSF, we believe that the finding that the functional form obtained in (1.4) generates the RF CFS is important. In the following, we propose an alternative theoretical justification for this functional form as well as a functional form that is more general than the one proposed in Corchon and Dahm (2011) . We show that given a sufficient condition describe below, the optimization of a utility function that is similar to the prospect value function results in the Tullock CSF. Specifically, the condition is that the first derivative of the utility function should be

[4]Neither when p_i is a share of a resource, nor when p_i is a probability of winning. We, however, focus of its interpretation as shares.

homogeneous of a certain degree k.

Let us consider an example that helps us to illustrate our findings. In this example, we no longer require the resource level to be normalized to 1, as it is in the Definition 4, however we still require that the sum of shares to be equal to 1. Suppose there is a venture investor facing a portfolio allocation problem. Several startups competing for capital each have e_i employees, where each employee contributes to the production of an output. Let the profit of each firm i be equal to:

$$U_i = g_i(e_i) * h_i(p_i * V),$$

where e_i is the number of employees working for the startup (assume $g_i(0) = 0$, and also assume $g(e_i) > 0,$ if $e_i > 0$), V is the total amount of invested capital, and $p_1 \in [0, 1]$ is the share of the capital invested by the venture investor in Firm 1. Then the final value of the portfolio for the investor is similar the value described in (1.4).

$$\begin{aligned} \max_{p_1, p_2} \quad & U = g(e_1) * h(p_1 * V) + g(e_2) * h(p_2 * V) \\ \text{subject to} \quad & \sum_{i=1}^{2} p_i = 1,\ p_i \geq 0. \end{aligned} \tag{1.6}$$

The investor chooses the profit maximizing portfolio shares p_i as follows.

FOC_{wrtp_i}:

$$g(e_1) * h'(p_1 * V) * V - g(e_2) * h'((1 - p_1) * V) * V = 0;$$

$$g(e_1) * h'(p_1 * V) - g(e_2) * h'((1 - p_1) * V) = 0;$$

$$\frac{g(e_1)}{g(e_2)} = \frac{h'((1 - p_1) * V)}{h'(p_1 * V)}.$$

Assume that $h'(1) \neq 0$, and $h'()$ is homogeneous of degree $-k$, where $k \in R_+$, then

$$h'(p_1 * V) = p_1^{-k} * V^{-k} * h'(1);$$

$$\frac{g(e_1)}{g(e_2)} = \frac{h'(1 - p_1) * V}{h'(p_1) * V} = \frac{V^{-k} * (1 - p_1)^{-k} * h'(1)}{V^{-k} * p_1^{-k} * h'(1)} = (\frac{1 - p_1}{p_1})^{-k};$$

$$(\frac{g(e_1)}{g(e_2)})^{-\frac{1}{k}} = \frac{1 - p_1}{p_1};$$

$$p_1 = \frac{1}{1 - (\frac{g(e_1)}{g(e_2)})^{-\frac{1}{k}}} = \frac{1}{1 - (\frac{g(e_2)}{g(e_1)})^{\frac{1}{k}}}.$$

Simplifying yields:

$$p_1 = \frac{g(e_1)^{\frac{1}{k}}}{g(e_1)^{\frac{1}{k}} + g(e_2)^{\frac{1}{k}}}. \tag{1.7}$$

Notice that we can extend the problem to more than two players and to player-specific functions for $g()$ and $h()$ as follows:

$$\begin{aligned} &\underset{p_1, p_2, \ldots, p_n}{\text{maximize}} \quad U_{\text{Principal}} = \sum_{i=1}^{n} g_i(e_i) * h_i(p_i) \\ &\text{subject to} \quad \sum_{i=1}^{n} p_i = 1, \; p_i \geq 0; \end{aligned} \tag{1.8}$$

Here we assume without loss of generality that $V = 1$. The Lagrangian, then, equals:

$$\mathcal{L} = \sum_{i=1}^{n} g_i(e_i) * h_i(p_i) + \lambda(1 - \sum_{i=1}^{n} p_i).$$

Thus, the Kuhn-Tucker conditions are:

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial p_i} = \frac{\partial [g_i(e_i) * h_i(p_i)]}{\partial p_i} - \lambda \leq 0, \quad p_i \geq 0, \quad p_i \frac{\partial \mathcal{L}}{\partial p_i} = 0 \quad \forall i = 1, \ldots, n; \\ &\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - \sum_{i=1}^{n} p_i = 0. \end{aligned} \tag{1.9}$$

Note, that $g_i(e_i)$ is a constant at the time of the resource allocation, thus:

$$\frac{\partial \mathcal{L}}{\partial p_i} = \frac{\partial [g_i(e_i) * h_i(p_i)]}{\partial p_i} - \lambda = g_i(e_i) * \frac{\partial h_i(p_i)}{\partial p_i} - \lambda$$

Slackness conditions are important for this problem: a contestant that exerts zero effort, gets no resources. Let $e_i = 0$, where $i \neq j$, then $g_i(0) = 0$, hence $\frac{\partial \mathcal{L}}{\partial p_i} = -\lambda < 0$. Thus, with the assumption $g_i(0) = 0$, $p_i(0) = 0$.

If all but one contestants exert zero effort, then the zero effort contestants receive no resources, while the contestant, who exerted a positive effort gets all resources. If no contestant exerts an effort then any allocation of a resource is equally optimal. In this case the Tullock CSF assumes that the resource is distributed equally:

$$\text{if } e_i = 0 \, \forall i, \text{ then } p_i = \frac{1}{n} \, \forall i.$$

Let us assume that contestants $(1, \dots, m)$ exerted positive efforts. Thus for these m contestants, $g_i(e_i) > 0$. Also, as before, assume that $h_i'()$ is homogeneous of degree $-k$. Then

$$\frac{\partial \mathcal{L}}{\partial p_1} = g_1(e_1)p_1^{-k}h_1'(1) - \lambda = 0;$$

$$\frac{\partial \mathcal{L}}{\partial p_2} = g_2(e_2)p_2^{-k}h_2'(1) - \lambda = 0;$$

$$...$$

$$\frac{\partial \mathcal{L}}{\partial p_m} = g_m(e_m)p_m^{-k}h_m'(1) - \lambda = 0;$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - \sum_{i=1}^{m} p_i = 0. \tag{1.10}$$

Note, that $\lambda > 0$. Then,

$$\frac{1}{\lambda} = \frac{1}{g_1(e_1)p_1^{-k}h_1'(1)} = \frac{1}{g_2(e_2)p_2^{-k}h_2'(1)} = ... = \frac{1}{g_m(e_m)p_m^{-k}h_m'(1)};$$

Taking $-k$ root of all parts and rearranging:

$$\frac{1}{(h_1'(1)g_1(e_1))^{-\frac{1}{k}}}\frac{1}{p_1} = \frac{1}{(h_2'(1)g_2(e_2))^{-\frac{1}{k}}}\frac{1}{p_2} = ... = \frac{n}{(h_m'(1)g_m(e_m))^{-\frac{1}{k}}}\frac{1}{p_m}.$$

Let $\frac{1}{(g_i(e_i))^{-\frac{1}{k}}} = x_i(e_i)$ and $\frac{1}{(h_i'(1))^{-\frac{1}{k}}} = a_i$ and $f_i(e_i) = a_i * x_i(e_i)$, then

$$a_1 x_1(e_1)\frac{1}{p_1} = a_2 x_2(e_2)\frac{1}{p_2} = ... = a_m x_m(e_m)\frac{1}{p_m}.$$

Then,

$$p_2 = \frac{a_2 x_2(e_2)}{a_1 x_1(e_1)} p_1;$$

$$p_3 = \frac{a_3 x_3(e_3)}{a_1 x_1(e_1)} p_1;$$

...

$$p_m = \frac{a_m x_m(e_m)}{a_1 x_1(e_1)} p_1.$$

Then substituting them into (1.10):

$$1 = \sum_{i=1}^{n} p_i = p_1 + \frac{a_2 x_2(e_2)}{a_1 x_1(e_1)} p_1 + \frac{a_3 x_3(e_3)}{a_1 x_1(e_1)} p_1 + ... + \frac{a_m x_m(e_m)}{a_1 x_1(e_1)} p_1;$$

$$1 = p_1(1 + \frac{a_2 x_2(e_2) + a_3 x_3(e_3) + ... + a_m x_m(e_m)}{a_1 x_1(e_1)}) = p_1(\frac{\sum_{i=1}^{n} a_i x_i(e_i)}{a_1 x_1(e_1)});$$

$$p_1 = \frac{a_1 x_1(e_1)}{\sum_{i=1}^{n} a_i x_i(e_i)} = \frac{f_1(e_1)}{\sum_{i=1}^{n} f_i(e_i)}.$$

The results related to other contestants are similar. Thus, as we showed, a utility function where probabilities enter nonlinearly leads to the RF CSF under the assumption that $h_i'(p_i)$ is homogeneous of degree $-k$.

Properties

Let us consider the properties of the mentioned functions:

Since $h'()$ is homogeneous of degree $-k$:

$$h_i'(p_i) = p_i^{-k} h_i'(1);$$

then,

$$h_i''(p_i) = -k(p_i)^{-k-1} h_i'(1). \tag{1.11}$$

Since p_i is in the range from 0 to 1 $\forall p_i$ it follows that

$$p_i^{-k-1} > 0. \tag{1.12}$$

By (1.11) and (1.12), $h''()$ has the opposite sign than $k * h_i'(1)$:

$$sign(h_i''(p_i)) = sign(h_i''(1-p_i)) = sign(h_i''()) = -sign(k * h_i'(1)). \tag{1.13}$$

So all h_i'' have the same sign.

By the definition of a_i:

$$sign(a_i) = sign(h_i'(1)^{1/k}) \tag{1.14}$$

Let us check properties of $f()$ function:

$$x_i(e_i) = (g_i(e_i))^{1/k};$$

$$x_i'(e_i) = \frac{1}{k}(g_i(e_i))^{\frac{1}{k}-1} g_i'(e_i).$$

The numerator of the RF CSF is increasing in e_i, if $f_i'(e_i) = a_i * x_i'(e_i) > 0$. It is satisfied, when:

$$f_i'(e_i) = a_i x_i'(e_i) = \frac{1}{k} a_i (g_i(e_i))^{\frac{1}{k}-1} g_i'(e_i) > 0. \tag{1.15}$$

The solution (1.7), which is a standard Tullock CSF, is a solution to the maximization problem if the following SOC is satisfied:

$$\frac{\partial^2 u}{\partial p_1^2} = g_i(e_i) * h_i''(p_1) + g_i(e_i) * h_i''(1 - p_1) < 0. \tag{1.16}$$

There are two possible cases:

Case 1: If $g_i() > 0$, then by (1.16), $h_i''() < 0$. In other words, $h_i()$ is concave. Thus, by (1.13) and by (1.14), $sign(k) = sign(h_i'(1)) = sign(a_i)$. Hence, $\frac{1}{k} a_i > 0$. In this case, equation (1.15) is satisfied if $g_i'() > 0$. This case describes a situation in which efforts of contestants increase the principal's utility. This utility is maximized by the RF CSF. An example of this case is a manager distributing equipment as discussed later in the discussion part.

Case 2: If assumption that $g_i() > 0$ is violated, specifically if $g_i() < 0$, then by (1.16),

$h''_i() > 0$, or in other words, $h_i()$ is convex. Then, by (1.13) and by (1.14), $sing(k) = -sign(a_i)$. Hence, $\frac{1}{k}a_i < 0$. In this case, equation (1.15) is satisfied if $g'_i() < 0$. This case describes a situation in which contestants' efforts decrease principal's utility function. Thus, the RF CSF is an allocation rule that minimizes the Principal's disutility. An example of this case is a fearful bribe-taker as discussed later in the discussion part.

The following proposition summarizes the results of this subsection.

Proposition 3 *A sufficient condition of the principal's monotonic utility function to produce a strongly credible RF CSF: for the game G in the following maximization problem*

$$\begin{aligned} &\underset{p_1,p_2,\ldots,p_n}{\textit{maximize}} && U_{\textit{Principal}} = \sum_{i=1}^{n} g_i(e_i) * h_i(p_i) \\ &\textit{subject to} && \sum_{i=1}^{n} p_i = 1,\ p_i \geq 0, \end{aligned} \tag{1.17}$$

functions $h_i(p_i)$ should be homogeneous of the same degree $-k \neq 0$.

1.3.3 Non-monotonic utility function

In this subsection we provide a back engineering approach of deriving one possible family of non-monotonic utility functions for the principal from the RF CSF. It is very important to note that for this derivation we do not assume that the principal considers a constraint $\sum p_i = 1$ in her maximization problem. This approach suggests a utility function with a bliss point, where by construction the bliss point satisfies the aforementioned constraint.

Derivation

Since the standard RF CSF is the result of an optimization problem, we can integrate it in order to obtain a possible initial optimization problem. We focus on additive separable utility functions:

Assumption 1 *The principal's utility function is additive separable in p_i.*

The RF CSF can be rearranged as follows:

$$p_i = \frac{f(e_i)}{\sum_{j=1}^{n} f(e_j)}, \quad \text{with} \quad f(0) = 0;$$

$$f(e_i) - p_i \sum_{j=1}^{n} f(e_j) = 0.$$

If this expression is a FOC with respect to p_i of an unconstrained maximization problem, then

$$\frac{\partial U}{\partial p_i} = f(e_i) - p_i \sum_{j=1}^{n} f(e_j) = 0,$$

given the Assumption 1 initial utility function could be derived by summing up parts obtained by integrating FOCs with respect to each variable p_i[5]. Integrating FOC with respect to p_i yields:

$$U^i_{Principal} = p_i f(e_i) - \frac{1}{2} p_i^2 \sum_{j=1}^{n} f(e_j) + c_i.$$

Again, this formula is the part of the principal's utility function related to the contestant i.

[5]Remember, that throughout the paper effort levels e_i are fixed.

The complete utility function is obtained when summing up across all contestants i. We obtain:

$$U_{Principal} = \sum_{i=1}^{n} p_i f(e_i) - \frac{1}{2}(\sum_{i=1}^{n} p_i^2)(\sum_{i=1}^{n} f(e_i)) + \sum_{i=1}^{n} c_i. \tag{1.18}$$

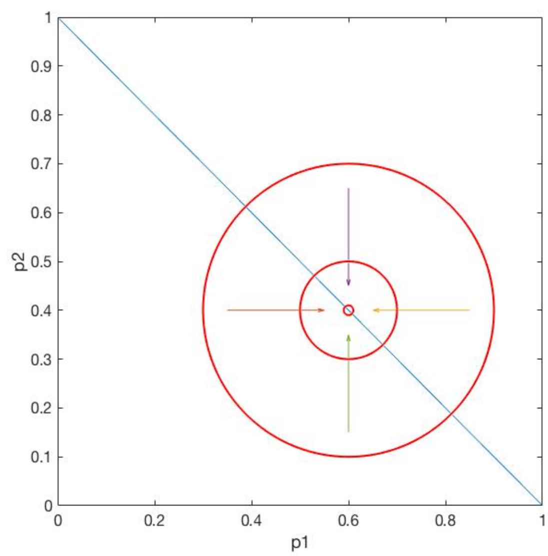

Figure 1.1: Indifference Curves, a Bliss Point and an Imaginary Budget Line of the Principal

For the case of two contestants, the optimization problem of the principal is depicted in Figure 1.1. The optimal point is a result of the unconstrained maximization problem of the principal, which represents preferences with a bliss point. In this problem, indifference curves are circles; the closer a point to the bliss point, the higher is the utility level. Note that the "budget line" is the line, where the sum of shares is equal to 1. Notice that the bliss point is on the "budget line" by construction as we reverse engineer an allocation that is assumed to be optimal.

Properties

The utility function obtained by the reverse engineering approach is concave (with a clear peak). It represents convex preferences with a bliss point. Notice U has a unique maximum if $e_i > 0$ for at least one contestant. Notice preferences here are not monotonic in p. An increase in p_i up to the peak, increases the utility. After the peak, an increase in p_i decreases utility. These are not a typical preferences, because often the utility function is assumed to be monotonic. However, there are some real life situations, where such preferences can be observed. For example, expression (1.18) may describe a quotas allocation between producers by a government. Producers submit bids that correlate with their efficiencies or technologies. The government prefers to allocate larger quotas to more efficient producers, thus $\sum_{i=1}^{n} p_i f(e_i)$ enters into its utility function. In addition, the government dislikes industry concentration, so it subtracts a Herfindahl index $(\sum_{i=1}^{n} p_i^2)$ multiplied by some parameter $\frac{1}{2}(\sum_{i=1}^{n} f(e_i))$. This parameter includes efforts to guarantee that scaling up efforts will not diminish the significance of the concentration term in the government's utility function. In such a setting the RF CSF is credible and efficient allocation rule of such quotas.

Another example for such preferences could refer to an international aid agency (referred to as donor). Aid recipient countries move first by exerting efforts towards reforms, e_i, which produce some results for the economy or society measured by function $f(e_i)$. After observing these results, the donor distributes aid in the form of shares of its aid budget, p_i. The donor tries to allocate aid where aid effectiveness is high, but the donor also tries to avoid aid concentration (for political, humanitarian or other reasons). So, the donor's utility function is a weighted sum of aid effectiveness and inequality aversion:

$$U_{\text{Donor}} = Aid_{\text{Effectiveness}} - \theta * Aid_{\text{Inequality}}. \tag{1.19}$$

Aid effectiveness depends on reforms — it was shown in the literature, that aid combined with reforms is more effective (Burnside and Dollar, 2000). A simple way to capture this is to multiply aid with reform outcomes:

$$Aid_{\text{Effectiveness}} = \sum_{i=1}^{n} p_i f(e_i). \tag{1.20}$$

Aid inequality can be measured by using the variance:

$$\begin{aligned} Aid_{\text{Inequality}} &= \frac{1}{n}\sum_{i=1}^{n}(p_i - \mu)^2 = \frac{1}{n}\sum_{i=1}^{n}(p_i - \frac{1}{n})^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}(p_i^2 - \frac{2}{n}p_i + \frac{1}{n^2}) \\ &= \frac{1}{n}\sum_{i=1}^{n}p_i^2 - \frac{2}{n^2}\sum_{i=1}^{n}p_i + \frac{1}{n^3}\sum_{i=1}^{n}1 \\ &= \frac{1}{n}\sum_{i=1}^{n}p_i^2 - \frac{1}{n^2}. \end{aligned} \tag{1.21}$$

Substitution of (1.20) and (1.21) into (1.19) results in:

$$U_{Donor} = \sum_{i=1}^{n} p_i f(e_i) - \theta * (\frac{1}{n}\sum_{i=1}^{n} p_i^2 - \frac{1}{n^2}) = \sum_{i=1}^{n} p_i f(e_i) - \frac{\theta}{n}\sum_{i=1}^{n} p_i^2 + \frac{\theta}{n^2}. \tag{1.22}$$

For following the values of parameters, equation (1.22) is identical to equation (1.18):

$$\begin{aligned}\theta &= \frac{n}{2}\sum_{i=1}^{n} f(e_i),\\ \sum_{i=1}^{n} c_i &= \frac{1}{2n} f(e_i).\end{aligned}$$

Substituting θ and $\sum_{i=1}^{n} c_i$ into (1.18) will result in:

$$U_{Donor} = \sum_{i=1}^{n} p_i f(e_i) - \frac{1}{2}(\sum_{i=1}^{n} p_i^2)(\sum_{i=1}^{n} f(e_i)) + \frac{1}{2n} f(e_i).$$

As we have shown, a donor who prefers allocating more aid to governments that do more reforms and dislikes aid inequality can be accurately described by our model. Hence, for such a donor the RF CSF will be the most efficient aid allocation rule and this allocation rule will be strongly credible.

Proposition 4 *For the game G it is sufficient to have a non-monotonic utility function*

$$U_{Principal} = \sum_{i=1}^{n} p_i f(e_i) - \frac{1}{2}(\sum_{i=1}^{n} p_i^2)(\sum_{i=1}^{n} f(e_i)) + \sum_{i=1}^{n} c_i, \tag{1.23}$$

to produce a strongly credible RF CSF.

1.4 Discussion

1.4.1 Comparison of derivation approaches

In the previous chapter we considered two cases of generating the RF CSF as a strongly credible allocation rule: for a monotonic and for a non-monotonic utility functions.

The two approaches produce utility functions that are homogeneous of degree 1 in efficiency functions $f_i(e_i)$ or $g_i(e_i)$. Also, both of principal's utility functions are nonlinear in p. As discussed in the literature review, if U is linear in p as in the expected utility framework, then the only strongly credible allocation rule is the all-pay auction.

In the definition of our game (Definition 4) we assume that a resource is normalized to 1. However, even if the resource in the utility function is not normalized to 1 (as it is in the case where p_i stands for the actual level of the resource), the optimal allocation of the resource between contestants stays the same: just sum of p_i will be equal to the overall level of the resource.

The difference between these two approached is that the monotonic utility function is assumed first, and then it's properties are derived, while the non-monotonic utility function is reverse engineered assuming additive separability in p_i.

1.4.2 Relation of our findings to the literature

Our model to our knowledge is the first attempt to backward engineer a non-monotonic principal's utility function from the RF CSF. Also, our model suggests sufficient conditions of the monotonic utility function to produce RF CSF. This utility function is consistent with the utility function of a principal proposed by Corchon and Dahm (2011) as discussed earlier. Their results were motivated by the Prospect Theory. We, however, show that the RF CSF cannot be motivated by the Prospect Theory. Also, the work of Corchon and Dahm (2011) can be seen as providing an example of a utility function that yields the RF CSF as the unique optimal allocation rule. We, instead, focus on the interpretation of p_i as shares, which is limiting compared to the literature, but our method allows us to find a more general form of the principal's utility function, than the utility function discussed by Corchon and Dahm (2011). Assuming $g_i(e_i) = f_i(e_i)^{1-\alpha}$, $h_i(p_i) = p_i^{\alpha}$, the utility function we propose:

$$U = \sum_{i=1}^{n} g_i(e_i)h_i(p_i),$$

can be simplified to the one proposed by Corchon and Dahm (2011):

$$W(p, e) = \sum_{i=1}^{n} p_i^{\alpha} f_i(e_i)^{1-\alpha}.$$

Also, our approach reveals that the RF CSF can be motivated by the utility function with a bliss point (1.18):

$$U_{Principal} = \sum_{i=1}^{n} p_i f(e_i) - \frac{1}{2}(\sum_{i=1}^{n} p_i^2)(\sum_{i=1}^{n} f(e_i)) + \sum_{i=1}^{n} c_i.$$

In a nutshell, Corchon and Dahm (2011) found an important relation between a social planner's utility function and the RF CSF in a situation where the planner cannot commit to a allocation rule. In our paper, however, we show that the RF CSF can be motivated by a more general family of monotonic utility functions, and a new family of non-monotonic utility functions.

1.4.3 Drivers of credibility and applications

This subsection provides examples to give intuitions for our findings. We divide them into intrinsic and extrinsic origins of the CSF credibility.

Intrinsic origins

Intrinsic origins of the CSF credibility can have multiple scenarios. The scenario of a central planner's problem provided earlier has an intrinsic origin because the principal intrinsically cares about the well-being of contestants. Contestants have preferences over the resource allocation, thus the principal's utility function is:

$$U_{\text{planner}} = U(u_1(p_1, V_1, e_1), ..., u_i(p_i, V_i, e_i), ..., u_n(p_n, V_n, e_n)). \tag{1.24}$$

Another example of an intrinsic credibility driver could be a reciprocal principal. This principal cares about exerted efforts and feels good in rewarding contestants for their efforts. The higher the effort, the higher is the reward. The difference of this mechanism and the central planner problem is that here the principal does not need to know or assume anything about the contestants' utility functions. She reciprocates for her own pleasure.

$$U_{\text{reciprocate}} = U(e, p). \tag{1.25}$$

For example, this case describes a situation when an international aid agency aims to have the highest use of its limited funds. The government of each country exerts an effort to fight the corruption that leads to an increase in the fund efficiency. Thus, the agency's problem is:

$$\max U_{\text{agency}} = \sum_{i=1}^{n} V * p_i * e_i,$$

with

$$\sum_{i=1}^{n} p_i = 1.$$

Then, the agency maximizes its utility by providing all the funds to the government, which exerted the highest effort.

However, if the agency has some degree of inequality aversion, then it's utility function must reflect decreasing marginal benefits of the aid. It can be reflected by raising the share of the prize to a power lower than 1:

$$\alpha < 1.$$

In this case the international aid agency's utility function is:

$$maxU_{\text{agency}} = \sum_{i=1}^{n} e_i * V * p_i^{\alpha}. \tag{1.26}$$

This utility function will lead to the Tullock CSF:

$$p_i = \frac{e_i^{1/(1-\alpha)}}{\sum\limits_{j=1}^{n} e_j^{1/(1-\alpha)}}. \tag{1.27}$$

Extrinsic origins

Despite many possible scenarios for an intrinsic origin for a credible CSF, there is a possibility that a principal will not have any of them. In this case she cannot strongly credibly commit to any contest. Thus, a contest will no longer be a Subgame Perfect Equilibrium and the contestants will no longer be interested in exerting efforts. However, in a repeated game setting the RF CSF equilibrium may stay as long as both sides, the principal and contestants, play along the equilibrium, and contestants execute a trigger strategy of not providing any bribes if the principal deviates from the promised RF CSF. For non-repeated games, however, adding more periods to the game could create an extrinsic origin for credibility.

Let us consider a principal, who is a fearful bribe-taker. The principal is afraid of retaliation of contestants.[6]. Higher bidders, expect a larger share. The higher the difference from what they expected and what they got may result in higher retaliation efforts by contestants. If, for example, the anger (that affects retaliation) of contestants is measured by a ratio of the effort to the reward, then a loss minimizing principal will choose the Tullock CSF as her optimal allocation rule. Modifications in the retaliation strategies result in modifications of

[6]There is a possibility that the retaliation is not credible as well. We, however, focus on the credibility of principal's actions without questioning the credibility of contestants

the CSF. This mechanism differs from those mentioned above because it requires at least one more stage of the game. We need an additional action for contestants after rewards have been distributed.

$$U_{\text{bribe}} = U(\sum_{i=1}^{n} \frac{e_i}{V * p_i}). \tag{1.28}$$

In the linear case with two contestants:

$$U_{\text{bribe}} = U(-\frac{e_1}{V * p_1} - \frac{e_2}{V * p_2}) = U(-\frac{e_1}{V * p} - \frac{e_2}{V * (1-p)}).$$

FOC:

$$\frac{\partial U_{\text{bribe}}}{\partial p} = U' * (\frac{e_1}{V * p^2} - \frac{e_2}{V * (1-p)^2}) = 0;$$

$$\frac{e_1}{e_2} = \frac{(1-p)^2}{p^2};$$

$$p = \frac{e_1^{1/2}}{e_1^{1/2} + e_2^{1/2}}. \tag{1.29}$$

As an example for such a retaliation strategy consider a voter, who contemplates to vote or volunteer for another politician in the next election. In order to maintain a network of contestants, the bribe-taker provides recognition prizes. This situation could be modeled in a similar way. Thus, it falls under the same category.

Utility function (1.7) depends on the relation between e and p. As we discussed earlier, e and p are also interrelated. But now let us focus on how these two vectors may affect the principal's utility function and why. The motivation of the function $f(e)$, which represents how efforts enter the principal's utility function, can be as follows:

1. Efforts are part of a production function and the principal cares about the output. In this case higher return to scale production functions will result in contests with higher competitiveness.

2. Outputs of the production function nonlinearly affect the principal. For example, ceteris paribus, per unit costs for the delivery of a big order may be lower, which drives the principal to favour larger producers. Another example is a bribe taker who prefers to maintain better relations with higher bribe givers.

In contrast, the function $h(p)$ represents how allocation of the resource affects the principal's utility. This function can have the following justifications:

1. A contested resource is another part of a production function. For example if a production function is the sum of Cobb-Douglas production functions with effort levels and resources as inputs, then the optimal allocation p will be the RF CSF. For example, a government that cares about gross agricultural production is interested in building better roads in more productive regions.

2. A principal has preferences over utilities of contestants. The best example is a utilitarian central planner, where contestants' utility functions contain efforts and resources as described in Proposition 2.

Let us consider an example of a profit maximizing manager, who allocates bonus funds to enhance the productivity after observing employees' efforts. The funds can be either spend on productivity enhancing equipment (for example in the form of access to a super-computer), or on additional training for the employees.

Assuming a Cobb-Douglas production function for every employee that is independent of each other:

$$u_i = e_i^{\alpha} * K_i^{1-\alpha},$$

where e_i is employee's i effort level, and K_i is capital. Then, the manager's utility function is the sum of individual Cobb-Douglas production functions:

$$U_{\text{manager}} = \sum_{i=1}^{n} e_i^{\alpha} * K_i^{1-\alpha}.$$

The Tullock CSF is an optimal allocation of manager's funds as follows.

For the case where the bonus funds are used to increase efficiency of the equipment, the manager's utility equals:

$$U_{\text{manager}} = \sum_{i=1}^{n} e_i^{\alpha} * (K_i * V * p_i)^{1-\alpha}. \tag{1.30}$$

A profit maximizing behaviour will lead to the Tullock CSF:

$$p_i = \frac{e_i}{e_i + e_j}. \tag{1.31}$$

For the case where the bonus funds are used to increase efficiency of the employees, the manager's utility equals:

$$U_{\text{manager}} = \sum_{i=1}^{n} (e_i * V * p_i)^{\alpha} * K_i^{1-\alpha}. \tag{1.32}$$

In this case a profit maximizing behaviour will lead to the following Tullock CSF:

$$p_i = \frac{e_i^{\frac{\alpha}{1-\alpha}}}{e_i^{\frac{\alpha}{1-\alpha}} + e_j^{\frac{\alpha}{1-\alpha}}}. \tag{1.33}$$

Note, that depending on where the bonus funds are allocated, the CSF changes. Depending on the incentive structure of the contestants (which are not modeled here), equilibrium effort levels may differ depending on the RF CSF and thus the fund manager may ultimately be better off allocating the funds through one or the other. For example, if contestants play the Tullock contest, the more competitive the contest, the higher the effort levels in equilibrium. If $\alpha > 0.5$, then the contest is more competitive if the funds are allocated for the training.

Another example of a principal who allocates a resource according to the RF CSF is an investor who constructs a portfolio using a fundamental indexing method. A fundamental indexing method assigns weights to assets in the portfolio based on the weights of metrics such as a book value, free cash flows, dividends, number of employees, salaries, sales, etc.

A simple, but efficient portfolio selection strategy provided by Arnott et al. (2005) assigns the following weights to stocks in the investment portfolio (notations are adjusted to our paper):

$$p_i = \frac{m_i}{\sum_{j=1}^{n} m_j}, \tag{1.34}$$

where p_i is a share of the capital allocated to the stock i, and m_j is some metrics of a company j.

This approach is similar to the optimization problem of the principal analyzed in our paper. Companies exert efforts to earn profit, which attracts investors. Even though it is not possible to observe efforts directly, efforts are realized through the efficiency functions $m_i = f_i(e_i)$, where m_i is sales or number of employees, etc. A principal (investor) observes m_i and allocates her resource (the capital).

Empirically it was shown that, the allocation rule proposed by Arnott et al. (2005) outperforms conventional allocation rules in finance for the period they studied. According to Arnott et al. (2005, p. 86), one dollar invested in a SP500 index and in the market overall in 1964 would result in \$73.98 and \$68.95 respectively by 2004. While the proposed allocation rule based on sales, would result in \$184.95 for the same period. The proposed allocation rule based on employment would result in \$156.83 for the same period.

The allocation rule described in (1.34) was reexamined by Walkshäusl and Lobe (2010) for a sample from 1982 to 2008 for 50 country specific markets and combined global portfolio. Global portfolio and portfolio for 14 out of 50 country markets (including the Toronto Stock Exchange) showed that the allocation rule in (1.34) outperforms conventional capital weighted portfolios with the level of statistical significance of 5 percent or better. This allocation rule did not suffer statistically significant underperformance on any of the analyzed markets. These results illustrate that a resource (portfolio) allocation made by a principal (investor) in accordance to the allocation rule discussed in the paper can create a significant gain in financial markets.

1.4.4 Possible avenues for future research

It could be beneficial to investigate the structure of the matrix of principal's utility function second derivatives with respect to the efforts and the resource shares. We noticed (not presented here) that the monotonic utility function we consider produces a diagonal second derivatives matrix. The reverse engineering approach produces a different matrix, with diagonal elements having a different structure from non diagonal elements. It remains unknown which structures of the second derivative matrices (w.r.t. to efforts and shares) can generate the RF CSF, a difference form CSF or other allocation rules.

This research considers a situation in which there are no institutional rules to enforce the contest and the principal is the ultimate decision maker. However, if there are some rules in place, the principal would incorporate them in the decision making. If expected costs of breaking rules are higher than gains, then the principal would follow the rules. Otherwise, she will break them. Thus our study can be applied to a more realistic situation of a principal operating in a regulated environment. It would be interesting to see how different rules translate into the principal's problem constraints and whether a principal will be able to strongly commit to an allocation rule. Also, it would be interesting to study how different types of constraints affect the form of the CSF.

It is also interesting to look closer into the discussed example related to investments. Despite the convincing performance of such strategy compared to CAPM methods of portfolio allocation, it is not clear whether this utility function is realistic. Thus, it is not clear whether it produces optimal or suboptimal results. Consequently, it would be interesting to see whether an application of a more realistic investor's utility function and the allocation rule discussed in this paper could outperform other portfolio allocation techniques.

1.5 Conclusions

The main purpose of this paper was to investigate the RF CSF resource allocation rule when a principal cannot commit. Credibility issues are very natural to any contest, where a principal distributes a resource after observing efforts of contestants. By being the last mover she may change her mind about the contest and distribute a resource differently.

We introduced a distinction between weak credibility and strong credibility. Under weak credibility a principal may employ a CSF allocation rule of the resource, but one cannot be confident in such rule: even though the CSF is the best response, but it is not necessarily the unique one. Under strong credibility the principal definitely distributes the resource according to the CSF because it is the unique best response. In propositions 1 and 2 we provided theoretical foundations for the principal's preferences to be able to commit to a contest with a strong credibility. Strong credibility comes as a result of her own optimization process. These theoretical foundations have a very general form and do not require differentiability of the principal's utility function. As contestants' efforts (e), change, the effect of resource shares (p) on the principal's utility (U) should change.

We provide an interesting observation: When a bureaucrat cares only about bribes, no CSF is guaranteed. The explanation of the paradox is intuitive: because the bureaucrat cannot credibly commit to the allocation rule, none of them is guaranteed. Another example included an international aid agency (Donor) that is interested in conditional aid and equality. Also, we provided an example of distributing quotas, when the government is interested in efficiency and industry Herfindahl index. All these examples illustrated the utility function that the principal could have in order to have the RF CSF as a credible allocation rule.

Furthermore, we derived sufficient conditions for the principal's utility function to produce the RF CSF as the unique allocation rule for a non-monotonic utility function, as well as for a larger family of monotonic utility functions compared to the literature. In addition, using our findings we provided a microeconomic justification for fundamental indexation investment strategies used by finance practitioners. Finally, we discussed examples of game structure requirements that lead to credible contests.

Chapter 2

Predictive Blends: Fundamental Indexing meets Markowitz

2.1 Introduction

The following analogy will help motivate our argument. Metallurgy teaches us that blending different metals produces alloys with better properties than their pure constituents. Even if new additions represent a very small percentage of the new alloy, its properties can change dramatically. For instance, duralumin, contains less than 6% of additives to 94% aluminium, but these additives dramatically change the properties of otherwise soft aluminium to an aircraft-grade strong alloy. We show that in composing stock portfolios the same phenomenon exists: blending portfolio construction approaches results in "blended" portfolios that outperform the benchmarks that sole-approach portfolios do not beat.

In this paper, we propose an innovative portfolio blending technique, combining the efficient portfolio selection method of Markowitz (1952) that takes into account the covariance structure of portfolio holdings and the fundamental indexing (FI) approach that favours investments with sound economic, financial, and managerial features.

Markowitz (1952) distinguishes between two stages in the portfolio selection process. The first stage is about forming beliefs about future performance. In practice, this often translates into reliance on historical data in estimating future rates of returns and their correlations. The second stage relies on the beliefs formed in the first stage and involves selecting a portfolio. Focusing only on the second stage, Markowitz (1952) introduces the mean-variance optimization (MVO) method for portfolio selection recommending that the choice of appropriate expected return and variance-covariance matrix "...should combine statistical techniques and the judgment of practical men..." (Markowitz, 1952, p.91). The conventional approach often ignores the need to develop appropriate beliefs. As Markowitz emphasizes, it is our responsibility to use "observation and experience" to develop "beliefs about the future performances" (Markowitz, 1952, p.77). While predicting future performance of stocks may be a daunting task, there is strong evidence that fundamental analysis may have some merit (Arnott et al. (2005); Walkshäusl and Lobe (2010); Basu and Forbes (2014)). As discussed in the forecast combination literature (Eklund and Karlsson, 2007; Smith and Wallis, 2009, etc.), we believe that fundamental analysis may improve the out-of-sample performance of MVO portfolios.

In practice, to predict expected returns and estimate correlations, the MVO method relies on past returns. Past correlations predict future correlations much better that past returns predict future returns (Cuthbertson and Nitzsche, 2005, p.158). Moreover, past returns fail to predict future returns in the long-run (Jorion, 1986; Poterba and Summers, 1988). Given the volatile nature of these underlying processes, the MVO method likely produces superior out-of-sample results only for short-term investments. To mitigate this, frequent portfolio rebalancing based on the latest historical data is recommended for consistent superior results, but leads to high portfolio turnover and increased transaction costs. Transaction costs are of particular concern for funds with long-term performance objectives. Thus, in the industry, long-term investments are often based on "the judgment of practical men", rooted in fundamental analysis. In turn, fundamental analysis focuses on financial statements and the economic health of a company in an attempt to evaluate its long-term economic prospects, assessing its future growth, and investment potential.

Taken separately, both the classical MVO and the FI methods have their own limitations: the FI approach ignores the correlation structure of stocks' returns, while the classic MVO method is silent about the firms' fundamentals, which may well be the driving factors of the stocks' future performance. Berger et al. (2013) have also shown empirically that the MVO technique provides some diversification gains. Our blending technique combines the classical MVO method and the FI approach, by bridging the two stages of portfolio construction mentioned in Markowitz (1952). Relying on 29 years of historical data we backtest and analyze out-of-sample performance of our proposed blending method and show that our blended portfolios are superior to conventional benchmarks as well as portfolios based on each method alone. Heteroskedasticity and autocorrelation (HAC) robust inference tests developed by Ledoit and Wolf (2008) show that our technique delivers statistically significantly higher Sharpe ratios than the (value weighted) S&P 500 and the Equally-Weighted S&P 500.

Currently the MVO and the FI literatures are isolated from each other.[1] Each of these literature streams considers stocks through a specific "oculus" described in the next two paragraphs. Up until now stocks have been considered separately through either one of these oculi.

In the first "oculus" considered, the MVO method, the expected returns and the variance-matrix are calculated based on in-sample information. Securities are sorted according to the MVO procedure, by maximizing the expected portfolio return while attaining a specific level of standard deviation. Since the introduction of the MVO by Markowitz (1952), a myriad of methods have been proposed in an attempt to refine this approach and offer superior out-of-sample performance. Among the most noticeable and practical extensions of the MVO method are those that control for outliers. Outliers often result in biased estimates of sample statistics translating in disproportionate portfolio holding weights. Several

[1] The FI approach was first proposed in Arnott et al. (2005) for US data; methodological improvement and empirical evidence can be found in Treynor (2005); Dopfel (2008). Walkshäusl and Lobe (2010) and Basu and Forbes (2014) provide international evidence for the FI approach. Extensions and/or empirical evidence in favour of the MVO approach are too numerous to be listed here, however, for excellent surveys of the literature please refer to Markowitz et al. (2000) and Rubinstein (2002). In a recent paper, Domowitz and Moghe (2018) consider a case where an exogenously pre-chosen "core" portfolio is complemented with other stocks based on the MVO method, without specifying how the "core" portfolio is constructed, and relying on expected returns of the individual components. To the best of our knowledge, no paper considers a portfolio construction strategy that combines the FI and MVO approaches. In our paper, we also propose the blending methodology based on economic conditions without relying on hard-to-predict expected returns of individual components.

prominent robust techniques have been proposed to take this into account. For example, Ledoit and Wolf (2004) introduce a method that shrinks the sample covariance matrix to a well-conditioned parsimonious structure to reduce estimation errors that were shown to bias the classic MVO method. As an alternative to shrinkage methods, limiting portfolio holdings only to long positions, can produce similar results (Jagannathan and Ma, 2003). However, Jagannathan and Ma (2003) note that such methods might lead to poor diversification, with only 20-25 stocks in the portfolio. Thus, to increase diversification and reduce the effect of measurement errors, it is possible to set up an upper bound on weights (e.g., 5-10%)[2]. Since the MVO method suffers from the negative effects caused by measurement errors, outliers and *blindness* to firms' fundamentals (which are our second "oculus"), the performance of the classic MVO method, even with adjustments for outlier effects, often does not exceed market benchmarks such as equally- or capitalization-weighted portfolios in out-of-sample tests[3]. Hence, if the blended approach shows statistically significant results, they cannot be attributed to the MVO part of the technique alone.

We now shift our focus to the other "oculus", the FI approach, pioneered by Arnott et al. (2005). In this approach, firms are ranked based on their fundamentals and securities are allocated proportionally to their overall fundamental scores. The fundamentals might include book value, free cash flow, revenue, sales, dividends, total employment, etc. In a recent paper, Asness et al. (2015) argue that the FI indexing is, basically, systematic

[2]Coincidentally, these weight recommendations are in accord with guidelines of many investment funds that try to avoid excessive dominance of a single security.

[3]The *p*-value for the tangency MVO portfolio vs the Equally-Weighted S&P 500 is 0.543; the *p*-value for the GMV portfolio against the Equally-Weighted S&P 500 is 0.098. We show *p*-values of all portfolios against the benchmarks in Table 2.2.

value investing. The FI approach significantly outperforms major benchmarks based on US market data (Arnott et al., 2005). Walkshäusl and Lobe (2010) apply the FI approach to stocks from 50 countries and find that the FI approach outperforms capitalization-weighted portfolios in most countries. However, after applying the robust-to-fat-tails performance test proposed by Ledoit and Wolf (2008), the FI portfolios in only 6 countries and the global FI portfolio have statistically significant positive differences in Sharpe ratios. Our empirical results confirm that in the US, the FI portfolio outperforms the cap-weighted portfolio, but these results are not statistically significant[4]. Hence, if the blended approach shows statistically significant results in our US-based study, they cannot be attributed to the FI part of the technique alone.

Out of all portfolios constructed with the MVO method, the richest information about the correlation structure is contained in the Global Minimum Variance (GMV) portfolio, which is based solely on the variance-covariance matrix and achieves the highest level of diversification. More importantly, construction of the GMV portfolio does not rely on often noisy estimates of individual expected returns, which makes it the portfolio of choice in blending with the FI portfolio. Firms' fundamentals help us detect and concentrate on 'healthy' stocks that are likely to grow in the long-run, while the assessment of the correlation structure allows us to construct well-diversified portfolios.

Before we discus the "how" in our next section, one question remains: In what proportion do we combine the GMV and FI portfolios? Given that the FI approach is relatively

[4]The *p*-value for the difference in Sharpe ratios of FI portfolio vs the Equally-Weighted S&P 500 is 0.1156, which is not statistically significant at conventional levels.

new, and is profoundly different from the MVO method, these two approaches have not yet been combined, even though each method offers distinctive benefits for portfolio choice problems. In fact, Hong and Wu (2016) show empirically that information on past returns and on the firms' fundamentals are complementary. They show that in "good times", when volatility is low, past returns provide better information about future returns. However, fundamentals perform better in "bad times", when volatility in the market is high. In such periods, past returns are not that informative and investors are forced to rely on firms' fundamentals. Thus, a portfolio allocation strategy should rely more on past returns (the GMV portfolio) in times of low volatility and rely more on the firms' fundamentals (the FI portfolio) in times of high volatility. It is a daunting task to predict "good" and "bad" times. We, however, use a metric often mentioned by Warren Buffett as a lead indicator of a stock market "bubble" - the market capitalization to nominal GDP ratio.[5] This approach is in the same spirit as Shiller's cyclically adjusted price-to-earnings (CAPE) ratio (Campbell and Shiller, 1988), where earnings per share are averaged over a long period. When this ratio indicates overpricing, and the likelihood of "bad times" is higher, we tilt the blend of our portfolio closer to the FI and away from the GMV portfolio. We discuss this in more detail in the methodology section.

The rest of the paper is organized as follows. We introduce the method of blended portfolios in Section 2.2. We summarize our data and empirical findings in Sections 2.3 and 2.4. Finally, Section 2.5 concludes.

[5] We use nominal GDP since we employ nominal market capitalization.

2.2 Methodology

The FI and the GMV portfolios are depicted in Figure 2.1, which illustrates our proposed technique of blending these two portfolios into one. First, the FI portfolio is constructed based on firms' fundamentals using the FI approach. Second, the GMV portfolio is identified on the mean-variance portfolio frontier. We construct 101 blended combinations (in one percent increments) of these two portfolios, which generate the new, blended GMV/FI mean-variance frontier (in red). On the blended GMV/FI portfolio frontier, we select a portfolio depending on prediction of stock market correction (captured by the Buffett Indicator Index, which is discussed in more detail in Subsection 2.2.3). This Predictive Blended (PB) portfolio is the final outcome of our blended GMV/FI technique. It is the performance of this portfolio that we compare to our benchmarks, the S&P 500 index and the S&P 500 Equally-Weighted index. Next, we describe several desirable features of our proposed technique.

First, the two initial portfolios are formed using profoundly different methods, that should result in better performance of the combined model. Since we are concerned with out-of-sample performance of our portfolios in mean-variance space, our blended approach is inspired by methods proposed in the forecast combination literature. Models with combined forecasts have been shown to outperform individual forecasts (Bates and Granger, 1969; Ericsson, 2017).[6]

Second, since portfolios constructed based on the classic Markowitz MVO (e.g., GMV)

[6] For an excellent survey of the literature, see Hamilton (1994).

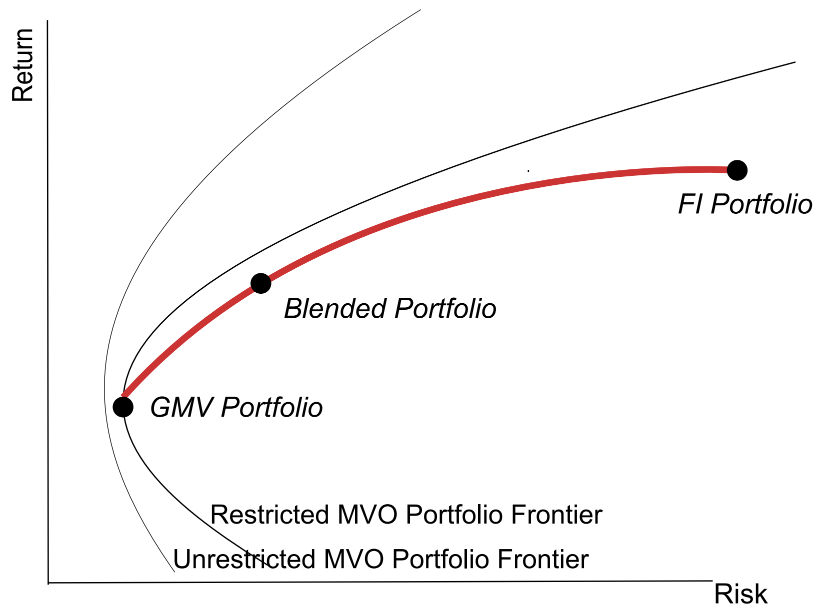

Figure 2.1: Bridging MVO and FI approaches.
The figure illustrates hypothetical unrestricted and restricted minimum variance sets (MVS) based on Markowitz mean–variance optimization, incorporating short-sale and no short-sale constraints, respectively. The FI portfolios are constructed with long positions only, thus appearing in the interior of the restricted MVS. Typically, construction of the GMV and FI portfolios result in conceptually different asset allocation which allows for nontrivial correlation, and results in the MVS being located between these two portfolios, as depicted by the bold red curve.

and FI approaches (e.g., Arnott FI), are most likely not perfectly correlated, the mean-variance optimal frontier (red curve in Figure 2.1) will not result in a straight line. This "second-stage" (blended GMV/FI) mean-variance frontier offers further refinement combining the weights of the GMV and the FI portfolios proportionally as in Figure 2.1. Since the FI portfolio brings additional forward-looking information which was not included in the estimated mean-variance frontier, the new blended portfolio may generate a frontier that outperforms the MVO efficient frontier in out-of-sample tests.

Third, construction of the GMV and FI portfolios does not depend on individual stocks' expected returns, which, as we mentioned earlier, is a major source of error in portfolio optimization problems. Blending the GMV and FI together also does not depend on their expected returns. Instead, we employ the Buffett Indicator Index discussed below.

2.2.1 Construction of the Global Minimum Variance (GMV) portfolio

The GMV portfolio carries the most information about the diversification structure. In general, it is obtained from the optimization problem:

$$w^{GMV} = \arg\min_{w} w'\Omega w \qquad \text{s.t.} \qquad w'e = 1, \tag{2.1}$$

where, Ω is the $N \times N$ variance-covariance matrix of stocks' returns, N is the number of assets, e is the $N \times 1$ column vector of ones, and w is the $N \times 1$ vector of weights, w^{GMV} is a vector of individual asset weights in the GMV portfolio.

Note, that we calculate weight-restricted portfolios, with no short sales and a maximum weight of 10%. The restricted GMV portfolio is obtained by solving the optimization problem (2.1) with the added constraint of $0 \leq w \leq 0.1$.

2.2.2 Construction of the Fundamental Indexing (FI) portfolio

Previous literature (Arnott et al., 2005; Walkshäusl and Lobe, 2010) considers fundamental indexes based on a single metric or an average of a number of fundamental factors.

A single metric fundamental index can be calculated as:[7]

$$FI_i^X = \frac{max\{0, X_i\}}{\sum_{j=1}^{n} max\{0, X_j\}}, \tag{2.2}$$

where X_i is a numeric value for the considered fundamentals for stock i, e.g., book value *(BV),* dividends paid *(D),* free cash flows *(FCF),* revenues *(REV),* among others.[8] We side with Arnott et al. (2005)'s composite approach in constructing our FI portfolios as follows:

$$FI_i^{COMP} = \begin{cases} \frac{1}{4}(FI_i^{BV} + FI_i^{D} + FI_i^{FCF} + FI_i^{REV}), & \text{in the presence of dividends for } i; \\ \frac{1}{3}(FI_i^{BV} + FI_i^{FCF} + FI_i^{REV}), & \text{otherwise.} \end{cases} \tag{2.3}$$

Then, the weights in the FI portfolio are normalized values of the fundamental index constructed above:

$$w_i^{FI} = \frac{FI_i}{\sum_{j=1}^{n} FI_j}. \tag{2.4}$$

Similarly to Arnott et al. (2005), we use book value for the preceding fiscal year, and trailing five-year averages of free cash flows, revenues and dividends. Combined with equation (2.2), equation (2.4) ensures no-short sales, full investment and under-weighting

[7] The use of $max()$ in Equation 2.2 ensures no short sales in the FI portfolios.

[8] Other fundamentals might include employment, income, sales (see Arnott et al., 2005; Basu and Forbes, 2014). However, evidence on outperformance of these alternative FI portfolios relative to the originally proposed baseline, FI by Arnott et al. (2005), is mixed.

of stocks with non-positive fundamentals.

Arnott's portfolio consists of 1000 stocks; in Walkshäusl and Lobe (2010) portfolio sizes vary. To make sure that the performance of our blending method compared with the S&P 500 is not driven by mid- or small-cap stocks, we include only the top 500 stocks ranked by their market capitalization.[9]

Arnott et al. (2005) rebalance portfolios on January 1st. Since the fundamentals of the preceding fiscal year might be unavailable by January 1st, we follow the Walkshäusl and Lobe (2010) methodology to rebalance portfolios on July 1st, using the data for the preceding fiscal year.

2.2.3 Construction of Predictive Blended Portfolios

We define our blended portfolios as the portfolios based on the two risky assets - the GMV and FI portfolios. We consider 101 combinations of GMV and FI portfolios: (0% FI & 100% GMV), (1% FI &99% GMV), ... , (100% FI & 0% GMV).

Our in-sample results suggest that the optimal blend depends on whether or not financial markets are in turmoil. To avoid look-ahead bias but incorporate this feature, as a proxy for a looming crisis, we use a metric often mentioned by Warren Buffett: Total Market Capitalization divided by GDP. Buffett and Loomis (2001, p.93) argue that this is "the best single measure of where valuations stand at any given moment". We will refer to this ratio

[9] Although the list of top-500 stocks by market capitalization is not identical to the list of the S&P 500, it mimics it closely.

as the Buffett Indicator (*BI*):

$$BI_t = \frac{\text{Wilshire } 5000_t}{\sum_{\tau=t-4}^{t} GDP_\tau / 5} \ , \tag{2.5}$$

where, the Wilshire 5000 is a market capitalization-weighted index of the market value of all stocks actively traded in the US (the actual number of stocks in the index may vary), and GDP is annualized US nominal GDP in the last five years. Similar to recent literature we favour GDP over GNP.[10] Nominal GDP is chosen because the Wilshire 5000 is also nominal. The Wilshire 5000 is highly correlated with the S&P 500 but more commonly used in the literature for calculating Market Capitalization-to-GDP ratio.

To adjust BI for cycles, in the spirit of Campbell and Shiller (1988), we test the BI ratio, taking ten-, five-, and one- year US GDP. The time horizon for the GDP average in BI calculations does not play a crucial role, producing similar results. Thus, we take the average GDP over a time span of five years.

According to Warren Buffett, chances of market crash increase if the market valuation is too high. He, however, does not provide any specific guidelines on which market valuation is too high or too low. For our purpose, we need some Buffett Indicator Index (BII, proposed later) to make a judgment on whether the Buffett Indicator is too high. Taking absolute values of the BI might not be universal enough as in various countries BI values

[10] The appropriateness of GDP vs GNP in equation (2.5) is contentious. Some implementations with GDP can be found in the World Bank and World Federation of Exchanges databases as well as among the Corporate Finance Institute (CFI)'s resources at `https://corporatefinanceinstitute.com/resources/knowledge/valuation/market-cap-to-gdp-buffett-indicator/`.

differ dramatically (995% for Hong Kong, vs. 10% for Egypt in 2016 [11]). Instead, for the BII we suggest to use a relative value - to compare the BI to its historical values. Comparing the BI for the long run historical data might not account for permanent changes in market structure, particularly as more companies become public, the Buffett Index has an upward sloping trend. Thus, for the BII we suggest comparing the current BI value to its recent values. Since we are blending two portfolios (described below), with no short sales, we restrict the BII to be between 0 and 1. In accord with aforementioned reasoning we propose to use the following formula for the Buffett Indicator Index:[12]

$$BII_t = \frac{BI_t - \min \{BI_\tau\}_{\tau=t-4}^{t}}{\max \{BI_\tau\}_{\tau=t-4}^{t} - \min \{BI_\tau\}_{\tau=t-4}^{t}} * 100\% \tag{2.6}$$

This index is confined between 0% and 100%. We choose a five-year window to calculate the BII, so as to maintain consistency within our analysis (as most of the variables are analyzed for this period), as well as to mimic political cycles. Table 2.9 provides robustness results for the cases when number of years to calculate average GDP and BII varies, this table shows that results are similar to our suggested framework discussed later. The BII is defined for a situation when the market is not flat, meaning that some changes happened in the market valuations during the time window. In the unlikely event, when market valuations did not change, the BII is not defined: maximum equals minimum, thus both numerator and denominator in Equation 2.6 are zeros. In our time period, as well as for

[11]Information on other countries could be found at www.theglobaleconomy.com/rankings/Stock-market-capitalization.

[12]Note, this formula is similar to the Dimension Index (attainment levels) in the Human Development Index (Sen, 1994, p.8)

the all time the US stock market exists, the five year minimum was never equal the five year maximum. Furthermore, in the unlikely case when GDP does not change, but market valuations do change, Equation 2.6 can be simplified to:

$$BII_t = \frac{\text{Wilshire 5000}_t - \min\left\{\text{Wilshire 5000}\right\}_{\tau=t-4}^{t}}{\max\left\{\text{Wilshire 5000}\right\}_{\tau=t-4}^{t} - \min\left\{\text{Wilshire 5000}\right\}_{\tau=t-4}^{t}} * 100\% \tag{2.7}$$

However, for the period considered, US GDP varies, so we use Equation 2.6 to calculate BII.

We propose[13] to choose the optimal blend proportionally to BII:

$$w_t^{PB} = BII_t * w_t^{FI} + (1 - BII_t) * w_t^{GMV}, \tag{2.8}$$

However, in this paper we round the exact value of BII to the nearest percentage point to improve calculation speed, obtaining:

$$w_t^{PB} = \alpha_t w_t^{FI} + (1 - \alpha_t)\, w_t^{GMV}, \tag{2.9}$$

where $\alpha = \underset{\alpha}{argmin}|BII - \alpha|$ is the proportion of the FI portfolio in a blended portfolio strategy.

When the market is likely to be undervalued, and the likelihood of growth increases, it

[13] We focus on the linear relation between BII and the optimal blending proportion. In our future research we will consider alternatives for $\alpha = f\,(BII)$, e.g., sigmoid functions for $f()$ as a smoothing alternative.

is prudent to invest in a well-diversified portfolio, best captured by the GMV portfolio. If the current BI is at its lowest point ($BII = 0\%$), we suggest that an investor should invest fully in the GMV portfolio.

When the market is likely to be overvalued, and the likelihood of a market crash increases, it is prudent to invest based on the economic footprint of companies, which is best captured by the FI portfolio. If the current BI is at its highest point ($BII = 100\%$), we suggest that an investor should invest fully in the FI portfolio.

When the market is neither undervalued nor overvalued, the likelihood of a crash or expected boom are unclear. This situation is somewhere between the two extremes, expected crash or expected boom. Thus, a blended portfolio constructed from the GMV and FI should be proportional to how close to either extremes the market happens to be.

For example, on July 3rd, 2017 [14] the BI metric was 141%; in the preceding five years the minimum BI was 109%, the maximum BI was 141%, thus according to equation 2.6, the Buffett Indicator Index is equal to 100%. In such a case, we argue that the PB portfolio should be the 100% FI portfolio.

Another widely used market timing tool is the gap between the yields of long-term and short-term rates. There is some empirical evidence (Harvey, 1989; Ang et al., 2006; Chinn and Kucko, 2015), that when the yield curve inverts an economic crisis is about to unfold. We follow Harvey (1989) methodology and use the difference between long-term (US Treasury Benchmark 10 Year Bonds) and short-term rates (US 3 months T-Bill Rates).

[14] Since scheduled rebalancing day July 1st, 2017 was a Saturday, the actual rebalancing day was the first following trading day, Monday July 3rd, 2017

Using this predictor we argue that when the gap between these two rates is less than one percentage point, a crisis is on the horizon and we should be switching from the GMV to the FI portfolio. On the other hand when the gap is one percentage point or more we recommend switching from the GMV to the FI portfolio.

In this section, we analyzed stocks in-sample and constructed the GMV, FI and PB portfolios out-of-sample. Before we perform the empirical investigation of our technique in Section 2.4, we describe our data and data preparation procedures in the following section.

2.3 Data Description and Preparation

2.3.1 Data Description

Our investable universe consists of the S&P 500 constituents listed on the NYSE, NASDAQ and AMEX from January 1990 to January 2018. To avoid survivorship bias we include delisted stocks in our analysis (see Brown et al., 1992). We obtain daily market values (MV) and return indices (RI), which are price index plus dividend disbursements. We collect annual data on book values (BV), dividends (Div), free cash flows (FCF) and revenues (Rev). We also consider the Wilshire 5000[15] (daily) and nominal GDP (annual) data from 1971 to 2018 to construct the Buffett Indicator. These data are sourced from Thomson Reuters Datastream.

To test our approach we construct 22 trailing sub-samples of six years each: five years

[15]The Wilshire 5000 is a market capitalization index.

Table 2.1: Descriptive Statistics for the period from July 1, 1990 to July 1, 2017. All values are in billions of USD.

\$bn	Mean	StDev	5%	50%	95%	Skew	Kurt
Market Value (MV)	10.20	29.16	0.08	2.36	41.42	8.00	97.91
Book Value (BV)	3.99	12.93	0.03	0.97	15.30	10.28	146.70
Total Dividends (Div)	0.20	0.77	0.00	0.02	0.84	11.34	260.83
Free Cash Flows (FCF)	0.99	3.84	-0.01	0.21	3.97	9.34	256.54
Revenue (Rev)	7.25	20.62	0.06	1.72	29.50	9.59	147.86

are used for estimation (July 1, 1990 - June 30, 1995; July 1, 1991 - June 30, 1996 etc.) with the remaining one year for out-of-sample performance (July 1, 1995 - June 30, 1996; July 1, 1996 - June 30, 1997, etc.). Portfolios are rebalanced on July 1 (or the next available trading day) of every year to ensure availability of fundamental data from previous calendar years. In each in-sample sub-period we select 500 stocks with the highest market values on the date of portfolio construction; these are closely related to our main benchmark, S&P 500.[16] Please see Table 2.1 for descriptive statistics of the data for stocks that are included at least once in our sample (1095 stocks, for the period of 27 years).[17]

[16] We find a high degree of concordance between the market values and free float market capitalization resulting in minimal changes in composition of our universe of 500 stocks.

[17] We do not require normality for the distribution of returns, as we use the Ledoit and Wolf (2008) test to calculate heteroskedasticity and autocorrelation-consistent p-values for statistical significance tests of portfolios' Sharpe ratios.

2.3.2 Data Preparation

Since the total return index (RI) reflects both the price of an asset and any dividend disbursements, we obtain daily stock returns as follows:

$$r_{i,t} = \frac{RI_{i,t} - RI_{i,t-1}}{RI_{i,t-1}} \tag{2.10}$$

Note, that using the simple return formula is essential for accurate aggregation of assets in portfolios, whereas log returns are convenient for time aggregation but result in inaccurate estimates when aggregated across several securities.

Our next section discusses the results of out-of-sample tests on the proposed blended portfolios comparing their performance to common market benchmarks, namely the S&P 500 Index, the Equally-Weighted portfolio comprised of the S&P 500 constituents, the GMV and Arnott's FI portfolios.

2.4 Results

We analyse portfolios when a "no short-sales" constraint is implemented with maximum holding weights of at most 10% of the portfolio at the time of construction. Table 2.2 shows the following central findings of our paper.

The first, and the most important result of this study is that over the period 1995-2017 in out-of-sample tests, the Predictive Blended (PB) portfolio, based on the Buffett Indicator discussed in Section 2.2.3, outperforms in terms of Sharpe ratio scores the Markowitz

Tangency, GMV, Arnott FI portfolios and any fixed blend of the GMV and FI portfolios. In Table 2.2 refer to the second column and the first row: the Sharpe ratio of the PB portfolio is 0.647; this is the highest in the out-of-sample calculation. The PB portfolio is the only portfolio that has a statistically significant outperformance compared to the Equally-Weighted portfolio. Given that all these methods use the same universe of stocks (the S&P 500 constituents lists), the only source of better performance is likely to be a superior methodological approach.

Second, even if the Predictive Blended approach is not applied, blending the GMV and FI portfolios in fixed proportions (for example 25%FI + 75%GMV) produces results stronger (Sharpe ratio is 0.566) than those of the Markowitz's tangency portfolio (0.558), S&P 500 (0.331), Equally-Weighted S&P 500 (0.505), or the FI portfolio (0.446).[18] This confirms the point we made earlier in Section 2.2 that blending portfolios produce better results than the pure Markowitz MVO or Arnott's FI approaches.

Third, even if the Predictive Blended portfolio is based on consistently flawed forecasts the result would not be much different from capital-weighted S&P 500: Table 2.2 shows that even when we consistently choose the worst blend, the since-inception Sharpe ratio is 0.295, compared to 0.331 for the S&P 500. In contrast, in the equally unrealistic case, when our forecasts are consistently right (the best blend), the Sharpe ratio is 0.791, compared to 0.331 for the S&P 500.

Fourth, the Predictive Blended portfolio produces higher since-inception return (13.22%)

[18] However, the GMV portfolio outperforms fixed blends, having a Sharpe ratio of 0.591.

than the GMV (10.84%) and FI (12.01%) portfolios taken separately. The S&P 500 and GMV portfolios have the lowest returns since inception: 10.11% and 10.84% respectively; in contrast, returns on the FI (12.01%), PB (13.22%), Equally-Weighted S&P 500 (13.39%), and Tangency (13.79%) portfolios are similar, but higher. The Predictive Blended portfolio provides marginally lower returns than the Tangency and the Equally-Weighted S&P 500 portfolios, but with the benefit of much lower volatility.

Fifth, the PB portfolio is less volatile (σ =14.33%) than the Tangency (σ =17.62%), FI (σ =18.06%), S&P 500 (σ =18.60%), and Equally-Weighted S&P 500 (σ =18.70%) over the period 1995 - 2017. This property makes the PB portfolio the portfolio of choice for investors with high aversion to volatility, but who still would like to make returns higher than those of the GMV portfolio (with the lowest volatility of σ =11.65%).

Interestingly, the Sharpe ratio in out-of-sample tests of the GMV portfolio (0.591) is superior to the Tangency portfolio (0.558). It may be due to the fact that in out-of-sample-tests the Tangency portfolio moves further inside the Minimum Variance Set than the GMV portfolio. This illustrates the point we made earlier that the GMV portfolio does not suffer as much from estimation errors of its inputs: the covariance structure needed for both of them is more robust than the hard-to-predict expected returns needed only for the Tangency portfolio.

Table 2.3 shows that predictive blended is the best strategy over the long-term, even though, there is a possibility that some other strategy might be better in specific years (see Table 2.3 for year-by-year performance). In fact, the PB strategy has the highest Sharpe

Table 2.2: Out-of-sample Sharpe ratio analysis.

This table outlines results of significance tests for the difference in Sharpe ratios (Sharpe ratios are highlighted in bold) of various portfolios (in rows) against the two benchmarks (in columns 2-3-4-5) for the period from July 1, 1995 to June 30, 2017. We apply the methodology in Ledoit and Wolf (2008) to calculate heteroskedasticity and autocorrelation-consistent (HAC) *p*-values for the difference in Sharpe ratios of two portfolios. (*) and (**) represent the 95% and 99% significance levels, respectively. The out-of-sample Sharpe ratios of our constructed portfolios are, generally, higher than those of the two benchmarks considered. The bottom two rows contain the best and the worst blends of FI and GMV portfolios under unrealistic perfect foresight scenarios, representing the most liberal and conservative thresholds.

	Portfolios:		S&P 500		Eq.-Weighted	
		Sharpe ratio	**0.331**		**0.505**	
			p-values		*p*-value	
			HAC	HAC (pre-whitened)	HAC	HAC (pre-whitened)
Out-of-sample	PB	**0.647**	0.001**	0.000**	0.029*	0.031*
	Tangency	**0.558**	0.058	0.030*	0.543	0.550
	GMV	**0.591**	0.014*	0.008**	0.098	0.099
	25%FI+75%GMV	**0.566**	0.007**	0.002**	0.073	0.075
	50%FI+50%GMV	**0.527**	0.008**	0.001**	0.136	0.140
	75%FI+25%GMV	**0.485**	0.025*	0.002**	0.705	0.710
	FI	**0.446**	0.116	0.019*	0.240	0.247
	Best Blend	**0.791**	0.000**	0.000**	0.001**	0.001**
	Worst Blend	**0.295**	0.853	0.819	0.006**	0.007**

ratios since inception in 16 out of the 22 years we considered (see Table 2.4. Even though the year-by-year Table 2.3. shows that the GMV portfolio outperforms other portfolios in 12 out of 22 years taken separately, it is not a reliable strategy in the long term. For example, during the Asian and the Long Term Capital Management crises in 1998-1999, the GMV portfolio was the only portfolio in our set to show negative returns (the GMV return was -2.11%, while the S&P 500 return was +22.45%, and FI +17.37%) and consequently, Sharpe ratios (for the GMV it was 2.300, while for the S&P 500 Sharpe ratio was 0.796, and for the FI it was 0.686). The following year, in 1999 to 2000, this situation was similar: returns were -1.03% for the GMV vs +8.58% for the S&P 500, and +0.79% for the FI portfolio.[19]

The fixed blends (25% GMV & 75% FI; 50% GMV & 50% FI; 75% GMV & 25% FI) outperformance compared to the benchmarks relies on the stock selection technique for the GMV and FI portfolios: Table 2.2 shows that these fixed blends have statistically significantly higher Sharpe ratios over the S&P 500, but not the Equally-Weighted S&P 500. On the other hand, the Predictive Blended portfolio outperformance over any fixed blend does rely on market timing - the Sharpe ratio for this portfolios is the only Sharpe ratio that is statistically significantly higher than the Equally-Weighted S&P 500 with a p value less than 0.05. Thus, the question of the optimal formula for market timing arises. The Predictive Blended portfolio based on the BII outperforms the Market Timing Tool based on the yield curve gap. The out-of-sample tests using the Yield-Gap MTT are presented in Tables 2.7 and 2.8. The Sharpe ratio since-inception using the Yield-Gap MTT is 0.42

[19]Refer to Table 2.6.

Table 2.3: Out-of-sample Sharpe ratios for portfolios established at the beginning of the period, in 1995, and ending in various years, assuming annual rebalancing on July 1 of each year.
The bottom row represents the number of years a portfolio had the highest Sharpe ratio among the benchmarks considered.

Period		Out-of-sample Sharpe ratios						Perfect foresight	
Start	End	PB	Tangency	GMV	FI	S&P 500	Eq.Weight	Best	Worst
1995	1996	1.899	2.847	**3.123**	1.899	1.718	2.576	3.123	1.899
1995	1997	1.947	2.430	**2.602**	1.947	1.771	2.155	2.598	1.947
1995	1998	1.720	2.096	**2.451**	1.720	1.499	1.725	2.469	1.720
1995	1999	**1.352**	1.281	1.240	**1.352**	1.225	1.222	1.405	1.172
1995	2000	**0.949**	0.904	0.737	**0.949**	0.952	0.904	0.882	0.834
1995	2001	**0.878**	0.695	0.817	**0.878**	0.589	0.877	0.914	0.767
1995	2002	**0.757**	0.416	0.701	0.619	0.286	0.678	0.814	0.482
1995	2003	**0.703**	0.376	0.614	0.501	0.273	0.573	0.738	0.378
1995	2004	**0.780**	0.461	0.741	0.565	0.312	0.661	0.828	0.460
1995	2005	0.831	0.518	**0.840**	0.564	0.313	0.670	0.901	0.468
1995	2006	0.815	0.505	**0.830**	0.557	0.317	0.669	0.886	0.469
1995	2007	**0.866**	0.522	0.857	0.612	0.371	0.707	0.942	0.499
1995	2008	0.662	0.484	**0.665**	0.465	0.255	0.551	0.775	0.370
1995	2009	**0.495**	0.285	0.423	0.266	0.116	0.332	0.544	0.180
1995	2010	**0.531**	0.336	0.474	0.303	0.152	0.379	0.583	0.225
1995	2011	**0.593**	0.437	0.550	0.370	0.226	0.455	0.648	0.300
1995	2012	**0.572**	0.468	0.563	0.351	0.226	0.421	0.654	0.285
1995	2013	**0.625**	0.510	0.598	0.401	0.264	0.468	0.712	0.315
1995	2014	**0.668**	0.577	0.607	0.440	0.308	0.511	0.758	0.327
1995	2015	**0.653**	0.593	0.572	0.433	0.311	0.502	0.737	0.310
1995	2016	**0.622**	0.566	0.607	0.419	0.305	0.484	0.761	0.300
1995	2017	**0.647**	0.558	0.591	0.446	0.331	0.505	0.791	0.295
No. of superior years		16	0	6	3	0	0		

Table 2.4: One-year out-of-sample Sharpe ratios for portfolios established in various periods, starting on July 1 of each year.
The bottom row represents the number of years a portfolio had the highest Sharpe ratio among the benchmarks considered.

Period		Out-of-sample						Perfect foresight	
Start	End	PB	Tangency	GMV	FI	S&P 500	Eq.Weight	Best	Worst
1995	1996	1.899	2.847	**3.123**	1.899	1.718	2.576	3.123	1.899
1996	1997	2.019	2.137	**2.175**	2.019	1.848	1.908	2.263	2.019
1997	1998	1.462	1.689	**2.300**	1.462	1.184	1.248	2.300	1.462
1998	1999	0.686	0.161	-0.659	0.686	**0.796**	0.463	0.686	-0.659
1999	2000	-0.298	-0.037	-0.672	-0.298	**0.125**	0.065	-0.298	-0.672
2000	2001	0.514	0.059	**1.221**	0.514	-0.906	0.758	1.221	0.514
2001	2002	-0.343	-1.412	**-0.033**	-0.764	-1.438	-0.326	-0.033	-0.764
2002	2003	**0.318**	0.166	**0.318**	0.095	0.223	0.230	0.318	0.095
2003	2004	**1.611**	1.474	**1.611**	1.312	0.860	1.539	1.660	1.312
2004	2005	1.611	1.332	**1.761**	0.602	0.372	0.819	1.761	0.602
2005	2006	0.640	0.354	**0.727**	0.500	0.440	0.685	0.727	0.500
2006	2007	**1.674**	0.744	1.249	**1.674**	1.480	1.362	1.674	1.249
2007	2008	-0.907	**0.265**	-0.864	-0.907	-0.918	-0.822	-0.864	-0.920
2008	2009	**-0.273**	-0.623	**-0.273**	-0.439	-0.530	-0.411	-0.273	-0.439
2009	2010	**1.260**	1.023	**1.260**	0.775	0.708	0.953	1.260	0.775
2010	2011	2.074	**2.391**	2.097	1.720	1.864	1.901	2.097	1.720
2011	2012	0.324	**0.887**	0.750	0.117	0.227	0.014	0.750	0.117
2012	2013	1.763	1.421	1.378	**1.766**	1.404	1.764	1.766	1.378
2013	2014	1.884	2.031	0.851	1.884	1.929	**2.042**	1.884	0.851
2014	2015	0.296	**1.077**	-0.233	0.296	0.451	0.298	0.296	-0.233
2015	2016	0.098	-0.021	**1.246**	0.098	0.163	0.060	1.246	0.098
2016	2017	1.668	0.370	0.161	1.758	**1.766**	1.575	1.758	0.161
No. of superior years		5	3	12	2	3	1		

(annualized return of 11.19% and standard deviation of 17.21%), compared to a PB based on the Buffett Index of 0.65 (Table 2.5). Thus, the PB market timing holds merit compared to the Yield-Gap market timing. In this paper, however, we do not focus on comparing different market timing models in light of optimal blending, which will be a question for future research.

Table 2.5 presents out-of-sample year-by-year returns, standard deviations and Sharpe ratios for the six portfolios we study: Global Minimum Variance (GMV), Arnott Fundamental Index (FI), Predictive Blended (PB), Tangency based on the restricted MVO frontier, the S&P 500 index, and Equally-Weighted S&P 500. We note in the bottom row (1995-2017) that in forward tests the PB portfolio outperforms all other portfolios as measured by Sharpe ratios, and the difference in performance is statistically significant at the 99% level compared with the S&P 500, 95% level compared with the Equally-Weighted S&P 500 as our main benchmarks. Taking year-by-year changes in risk-adjusted return performance, the PB underperformed GMV or other portfolios in certain years, but over the long term the PB proved to be the most successful portfolio. Even though the cumulative over-performance of the PB over the GMV portfolio is not statistically significant, cumulative returns of the PB dominate those of the GMV portfolio in 16 out of 22 years, not a bad property for practitioners.

2.5 Conclusion

In this paper, we propose a new portfolio construction technique that combines the benefits of Mean-Variance Optimization (MVO) and Fundamental Indexing (FI). Given that the FI approach is relatively new, and is profoundly different from the MVO, these two approaches have not yet been combined, even though each method offers distinctive benefits for portfolio choice problems. Our paper fills this gap in the literature. Our results attest to the superior performance of the proposed Predictive Blended (PB) portfolio compared to two hard-to-beat benchmarks, the S&P 500 and the Equally-Weighted S&P 500.

Applying the MVO method proposed by Markowitz (1952), we find the portfolio that contains the most information about the variance-covariance structure of stock returns - the Global Minimum Variance portfolio (GMV). Applying the FI method proposed by Arnott et al. (2005), we construct a portfolio from stocks that are in sound financial health. Blending these two portfolios generates a portfolio that has better diversification than the FI portfolio and better risk-adjusted return characteristics than the GMV portfolio. Although, ad-hoc static fixed-proportion blends provide promising results compared to the benchmarks, we find that the dynamic Predictive Blended portfolio is remarkably superior.

We test the out-of-sample performance of the predictive and fixed blends (for example 25% FI and 75% GMV) using 29 years worth of data from S&P 500 companies. The suggested PB approach is the only portfolio that provides statistically significant superior (over the S&P 500 and Equally-Weighted S&P 500 benchmarks) Sharpe ratios in out-of-sample tests. The FI, GMV or classic Markowitz Tangency portfolios taken separately do not have statistically significant Sharpe ratios over the hard-to-beat Equally-Weighted S&P 500 benchmark.

The second major result of our paper is that almost any fixed blend between the GMV and FI portfolios performs better than the S&P 500 (but not necessarily better than the Equally-Weighted benchmark).

Our future research will focus on finding improved FI techniques that would enhance our predictive blended portfolios even further. In particular, within-industry analysis of the FI portfolios could enable portfolio managers to fine-tune prediction metrics and optimal blends during industry-specific crises vs market-wide turmoils. In addition, given the limited number of studies on FI strategies for non-US markets, a comparative study assessing predictive blended portfolios in global markets is worth pursuing.

Appendix

2.5.1 Additional tables

Table 2.5: Annualized Out-of-Sample Portfolio Performance Statistics beginning in 1995 and ending in various years. All portfolios are based on S&P 500 stocks. Portfolios are rebalanced every year on July 1. GMV is the Global Minimum Variance portfolio, FI is Arnott's FI, PB is Predictive Blended portfolio (which blends the GMV and FI), Tangency portfolio is a portfolio tangent to the MVO frontier, S&P 500 is market capitalization-weighted portfolio, while Equally-Weighted is a portfolio, where each stock has equal presence. We estimate annualized average portfolio returns, $\bar{r}$, portfolio standard deviations, $\hat{\sigma}$, and Sharpe ratios, S, for various portfolio strategies and benchmarks. Figures in bold represent the lowest portfolio risk, the highest portfolio return and Sharpe ratio among the portfolio strategies and benchmarks for a given period under consideration. It is not surprising that the GMV portfolio consistently results in the lowest risk, even in out-of-sample. However, the average portfolio return and Sharpe ratio suggest that the PB portfolio exhibits superior out-of-sample performance.

July 1, *yyyy*		GMV			FI			PB			Tangency			S&P 500			Equally-Weighted		
Start	End	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S
1995	1996	21.28	**4.88**	**3.12**	22.79	8.83	1.90	22.79	8.83	1.90	27.18	7.43	2.85	**22.94**	9.85	1.72	24.45	7.15	2.58
1995	1997	20.05	**5.32**	**2.60**	26.57	10.46	1.95	26.57	10.46	1.95	26.93	8.53	2.43	**27.12**	11.81	1.77	24.99	8.71	2.16
1995	1998	20.83	**6.04**	**2.45**	**26.82**	12.09	1.72	**26.82**	12.09	1.72	26.56	9.80	2.10	26.73	13.81	1.50	24.07	10.47	1.72
1995	1999	15.09	**7.53**	1.24	24.45	13.83	**1.35**	24.45	13.83	**1.35**	21.91	12.61	1.28	**25.66**	16.24	1.23	21.38	12.79	1.22
1995	2000	11.85	**8.21**	0.74	19.70	14.64	**0.95**	19.70	14.64	**0.95**	18.57	14.12	0.90	**22.23**	17.25	**0.95**	18.53	14.08	0.90
1995	2001	12.43	**8.21**	0.82	**18.54**	14.61	**0.88**	**18.54**	14.61	**0.88**	16.58	15.63	0.69	16.28	17.92	0.59	18.30	14.35	**0.88**
1995	2002	11.32	**8.16**	0.70	14.81	14.88	0.62	**16.16**	13.94	**0.76**	12.03	15.46	0.42	10.77	18.08	0.29	15.60	14.74	0.68
1995	2003	10.93	**9.03**	**0.61**	13.75	16.71	0.50	**15.16**	13.91	0.70	11.41	16.07	0.38	10.64	19.29	0.27	14.88	16.57	0.57
1995	2004	12.09	**9.23**	0.74	14.42	16.24	0.57	15.85	13.58	**0.78**	12.42	15.59	0.46	11.05	18.61	0.31	**15.99**	16.25	0.66
1995	2005	12.86	**9.20**	**0.84**	14.01	15.75	0.56	**16.09**	13.18	0.83	12.99	15.15	0.52	10.75	17.96	0.31	15.75	15.85	0.67
1995	2006	12.66	**9.13**	**0.83**	13.60	15.31	0.56	**15.54**	12.84	0.82	12.60	14.90	0.51	10.59	17.41	0.32	15.46	15.52	0.67
1995	2007	12.75	**8.99**	0.86	14.17	14.91	0.61	**15.95**	12.59	**0.87**	12.72	14.71	0.52	11.31	16.91	0.37	15.77	15.18	0.71
1995	2008	11.21	**9.39**	**0.67**	12.06	15.27	0.46	**13.71**	13.21	0.66	12.55	15.68	0.48	9.34	17.17	0.25	13.58	15.63	0.55
1995	2009	10.03	**12.27**	0.42	9.97	19.29	0.27	**12.35**	15.19	**0.49**	10.11	18.49	0.28	7.21	20.40	0.12	11.43	19.84	0.33
1995	2010	10.52	**12.19**	0.47	10.62	19.40	0.30	**12.68**	14.94	**0.53**	10.99	18.55	0.34	7.82	20.26	0.15	12.35	20.05	0.38
1995	2011	11.26	**12.02**	0.55	11.72	19.14	0.37	13.33	14.67	**0.59**	12.65	18.33	0.44	9.14	19.93	0.23	**13.64**	19.81	0.45
1995	2012	11.32	**12.13**	0.56	11.31	19.44	0.35	13.02	14.93	**0.57**	**13.17**	18.56	0.47	9.03	20.12	0.23	12.98	20.16	0.42
1995	2013	11.52	**12.00**	0.60	12.00	19.12	0.40	13.58	14.79	**0.63**	**13.71**	18.38	0.51	9.56	19.76	0.26	13.62	19.83	0.47
1995	2014	11.45	**11.85**	0.61	12.49	18.75	0.44	13.99	14.57	**0.67**	**14.76**	18.23	0.58	10.21	19.37	0.31	14.19	19.46	0.51
1995	2015	10.87	**11.75**	0.57	12.15	18.46	0.43	13.57	14.44	**0.65**	**14.81**	17.99	0.59	10.08	19.06	0.31	13.76	19.14	0.50
1995	2016	11.20	**11.80**	0.61	11.74	18.38	0.42	13.09	14.56	**0.62**	**14.18**	17.93	0.57	9.82	18.96	0.30	13.25	19.03	0.48
1995	2017	10.84	**11.65**	0.59	12.01	18.06	0.45	13.22	14.33	**0.65**	**13.79**	17.62	0.56	10.11	18.60	0.33	13.39	18.70	0.50

Table 2.6: Annualized Out-of-Sample Portfolio Performance Statistics of year-by-year portfolios.
GMV is the Global Minimum Variance portfolio, FI is Arnott's FI, PB is Predictive Blended portfolio (which blends the GMV and FI), Tangency portfolio is a portfolio tangent to the MVO frontier, S&P 500 is market capitalization-weighted portfolio, while Equally-Weighted is a portfolio, where each stock has equal presence. We estimate annualized average portfolio returns, $\bar{r}$, portfolio standard deviations, $\hat{\sigma}$, and Sharpe ratios, S, for various portfolio strategies and benchmarks. Figures in bold represent the lowest portfolio risk, the highest portfolio return and Sharpe ratio among the portfolio strategies and benchmarks for a given period under consideration. It is not surprising that GMV portfolio consistently results in the lowest risk, even in out-of-sample. The evidence of out-of-sample superior performance of PB portfolios is less apparent from year-by-year results compared to results in Tables 2.5, suggesting that this strategy is more appropriate for long-term investments. Figures in bold represent the lowest portfolio risk, the highest portfolio return and Sharpe ratio among the portfolio strategies and benchmarks for a given period under consideration.

July 1, *yyyy*		GMV			FI			PB			Tangency			S&P500			Equally-Weighted		
Start	End	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S
1995	1996	21.28	**4.88**	**3.12**	22.79	8.83	1.90	22.79	8.83	1.90	**27.18**	7.43	2.85	22.94	9.85	1.72	24.45	7.15	2.58
1996	1997	18.83	**5.72**	**2.17**	30.33	11.86	2.02	30.33	11.86	2.02	26.68	9.50	2.14	**31.29**	13.48	1.85	25.52	10.03	1.91
1997	1998	22.38	**7.27**	**2.30**	**27.31**	14.81	1.46	**27.31**	14.81	1.46	25.82	11.94	1.69	25.94	17.12	1.18	22.25	13.29	1.25
1998	1999	-2.11	**10.75**	-0.66	17.37	18.06	0.69	17.37	18.06	0.69	7.97	18.62	0.16	**22.45**	21.97	**0.80**	13.32	18.02	0.46
1999	2000	-1.03	**10.45**	-0.67	0.79	17.45	-0.30	0.79	17.45	-0.30	5.28	18.95	-0.04	**8.58**	20.76	**0.12**	7.19	18.32	0.07
2000	2001	15.36	**8.24**	1.22	12.73	14.43	0.51	12.73	14.43	0.51	6.58	21.65	0.06	-13.57	20.84	-0.91	**17.14**	15.62	**0.76**
2001	2002	**4.60**	**7.79**	**-0.03**	-7.65	16.37	-0.76	1.79	8.91	-0.34	-15.36	14.31	-1.41	-22.38	18.93	-1.44	-0.67	16.91	-0.33
2002	2003	8.20	**13.68**	**0.32**	6.35	26.12	0.10	8.20	**13.68**	**0.32**	7.13	19.78	0.17	9.72	26.23	0.22	**9.83**	25.98	0.23
2003	2004	21.30	**10.63**	1.61	19.77	11.88	1.31	21.30	**10.63**	1.61	20.46	11.04	1.47	14.32	11.81	**0.86**	**24.83**	13.42	1.54
2004	2005	**19.87**	8.93	**1.76**	10.31	10.26	0.60	18.28	**8.78**	1.61	18.08	10.47	1.33	8.04	10.49	0.37	13.64	11.60	0.82
2005	2006	10.60	**8.41**	**0.73**	9.50	10.03	0.50	10.09	8.76	0.64	8.74	12.03	0.35	9.03	10.33	0.44	**12.51**	11.71	0.69
2006	2007	13.80	**7.33**	1.25	20.41	9.41	**1.67**	**20.41**	9.41	**1.67**	13.97	12.54	0.74	19.27	9.88	1.48	19.20	10.68	1.36
2007	2008	-7.32	**13.15**	-0.86	-13.18	18.99	-0.91	-13.18	18.99	-0.91	**10.51**	24.45	**0.26**	-14.34	20.03	-0.92	-12.62	20.26	-0.82
2008	2009	**-5.25**	**30.98**	**-0.27**	-17.25	46.59	-0.44	**-5.25**	**30.98**	**-0.27**	-21.59	39.82	-0.62	-20.43	44.61	-0.53	-16.59	48.24	-0.41
2009	2010	17.29	**10.98**	**1.26**	19.72	20.98	0.77	17.29	**10.98**	**1.26**	23.27	19.36	1.02	16.34	18.19	0.71	**25.19**	22.79	0.95
2010	2011	22.36	**9.23**	2.10	28.17	14.62	1.72	23.09	9.68	2.07	**37.57**	14.46	**2.39**	28.98	13.93	1.86	33.13	15.84	1.90
2011	2012	12.33	**13.73**	**0.75**	4.81	23.74	0.12	8.06	18.60	0.32	**21.50**	21.94	0.89	7.24	22.98	0.23	2.39	25.10	0.01
2012	2013	14.87	**9.50**	1.38	23.78	12.46	1.77	23.08	12.08	**1.76**	22.90	14.86	1.42	18.50	11.90	1.40	**24.48**	12.86	**1.76**
2013	2014	10.19	**8.85**	0.85	21.31	9.90	1.88	21.31	9.90	1.88	**33.69**	15.27	2.03	21.98	10.01	1.93	24.44	10.67	**2.04**
2014	2015	-0.04	**9.61**	-0.23	5.61	11.53	0.30	5.61	11.53	0.30	15.83	12.65	**1.08**	**7.55**	11.87	0.45	5.65	11.60	0.30
2015	2016	**17.72**	**12.63**	**1.25**	3.64	16.80	0.10	3.64	16.80	0.10	1.64	16.66	-0.02	4.73	16.83	0.16	3.00	16.75	0.06
2016	2017	3.36	**7.92**	0.16	**17.62**	8.84	1.76	15.98	8.33	1.67	5.45	9.10	0.37	16.25	8.02	**1.77**	16.33	9.05	1.57

Table 2.7: Annualized Out-of-Sample Portfolio Performance Statistics beginning in 1995 - ending in different years to 2017.
First three portfolios are blended portfolios constructed in fixed proportions; the Worst and the Best Blends are constructed given unrealistic perfect foresight scenarios, and the Yield-Gap-Market Timing Tool (Yield-Gap MTT). We estimate annualized average portfolio returns, $\bar{r}$, portfolio standard deviations, $\hat{\sigma}$, and Sharpe ratios, S, for various portfolio strategies and benchmarks.

July 1, *yyyy*		25%FI+75%GMV			50%FI+50%GMV			75%FI+25%GMV			Worst Blend			Best Blend			Yield-Gap-MTT		
Start	End	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S
1995	1996	21.61	5.29	2.95	21.98	6.21	2.57	22.37	7.44	2.20	22.79	8.83	1.90	21.29	4.89	3.12	21.28	4.88	3.12
1995	1997	21.68	6.09	2.54	23.31	7.34	2.33	24.94	8.84	2.12	26.57	10.46	1.95	21.35	5.83	2.60	25.81	9.08	2.16
1995	1998	22.30	7.06	2.31	23.79	8.53	2.08	25.29	10.24	1.88	26.82	12.09	1.72	21.69	6.35	2.47	26.31	11.32	1.79
1995	1999	17.46	8.47	1.38	19.80	9.99	1.41	22.13	11.82	1.39	19.58	11.79	1.17	20.61	10.57	1.40	19.20	11.20	1.20
1995	2000	13.80	9.06	0.88	15.76	10.58	0.94	17.72	12.50	0.95	15.44	11.55	0.83	16.63	12.28	0.88	15.50	12.72	0.76
1995	2001	13.93	9.06	0.91	15.45	10.56	0.92	16.99	12.47	0.90	14.99	12.07	0.77	16.42	11.70	0.91	15.48	12.09	0.81
1995	2002	12.17	9.06	0.73	13.03	10.64	0.70	13.91	12.64	0.66	11.76	12.78	0.48	14.74	11.23	0.81	12.19	12.80	0.51
1995	2003	11.58	10.19	0.61	12.27	12.00	0.57	12.99	14.23	0.54	11.09	15.11	0.38	13.92	11.57	0.74	11.45	15.12	0.40
1995	2004	12.61	10.17	0.72	13.18	11.80	0.67	13.78	13.88	0.61	12.05	14.79	0.46	14.69	11.41	0.83	12.38	14.80	0.48
1995	2005	13.10	10.04	0.79	13.37	11.54	0.71	13.68	13.50	0.63	11.88	14.40	0.47	15.21	11.18	0.90	12.18	14.41	0.49
1995	2006	12.85	9.91	0.78	13.07	11.33	0.71	13.32	13.18	0.63	11.66	14.06	0.47	14.79	10.96	0.89	12.03	13.97	0.5
1995	2007	13.07	9.74	0.82	13.41	11.09	0.75	13.77	12.86	0.68	11.84	13.63	0.50	15.25	10.84	0.94	12.18	13.54	0.53
1995	2008	11.38	10.17	0.63	11.58	11.51	0.57	11.81	13.26	0.52	10.09	13.85	0.37	13.52	11.04	0.77	10.68	13.52	0.42
1995	2009	9.96	13.34	0.38	9.92	14.94	0.34	9.92	16.95	0.30	8.13	18.26	0.18	12.18	13.49	0.54	8.68	18.03	0.21
1995	2010	10.48	13.32	0.43	10.48	14.98	0.38	10.52	17.04	0.34	8.91	18.46	0.23	12.52	13.34	0.58	9.42	18.24	0.26
1995	2011	11.31	13.15	0.51	11.40	14.79	0.46	11.54	16.82	0.41	10.11	18.24	0.30	13.13	13.12	0.65	10.59	18.04	0.33
1995	2012	11.25	13.32	0.51	11.23	15.01	0.45	11.24	17.08	0.40	9.80	18.61	0.29	13.09	13.15	0.65	10.25	18.42	0.31
1995	2013	11.58	13.15	0.55	11.68	14.79	0.50	11.82	16.81	0.44	10.08	18.23	0.32	13.68	13.12	0.71	11.00	18.14	0.37
1995	2014	11.66	12.95	0.57	11.90	14.54	0.53	12.17	16.50	0.48	10.09	17.86	0.33	14.08	12.97	0.76	11.54	17.81	0.41
1995	2015	11.14	12.81	0.55	11.44	14.35	0.51	11.77	16.26	0.47	9.58	17.54	0.31	13.66	12.90	0.74	11.25	17.55	0.40
1995	2016	11.29	12.82	0.57	11.41	14.33	0.51	11.56	16.21	0.46	9.30	17.50	0.30	13.85	12.89	0.76	10.88	17.51	0.39
1995	2017	11.10	12.62	0.57	11.38	14.09	0.53	11.68	15.93	0.48	9.03	17.18	0.30	14.02	12.73	0.79	11.19	17.21	0.42

Table 2.8: Annualized Out-of-Sample Portfolio Performance Statistics Portfolio of year-by-year portfolios.
First three portfolios are blended portfolios constructed in fixed proportions; the Worst and the Best Blends are constructed given unrealistic perfect foresight scenarios, and Yield-Gap-Market Timing Tool (Yield-Gap MTT). We estimate annualized average portfolio returns, $\bar{r}$, portfolio standard deviations, $\hat{\sigma}$, and Sharpe ratios, S, for various portfolio strategies and benchmarks.

July 1, *yyyy*		25%FI+75%GMV			50%FI+50%GMV			75%FI+25%GMV			Worst Blend			Best Blend			Yield-Gap MTT		
Start	End	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S	$\bar{r}$(%)	$\hat{\sigma}$ (%)	S
1995	1996	21.61	5.29	2.95	21.98	6.21	2.57	22.37	7.44	2.20	22.79	8.83	1.90	21.29	4.89	3.12	21.28	4.88	3.12
1996	1997	21.75	6.79	2.26	24.64	8.31	2.20	27.50	10.04	2.10	30.33	11.86	2.02	21.40	6.63	2.26	30.33	11.86	2.02
1997	1998	23.54	8.68	2.06	24.74	10.50	1.82	26.00	12.58	1.62	27.31	14.81	1.46	22.38	7.27	2.30	27.31	14.81	1.46
1998	1999	2.94	11.68	-0.17	7.86	13.42	0.22	12.67	15.62	0.49	-2.11	10.75	-0.66	17.37	18.06	0.69	-2.11	10.75	-0.66
1999	2000	-0.76	11.06	-0.61	-0.36	12.63	-0.50	0.16	14.85	-0.39	-1.03	10.45	-0.67	0.79	17.45	-0.30	0.79	17.45	-0.30
2000	2001	14.60	9.02	1.03	13.91	10.44	0.82	13.29	12.30	0.65	12.73	14.43	0.51	15.36	8.24	1.22	15.36	8.24	1.22
2001	2002	1.55	9.04	-0.37	-1.51	11.07	-0.57	-4.58	13.57	-0.69	-7.65	16.37	-0.76	4.60	7.79	-0.03	-7.65	16.37	-0.76
2002	2003	7.48	15.99	0.23	6.92	18.95	0.16	6.54	22.36	0.12	6.35	26.12	0.10	8.20	13.68	0.32	6.35	26.12	0.10
2003	2004	20.82	10.03	1.66	20.41	10.05	1.61	20.06	10.70	1.48	19.77	11.88	1.31	20.80	10.02	1.66	19.77	11.88	1.31
2004	2005	17.52	8.75	1.53	15.15	8.92	1.23	12.74	9.43	0.91	10.31	10.26	0.60	19.87	8.93	1.76	10.31	10.26	0.60
2005	2006	10.30	8.54	0.68	10.02	8.87	0.62	9.75	9.38	0.56	9.50	10.03	0.50	10.60	8.41	0.73	10.60	8.41	0.73
2006	2007	15.47	7.60	1.42	17.13	8.06	1.55	18.77	8.68	1.63	13.80	7.33	1.25	20.41	9.41	1.67	13.80	7.33	1.25
2007	2008	-8.82	14.24	-0.90	-10.30	15.61	-0.92	-11.76	17.20	-0.92	-10.95	16.28	-0.92	-7.32	13.15	-0.86	-7.32	13.15	-0.86
2008	2009	-8.52	33.87	-0.35	-11.66	37.42	-0.40	-14.59	41.66	-0.43	-17.25	46.59	-0.44	-5.25	30.98	-0.27	-17.25	46.59	-0.44
2009	2010	17.75	13.06	1.09	18.32	15.56	0.95	18.98	18.24	0.85	19.72	20.98	0.77	17.29	10.98	1.26	19.72	20.98	0.77
2010	2011	23.77	10.20	2.04	25.21	11.50	1.93	26.68	13.00	1.82	28.17	14.62	1.72	22.36	9.23	2.10	28.17	14.62	1.72
2011	2012	10.36	15.71	0.53	8.44	18.09	0.35	6.58	20.78	0.22	4.81	23.74	0.12	12.33	13.73	0.75	4.81	23.74	0.12
2012	2013	17.12	9.75	1.57	19.35	10.39	1.69	21.57	11.32	1.75	14.87	9.50	1.38	23.78	12.46	1.77	23.78	12.46	1.77
2013	2014	13.05	8.66	1.20	15.86	8.81	1.50	18.61	9.24	1.73	10.19	8.85	0.85	21.31	9.90	1.88	21.31	9.90	1.88
2014	2015	1.36	9.75	-0.09	2.77	10.13	0.06	4.19	10.74	0.19	-0.04	9.61	-0.23	5.61	11.53	0.30	5.61	11.53	0.30
2015	2016	14.32	13.14	0.94	10.84	14.01	0.63	7.28	15.25	0.35	3.64	16.80	0.10	17.72	12.63	1.25	3.64	16.8	0.10
2016	2017	7.05	7.27	0.68	10.66	7.29	1.18	14.18	7.87	1.54	3.36	7.92	0.16	17.62	8.84	1.76	17.62	8.84	1.76

Table 2.9: Annualized Out-of-Sample Portfolio Performance Statistics For Alternative Buffett Indicator Index Calculation Methods beginning in 1995 - ending in different years to 2017.

GDP X,Window Y, where X is a number of years taken to calculate average GDP for Buffett Indicator, and Window Y is a number of years in a window to calculate Buffett Indicator Index. We estimate annualized average portfolio returns, $\bar{r}$, portfolio standard deviations, $\hat{\sigma}$, and Sharpe ratios, S, for these portfolio strategies. All portfolios produce similar results, with (GDP 5, Window 5) and (GDP 10, Window 5) having somewhat higher Sharpe ratio. The strategy of choice for BP portfolio is (GDP 5, Window 5), which is consistent with the rest of the paper.

July 1, *yyyy*		GDP 1, Window 5			GDP 1, Window 10			GDP 5, Window 5			GDP 5, Window 10			GDP 10, Window 5			GDP 10, Window 5		
Start	End	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S	$\bar{r}(\%)$	$\hat{\sigma}$ (%)	S
1995	1996	22.79	8.83	1.90	22.79	8.83	1.90	22.79	8.83	1.90	22.79	8.83	1.90	22.79	8.83	1.90	22.79	8.83	1.90
1995	1997	26.57	10.46	1.95	26.57	10.46	1.95	26.57	10.46	1.95	26.57	10.46	1.95	26.57	10.46	1.95	26.57	10.46	1.95
1995	1998	26.82	12.09	1.72	26.82	12.09	1.72	26.82	12.09	1.72	26.82	12.09	1.72	26.82	12.09	1.72	26.82	12.09	1.72
1995	1999	24.45	13.83	1.35	24.45	13.83	1.35	24.45	13.83	1.35	24.45	13.83	1.35	24.45	13.83	1.35	24.45	13.83	1.35
1995	2000	19.70	14.64	0.95	19.70	14.64	0.95	19.70	14.64	0.95	19.70	14.64	0.95	19.70	14.64	0.95	19.70	14.64	0.95
1995	2001	18.54	14.61	0.88	18.54	14.61	0.88	18.54	14.61	0.88	18.54	14.61	0.88	18.54	14.61	0.88	18.54	14.61	0.88
1995	2002	16.07	13.97	0.75	16.07	13.97	0.75	16.16	13.94	0.76	15.48	14.30	0.69	16.14	13.95	0.76	15.48	14.30	0.69
1995	2003	15.09	13.94	0.70	15.09	13.94	0.70	15.16	13.91	0.70	14.47	14.57	0.62	15.15	13.92	0.70	14.47	14.57	0.62
1995	2004	15.78	13.61	0.77	15.78	13.61	0.77	15.85	13.58	0.78	15.17	14.13	0.70	15.83	13.59	0.78	15.17	14.13	0.70
1995	2005	16.04	13.21	0.83	16.04	13.21	0.83	16.09	13.18	0.83	15.37	13.69	0.75	16.09	13.19	0.83	15.37	13.69	0.75
1995	2006	15.51	12.86	0.81	15.51	12.86	0.81	15.54	12.84	0.82	14.91	13.30	0.74	15.55	12.84	0.82	14.91	13.30	0.74
1995	2007	15.91	12.61	0.86	15.91	12.61	0.86	15.95	12.59	0.87	14.97	12.93	0.77	15.95	12.60	0.87	14.97	12.93	0.77
1995	2008	13.67	13.22	0.66	13.67	13.22	0.66	13.71	13.21	0.66	13.03	13.14	0.61	13.71	13.21	0.66	13.03	13.14	0.61
1995	2009	12.32	15.20	0.49	12.32	15.20	0.49	12.35	15.19	0.49	11.62	15.31	0.44	12.35	15.19	0.49	11.62	15.31	0.44
1995	2010	12.65	14.95	0.53	12.65	14.95	0.53	12.68	14.94	0.53	12.00	15.06	0.48	12.68	14.94	0.53	12.00	15.06	0.48
1995	2011	13.31	14.68	0.59	13.31	14.68	0.59	13.33	14.67	0.59	12.70	14.78	0.54	13.33	14.67	0.59	12.70	14.78	0.54
1995	2012	12.97	14.99	0.57	12.97	14.99	0.57	13.02	14.93	0.57	12.42	15.04	0.53	13.04	14.88	0.57	12.42	15.04	0.53
1995	2013	13.51	14.83	0.62	13.51	14.83	0.62	13.58	14.79	0.63	12.82	14.82	0.57	13.53	14.71	0.62	12.82	14.82	0.57
1995	2014	13.92	14.62	0.66	13.92	14.62	0.66	13.99	14.57	0.67	13.16	14.58	0.61	13.94	14.50	0.67	13.16	14.58	0.61
1995	2015	13.51	14.48	0.65	13.51	14.48	0.65	13.57	14.44	0.65	12.78	14.45	0.60	13.52	14.37	0.65	12.78	14.45	0.60
1995	2016	13.04	14.60	0.62	13.04	14.60	0.62	13.09	14.56	0.62	12.34	14.57	0.57	13.05	14.49	0.62	12.34	14.57	0.57
1995	2017	13.19	14.38	0.64	13.19	14.38	0.64	13.22	14.33	0.65	12.54	14.35	0.60	13.19	14.27	0.65	12.54	14.35	0.60

2.5.2 Notations and Figures

Variable	Description
i and t	subscripts denoting stock i and period t
RI	Total Return Index (includes change in price and dividends)
r_{it}	Simple return (based on RI)
FI	Fundamental Index
w^{FI}	Vector of weights of the FI portfolio
w^{GMV}	Vector of weights of the GMV portfolio
w^{PB}	Vector of weights of the Predictive Blended portfolio
Ω	Expected variance-covariance matrix of stocks
BI	Buffett Indicator = Wilshire 5000 / nominal GDP
BII	Buffett Indicator Index = (BI - min(BI)/(max(BI) - min (BI))
BV	Book Value
Div	Dividends for the last year
FCF	Free Cash Flows
MV	Market Value, capitalization
Rev	Revenue for the last year

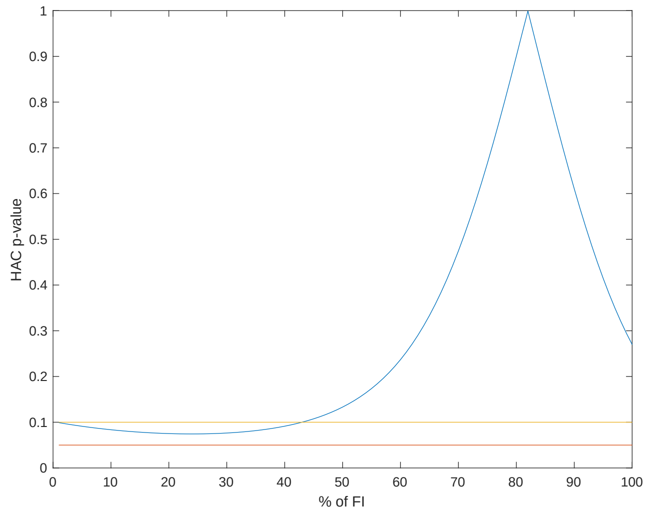

Figure 2.2: Statistical Significance of Fixed-Blended Portfolios Sharpe Ratios Difference Against the Equally-Weighted S&P 500 Benchmark.
Unlike the PB portfolios no Fixed-Blended portfolio is statistically significant on 95% level. However, most of Fixed-Blended portfolios with share of FI less than 40% achieve 90% significance level.

2.5.3 Algorithms

Data pre-processing procedures are detailed in Algorithm 1. Steps for blended portfolio construction are summarized in Algorithm 2. Performance evaluation procedures for our proposed approach are laid out in Algorithm 3.

Algorithm 1 Data Preparation

1. Raw data are from the Thomson Reuters Datastream.
   (a) Daily: MV, RI for each stock, and for S&P 500; MV for the Wilshire 5000
   (b) Annual: BV, DIV, FCF, REV, GDP (nominal).
2. Full sample filtering: bankruptcy filter. Upon the first occurrence of RI or PI =0, a NaN (Not-a-Number) is assigned to all subsequent PI or RI values.
3. Creating the matrix of simple returns:
$$r_{(\tau_{sample}-1)\times n} = (RI_{2:\tau\times 1:n} - RI_{1:(\tau-1)\times 1:n}) \oslash RI_{1:(\tau-1)\times 1:n} - e_{(\tau-1)\times 1} e'_{1\times n}, \quad (2.11)$$
   where τ_{sample} is the length of the sample dates
4. The expected annualized return for the each stock is calculated as the geometric return average from the in-sample RI data:
$$\mu_{n\times 1.daily} = (RI'_{in-sample\ end} \oslash RI'_{in-sample\ beginning})^{\circ \frac{1}{NumberOfDays}} - e_{n\times 1}, \quad (2.12)$$
$$\mu_{n\times 1} = (e_{n\times 1} + \mu_{n\times 1.daily})^{\circ 260} - e_{n\times 1}, \quad (2.13)$$
   where $e_{n\times 1}$ is the vector of ones.
5. The in-sample variance-covariance matrix calculation is based on the daily returns data for the 5 in-sample years
$$\Omega_{n\times n} = X \times X' \ , \ X = [r'_{n\times(\tau_{in-sample}-1)} - \mu_{n\times 1,daily} e_{1\times(\tau_{in-sample}-1)}], \quad (2.14)$$
   where $\tau_{in-sample}$is the length of in-sample dates
6. The Buffet Indicator calculation:
$$BI_t = \frac{\text{Wilshire } 5000_t}{\sum_{\tau=t-4}^{t} GDP_\tau/5}, \quad (2.15)$$
7. The Buffett Indicator Index calculation:
$$BII_t = \frac{BI_t - \min\{BI_\tau\}_{\tau=t-4}^{t}}{\max\{BI_\tau\}_{\tau=t-4}^{t} - \min\{BI_\tau\}_{\tau=t-4}^{t}} * 100\% \quad (2.16)$$

Algorithm 2 Blended portfolio construction.

1. The Global Minimum Variance (GMV) portfolio construction.

   (a) The weights are:

$$w_{GMV\ n\times 1} = \frac{\Omega^{-1}_{n\times n} e_{n\times 1}}{e'_{1\times n}\Omega^{-1}_{n\times n} e_{n\times 1}} \tag{2.17}$$

   (b) The expected return is:

$$\mu_{GMV\ 1\times 1} = \frac{\mu'_{1\times n}\Omega^{-1}_{n\times n} e_{n\times 1}}{e'_{1\times n}\Omega^{-1}_{n\times n} e_{n\times 1}} \tag{2.18}$$

   (c) The variance is:

$$\sigma^2_{GMV\ 1\times 1} = \frac{1}{e'_{1\times n}\Omega^{-1}_{n\times n} e_{n\times 1}} \tag{2.19}$$

2. The Fundamental Index (FI) portfolio construction

   (a) The weights are:

$$w_{FI\ n\times 1} = \frac{FI_{n\times 1}}{e_{1\times n} FI_{n\times 1}} \tag{2.20}$$

   (b) The expected return is:

$$\mu_{FI\ 1\times 1} = \frac{\mu'_{1\times n} FI_{n\times 1}}{e'_{1\times n} FI_{n\times 1}} \tag{2.21}$$

   (c) The variance is:

$$\sigma^2_{FI\ 1\times 1} = w'_{FI\ 1\times n}\Omega_{n\times n} w_{FI\ n\times 1} \tag{2.22}$$

3. The Blended Portfolios construction

   (a) Fixed proportion ($\alpha = 0, 1, 2, ..., 100\%$) Blended Portfolios construction:

$$w_B = \alpha w_{FI} + (1-\alpha) w_{GMV} \tag{2.23}$$

   (b) Predictive Blended portfolio construction:

$$w_{PB} = BII * w_{FI} + (1 - BII) * w_{GMV} \tag{2.24}$$

   (c) Calculation of the expected return and variance of the PB portfolio.

      i. $\mu_{PB\ 1\times 1} = w'_{PB\ 1\times n}\mu_{n\times 1}$

      ii. $\sigma^2_{PB\ 1\times 1} = w'_{PB\ 1\times n}\Omega_{n\times n} w_{PB\ n\times 1}$

Algorithm 3 Performance and Out-Of-Sample Tests

1. Out-of-sample performance for every sample (note: τ_{out} is the length of out-of-sample dates). For every portfolio X, $X \in \{PBP,\, FI,\, GMV,\, Equally-Weighted,\; etc\}$ we calculate returns for every sample (from July 1, 1995 to July 1, 1996; from July 1, 1996 to July 1, 1997;...; from July 1, 2016 to July 1, 2017):

$$r_{X\ \tau_{out}\times 1} = r_{\tau_{out}\times n} w_{X\ n\times 1}. \tag{2.25}$$

2. Out-of-sample returns of the portfolio $X \in \{PBP,\, FI,\, GMV,\, Equally-Weighted,\; etc\}$ across the samples (from July 1, 1995 to July 1, 2017)

$$r^{across}_{X\ T\times 1} = [r^{sample1}_{X\ \tau_{out}\times 1}; r^{sample2}_{X\ \tau_{out}\times 1}; ...; r^{sampleLast}_{X\ \tau_{out}\times 1}], \tag{2.26}$$

   where $T_{out-of-sample} = \tau^{sample1}_{out} + \tau^{sample2}_{out} + ... + \tau^{sampleLast}_{out}$

3. Standard deviation of the portfolio $X \in \{PBP,\, FI,\, GMV,\, Equally-Weighted,\; etc\}$ across samples:

$$\sigma^{2\ realised}_{X\ 1\times 1} = [r_{X\ \tau_{out}\times 1} - e_{\tau_{out}\times 1}\mu^{realised}_{X\ daily\ 1\times 1}]'[r_{X\ \tau_{out}\times 1} - e_{\tau_{out}\times 1}\mu^{realised}_{X\ daily\ 1\times 1}] \tag{2.27}$$

4. Sharpe ration of the portfolio $X \in \{PBP,\, FI,\, GMV,\, Equally-Weighted,\; etc\}$ across samples:

$$Sharpe^{overall}_{X} = \frac{FV^{realised}_{X\ 1\times 1} - \prod_{i=1}^{number\ of\ samples}(1 + r_{f\ i})_{1\times 1}}{\sigma^{realised}_{X\ 1\times 1}} \tag{2.28}$$

Chapter 3

Price-adjusted fundamental indexing

3.1 Introduction

The semi-strong form version of the efficient market hypothesis claims that all publicly available information is quickly recognized by investors and incorporated in stock prices (Malkiel and Fama, 1970; Fama, 1991). Our paper, however, suggests methods on how to combine publicly available information on firms' fundamentals with the information on past correlations to produce portfolios that may outperform the market index. The previous chapter is focused on the blending technique using the constituents of the S&P 500 index, and shows statistically significant outperformance over the S&P 500 and the equally-weighted S&P 500. This chapter focuses on possible extensions of the blending technique.

First, instead of considering all 500 stocks, pooling cyclical and non-cyclical industries together, we explore how portfolio allocation may be improved by considering separately different types of industries.

Second, instead of using the standard Arnott's fundamental index (FI) portfolio, which ignores the current price of stocks, we suggest and test an alternative FI portfolio, which underweights possibly overpriced stocks and overweights possibly underpriced stocks.

Third, since information on the possible correlations between the Global Minimum Variance (GMV) and Arnott FI portfolios was not utilized, we test blends that are constructed based on this information. Specifically, instead of using the Buffett Indicator Index to blend the GMV and Arnott FI portfolios, we blend these two portfolios using the Markowitz Mean-Variance Optimization (MVO) method[1]. Construction of the GMV and FI portfolios does not depend on individual stocks' expected returns, which as we discussed in the previous chapter, is a major source of error (Jorion, 1986; Poterba and Summers, 1988). We, however, estimate the expected return of the GMV and FI portfolios. In this case, individual errors are pooled, which results in much lower estimation errors of the portfolios' expected returns; as we show in the empirical section of this chapter this may improve the performance of the resulting portfolio.

The rest of the paper is organized as follows. We introduce the methodology in Section 3.2. We summarize the data and discuss the data preparation procedures in Section 3.3. We discuss our empirical findings in Section 3.4. Finally, Section 3.5 concludes.

[1]In fact, it is a second stage Markowitz mean-variance optimization since the first stage is the GMV portfolio

3.2 Methodology

The construction of the Global Minimum Variance (GMV) and Arnott's Fundamental Indexing (FI) portfolio is the same as in the previous chapter.

3.2.1 Equity classification structure

To classify S&P 500 stocks, we first follow the Industry Classification Benchmark (ICB) produced by FTSE International, which is based on the reported primary source of operating revenue. The ICB includes 10 industries (the industry code is in parenthesis): Oil & Gas (0001), Basic Materials (1000), Industrials (2000), Consumer Goods (3000), Health Care (4000), Consumer Services (5000), Telecommunications (6000), Utilities (7000), Financials (8000), Technology (9000). Since the S&P 500 includes fewer than ten stocks in Telecommunications, during our analysis, we combine this industry with the closely related industry: Technology.

However, some of these industries (or even their sectors and sub-sectors) might exhibit pro-cyclical behavior, while others do not. Since the seminal paper by Farrell (1974), industries and sectors have been classified according to their cyclical nature. To arrange the ICB industries into the groups depending on their cyclical nature, we follow the recent classification produced by Morningstar Research, - the Morningstar Global Equity Classification Structure. They divide the universe of stocks into three broad categories: cyclical (typical beta higher than 1), defensive (typical beta lower than 1), and sensitive (typical beta is around 1). We match their description of sectors with our data containing ICB in the following manner:

The cyclical super industry includes the following items from the ICB (the ICB codes are in parenthesis): Basic Materials (1000), Automobiles & Parts (3300), Household Goods & Home Construction (3720), Leisure Goods (3740), Personal Goods (3760), General Retailers (5370), Media (5500), Travel & Leisure (5700), Financials (8000).

The defensive super industry includes: Consumer Goods: Food and Beverage (3500), Tobacco (3780), Health Care (4000), Food & Drug Retailers (5330), Utilities (7000).

Sensitive super industry includes: Oil & Gas (0001) Industrials (2000), Telecommunications (6000) and Technology (9000).

We analyze how the portfolios based on single fundamentals and the GMV, FI, Tangency portfolios, and PBP perform in every industry and super-industry.

3.2.2 Construction of the Price-Adjusted Fundamental Indexing (PAFI) portfolio

Previous literature (Arnott et al., 2005; Walkshäusl and Lobe, 2010) considers fundamental indexes based on a single metric or an average of a number of fundamental factors. A single metric fundamental index can be calculated as[2]:

$$FI_i^X = \frac{max\{0, X_i\}}{\sum_{j=1}^{n} max\{0, X_j\}}, \tag{3.1}$$

[2]Note: this formula ensures that there are no short sales in the FI portfolio.

where X_i is a numeric value for the considered fundamentals for stock i, e.g., book value *(BV),* dividends paid *(D),* free cash flows *(FCF),* revenues *(REV),* among others. However, this approach ignores the current valuation of stocks.

We propose to underweight possibly overpriced stocks and overweigh possibly underpriced stocks. To do so, we weigh the fundamental by the ratio of the current fundamental divided by the current market valuation, which creates the Price-Adjusted Fundamental (PAF):

$$PAF_i^X = X_i * \frac{X_i}{MV_i} \ , \tag{3.2}$$

where MV_i is the market capitalization (market value) of the company i. Similarly to 3.1 we create the Price-Adjusted Fundamental Index for variable X:

$$PAFI_i^X = \frac{max\{0, PAF_i^X\}}{\sum_{j=1}^{n} max\{0, PAF^X{}_j\}} \ , \tag{3.3}$$

Similarly to Arnott et al. (2005)'s composite approach the price-adjusted composite fundamental index is:

$$PAFI_i^{COMP} = \begin{cases} \frac{1}{4}(PAFI_i^{BV} + PAFI_i^{D} + PAFI_i^{FCF} + PAFI_i^{REV}), & \text{if } PAFI_i^D > 0; \\ \frac{1}{3}(PAFI_i^{BV} + PAFI_i^{FCF} + PAFI_i^{REV}), & \text{otherwise.} \end{cases} \tag{3.4}$$

We normalize the weights in the PAFI portfolio:

$$w_i^{PAFI} = \frac{PAFI_i}{\sum_{j=1}^{n} PAFI_j}. \tag{3.5}$$

The PAFI methodology has the following properties:

First, if all companies in the market are valued proportionally to their fundamentals, then the PAFI is simplified to the classic Arnott FI:

$$\frac{X_i}{MV_i} = \frac{X_j}{MV_j}\,\forall i,j \;\therefore PAFI_i = FI_i$$

Second, in a less realistic case, if all companies have the same market valuation, and different values of fundamentals X, then w_i is proportional to the squared values of X_i. If X_i stands for a company's market share, then the PAFI's is related to the Herfindahl-Hirschman index (HHI) as follows:

$$MV_i = MV_j = MV\,\forall\, i,j \;\therefore\; PAFI_i = \frac{X_i / \sum_{j=1}^{n} X_j}{HHI},$$

where HHI is the Herfindahl-Hirshman Index in terms of fundamental X.

In our case, however, the PAFI composite, $PAFI^{COMP}$, is based on four fundamentals discussed earlier, making HHI represent concentration of the composite fundamental.

Third, in an equally unrealistic case, if all companies have an identical fundamental value X, the PAFI portfolio allocation will be the opposite to that of the Market Capitalization-based approach: the larger the market capitalization of a company, the smaller the PAFI and the company's weight in the portfolio. In this situation PAFI is related to the harmonic mean of MV as follows:

$$X_i = X_j = X \,\forall i, j \therefore \, PAFI_i = \frac{\frac{1}{MV_i}}{\sum_{j=1}^{n} \frac{1}{MV_j}} = \frac{H}{n * MV_i},$$

where H is a harmonic mean of Market Values.

This is a desirable property, because such a situation highlights the advantage of our approach: with identical economic footprints of companies, our approach allocates a smaller weight to overpriced companies.

3.2.3 Construction of the Tangency Blended (TB) portfolio

We define the Tangency Blended (denoted with a subscript TB) portfolio as the tangency portfolio between the capital market line originating at r_f and the efficient frontier based on the two risky assets - the GMV and FI (or PAFI) portfolios. We then compare this portfolio to the benchmarks in out-of-sample tests.

There are two ways to calculate the weights of the blended portfolio. The first, is to calculate the weights of the GMV and FI portfolios in the TB portfolio:

$$\alpha = \arg\max_{\alpha} \frac{r_B - r_f}{\sigma_B} \tag{3.6}$$

subject to:

$$\begin{cases} r_B = [\mu_{FI}\ \mu_{GMV}]'[\alpha\ (1-\alpha)] \\ \sigma_p^2 = (\alpha\sigma_{FI})^2 + ((1-\alpha)\sigma_{GMV})^2 + 2\alpha(1-\alpha)cov(r_{FI}, r_{GMV}) \end{cases}$$

where μ_{GMV} and μ_{FI} are expected returns of the GMV and FI portfolios, and σ_{GMV} and σ_{FI} are the standard deviations of these portfolios and $cov(r_{FI}, r_{GMV})$ is the covariance between the return of the GMV and FI portfolios. The maximization problem (3.6) maximizes the Sharpe ratio, which we choose due to its wide applicability in practice. Maximizing the Sharpe ratio finds the best allocation of stocks for the tangency portfolio which, combined with a risk-free investment, would maximize an investor's utility function in mean return - variance space. In turn, Markowitz (2014) shows[3] that mean-variance optimization maximizes a wide range of an investor's utility functions or their approximations. The solution for this maximization problem is:

$$\begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix} = \frac{\begin{pmatrix} \sigma_{GMV}^2 & Cov_{GMV,FI} \\ Cov_{GMV,FI} & \sigma_{FI}^2 \end{pmatrix}^{-1} \begin{pmatrix} \mu_{GMV} - r_f \\ \mu_{FI} - r_f \end{pmatrix}}{[1\ 1] \begin{pmatrix} \sigma_{GMV}^2 & Cov_{GMV,FI} \\ Cov_{GMV,FI} & \sigma_{FI}^2 \end{pmatrix}^{-1} \begin{pmatrix} \mu_{GMV} - r_f \\ \mu_{FI} - r_f \end{pmatrix}}, \tag{3.7}$$

[3]This property of the MVO method combined with its relative simplicity made us chose it to optimize the first two moments of the investor's utility function over more sophisticated Stochastic Dominance approaches initially proposed by Levy (1992).

or alternatively Bodie et al. (2014, p.217):

$$\alpha = 1 - \frac{[\mu_{GMV} - r_f]\sigma^2_{FI} - [\mu_{FI} - r_f]Cov_{GMV,FI}}{[\mu_{GMV} - r_f]\sigma^2_{FI} + [\mu_{FI} - r_f]\sigma^2_{GMV} - [\mu_{GMV} - r_f + \mu_{FI} - r_f]Cov_{GMV,FI}} \tag{3.8}$$

Or,

$$\alpha = 1 - \frac{[\mu_{GMV} - r_f]\sigma^2_{FI} - [\mu_{FI} - r_f]\sigma^2_{GMV}}{[\mu_{GMV} - r_f]\sigma^2_{FI} - [\mu_{GMV} - r_f]\sigma^2_{GMV}} = \frac{[\mu_{GMV} - r_f]\sigma^2_{FI} - [\mu_{FI} - r_f]\sigma^2_{GMV}}{[\mu_{GMV} - r_f][\sigma^2_{FI} - \sigma^2_{GMV}]} \tag{3.9}$$

Calculating the weights of the GMV and FI portfolios in the Tangency Blended (TB) portfolio allow us to find the final TB portfolio composition in terms of stocks:

$$w_{TB} = \alpha w_{FI} + (1 - \alpha) w_{GMV} \tag{3.10}$$

Alternatively, we can apply a closed form solution for the individual stocks' weights in the TB portfolio as a function of the initial data. In other words, the two steps procedure can be reduced to one step (please see the proof in the appendix 3.5.2):

$$w_{TB} = \frac{[(\mu' - r_f e')(\Omega^{-1} e FI'\Omega - FIe')FI\mathbf{I} + FI\mu'(\mathbf{I}e'FI - FIe')]\Omega^{-1}e}{e'(\Omega^{-1}eFI'\Omega - FIe')FI(\mu' - r_f e')\Omega^{-1}e}, \tag{3.11}$$

where μ is a column vector of individual stocks' column vector, r_f is a scalar representing the risk-free rate, e is a column vector of ones, FI is a column vector of individual stocks' fundamental indexes; all vectors have the same dimensions equal to the number of stocks under consideration; Ω is the variance-covariance matrix of individual stocks under consideration, with the number of rows and columns identical to the length of the vectors in the equation. These formulas are applicable for unconstrained optimization problems. Constrained optimization problems have no analytical solution and require numerical methods.

In this section, we divided stocks into industries, and super-industries, proposed a new Price-Adjusted Fundamental Indexing (PAFI) portfolio and analyzed stocks in-sample and constructed the GMV, FI and TB portfolios out-of-sample. Before we perform the empirical investigation of our technique in Section 3.4, we describe our data in the following section.

3.3 Data Description and Preparation

The descriptive statistics for the S&P 500 data set is presented in the previous chapter. In this chapter, however, we present values of major fundamentals for each industry, and super-industries. As before our investable universe consists of the S&P 500 constituents listed on the NYSE, NASDAQ, and AMEX from January 1990 to January 2018. Since the last portfolio we analyze is constructed on July 1, 2016, the Table 3.1 is presented for this last period to illustrate the relative sizes of fundamentals for different industries.

Table 3.1: Industry specific data description for 2016.
Number of stocks (N), book value (BV), and Market Value (MV) are presented as of July 1, 2016; dividends (D), Free Cash Flows (FCF), and Revenues (Rev) are average values of reported annual values for 2011-2015.

	N	BV	D	FCF	Rev	MV
S&P 500	500	6,959	1657	349	10,444	18,614
Cyclical	215	3,371	683	122	4,148	6,993
Defensive	109	1,266	280	82	2,429	4,711
Sensitive	176	2,322	695	144	3,868	6,910
Oil & Gas	51	829	226	48	1,440	1,548
Basic Materials	19	126	40	12	293	421
Industrial	66	511	154	39	1,192	2,090
Consumer Goods	59	413	122	44	1,174	2,102
Health Care	47	674	128	34	899	2,491
Consumer Services	76	604	191	37	2,421	2,874
Utilities	27	316	70	18	309	643
Financials	96	2,503	412	60	1,480	3,173
Technology and Telecommunications	59	983	315	57	1236	3,272

3.4 Results

We analyze the data for July 1, 1995 to June 30, 2017 with annual portfolio rebalancing. This produces the Sharpe Ratios presented in Table 3.2.

First, let us consider the universe of all S&P 500 stocks. The pure Price-Adjusted Fundamental Index (PAFI) portfolio and the TB portfolio based on PAFI have somewhat higher Sharpe ratios over the period 1995 - 2017 compared to the pure Arnott FI portfolio and the TB based on the FI portfolio. However, these differences are not statistically significant. Also for the predictive blend both FI and PAFI produce similar results.

Second, we consider super-industries (or industry aggregates). For defensive and sensitive industry aggregates, the PAFI methodology provides statistically significant improvements compared to Arnott's FI methodology in pure PAFI and FI portfolios (Sharpe ratios 0.69 vs. 0.55 for defensive and 0.51 vs. 0.43 for sensitive super-industries). For sensitive and cyclical industry aggregates, the PAFI methodology provides statistically significant improvements compared to Arnott's FI methodology in the TB case (Sharpe ratios 0.59 vs. 0.50 and 0.52 vs. 0.38 respectively). However, for Predictive Blends PAFI methodology does not provide statistically significant improvements over Arnott's FI methodology.

Third, we consider every industry taken separately. Pure portfolios based on PAFI have somewhat higher Sharpe ratios than pure portfolios based on FI in six out of eight industries, most statistically significantly in Oil & Gas, (Sharpe ratio 0.41 vs. 0.37), Health Care (Sharpe ratio 0.71 vs. 0.50), and Technology & Telecommunications (Sharpe ratio 0.51 vs. 0.41). Predictive blends based on PAFI outperform those based on FI in Health Care (Sharpe ratio 0.76 vs. 0.55), and Technology & Telecommunications (Sharpe ratio 0.56 vs. 0.48). The underperformance of PAFI compared to FI portfolios is evident only in the Consumer Goods industry (Sharpe ratio 0.38 vs. 0.45). Results are similar for the BP and TB: proposed PAFI-based blends have higher Sharpe ratios than Arnott's FI-based blends in most industries.

Table 3.2: Industry specific Sharpe ratios for 1995 - 2017 out-of-sample tests. The proposed PAFI portfolios are compared to Arnott's FI (FI) for three different cases: taken purely, blended with the GMV in the predictive blended (PB) portfolio, and blended with the GMV in the tangency blended (TB) portfolio. We apply the methodology in Ledoit and Wolf (2008) to calculate heteroskedasticity and autocorrelation-consistent (HAC) p-values in comparing Sharpe ratios of portfolios based on FI against PAFI. For each of three cases, the highest Sharpe ratio is in underlined-bold if it is statistically significantly higher than its counterpart, p-value is provided. SR is Sharpe Ratio. EW is the equally-weighted portfolio.

	Pure portfolio:			PB based on:			TB based on:			
	SR		p	SR		p	SR		p	SR
	FI	PAFI		FI	PAFI		FI	PAFI		EW
S&P 500	0.44	0.47	0.99	0.64	0.64	0.93	0.55	0.58	0.43	0.50
Defensive	0.55	**0.69**	0.05	0.65	0.68	0.36	0.52	0.54	0.47	0.43
Sensitive	0.43	**0.51**	0.07	0.58	0.61	0.38	0.50	**0.59**	0.04	0.64
Cyclical	0.35	0.35	0.67	0.49	0.47	0.45	0.38	**0.52**	0.01	0.47
Oil & Gas	0.37	**0.41**	0.09	0.46	0.48	0.37	0.49	0.49	0.50	0.34
Basic Materials	0.23	0.23	0.86	0.32	0.28	0.24	0.32	0.33	0.73	0.30
Industrial	0.41	0.46	0.41	0.45	0.45	0.96	0.48	0.53	0.18	0.47
Consumer Goods	0.45	0.38	0.17	0.56	0.49	0.13	0.58	0.56	0.47	0.46
Health Care	0.50	**0.71**	0.05	0.55	**0.76**	0.04	0.56	0.59	0.30	0.65
Consumer Services	0.49	0.53	0.70	0.41	0.47	0.53	0.36	0.36	0.79	0.53
Utilities	0.39	0.43	0.54	0.48	0.50	0.31	0.36	0.42	0.14	0.43
Financials	0.33	0.36	0.83	0.47	0.50	0.23	0.52	0.51	0.89	0.41
Technology and Telecommunications	0.41	**0.51**	0.02	0.48	**0.56**	0.04	0.36	0.42	0.14	0.43

Fourth, for S&P 500 stocks predictive blends (PB) based on both FI and PAFI based portfolios are superior compared to the equally-weighted approach, a hard-to-beat benchmark. The PB and TB and pure portfolios are also superior to the Sharpe ratio of the S&P 500 market capitalization index, which is 0.33 for the period considered (not in the table). Either of the blended approaches provides a substantial improvement to the Sharpe ratios of Defensive stocks, and to one of the major industries (please see 3.1), Financials.

Fifth, there is no industry or super-industry where classic Arnott's FI approach provides statistically significantly higher Sharpe ratio against Price-Adjusted FI proposed in this paper.

In future research we will focus on ways of combining industry-based portfolios.

3.5 Conclusion

In this paper, we introduce a Price-Adjusted Fundamental Indexing (PAFI) portfolio by weighting Arnott's fundamentals with price - adjusted measures. We find, that this new portfolio methodology improves the Sharpe ratios on Arnott's FI portfolios for some US industries and super-industries. In addition, the Predictive Blended portfolios based on the PAFI and GMV outperform Predictive Blended portfolios based on Arnott's FI and the GMV. We also check, whether the Tangency Blended portfolio, which is produced by blending the GMV and FI (or PAFI) portfolios using Markowitz mean-variance optimization can produce superior Sharpe ratios. We show that for Tangency Blended portfolios blends based on PAFI outperform those based on Arnott's FI portfolios in sensitive and cyclical super-industries over the period 1995-2017.

Appendix

3.5.1 Notation

Variable	Description
PI	price index
RI	total return index (includes change in price and dividends)
r_{it}	simple return for stock i in day t (based on RI)
μ_i;μ	expected return for stock i; vector of r_i's
r_f	risk-free rate
R_{it}	gross return for the stock i on day t
FI_i	fundamental index of stock i
w_{FI}	vector of weights in a portfolio obtained using the FI approach
w_{GMV}	vector of weights in a portfolio obtained using the GMV method
w_B	vector of weights in a portfolio obtained using the blending technique
Ω	expected variance covariance matrix of stocks
BI	Buffett indicator = Wilshire 5000 / nominal GDP
BII	Buffett indicator index = (BI - min(BI)/(max(BI) - min (BI))
BV	book value
Div, D	dividends for the last year
FCF	free cash flows
MV	market value, capitalization
Rev, REV	revenue for the last year

3.5.2 Proofs

Weights of the GMV $(1-\alpha)$ and FI (α) portfolios in the TB are:

$$\begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix} = \frac{\Omega^{-1}_{GMV,FI}(\mu_{GMVFI} - r_f e)}{e'\Omega^{-1}_{GMV,FI}(\mu_{GMVFI} - r_f e)} \tag{3.12}$$

Note, that the inversion of variance-covariance matrix between two assets, the GMV and FI yields:

$$\Omega^{-1}_{GMV,FI} = \begin{pmatrix} \sigma^2_{GMV} & \sigma_{GMV,FI} \\ \sigma_{GMV,FI} & \sigma^2_{FI} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} \sigma^2_{FI} & -\sigma_{GMV,FI} \\ -\sigma_{GMV,FI} & \sigma^2_{GMV} \end{pmatrix}}{\sigma^2_{GMV}\sigma^2_{FI} - \sigma^2_{GMVFI}\sigma^2_{GMVFI}} \tag{3.13}$$

Thus, we rewrite 3.12:

$$\begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix} = \frac{\begin{pmatrix} \sigma^2_{FI} & -\sigma_{GMV,FI} \\ -\sigma_{GMV,FI} & \sigma^2_{GMV} \end{pmatrix} \frac{\mu_{GMVFI} - r_f e}{\sigma^2_{GMV}\sigma^2_{FI} - \sigma^2_{GMVFI}\sigma^2_{GMVFI}}}{e' \begin{pmatrix} \sigma^2_{FI} & -\sigma_{GMV,FI} \\ -\sigma_{GMV,FI} & \sigma^2_{GMV} \end{pmatrix} \frac{\mu_{GMVFI} - r_f e}{\sigma^2_{GMV}\sigma^2_{FI} - \sigma^2_{GMVFI}\sigma^2_{GMVFI}}} \tag{3.14}$$

Cancelling the scalar, non-zero determinant of the variance covariance matrix:

$$\begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix} = \frac{\begin{pmatrix} \sigma^2_{FI} & -\sigma_{GMV,FI} \\ -\sigma_{GMV,FI} & \sigma^2_{GMV} \end{pmatrix} (\mu_{GMVFI} - r_f e)}{e' \begin{pmatrix} \sigma^2_{FI} & -\sigma_{GMV,FI} \\ -\sigma_{GMV,FI} & \sigma^2_{GMV} \end{pmatrix} (\mu_{GMVFI} - r_f e)} \tag{3.15}$$

Now, let us expand each element of the matrix in the equation:

$$\sigma^2_{FI} = w'_{FI} \Omega w_{FI} \tag{3.16}$$

Please note, for that the three remaining matrix elements have similar form. Let us analyze it:

The covariance of the GMV portfolio with any other portfolio is equal to the variance of the GMV portfolio [4]. Noting that we require that the sum of weights in any portfolio X be equal to one,

$$\sigma_{X,GMV} = w'_X \Omega w_{GMV} = w'_X \Omega \Omega^{-1} \frac{e}{e'\Omega^{-1}e} = w'_X e \frac{1}{e'\Omega^{-1}e} = \frac{1}{e'\Omega^{-1}e}. \tag{3.17}$$

[4] No academic citation related to this property was not found. However, the existence of this property is proven in a non-academic source (https://quant.stackexchange.com/questions/8776/covariance-of-a-gmv-portfolio-with-any-asset). We, however, rewrite the property with a proof in our notations.

Applying this property to the each remaining element of the matrix discussed:

$$\sigma^2_{GMV} = \sigma_{GMV,GMV} = \sigma_{GMV,FI} = \sigma_{X,GMV} = \frac{1}{e'\Omega^{-1}e} \tag{3.18}$$

Let us take the scalar from 3.18 out of the matrix in 3.15:

$$\begin{pmatrix} \sigma^2_{FI} & -\sigma_{GMVFI} \\ -\sigma_{GMVFI} & \sigma^2_{GMV} \end{pmatrix} = \begin{pmatrix} w'_{FI}\Omega w_{FI} & -\frac{1}{e'\Omega^{-1}e} \\ -\frac{1}{e'\Omega^{-1}e} & \frac{1}{e'\Omega^{-1}e} \end{pmatrix} = \frac{1}{e'\Omega^{-1}e} \begin{pmatrix} w'_{FI}\Omega w_{FI} e'\Omega^{-1}e & -1 \\ -1 & 1 \end{pmatrix} \tag{3.19}$$

Now, let us rewrite 3.15 and cancel a positive scalar from the numerator and denominator:

$$\begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix} = \frac{\frac{1}{e'\Omega^{-1}e} \begin{pmatrix} w'_{FI}\Omega w_{FI} e'\Omega^{-1}e & -1 \\ -1 & 1 \end{pmatrix} (\mu_{GMVFI} - r_f e)}{e' \frac{1}{e'\Omega^{-1}e} \begin{pmatrix} w'_{FI}\Omega w_{FI} e'\Omega^{-1}e & -1 \\ -1 & 1 \end{pmatrix} (\mu_{GMVFI} - r_f e)} =$$

$$= \frac{\begin{pmatrix} w'_{FI}\Omega w_{FI} e'\Omega^{-1}e & -1 \\ -1 & 1 \end{pmatrix} (\mu_{GMVFI} - r_f e)}{e' \begin{pmatrix} w'_{FI}\Omega w_{FI} e'\Omega^{-1}e & -1 \\ -1 & 1 \end{pmatrix} (\mu_{GMVFI} - r_f e)} \tag{3.20}$$

Let us analyse the denominator of equation 3.20:

$$e'\begin{pmatrix} w'_{FI}\Omega w_{FI}e'\Omega^{-1}e & -1 \\ -1 & 1 \end{pmatrix}(\mu_{GMVFI} - r_f e) =$$

$$= [1\ 1]\begin{pmatrix} w'_{FI}\Omega w_{FI}e'\Omega^{-1}e & -1 \\ -1 & 1 \end{pmatrix}(\mu_{GMVFI} - r_f e) =$$

$$= (w'_{FI}\Omega w_{FI}e'\Omega^{-1}e - 1\ \ 0)(\mu_{GMVFI} - r_f e) = (w'_{FI}\Omega w_{FI}e'\Omega^{-1}e - 1\ \ 0)(\begin{matrix} \mu_{GMV} - r_f \\ \mu_{FI} - r_f \end{matrix}) =$$

$$= (w'_{FI}\Omega w_{FI}e'\Omega^{-1}e - 1)(\mu_{GMV} - r_f) = (w'_{FI}\Omega w_{FI}e'\Omega^{-1}e - 1)(\mu_{GMV} - r_f) \quad (3.21)$$

Let us substitute 3.21 into 3.20

$$\begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix} = \frac{\begin{pmatrix} w'_{FI}\Omega w_{FI}e'\Omega^{-1}e & -1 \\ -1 & 1 \end{pmatrix}(\mu_{GMVFI} - r_f e)}{(w'_{FI}\Omega w_{FI}e'\Omega^{-1}e - 1)(\mu_{GMV} - r_f)} =$$

$$= \frac{\begin{pmatrix} w'_{FI}\Omega w_{FI}e'\Omega^{-1}e & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \mu_{GMV} - r_f \\ \mu_{FI} - r_f \end{pmatrix}}{(w'_{FI}\Omega w_{FI}e'\Omega^{-1}e - 1)(\mu_{GMV} - r_f)} =$$

$$= \frac{\begin{pmatrix} w'_{FI}\Omega w_{FI}e'\Omega^{-1}e(\mu_{GMV}-r_f)-(\mu_{FI}-r_f) \\ -(\mu_{GMV}-r_f)+(\mu_{FI}-r_f) \end{pmatrix}}{(w'_{FI}\Omega w_{FI}e'\Omega^{-1}e-1)(\mu_{GMV}-r_f)} =$$

$$= \frac{\begin{pmatrix} w'_{FI}\Omega w_{FI}e'\Omega^{-1}e(\mu_{GMV}-r_f)-(\mu_{FI}-r_f) \\ -\mu_{GMV}+\mu_{FI} \end{pmatrix}}{(w'_{FI}\Omega w_{FI}e'\Omega^{-1}e-1)(\mu_{GMV}-r_f)} =$$

$$= \frac{\begin{pmatrix} \frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e}-r_f)-(\frac{\mu'FI}{e'FI}-r_f) \\ -\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e}+\frac{\mu'FI}{e'FI} \end{pmatrix}}{(\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e-1)(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e}-r_f)} \tag{3.22}$$

Thus, combining weights of individual stocks in the blended portfolio we find:

$$w_{TB} = (1-\alpha)w_{GMV} + \alpha w_{FI} = (1-\alpha)\frac{\Omega^{-1}e}{e'\Omega^{-1}e} + \alpha\frac{FI}{e'FI}\upsilon_{FI} \tag{3.23}$$

First, we will simplify each part and then we will bring them back together.

$$w_{GMV}(1-\alpha) = \frac{\frac{\Omega^{-1}e}{e'\Omega^{-1}e}[\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e}-r_f)-(\frac{\mu'FI}{e'FI}-r_f)]}{(\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e-1)(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e}-r_f)} =$$

$$= \frac{\Omega^{-1}e[\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e}-r_f)-\frac{1}{e'\Omega^{-1}e}(\frac{\mu'FI}{e'FI}-r_f)]}{(\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e-1)(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e}-r_f)} \tag{3.24}$$

$$w_{FI}\alpha = \frac{\frac{FI}{e'FI}(-\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e} + \frac{\mu'FI}{e'FI})}{(\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e - 1)(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e} - r_f)} \tag{3.25}$$

Combining last two equations:

$$w_{TB} = \frac{\frac{\Omega^{-1}e}{e'\Omega^{-1}e}[\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e} - r_f) - (\frac{\mu'FI}{e'FI} - r_f)] - \frac{FI}{e'FI}(-\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e} + \frac{\mu'FI}{e'FI})}{(\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e - 1)(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e} - r_f)} =$$

$$= \frac{\frac{\Omega^{-1}e}{e'\Omega^{-1}e}[\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}(\mu'\Omega^{-1}e - r_f e'\Omega^{-1}e) - (\frac{\mu'FI}{e'FI} - r_f)] - \frac{FI}{e'FI}(-\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e} + \frac{\mu'FI}{e'FI})}{(\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e - 1)(\frac{\mu'\Omega^{-1}e}{e'\Omega^{-1}e} - r_f)} =$$

$$= \frac{\Omega^{-1}e[\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}(\mu'\Omega^{-1}e - r_f e'\Omega^{-1}e) - (\frac{\mu'FI}{e'FI} - r_f)] + \frac{FI}{e'FI}(\mu'\Omega^{-1}e - \frac{\mu'FI}{e'FI}e'\Omega^{-1}e)}{(\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e - 1)(\mu'\Omega^{-1}e - r_f e'\Omega^{-1}e)} =$$

$$= \frac{[[\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}(\mu'\Omega^{-1}e - r_f e'\Omega^{-1}e) - (\frac{\mu'FI}{e'FI} - r_f)]\mathbf{I} + \frac{FI}{e'FI}(\mu' - \frac{\mu'FI}{e'FI}e')]\Omega^{-1}e}{(\frac{FI'}{e'FI}\Omega\frac{FI}{e'FI}e'\Omega^{-1}e - 1)(\mu' - r_f e')\Omega^{-1}e} =$$

$$= \frac{[[FI'\Omega FI(\mu'\Omega^{-1}e - r_f e'\Omega^{-1}e) - (\mu'FI - r_f e'FI)e'FI]\mathbf{I} + FI(\mu'e'FI - \mu'FIe')]\Omega^{-1}e}{(e'\Omega^{-1}eFI'\Omega FI - e'FIe'FI)(\mu' - r_f e')\Omega^{-1}e} =$$

$$= \frac{[[(\mu' - r_f e')\Omega^{-1}eFI'\Omega - (\mu' - r_f e')FIe']FI\mathbf{I} + FI\mu'(e'FI - FIe')]\Omega^{-1}e}{e'(\Omega^{-1}eFI'\Omega - FIe')FI(\mu' - r_f e')\Omega^{-1}e} =$$

$$= \frac{[(\mu' - r_f e')(\Omega^{-1}eFI'\Omega - FIe')FI\mathbf{I} + FI\mu'(\mathbf{I}e'FI - FIe')]\Omega^{-1}e}{e'(\Omega^{-1}eFI'\Omega - FIe')FI(\mu' - r_f e')\Omega^{-1}e} \tag{3.26}$$

Table 3.3: Industry specific p-values for difference in Sharpe ratios against equally-weighted benchmarks for 1995 - 2017 out-of-sample tests.
The proposed PAFI portfolios are compared to Arnott's FI (FI) for three different cases: taken purely, blended with the GMV in the predictive blended (PB) portfolio, and blended with the GMV in the tangency blended (TB) portfolio. For each of three cases, the we apply the methodology in Ledoit and Wolf (2008) to calculate heteroskedasticity and autocorrelation-consistent (HAC) p-values for the difference in Sharpe ratios against equally-weighted portfolio for every category. For most of the industries and super-industries the results are not statistically significant. The PB portfolios have statistically significant different Sharpe ratio (against the benchmark) only for the S&P 500, Sensitive super-industry, and Oil & Gas industry. The TB portfolios have such results only for the Oil & Gas and Financials. For the pure FI portfolios, only the Financials have statistically significant different Sharpe ratio against equally-weighted Financials.

	Pure portfolio:		PB based on:		TB based on:	
	FI	PAFI	FI	PAFI	FI	PAFI
S&P 500	0.240	0.519	0.029	0.060	0.187	0.118
Defensive	0.141	0.619	0.717	0.392	0.152	0.479
Sensitive	0.827	0.373	0.063	0.042	0.311	0.077
Cyclical	0.051	0.198	0.274	0.499	0.947	0.076
Oil & Gas	0.553	0.225	0.057	0.031	0.026	0.024
Basic Materials	0.177	0.182	0.833	0.789	0.669	0.604
Industrial	0.293	0.711	0.909	0.891	0.580	0.192
Consumer Goods	0.916	0.297	0.252	0.836	0.116	0.192
Health Care	0.103	0.830	0.323	0.384	0.443	0.719
Consumer Services	0.632	0.969	0.316	0.634	0.205	0.334
Utilities	0.208	0.733	0.259	0.137	0.118	0.958
Financials	0.033	0.253	0.317	0.182	0.056	0.063
Technology and Telecommunications	0.894	0.255	0.339	0.113	0.752	0.702

References

Andrew Ang, Monika Piazzesi, and Min Wei. What does the yield curve tell us about gdp growth? *Journal of econometrics*, 131(1-2):359–403, 2006.

Robert D. Arnott, Jason Hsu, and Philip Moore. Fundamental indexation. *Financial Analysts Journal*, 61(2):83–99, 2005.

Clifford Asness, Andrea Frazzini, Ronen Israel, and Tobias Moskowitz. Fact, fiction, and value investing. *The Journal of Portfolio Management*, 42(1):34–52, 2015.

Anup K Basu and Brigette Forbes. Does fundamental indexation lead to better risk-adjusted returns? new evidence from australian securities exchange. *Accounting & Finance*, 54 (3):699–728, 2014.

John M Bates and Clive WJ Granger. The combination of forecasts. *Journal of the Operational Research Society*, 20(4):451–468, 1969.

Dave Berger, Kuntara Pukthuanthong, and J Jimmy Yang. Is the diversification benefit of frontier markets realizable by mean-variance investors? the evidence of investable funds. *Journal of Portfolio Management*, 39(4):36, 2013.

Zvi Bodie, Alex Kane, and Alan J Marcus. *Investments*. McGraw-Hill Irwin, revised 1973 edition edition, 2014.

Stephen J Brown, William Goetzmann, Roger G Ibbotson, and Stephen A Ross. Survivorship bias in performance studies. *Review of Financial Studies*, 5(4):553–580, 1992.

Warren Buffett and Carol Loomis. Warren Buffett on the stock market. *Fortune Investor's Guide*, pages 80–94, 2001.

Craig Burnside and David Dollar. Aid, growth, the incentive regime, and poverty reduction. *The World Bank: Structure and Policies*, 3:210, 2000.

John Y Campbell and Robert J Shiller. Stock prices, earnings, and expected dividends. *The Journal of Finance*, 43(3):661–676, 1988.

Menzie Chinn and Kavan Kucko. The predictive power of the yield curve across countries and time. *International Finance*, 18(2):129–156, 2015.

Lois Corchon and Matthias Dahm. Foundations for contest success functions. *Economics Theory*, 43:81–98, 2010.

Lois Corchon and Matthias Dahm. Welfare maximizing contest success functions when the planner cannot commit. *Journal of Mathematical Economics*, 47:309–317, 2011.

Keith Cuthbertson and Dirk Nitzsche. *Quantitative financial economics: stocks, bonds and foreign exchange*. John Wiley & Sons, 2005.

Ian Domowitz and Ameya Moghe. Donuts: A picture of optimization applied to fundamental portfolios. *The Journal of Portfolio Management*, 44(3):103–113, 2018. ISSN 0095-4918. doi: 10.3905/jpm.2018.44.3.103. URL `http://jpm.iijournals.com/content/44/3/103`.

Frederick E Dopfel. The arithmetic of fundamental indexing. *The Journal of Investing*, 17 (2):114–118, 2008.

Jana Eklund and Sune Karlsson. Forecast combination and model averaging using predictive measures. *Econometric Reviews*, 26(2-4):329–363, 2007.

Neil R Ericsson. Economic forecasting in theory and practice: An interview with David F. Hendry. *International Journal of Forecasting*, 2017.

Eugene F Fama. Efficient capital markets: Ii. *The journal of finance*, 46(5):1575–1617, 1991.

James L Farrell. Analyzing covariation of returns to determine homogeneous stock groupings. *The Journal of Business*, 47(2):186–207, 1974.

James Douglas Hamilton. *Time series analysis*, volume 2. Princeton University Press Princeton, 1994.

Jia Hao, Stergios Skaperdas, and Vaidya Samarth. Contest functions: Theoretical foundations and issues in estimation. *International Journal of Industrial Organization*, 31(3): 211–222, 2013.

Campbell R Harvey. Forecasts of economic growth from the bond and stock markets. *Financial Analysts Journal*, 45(5):38–45, 1989.

Jack Hirshleifer and Evan Osborne. Truth, effort, and the legal battle. *Public Choice*, 108(1-2):169–195, 2001. ISSN 0048-5829. doi: 10.1023/A:1017520505573. URL `http://dx.doi.org/10.1023/A%3A1017520505573`.

KiHoon Hong and Eliza Wu. The roles of past returns and firm fundamentals in driving US stock price movements. *International Review of Financial Analysis*, 43(C):62–75, 2016.

Ravi Jagannathan and Tongshu Ma. Risk reduction in large portfolios: Why imposing the wrong constraints helps. *The Journal of Finance*, 58(4):1651–1684, 2003.

Hao Jia. A stochastic derivation of the ratio form of contest success functions. *Public Choice*, 135(3):125–130, 2008.

Philippe Jorion. Bayes - Stein estimation for portfolio analysis. *Journal of Financial and Quantitative Analysis*, 21(3):279–292, 1986.

Daniel Kahneman and Amos Tversky. Prospect theory: an analysis of decisions under risk. *Econometrica*, 47:263–292, 1979.

Oliver Ledoit and Michael Wolf. Robust performance hypothesis testing with the Sharpe ratio. *Journal of Empirical Finance*, 15(5):850–859, 2008.

Olivier Ledoit and Michael Wolf. Honey, I shrunk the sample covariance matrix. *The Journal of Portfolio Management*, 30(4):110–119, 2004.

Haim Levy. Stochastic dominance and expected utility: Survey and analysis. *Management science*, 38(4):555–593, 1992.

Burton G Malkiel and Eugene F Fama. Efficient capital markets: A review of theory and empirical work. *The journal of Finance*, 25(2):383–417, 1970.

Harry Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952.

Harry Markowitz. Mean–variance approximations to expected utility. *European Journal of Operational Research*, 234(2):346–355, 2014.

Harry M Markowitz, G Peter Todd, and William F Sharpe. *Mean-variance analysis in portfolio choice and capital markets*, volume 66. John Wiley & Sons, 2000.

James M Poterba and Lawrence H Summers. Mean reversion in stock prices: Evidence and implications. *Journal of Financial Economics*, 22(1):27–59, 1988.

Mark Rubinstein. Markowitz's 'portfolio selection': A fifty-year retrospective. *The Journal of finance*, 57(3):1041–1045, 2002.

Amartya Sen. Human development index: Methodology and measurement. 1994.

Stergios Skaperdas. Contest success functions. *Economic Theory*, 7:283–290, 1996.

Stergios Skaperdas and Samarth Vaidya. Persuasion as a contest. *Economics Theory*, 51: 465–486, 2012.

Jeremy Smith and Kenneth F Wallis. A simple explanation of the forecast combination puzzle. *Oxford Bulletin of Economics and Statistics*, 71(3):331–355, 2009.

Jack Treynor. Why market-valuation-indifferent indexing works. *Financial Analysts Journal*, 61(5):65–69, 2005.

Gordon Tullock. Efficient rent seeking. In J.M. Buchanan, R.D. Tollison, and Gordon Tullock, editors, *Toward a theory of the rent-seeking society*, pages 97–112. College Station: Texas AM Press, 1980.

Christian Walkshäusl and Sebastian Lobe. Fundamental indexing around the world. *Review of Financial Economics*, 19(3):117–127, 2010.