Entanglement Breaking Quantum Channels, Stochastic Matrices, and Primitivity

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ABSTRACT

ENTANGLEMENT BREAKING QUANTUM CHANNELS, STOCHASTIC MATRICES, AND PRIMITIVITY

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Quantum entanglement is a fundamental phenomenon in quantum information where states of different quantum systems are connected in a way that they cannot be described independently of each other irrespective of the spatial distance between them. We study a class of quantum channels called entanglement breaking channels. These channels break the presence of entanglement when acting on bipartite states. We also study an underlying structure describing such channels known as their Holevo form and generate stochastic matrices from them. Upon examination, we find that the nonzero spectrum and the Jordan canonical forms of such channels and their associated stochastic matrices are closely related. Focusing on conditions for primitivity of our channels, we further investigate the relationship between their primitivity indices and those of their matrices and eventually provide a quantum bound for the primitivity index of these channels. Finally, we also introduce the notion of the Holevo rank and provide a bound on these entanglement breaking channels in terms of this rank.

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Chapter 1

Introduction

Motivated by limitations of the transmission of classical information through communication channels, quantum information theory emerged, and has seen a rise over the last few decades. Through quantum information theory, previously impossible tasks are not so far fetched in the future. A fundamental and yet important aspect of quantum information is the theory of entanglement. Its origin dates back to the 1930s when Einstein and Schrödinger first discovered it. Entanglement is a phenomenon that allows for different quantum systems to stay connected and interact with each other regardless of how far apart they are. With that, quantum states from these composite systems are referred to as entangled and therefore cannot be split into a product of states from the individual quantum systems. Over the years, there have been many applications of entanglement in different aspects of quantum information theory such as; quantum computation [7], teleportation [33], communication[30] and cryptography [18]. At the same time, the theory of entanglement remains a very deep and challenging subject in quantum information and modern science.

The effects of entanglement on quantum channels have been extensively studied. It was observed that there exist a class of quantum channels that completely break the entanglement in input states, hence the name entanglement breaking channels [15]. More research into these channels brought about equivalent characterizations. Holevo in [10] generalized the finite dimensional results of entanglement breaking channels obtained in [15]. In [11], Holevo provided an equivalent representation of these channels that generalizes their operator sum form with rank one Kraus operators which later became known as the Holevo form. Ruskai in [26] continued the study by considering qubit entanglement breaking channels and came up with some interesting results. These among others led to a growing interest in the subject area. This is the motivation of this thesis.

In this thesis, we contribute to this study of entanglement breaking channels by showing how stochastic matrix representations are obtained from the Holevo form and we find that these representations have the same nonzero spectrum as the channels. We also show how the Jordan canonical forms of these channels and their matrix representations are very closely related. Using matrix theoretic characterizations, we identify when such channels are primitive and prove that the primitivity of the channels depend on that of their associated stochastic matrices. We will also introduce the notion of the Holevo rank and find a new bound for the primitivity of the channel in terms of this rank.

The organization of this thesis is outlined as follows: In Chapter 2, we present mathematical preliminaries that are important for much of what follows in the subsequent Chapters. Our presentation of this introductory material is based on the contents in [22, 13, 21]. Chapter 3 introduces entanglement breaking channels and provides insight into the various characterizations for these channels including the Holevo form. Here, we will also identify the relationship between the channels and positive partial transpose channels as well as complementary channels. This Chapter is based on presentations from [12, 15, 26, 16]. In Chapter 4, we will discuss how stochastic matrix representations are generated from the Holevo form. We will also consider when a quantum channel is primitive and describe primitivity indices of the channels and their associated matrices. By doing so, we will examine the iterative behaviour of such channels and show how their primitivity is dependent on those of their associated stochastic matrices. Finally, we will introduce the notion of Holevo rank and provide a tighter bound for the primitivity index of entanglement breaking channels in terms of the Holevo rank. New results presented in Chapter 4 are discussed in the recently published paper [1].

Chapter 2

Preliminaries

We begin by highlighting concepts and tools that are relevant to this thesis [13, 21].

2.1 Linear Algebra

A basis of a vector space is defined to be a set of linearly independent vectors which span the vector space. The number of vectors in the basis set determines the dimension of the vector space. Thus, if the cardinality of the basis is not finite, then the vector space is said to be infinite-dimensional. For instance, the vector space $\mathbb{C}^2$ has the standard basis

$$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

and therefore has dimension 2. In the space of real or complex vector spaces, we can define a function on the vector space in the form of the **inner product** between two vectors in the space. Given vectors $a = (a_0, a_1, ..., a_{n-1})$ and $b = (b_0, b_1, ..., b_{n-1})$ in a finite-dimensional vector space V, we define the inner product of a and b as

$$\langle a, b \rangle = \overline{a_0} b_0 + \overline{a_1} b_1 + \cdots + \overline{a_{n-1}} b_{n-1} = \sum_i \overline{a_i} b_i,$$

where $\overline{a_i}$ is the complex conjugate of each element in a. We will consider the inner product to be linear in the second argument and conjugate linear in the first argument.

Definition 2.1.1 (Inner Product Properties)**.** *Let V be a real or complex vector space. The inner product on V is a function which assigns a scalar value $\langle a, b \rangle$ to any two elements,*

$a, b \in V$ such that;

1. *for $c \in V$, $\langle a+b, c\rangle = \langle a, c\rangle + \langle b, c\rangle$*

2. *for a scalar α, $\langle a, \alpha b\rangle = \alpha\langle a, b\rangle$*

3. *$\langle a, b\rangle = \overline{\langle b, a\rangle}$*

4. *$\langle a, a\rangle > 0$ for $a \neq 0$.*

Note that a vector space equipped with an inner product is referred to as an inner product space.

Definition 2.1.2 (Hilbert space)**.** *An inner product space with a complete norm induced by its inner product is known as a Hilbert space.*

In this thesis, we will restrict our discussion to finite dimensional Hilbert spaces denoted $\mathcal{H}$ with an associated space of continuous linear operators, $\mathcal{L}(\mathcal{H})$. Note that every finite dimensional inner product space is automatically complete and hence it is a Hilbert space.

We will also use the standard Dirac notation, $|\cdot\rangle$ known as 'ket' and $\langle\cdot|$, the 'bra', to represent vector elements in a given Hilbert space. Note that the standard orthonormal basis of a Hilbert space $\mathcal{H}$ is given by $\{|0\rangle, |1\rangle, ..., |n-1\rangle\}$, where $|i\rangle$ represents the $n \times 1$ dimensional vector with 1 in the *ith* entry and 0's elsewhere. Therefore, every vector $|a\rangle \in \mathcal{H}$ can be represented as some linear combination of the basis elements as

$$|a\rangle = \sum_i a_i |i\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix},$$

for some scalars a_i.

The corresponding dual vector $\langle a|$ which is just the complex conjugate transpose of $|a\rangle$ (that is, $\langle a| = \overline{|a\rangle}^T$) is given by

$$\langle a| = \begin{pmatrix} \overline{a_0} & \overline{a_1} & \dots & \overline{a_{n-1}} \end{pmatrix}.$$

Thus, the inner product $\langle a, b\rangle$ will then be written in 'bra-ket' notation as $\langle a|b\rangle$ .We can also define the **outer product** of elements in a Hilbert space to be the matrix product

$$|a\rangle\langle b| = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \begin{pmatrix} \overline{b_0} & \overline{b_1} & \dots & \overline{b_{n-1}} \end{pmatrix} = \begin{pmatrix} a_0\overline{b_0} & a_0\overline{b_1} & \dots & a_0\overline{b_{n-1}} \\ a_1\overline{b_0} & a_1\overline{b_1} & \dots & a_1\overline{b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1}\overline{b_0} & a_{n-1}\overline{b_1} & \dots & a_{n-1}\overline{b_{n-1}} \end{pmatrix}.$$

Example 2.1.3. *Consider the vectors* $|a\rangle = \begin{pmatrix} 1 & 2i \end{pmatrix}^T$ *and* $|b\rangle = \begin{pmatrix} -i & 1 \end{pmatrix}^T$ *in* $\mathbb{C}^2$*. The inner product* $\langle a|b\rangle$ *is computed as*

$$\langle a|b\rangle = \begin{pmatrix} 1 & \overline{2i} \end{pmatrix} \begin{pmatrix} -i \\ 1 \end{pmatrix} = (1)(-i) + (-2i)(1) = -3i.$$

The outer product $|a\rangle\langle b|$ *is also computed as*

$$|a\rangle\langle b| = \begin{pmatrix} 1 \\ 2i \end{pmatrix} \begin{pmatrix} i & 1 \end{pmatrix} = \begin{pmatrix} i & 1 \\ -2 & 2i \end{pmatrix}.$$

We see that a combination of outer products of the standard orthonormal basis elements form a basis for the set of 2×2 *matrices. Thus, every outer product in this space can be written as a linear combination of these matrix basis elements.*

Definition 2.1.4 (Linear Operator)**.** *A linear operator between vector spaces* V *and* W *is defined by a function* $A : V \longrightarrow W$ *which is linear in its inputs, such that for* $|v_i\rangle \in V$

$$A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A(|v_i\rangle).$$

The identity operator I_V is an example of a simple, yet important linear operator on any vector space V. It is defined by the equation $I_V |v\rangle = |v\rangle$ for all vectors $|v\rangle$. We will also use the equivalent matrix representation of linear operators.

2.2 Matrix Theory

In this section, we discuss characterizations of matrices that are of importance to the understanding of the thesis. By $M_n(\mathbb{C})$, we will denote the set of $n \times n$ complex matrices. Given that $M_n(\mathbb{C})$ is finite dimensional, we can fix an orthonormal basis $\{E_{ij} = |i\rangle\langle j|\}$ of rank-one operators where in matrix form E_{ij} represents a 1 in the (i, j) entry and 0 otherwise [13, 21].

Definition 2.2.1 (Characteristic Value)**.** *Let* V *be a vector space and let* A *be a linear operator on* V. *A characteristic value of* A *is a scalar* λ *such that there is a non-zero vector* $|v\rangle$ *in* V *with* $\mathrm{A}\,|v\rangle = \lambda\,|v\rangle$.

The characteristic value λ is also referred to as the eigenvalue of the operator A. The vector $|v\rangle$ is known as the characteristic vector or equivalently eigenvector of λ.

Definition 2.2.2 (Trace)**.** *Let n be a positive integer and A an $n \times n$ matrix with scalar entries (a_{ij}). The trace of A is the scalar value*

$$\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_i a_{ii}.$$

Note that if $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A, we can also define the trace of A to be the sum of these eigenvalues. That is, $\mathrm{Tr}(A) = \sum_i \lambda_i$. We can also define an inner product for matrices $A, B \in M_n(\mathbb{C})$ known as the Hilbert-Schmidt trace inner product, which is given by $\langle A|B\rangle = \mathrm{Tr}(A^*B)$ where $A^* = \overline{A}^T$, the conjugate transpose of A.

Definition 2.2.3. *Let A be an element in the space $M_n(\mathbb{C})$, then*

1. *A is normal if $AA^* = A^*A$, where A^* is the conjugate transpose of A*
2. *A is unitary if $AA^* = A^*A = I_n$, where I_n is the identity $n \times n$ matrix*
3. *A is self-adjoint/ Hermitian if $A = A^*$.*

Definition 2.2.4 (Positive Semidefinite Matrix)**.** *Let A be a Hermitian matrix in $M_n(\mathbb{C})$. Then A is said to be positive semidefinite if $\langle Av|v\rangle \geq 0$ for all $|v\rangle \in \mathbb{C}^n$.*

Positive semidefinite matrices have only nonnegative eigenvalues and this is especially useful in quantum information theory. In the case that $\langle Av|v\rangle > 0$ for all non-zero $|v\rangle \in \mathbb{C}^n$, we say A is a positive definite matrix. We will denote a positive semidefinite matrix A by $A \geq 0$.

Definition 2.2.5 (Nonnegative Matrix)**.** *A matrix is nonnegative if all of its entries are nonnegative real numbers.*

Definition 2.2.6 (Probability Vector)**.** *A probability vector is a row or column vector whose entries are nonnegative and sum up to exactly one.*

The probability vector is used to describe the probabilities of any possible outcome. For instance, the row vector $v = (0.5, 0.5)$ can have each entry representing the odds of obtaining one face of a coin, that is, a head or tail. Notice that the row vector has non-negative entries which add up to one and is therefore a probability vector. Sometimes the probability vector is referred to as a stochastic vector.

Definition 2.2.7 (Stochastic Matrix)**.** *An $n \times n$ real matrix $S = (s_{ij})$ is column stochastic if*

$$0 \leq s_{ij} \leq 1 \quad \text{and} \quad \sum_i s_{ij} = 1,$$

for $j = 1, 2, ..., n$.

This shows that each entry of the stochastic matrix must be a nonnegative real number, and for each column j, the sum of its entries equals 1. Thus, each column of the stochastic matrix is a probability vector.

Theorem 2.2.8 (Spectral Decomposition)**.** *For a Hermitian operator $A \in M_n(\mathbb{C})$, there is an orthonormal basis for $\mathbb{C}^n$ of eigenvectors $\{|e_i\rangle\}_{i=1}^n$ to which correspond real eigenvalues a_i, such that*

$$A = \sum_{i=1}^{n} a_i |e_i\rangle\langle e_i| \, .$$

Definition 2.2.9. *For an $m \times n$ matrix $A = (a_{ij})$ where a_{ij} are the matrix entries of A, we define*

$$vec(A) = [a_{11}, ..., a_{m1}, a_{12}, ..., a_{m2}, ..., a_{1n}, ..., a_{mn}]^T \, .$$

Definition 2.2.10 (Direct Sum of Matrices)**.** *Let $A = (a_{ij})$ be an $n \times m$ matrix and $B = (b_{ij})$*

be a $p \times q$ matrix. We define the direct sum of matrices A and B to be the block matrix

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1m} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} & 0 & \dots & 0 \\ 0 & \dots & 0 & b_{11} & \dots & b_{1q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & b_{11} & \dots & b_{pq} \end{pmatrix}.$$

Definition 2.2.11. *Let A and B be $n \times n$ matrices. We say that A is similar to B ($A \sim B$) if there exists an invertible $n \times n$ matrix P such that $A = P^{-1}BP$.*

Note that a matrix $A \in M_n(\mathbb{C})$ is a diagonalizable matrix if $A \sim D$, where $D \in M_n(\mathbb{C})$ is a diagonal matrix.

Definition 2.2.12. *Let $J_n(\lambda)$ be an $n \times n$ matrix with characteristic value λ. Then we define the elementary Jordan matrix with characteristic value λ (Jordan block) to be the matrix*

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & 0 & \lambda & 1 \\ 0 & \dots & 0 & 0 & \lambda \end{pmatrix}. \tag{2.1}$$

For any matrix $M \in M_n(\mathbb{C})$, we can define the direct sum

$$\bigoplus_{k=1}^{s} J_{n_k}(\lambda_k) = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \dots & 0 \\ 0 & J_{n_2}(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_{n_s}(\lambda_s) \end{pmatrix},$$

such that $\bigoplus_{k=1}^{s} J_{n_k}(\lambda_k) \sim M$ where $\sum_{k=1}^{s} n_k = n$. Thus, there exists an invertible $n \times n$ matrix P such that

$$M = P \left(\bigoplus_{k=1}^{s} J_{n_k}(\lambda_k) \right) P^{-1}.$$

This direct sum $\bigoplus_{k=1}^{s} J_{n_k}(\lambda_k)$ is the Jordan canonical form of M and is unique up to reordering of the characteristic value λ_k. This representation may require dummy blocks $J_0(0)$.

Example 2.2.13. *Let us consider the* 3×3 *matrix*

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that M has eigenvalues $\lambda_1 = 1, \lambda_2 = 2$ with multiplicity 2, and M is not diagonizable. Therefore for $\lambda_1 = 1$, we have the elementary Jordan matrix $J_1 = (1)$. For $\lambda_2 = 2$, we have the elementary Jordan matrix $J_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Thus, the matrix M has Jordan canonical form

$$\bigoplus_{k=1}^{2} J_{n_k}(\lambda_k) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will make use of the following important theorem on Jordan forms of products of matrices.

Theorem 2.2.14 (Flanders' Theorem)**.** *[8, 6, 19] If A is an $m \times n$ matrix and B is an $n \times m$ matrix, then there exist $\lambda_1, ..., \lambda_s \in \mathbb{C}$ and nonnegative integers $m_1, ..., m_s$ and $n_1, ..., n_s$ such that for all k with $1 < k < s$,*

$$AB \sim \bigoplus_{k=1}^{s} J_{m_k}(\lambda_k) \quad \text{and} \quad BA \sim \bigoplus_{k=1}^{s} J_{n_k}(\lambda_k),$$

with $m_k = n_k$ if $\lambda_k \neq 0$ and $|m_k - n_k| \leq 1$ if $\lambda_k = 0$.

Here, we mean that the Jordan form of the square matrices AB and BA are identical except where the Jordan blocks are nilpotent, that is, a square matrix with zero eigenvalues.

2.3 Quantum Information Theory

Analogous to the concept of bits, the basic unit of classical information and computing, quantum bits (usually called qubits) are those associated in the quantum sense [22]. In the classical sense, bits are represented as either a 0 or 1. We can talk about the state of a quantum system by way of qubits, which are represented mathematically as the vectors $|0\rangle$

or $|1\rangle$ in the two-dimensional Hilbert space $\mathbb{C}^2$. States of qubits are not limited to $|0\rangle$ or $|1\rangle$. We can also have a linear combination of these states to generate other states in the form

$$|\phi\rangle = \alpha |0\rangle + \beta |1\rangle ,$$

with the normalization condition $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha, \beta \in \mathbb{C}$. Thus, the set of states of a qubit has the standard basis $\{|0\rangle, |1\rangle\}$ whose elements are orthonormal.

2.3.1 Quantum Measurements

As quantum systems are physical, they evolve as they interact with their environment. Thus, we seek to understand the events that occur when measurements are undertaken within such systems [22]. Suppose we have a set of operators $\{M_m\}$ known as measurement operators which act on a state $|\psi\rangle$ of some quantum system which is being measured. Then the probability that we obtain some outcome, indexed m is

$$p(m) = \langle\psi| M_m^* M_m |\psi\rangle . \tag{2.2}$$

After measurement, the state $|\psi\rangle$ collapses into another state known as the post-measurement state which is given by

$$\frac{M_m |\psi\rangle}{\sqrt{\langle\psi| M_m^* M_m |\psi\rangle}}. \tag{2.3}$$

Now since the probabilities $p(m)$ must sum up to 1, we have

$$1 = \sum_m p(m) = \sum_m \langle\psi| M_m^* M_m |\psi\rangle , \tag{2.4}$$

and this is only possible if

$$\sum_m M_m^* M_m = I. \tag{2.5}$$

Equation 2.5 is known as the completeness equation.

Example 2.3.1. *Suppose the state of a qubit can be written as*

$$|\psi\rangle = \frac{1}{\sqrt{3}} |0\rangle + \sqrt{\frac{2}{3}} |1\rangle .$$

Notice the normalization condition holds valid since $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\sqrt{\frac{2}{3}}\right)^2 = 1$. *Let* $M_0 = |0\rangle\langle 0|\,, M_1 = |1\rangle\langle 1|$ *be measurement operators which act on* $|\psi\rangle$. *The probability of obtaining the outcome 0 is*

$$p(0) = \langle\psi|\, M_0^* M_0 \,|\psi\rangle = \langle\psi|\, M_0 \,|\psi\rangle = \frac{1}{3},$$

with post-measurement state

$$\frac{M_0\,|\psi\rangle}{\sqrt{\langle\psi|\, M_0\,|\psi\rangle}} = \sqrt{3}\,|0\rangle\langle 0| \left(\frac{1}{\sqrt{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|1\rangle\right) = \sqrt{3}\frac{1}{\sqrt{3}}\,|0\rangle = |0\rangle\,.$$

The same can be done for outcome 1. This tells us that there is a 0.33% *chance that* $|\psi\rangle$ *will collapse to* $|0\rangle$ *and a* 0.67% *chance of it collapsing to* $|1\rangle$ *when measured.*

When a qubit is measured in the standard basis, it has the tendency to collapse into either $|0\rangle$ or $|1\rangle$. The probability of a qubit collapsing into the state $|0\rangle$ is $|\alpha|^2$ and into the the state $|1\rangle$ is $|\beta|^2$. In general, quantum states are unit vectors from n-dimensional Hilbert spaces. The orthonormal basis set for $\mathbb{C}^n$, $\{|i\rangle : 0 \leq i \leq n-1\}$ then becomes the standard computational basis for these spaces.

Definition 2.3.2 (Positive Operator Valued Measure)**.** *[22] A set* $\{F_k\}$ *is a Positive Operator Valued Measure (POVM) if it is a set of Hermitian positive semidefinite operators satisfying* $\sum_k F_k = I$.

Suppose a measurement performed by a set of measurement operators $\{M_m\}$ act on a state $|\psi\rangle$ of a quantum system. Then the probability of obtaining the outcome m is given by $p(m) = \langle\psi|\, M_m^* M_m \,|\psi\rangle$. Suppose we define $F_m = M_m^* M_m$. Then the set of positive operators such that $\sum_m F_m = I$ is known as a Positive Operator Valued Measure (POVM).

2.3.2 Density Operators

Quantum states are best described by way of using density matrices or operators. They describe the probability distribution of ensembles of quantum states [22].

Definition 2.3.3 (Density operator)**.** *A density operator* ρ *of a quantum state is a finite sum of the form*

$$\rho = \sum_i p_i\,|\phi_i\rangle\langle\phi_i|\,,$$

where $|\phi_i\rangle$ *are states (unit vectors) in the state space of* ρ *with an associated probability* p_i *such that* $\sum_i p_i = 1$.

A density operator is a pure state if it is known to be of the form $\rho = |\psi\rangle\langle\psi|$ where $\langle\psi|\psi\rangle = 1$. Otherwise, it is a mixed state. Pure states have trace $\mathrm{Tr}(\rho^2) = 1$ while mixed states have $\mathrm{Tr}(\rho^2) < 1$.

Proposition 2.3.1. *An operator* ρ *is the density operator of some ensemble* $\{p_i, |\phi_i\rangle\}$ *if and only if it satisfies the following conditions:*

1. *(Trace)* ρ *has unit trace,* $\mathrm{Tr}(\rho) = 1$.

2. *(Positivity)* ρ *is a positive operator,* $\rho \geq 0$.

Proof. Assume ρ is a density operator, then

$$\mathrm{Tr}(\rho) = \sum_i p_i \, \mathrm{Tr}(|\phi_i\rangle\langle\phi_i|) = \sum_i p_i = 1.$$

Now consider any vector $|\psi\rangle$, then

$$\langle\psi| \, \rho \, |\psi\rangle = \sum_i p_i \, \langle\psi|\phi_i\rangle \, \langle\phi_i|\psi\rangle = \sum_i p_i | \, \langle\psi|\phi_i\rangle \, |^2 \geq 0,$$

since $p_i \geq 0$ and $| \, \langle\psi|\phi_i\rangle \, |^2$ is always positive.

For the converse direction we can apply the spectral theorem. □

2.3.3 Composite Systems

We can consider states from a composition of quantum systems by way of the tensor product of the components of the Hilbert spaces in which they belong. A new composite system generated from Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted $\mathcal{H}_1 \otimes \mathcal{H}_2$ [22].

Definition 2.3.4 (Vector Tensor Product)**.** *Let* $|v\rangle \in \mathcal{H}_1$ *and* $|w\rangle \in \mathcal{H}_2$ *such that*

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{and} \quad |w\rangle = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}.$$

Then the tensor product $|v\rangle \otimes |w\rangle$ in the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$ is an $nm \times 1$ vector given by

$$|v\rangle \otimes |w\rangle = \begin{pmatrix} v_1 \cdot |w\rangle \\ v_2 \cdot |w\rangle \\ \vdots \\ v_n \cdot |w\rangle \end{pmatrix},$$

where each $v_i \cdot |w\rangle$ is an $m \times 1$ vector for each $i = 1, 2, ..., n$.

If the sets $\{|i_1\rangle\}_i$ and $\{|i_2\rangle\}_i$ form orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then the orthonormal set $\{|i_1\rangle\}_i \otimes \{|i_2\rangle\}_i$ is a basis for the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$. We can also define a tensor product in the same way for matrices.

Definition 2.3.5 (Matrix Tensor Product)**.** *The tensor product of an $n \times n$ matrix $A = (a_{ij})$ and an $m \times m$ matrix $B = (b_{ij})$ is given by*

$$A \otimes B = \left(\begin{array}{c|c|c|c} a_{11}B & a_{12}B & \dots & a_{1n}B \\ \hline a_{21}B & a_{22}B & \dots & a_{2n}B \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{array}\right),$$

where each $a_{ij}B$ is an $m \times m$ block matrix and the tensor product $A \otimes B$ is an $nm \times nm$ matrix.

Example 2.3.6. *Consider the matrices $A \in M_2(\mathbb{C})$ and $B \in M_3(\mathbb{C})$ given by*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad \textit{and} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The matrix tensor $A \otimes B \in M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$ *is computed such that*

$$A \otimes B = \begin{pmatrix} 1 \cdot B & 0 \cdot B \\ 0 \cdot B & i \cdot B \end{pmatrix} \tag{2.6}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & i & 0 & i \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 \end{array}\right). \tag{2.7}$$

The matrix $A \otimes B$ *therefore consists of 4 blocks of* 3×3 *matrices.*

Definition 2.3.7 (Partial Trace)**.** *The partial trace is a linear mapping* $\mathrm{Tr}_B : \mathcal{H}_A \otimes \mathcal{H}_B \longrightarrow \mathcal{H}_A$ *such that*

$$\mathrm{Tr}_B(S \otimes T) = \mathrm{Tr}(T)S$$

for any matrix S *on* $\mathcal{H}_A$ *and* T *on* $\mathcal{H}_B$*.*

We can also talk about Pauli matrices, which are known to be a set of three Hermitian and unitary matrices in $M_2(\mathbb{C})$ given by

$$\sigma_{\mathrm{X}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_{\mathrm{Y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_{\mathrm{Z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They are sometimes simply denoted by X, Y and Z respectively and are known to form a basis set for the space of all real 2×2 Hermitian matrices and as such, they span the space of observables of complex two dimensional Hilbert spaces. Thus, they are particularly useful in aspects of quantum computing such as quantum error correction [22].

2.3.4 Quantum Entanglement

Quantum entanglement as described by Einstein in the 1930's as "spooky action at a distance", allows for quantum systems to behave as though they are connected irrespective of the spacial distance between them. It is an important resource in quantum theory as it is used in quantum key distribution and teleportation for instance. Thus, research in the study of entanglement is actively ongoing [22].

Definition 2.3.8 (Separable and Entangled States)**.** *Given two quantum systems A and B with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, a pure state, $|\psi\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, is said to be separable if*

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle\,, \tag{2.8}$$

where $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$. Hence, we say a pure state is separable if it is a tensor product of pure states from each of the Hilbert spaces. Otherwise, the state is said to be entangled.

The notions of separable and entangled extend from vector states to operators as follows.

Definition 2.3.9. *A quantum state with density operator ρ from the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ is separable if*

$$\rho = \sum_{i=1}^{k} p_i \rho_A^i \otimes \rho_B^i, \tag{2.9}$$

with $\sum_{i=1}^{k} p_i = 1$ and ρ_A^i, ρ_B^i are density operators defined on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Otherwise, ρ is entangled.

Example 2.3.10. *Consider the so-called Bell state $|\Phi^+\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ such that*

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{2.10}$$

where here we are using the notation $|00\rangle = |0\rangle \otimes |0\rangle$ and $|11\rangle = |1\rangle \otimes |1\rangle$. This 2-qubit state is an entangled pure state.

To show this, suppose $|\Phi^+\rangle$ is separable. Then, there exist some $(a, b), (c, d) \in \mathbb{C}^2$ such that

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Note from above that, we have that $ac \neq 0$ and $bd \neq 0$. This implies that $a, b, c, d \neq 0$. Hence, ad and bc must not be zero. However, we have that $ad = 0$ and $bc = 0$; and therefore we have a contradiction. We can also see that the two blocks $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ are linearly independent. Hence, the Bell state is entangled.

Example 2.3.11. *The following are examples of widely used entangled states [16]:*

1. *The maximally entangled 2-qubit Bell states, also known as EPR (Einstein-Podolsky-Rosen) states in* $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad \left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \qquad \left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

2. *W-States are* m*-qubit states,* $|W\rangle \in (\mathbb{C}^2)^{\otimes m}$ *with* $m \geq 3$ *defined as*

$$|W\rangle = \frac{1}{\sqrt{m}}(|100\cdots 0\rangle + |010\cdots 0\rangle + \cdots + |0\cdots 01\rangle).$$

3. *The GHZ (Greenberger-Horne-Zeilnger) 3-qubit states are given by [9]*

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

which is generalized for m *quantum systems* $(\mathbb{C}^2)^{\otimes m}$ *to*

$$|GHZ\rangle_d^{(m)} = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}(|i\rangle^{\otimes m}),$$

where d *is the dimension of each quantum system [16].*

4. *The* $d \otimes d$ *bipartite Werner states [31]*

$$W(p) = p\frac{2}{d^2 - d}P^{(-)} + (1-p)\frac{2}{d^2 + d}P^{(+)}, \quad 0 \leq p \leq 1,$$

where we define projectors $P^{(+)} = \dfrac{I + \mathbb{F}}{2}$ *and* $P^{(-)} = \dfrac{I - \mathbb{F}}{2}$ *with identity* I*, and "flip" operator* $\mathbb{F}$ *defined by* $\mathbb{F}\,|\phi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\phi\rangle$*. Here,* d *is the dimension of each quantum system with* $d \geq 2$*.*

2.4 Completely Positive Maps and Quantum Channels

In this section, we will focus on completely positive maps and their characterizations that are heavily influenced by [4]. We will also briefly introduce quantum channels and provide examples for illustration.

Definition 2.4.1 (Positive Map)**.** *A linear map* $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ *is positive if and only if* $\Phi(A) \geq 0$ *for* $A \geq 0$.

Definition 2.4.2 (Completely Positive Map)**.** *[4] Given a linear map* $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$, Φ *is completely positive if the map* $id_k \otimes \Phi$ *is positive for all* k, *and where* $id_k : M_k(\mathbb{C}) \longrightarrow M_k(\mathbb{C})$ *is the identity map.*

That is to say, given a positive $n \times m$ block matrix with entries $a_{ij} \in M_n(\mathbb{C})$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{12} & \dots & a_{nm} \end{pmatrix},$$

we have that the map Φ is completely positive if

$$(id_k \otimes \Phi)(A) = \begin{pmatrix} \Phi(a_{11}) & \Phi(a_{12}) & \dots & \Phi(a_{1m}) \\ \Phi(a_{21}) & \Phi(a_{22}) & \dots & \Phi(a_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(a_{n1}) & \Phi(a_{12}) & \dots & \Phi(a_{nm}) \end{pmatrix}$$

is a positive matrix.

The following fundamental result of Choi [4] gives us the most widely used form of such maps.

Theorem 2.4.3. *[4] Let* $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$. *Then* Φ *is completely positive if and only if* Φ *is of the form*

$$\Phi(A) = \sum_k V_k^* A V_k, \tag{2.11}$$

where each $V_k \in M_{n,m}(\mathbb{C})$.

Proof. Let's begin by considering the 'if' part. It is straightforward to see that for a matrix $A \geq 0$, $V_k^* A V_k \geq 0$ for $A \in M_n(\mathbb{C})$ and $V_k \in M_{n,m}(\mathbb{C})$. Thus, the composite map

$$(id \otimes \Phi)A = \sum_k (I \otimes V_k^*) A (I \otimes V_k) \tag{2.12}$$

$$= \sum_k (I \otimes V_k)^* A (I \otimes V_k). \tag{2.13}$$

Similarly, $(I \otimes V_k)^* A (I \otimes V_k) \geq 0$ for $A \geq 0$ and so is the finite sum. Thus, Φ is completely positive. Conversely, consider a $1 \times mn$ matrix $v = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}$, where each x_i entry is regarded as a $1 \times m$ matrix $x_i = \begin{pmatrix} x_i^1 & \dots & x_j^m \end{pmatrix}$, we can have a corresponding $n \times m$ matrix V given by

$$V = \begin{pmatrix} \text{---} & x_1 & \text{---} \\ \text{---} & x_2 & \text{---} \\ & \vdots & \\ \text{---} & x_m & \text{---} \end{pmatrix},$$

where each x_i is the ith row. By that, we can have the computation

$$(V^* E_{ij} V)_{1 \leq i,j \leq n} = (x_i^* x_j)_{1 \leq i,j \leq n} = v^* v.$$

Thus, assume that $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ is a completely positive map. Since the standard basis matrices $(E_{ij})_{1 \leq i,j \leq n}$ are always positive, we have that $\Phi(E_{ij})$ is positive. Therefore, there exists vectors v_k^* such that $(\Phi(E_{ij}))_{1 \leq i,j \leq n} = \sum_k v_k^* v_k = \sum_k (V^* E_{ij} V)_{1 \leq i,j \leq n}$. Hence, $\Phi(A) = \sum_k V_k^* A V_k$ for all $A \in M_n(\mathbb{C})$. □

The form of a completely positive map as shown in Equation (2.11) is known as the operator sum representation of the map.

Remark 2.4.4. *[4] Consider the proof of Theorem 2.4.3. We know that there exists vectors v_i^* such that $(\Phi(E_{jk}))_{1 \leq j,k \leq n} = \sum_i v_i^* v_i$. These v_i^*'s are not unique. So, their associated V_i, the $n \times m$ matrices are equally not unique. We assume that $\{v_i^*\}$ to be linearly independent, then $\{V_i\}$ must be linearly independent. $\{V_i\}_i^l$ ensures that $\Phi(A) = \sum_i^l V_i^* A V_i$ is a canonical expression for Φ, in the following sense:*

Let $\{W_p\}_p^{l'}$ be a class of $n \times m$ matrices, then Φ has the expression $\Phi(A) = \sum_p^{l'} W_p^* A W_p$ if and only if there exists an isometric $l' \times l$ matrix $(\mu_{pi})_{pi}$, such that $W_p = \sum_i \mu_{pi} V_i$ for all p. Moreover, if $\{W_p\}_p^{l'}$ is also a linearly independent set, then $l' = l$, and $(\mu_{pi})_{pi}$ is unitary.

Theorem 2.4.5. *[4] Let $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$. Then Φ is an extreme completely positive map if and only if $\Phi(A) = \sum_i^l V_i^* A V_i$ for all $A \in M_n$, V_i are $n \times m$ matrices, $\sum_i V_i^* V_i = I$ and $\{V_i^* V_j\}$ linearly independent.*

Proof. Suppose Φ is an extreme completely positive map with canonical form $\Phi(\rho) = \sum_i V_i^* \rho V_i$, and $\{V_i\}$ linearly independent. Thus, suppose $\sum_{ij} \lambda_{ij} V_i^* V_j = 0$, then we want to show that the matrix $(\lambda_{ij})_{ij} = 0$. Assume that $(\lambda_{ij})_{ij}$ is a Hermitian matrix and by a scalar multiplication $-I \leq (\lambda_{ij})_{ij} \leq I$.

Define $\Psi_\pm : M_n(\mathbb{C}) \longrightarrow M_m$ by

$$\Psi_\pm(\rho) = \sum_i V_i^* \rho V_i \pm \sum_{ij} \lambda_{ij} V_i^* \rho V_j,$$

then $\Psi_\pm(I) = \sum_i V_i^* V_i = I$. Let $I + (\lambda_{ij})_{ij} = (a_{ij})_{ij}^* (a_{ij})_{ij}$. We have $\Psi_+(\rho) = \sum_j W_j^* \rho W_j$ with $W_j = \sum_i a_{ij} V_i$ by direct computation. Then, Ψ_+ is completely positive. We can also conclude in the same manner that Ψ_- is completely positive. Notice that $\Phi = \frac{1}{2}(\Psi_+ + \Psi_-)$, but Φ is extreme and so, $\Phi = \Psi_+ = \Psi_-$. By Remark 2.4.4, $(a_{ij})_{ij}$ is an isometry. So, $(a_{ij})_{ij}^* (a_{ij})_{ij} = I + (\lambda_{ij})_{ij} = I$ i.e, $(\lambda_{ij})_{ij} = 0$ as required. Therefore, $\{V_i^* V_j\}$ is linearly independent.

Conversely, assume $\Phi(\rho) = \sum_i V_i^* \rho V_i$, $\sum_i V_i^* V_i = I$ with $\{V_i^* V_j\}$ linearly independent (As such $\{V_i\}$ is a linearly independent set). Now assume $\Phi = \frac{1}{2}(\Psi_+ + \Psi_-)$, where

$$\Psi_+(\rho) = \sum_p W_p^* \rho W_p, \quad \sum_p W_p^* W_p = I,$$

$$\Psi_-(\rho) = \sum_k Z_k^* \rho Z_k, \quad \sum_k Z_k^* Z_k = I.$$

Then, $\Phi = \frac{1}{2} \sum_p W_p^* \rho W_p + \frac{1}{2} \sum_k Z_k^* \rho Z_k$, such that W_p and $Z_k \in span(V_i)$. Let $W_p = \sum_i \mu_{pi} V_i$, then, we have that,

$$\sum_i V_i^* V_i = \sum_p W_p^* W_p = \sum_{pij} \overline{\mu_{pi}} \mu_{pj} V_i^* V_j.$$

This implies that, $\sum_p \overline{\mu_{pi}} \mu_{pj} = \delta_{ij}$. That is, $(\mu_{pi})_{pi}$ is an isometry. Therefore, $\Phi = \Psi_+$ and so, Φ is an extreme completely positive map. □

Definition 2.4.6 (Choi Matrix)**.** *[4] The Choi matrix of a linear map $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$*

is defined as $C_\Phi = \sum_{ij} E_{ij} \otimes \Phi(E_{ij})$*, where* E_{ij} *is the matrix with 1 in the* (i,j)*th entry and 0's elsewhere.*

Theorem 2.4.7 (Choi's Theorem on Completely Positive Linear Maps)**.** *[4] Let* $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ *be a linear map. The following are equivalent:*

1. Φ *is n-positive; in other words,* $id_n \otimes \Phi$ *is a positive map.*
2. *The matrix with operator entries*

$$C_\Phi = (id_n \otimes \Phi)\left(\sum_{ij} E_{ij} \otimes E_{ij}\right) = \sum_{ij} E_{ij} \otimes \Phi(E_{ij})$$

is positive, where $E_{ij} \in M_n(\mathbb{C})$ *is the matrix with 1 in the ij-th entry and 0 elsewhere;*

3. Φ *is completely positive.*

It is quite clear that every completely positive map is a positive map. However, the converse is not always true. For instance, the transpose map T on $M_n(\mathbb{C})$ is known to be a positive but not completely positive map. To show this, we consider the matrix

$$A = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right),$$

such that we have

$$(id \otimes T)A = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

We can easily verify that the eigenvalues of the matrix are $\frac{1}{2}$ with multiplicity 3 and $\frac{-1}{2}$. Since one of its eigenvalues is negative, the transpose map is not a completely positive map.

Definition 2.4.8 (Dual Map)**.** *[21] For every positive trace preserving linear map* Φ*, the dual map* Φ^* *is defined by*

$$\mathrm{Tr}(\Phi(\rho)X) = \mathrm{Tr}\left(\rho\, \Phi^*(X)\right).$$

Definition 2.4.9 (Trace Preserving Map). *A linear map* $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ *is trace-preserving if* $\mathrm{Tr}(\Phi(A)) = \mathrm{Tr}(A)$ *for all* $A \in M_n(\mathbb{C})$.

Definition 2.4.10 (Unital map). *A linear map* $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ *is unital if* $\Phi(I_n) = I_m$.

Given a linear map $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$, we define the Choi matrix for Φ as

$$C_\Phi := \sum_{i,j}^{n} |e_i\rangle\langle e_j| \otimes \Phi(|e_i\rangle\langle e_j|), \tag{2.14}$$

with $\{|e_i\rangle\}$ being the orthonormal basis of some Hilbert space $\mathcal{H}$ which has dimension n [5]. This establishes a correspondence between linear maps from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$ and operators in the space $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$. We define the associated linear map $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ as

$$\Phi(\rho) := \mathrm{Tr}_B(C_\Phi \cdot (I \otimes \rho^T)), \tag{2.15}$$

where $\rho \in M_n(\mathbb{C})$, $I \in M_m(\mathbb{C})$ and Tr_B is the partial trace. This correspondence as described by Equations (2.14) and (2.15) is known as the Choi-Jamiołkowski isomorphism [4] [17]. Due to this isomorphism, we can obtain specific classes of linear maps and their associated operators. According to [17], the map Φ is positive if and only if for every product state $|a\rangle |b\rangle$, its associated Choi operator C_Φ is block-positive, that is,

$$\langle a| \langle b| C_\Phi |a\rangle |b\rangle \geq 0.$$

However, this operator C_Φ is not, in general, positive semidefinite. On the contrary, Φ is completely positive if and only if the associated Choi matrix is positive semidefinite as we have seen in Theorem 2.4.7 [4].

The most important class of completely positive maps in quantum information are quantum channels. Quantum channels are communication channels that are essential in the transmission of both classical and quantum information, and they are described mathematically as follows.

Definition 2.4.11 (Quantum Channel). *[22] A quantum channel is a completely positive trace preserving map.*

This definition is motivated by the postulates of quantum mechanics and the description of time evolution of open quantum systems in particular [22, 12]. Following from Definition

2.4.2 and 2.4.9 and the results above, we have the following equivalent characterization of quantum channels:

Theorem 2.4.12 (Kraus Representation of Quantum Channels). *[22] A linear map $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ is a quantum channel if and only if there exist operators $\{V_i\}$ such that*

$$\Phi(A) = \sum_i V_i A V_i^* \quad \text{and} \quad \sum_i V_i^* V_i = I.$$

The condition $\sum_i V_i^* V_i = I$ is equivalent to the trace preserving condition in the following sense; A trace preserving channel has for all $A \in M_n(\mathbb{C})$,

$$\begin{aligned}
\mathrm{Tr}(A) &= \mathrm{Tr}(\Phi(A)) \\
&= \mathrm{Tr}\left(\sum_i V_i A V_i^*\right) \\
&= \mathrm{Tr}\left(\sum_i A V_i^* V_i\right) \quad \text{by the cyclic property of the trace function,} \\
&= \mathrm{Tr}\left(A \sum_i V_i^* V_i\right).
\end{aligned}$$

This is only true when $\sum_i V_i^* V_i = I$. The matrices V_i are known as Kraus operators of Φ and are not unique. It turns out that whenever $\{V_i\}_{i=1}^m$ and $\{W_j\}_{i=1}^n$ are a set of Kraus operators for a quantum channel Φ, there exist a unitary matrix $U = (u_{ij}) \in M_n(\mathbb{C})$ such that $V_i = \sum_j u_{ij} W_j$. This relation is only achieved by assigning zero operators to the set with a lesser dimension so that they can have equal dimension.

Example 2.4.13. *An example of a quantum channel is the phase-damping channel which has Kraus operators*

$$M_0 = \sqrt{1-p}I, \quad M_1 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix},$$

with $p \in [0, 1]$.

In the next chapter, we will turn to the class of channels that will be our focus for the rest of the thesis.

Chapter 3

Entanglement Breaking Channels

The theory of quantum entanglement has been a very important subject in quantum information and has gained much attention in the last two decades. Entanglement is an important resource in the creation of powerful private keys for data encryption and the teleportation of information, however there is still a lot to be discovered around it. As important as entanglement is in the transmission of information, there exist an important class of channels that break entanglement in the transmission process. In this chapter, we seek to understand the associated characterizations of these so-called Entanglement Breaking Channels.

3.1 Entanglement Breaking Channels

Definition 3.1.1. *[12]A quantum channel $\Phi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ is called entanglement breaking if for an arbitrary input state $\rho \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, the output state $(id_n \otimes \Phi)(\rho)$ is always separable.*

This means that every quantum state, whether entangled or not, is always mapped into a separable one under the composite map $(id \otimes \Phi)$ of an entanglement breaking channel. There are many equivalent characterizations of entanglement breaking channels, however that which is of relevance to this thesis include the following.

An entanglement breaking channel Φ as shown by Holevo in [27] can be written in the form,

$$\Phi(\rho) = \sum_k \operatorname{Tr}(F_k \rho) R_k, \tag{3.1}$$

where each R_k is a density operator and $\{F_k\}$ form a POVM, that is, $F_k \geq 0$, and $\sum_k F_k = I$. The form as shown in Equation 3.1 is known as the Holevo form of the quantum channel Φ.

This form is entirely dependent on the choice of $\{R_k\}$ and $\{F_k\}$ and as such, the Holevo form of a quantum channel is not unique. There are at least two subclassifications of entanglement breaking channels that are of interest [12]:

i. A classical-quantum (CQ) channel is a quantum channel that maps classical information into quantum information. Mathematically, we represent CQ channels as those channels with which each $F_k = |k\rangle\langle k|$ in Equation (3.1) is a one-dimensional projection, that is,

$$\Phi(\rho) = \sum_k R_k \operatorname{Tr}(|k\rangle\langle k| \rho),$$

where $\{|k\rangle\}$ is an orthonormal basis set. Note that an extreme CQ map is given by

$$\Phi(\rho) = \sum_k |\varphi_k\rangle\langle\varphi_k| \langle k| \rho |k\rangle , \tag{3.2}$$

where $R_k = |\varphi_k\rangle\langle\varphi_k|$ is a pure state and $\{k\}$ is an orthonormal set.

ii. A quantum-classical (QC) channel is a quantum channel that maps quantum information into a classical output such that if each density matrix $R_k = |k\rangle\langle k|$ is a one-dimensional projection and $\sum_k R_k = I$. Thus, QC channels have the form

$$\Phi(\rho) = \sum_k |k\rangle\langle k| \operatorname{Tr}(F_k\rho).$$

We can also outline some equivalent characterizations of entanglement breaking channels as follows.

Theorem 3.1.2. *[15]The following are equivalent*

1. *Φ has the Holevo form as in Equation 3.1.*

2. *Φ is entanglement breaking.*

3. *The Choi matrix $C_\Phi = (I \otimes \Phi)(|\beta\rangle\langle\beta|)$ is separable for $|\beta\rangle = d^{-1/2}\sum_j |j\rangle \otimes |j\rangle$ a maximally entangled state.*

4. *Φ can be written in operator sum form using only Kraus operators of rank one.*

5. *$\Upsilon \circ \Phi$ is completely positive for all positivity preserving maps Υ.*

6. $\Phi \circ \Upsilon$ is completely positive for all positivity preserving maps Υ.

Proof. 1 $\implies$ 2. Suppose Φ has Holevo form

$$\Phi(\rho) = \sum_k \text{Tr}(F_k \rho) R_k.$$

Let $\rho_{ij}, i, j = 1, ..., n$ be the (i, j)th block of ρ. Then,

$$\begin{aligned}
(id \otimes \Phi)(\rho) &= \begin{pmatrix} \Phi(\rho_{11}) & \dots & \Phi(\rho_{1n}) \\ \vdots & \ddots & \vdots \\ \Phi(\rho_{n1} & \dots & \Phi(\rho_{nn}) \end{pmatrix} \\
&= \sum_k \begin{pmatrix} \text{Tr}(F_k \rho_{11}) R_k & \dots & \text{Tr}(F_k \rho_{1n}) R_k \\ \vdots & \ddots & \vdots \\ \text{Tr}(F_k \rho_{n1}) R_k & \dots & \text{Tr}(F_k \rho_{nn}) R_k \end{pmatrix} \\
&= \sum_k \begin{pmatrix} \text{Tr}(F_k \rho_{11}) & \dots & \text{Tr}(F_k \rho_{1n}) \\ \vdots & \ddots & \vdots \\ \text{Tr}(F_k \rho_{n1}) & \dots & \text{Tr}(F_k \rho_{nn}) \end{pmatrix} \otimes R_k \\
&= \sum_k \text{Tr}_B((I \otimes F_k)\rho) \otimes R_k
\end{aligned}$$

since $\text{Tr}_B((I \otimes F_k)\rho) = (\text{Tr}(F_k\ \rho_{ij}))_{1 \leq i,j \leq n}$. Thus, Φ is entanglement breaking since the tensor product $(id \otimes \Phi)(\rho)$ is always separable.

2 $\implies$ 3. By Definition 3.1.1, it is quite straightforward that $C_\Phi = (I \otimes \Phi)(|\beta\rangle\langle\beta|)$ is separable whenever Φ is entanglement breaking.

3 $\implies$ 4. Assume the Choi matrix C_Φ is separable. Then C_Φ can be written as a separable state given by

$$C_\Phi = \sum_i \lambda_{ij} \rho_i \otimes \rho_j,$$

where $\lambda_{ij} \geq 0, \sum_{ij} \lambda_{ij} = 1$ and ρ_i, ρ_j are density operators. By the Choi-Jamiołkowski correspondence [4] [17] between operators and linear maps, we have a linear map

$$\begin{aligned}\Phi(X) &= \mathrm{Tr}_B(C_\Phi(I\otimes X^T))\\ &= \mathrm{Tr}_B\left(\sum_i \lambda_{ij}\rho_i\otimes\rho_j(I\otimes X^T)\right)\\ &= \sum_i \lambda_{ij}\,\mathrm{Tr}_B\left(\rho_i\otimes\rho_j X^T\right)\\ &= \sum_i \lambda_{ij}\,\mathrm{Tr}\big(\rho_j X^T\big)\rho_i.\end{aligned}$$

Assume ρ_i,ρ_j are rank one operators such that $\rho_i^T = |\psi_i\rangle\langle\psi_i|$ and $\rho_j = |\phi_j\rangle\langle\phi_j|$. Note that $\mathrm{Tr}\big(\rho_j X^T\big) = \mathrm{Tr}\big(\rho_j^T X\big)$. Therefore,

$$\begin{aligned}\Phi(X) &= \sum_i \lambda_{ij}\,\mathrm{Tr}(|\psi_j\rangle\langle\psi_j|\,X)\,|\phi_i\rangle\langle\phi_i|\\ &= \sum_i \lambda_{ij}\,\langle\psi_j|\,X\,|\psi_j\rangle\,|\phi_i\rangle\langle\phi_i|\\ &= \sum_i \lambda_{ij}\,|\phi_i\rangle\langle\psi_j|\,X\,|\psi_j\rangle\langle\phi_i|\\ &= \sum_i \left(\sqrt{\lambda_{ij}}\,|\psi_j\rangle\langle\phi_i|\right) X \left(\sqrt{\lambda_{ij}}\,|\psi_j\rangle\langle\phi_i|\right)^*.\end{aligned}$$

Thus, if we take $V_k = \left(\sqrt{\lambda_{ij}}\,|\psi_j\rangle\langle\phi_i|\right)$, then we have that $\Phi(X)$ can be written in Kraus representation form with only rank one Kraus operators.

$4 \implies 5$. Suppose Φ has Kraus representation, then Φ is a completely positive map and as such Φ is also positive. Now consider the Choi matrix

$$\begin{aligned}C_{\Upsilon\circ\Phi} &= (id\otimes\Upsilon\circ\Phi)(|\beta\rangle\langle\beta|)\\ &= \frac{1}{\sqrt{d}}\sum_{ij}|i\rangle\langle j|\otimes\Upsilon\circ\Phi(|i\rangle\langle j|)\\ &= \frac{1}{\sqrt{d}}\sum_{ij}|i\rangle\langle j|\otimes\Upsilon(\Phi(|i\rangle\langle j|)).\end{aligned}$$

Now since $\Phi(|i\rangle\langle j|)$ is positive and Υ is a positivity preserving map, $\Upsilon(\Phi(|i\rangle\langle j|))$ is positive. Each $|i\rangle\langle j|$ is positive as they are projectors and so each tensor product is also positive. Clearly, the sum of these positive operators yields a positive operator. Thus, the Choi matrix is positive. By Theorem 2.4.7, $\Upsilon\circ\Phi$ is completely positive.

$5 \implies 6$. Suppose $\Upsilon \circ \Phi$ is completely positive and Υ a positivity preserving map. Then the dual Υ^* is also a positivity preserving map. Similarly, if Φ has a Kraus representation, so does the dual map Φ^*. As such, the composite map $\Upsilon^* \circ \Phi^* = (\Phi \circ \Upsilon)^*$ is also completely positive. Therefore, the composite map $\Phi \circ \Upsilon$ is a completely positive map.

$6 \iff 2$. Suppose Φ is an entanglement breaking channel, then for any density matrix $\rho \in M_n(\mathbb{C})$, $(id \otimes \Phi)(\rho)$ is separable. We see that for

$$\begin{aligned}(id \otimes \Upsilon \circ \Phi)(\rho) &= (id \otimes \Upsilon) \circ (id \otimes \Phi)(\rho) \\ &= (id \otimes \Upsilon) \left(\sum_{ij} \lambda_{ij} \; \rho_i \otimes \rho_j \right) \\ &= \sum_{ij} \lambda_{ij} \; \rho_i \otimes \Upsilon(\rho_j).\end{aligned}$$

Therefore, we see $(id \otimes \Upsilon \circ \Phi)(\rho) \geq 0$ since it is a sum of positive operators $\rho_i \otimes \Upsilon(\rho_j)$ for Υ a positivity preserving map. Thus, $\Upsilon \circ \Phi$ is a completely positive map. However, we have already proven that $\Phi \circ \Upsilon$ is a completely positive map whenever $\Upsilon \circ \Phi$ for all positivity preserving maps Υ.

Conversely, suppose the composite map $\Phi \circ \Upsilon$ is a completely positive map for a positivity preserving map Υ. Assume that $X = (id \otimes \Phi)(\rho)$ is a density matrix whenever ρ is a density matrix. Then, $(id \otimes \Phi \circ \Upsilon)(X) \geq 0$. According to [14], a density matrix ρ is separable if and only if $(id \otimes \Psi)(\rho) \geq 0$ for all positivity preserving maps Ψ. As such, $X = (id \otimes \Phi)(\rho)$ is separable. Therefore, Φ is an entanglement breaking channel.

$4 \iff 1$. Suppose Φ has operator sum form $\Phi(X) = \sum_i V_i X V_i^*$, where the Kraus operators V_i are rank one. As such, take $V_i = |\phi_i\rangle\langle\psi_i|$. Then we have

$$\begin{aligned}\Phi(X) &= \sum_i V_i X V_i^* \\ &= \sum_i (|\phi_i\rangle\langle\psi_i|) X (|\phi_i\rangle\langle\psi|_i)^* \\ &= \sum_i |\phi_i\rangle\langle\psi_i| \, X \, |\psi_i\rangle\langle\phi_i| \\ &= \sum_i \langle\psi_i| \, X \, |\psi_i\rangle \, |\phi_i\rangle\langle\phi_i| \\ &= \sum_i \mathrm{Tr}(X \, |\psi_i\rangle\langle\psi_i|) \, |\phi_i\rangle\langle\phi_i| \, .\end{aligned}$$

Take $F_i = |\psi_i\rangle\langle\psi_i|$ and $R_i = |\phi_i\rangle\langle\phi_i|$. Notice that $\sum_i |\psi_i\rangle\langle\psi_i| = I$ and $R_i = |\phi_i\rangle\langle\phi_i|$ is a density matrix. Thus, Φ can be written in the Holevo form. □

Example 3.1.3. *An important entanglement breaking channel is known as the completely depolarizing channel Φ_{CD} which is given by*

$$\Phi_{CD}(\rho) = \frac{1}{N}\,\mathrm{Tr}(\rho) I_N, \tag{3.3}$$

which is implemented with Holevo form where $F_1 = I_N, R_1 = \frac{1}{N} I_N$. To see the rank one form of Φ_{CD}, take an orthonormal set $\{|i\rangle : 0 \le i \le N-1\}$. Then,

$$\begin{aligned}\Phi_{CD}(\rho) &= \frac{1}{N}\,\mathrm{Tr}(\rho) I_N \\ &= \frac{1}{N}\,\mathrm{Tr}(\rho) \sum_i |i\rangle\langle i| \\ &= \frac{1}{N} \sum_i \left(\mathrm{Tr}\left(\sum_j |j\rangle\langle j| \, \rho \right) \right) |i\rangle\langle i| \\ &= \frac{1}{N} \sum_i \left(\sum_j \langle j| \, \rho \, |j\rangle \right) |i\rangle\langle i| \\ &= \frac{1}{N} \sum_{i,j} \langle j| \, \rho \, |j\rangle \, |i\rangle\langle i| \, .\end{aligned}$$

This gives us

$$\begin{aligned}\Phi_{CD}(\rho) &= \frac{1}{N}\sum_{i,j} |i\rangle\langle j|\,\rho\,|j\rangle\langle i| \\ &= \sum_{i,j=0}^{N-1} V_{ij}\rho V_{ij}^*,\end{aligned}$$

where $V_{ij} = \frac{1}{\sqrt{N}}|i\rangle\langle j|$ *is the rank one Kraus operator of* Φ_{CD}.

Also, to check the trace preserving condition, notice that

$$\begin{aligned}\sum_{i,j} V_{ij}^* V_{ij} &= \frac{1}{N}\sum_{i,j=0}^{N-1} |j\rangle\langle i|\,\rho\,|i\rangle\langle j| \\ &= \frac{1}{N}\sum_{j=0}^{N-1} |j\rangle \left(\sum_i \langle i|i\rangle\right)\langle j| \\ &= \frac{1}{N}\sum_{j=0}^{N-1} |j\rangle \left(\sum_{i=0}^{N-1} 1\right)\langle j| \\ &= \sum_{j=0}^{N-1} |j\rangle\langle j| \\ &= I_N.\end{aligned}$$

We can also check to see that Φ_{CD} is entanglement breaking when $N = 2$ using the maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The density operator for the maximally entangled state is given by

$$\begin{aligned}\left|\Phi^+\right\rangle\!\left\langle\Phi^+\right| &= \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \\ &= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \\ &= \frac{1}{2}(|0\rangle\langle 0|\otimes|0\rangle\langle 0| + |0\rangle\langle 1|\otimes|0\rangle\langle 1| + |1\rangle\langle 0|\otimes|1\rangle\langle 0| + |1\rangle\langle 1|\otimes|1\rangle\langle 1|).\end{aligned}$$

So, we have

$$
\begin{aligned}
(id_2 \otimes \Phi_{CD})(|\Phi^+\rangle\langle\Phi^+|) &= \frac{1}{2}(|0\rangle\langle 0| \otimes \Phi_{CD}(|0\rangle\langle 0|) + |0\rangle\langle 1| \otimes \Phi_{CD}(|0\rangle\langle 1|) + |1\rangle\langle 0| \otimes \Phi_{CD}(|1\rangle\langle 0|) \\
&\quad + |1\rangle\langle 1| \otimes \Phi_{CD}(|1\rangle\langle 1|)) \\
&= \frac{1}{2}\left(|0\rangle\langle 0| \otimes \frac{1}{2}I_2 + |0\rangle\langle 1| \otimes \left(\frac{0}{2}I_2\right) + |1\rangle\langle 0| \otimes \left(\frac{0}{2}I_2\right) + |1\rangle\langle 1| \otimes \frac{1}{2}I_2\right) \\
&= \frac{1}{2}\left(|0\rangle\langle 0| \otimes \frac{1}{2}I_2 + |1\rangle\langle 1| \otimes \frac{1}{2}I_2\right) \\
&= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \otimes \left(\frac{1}{2}I_2\right) \\
&= \frac{1}{2}I_2 \otimes \frac{1}{2}I_2.
\end{aligned}
$$

Therefore, Φ_{CD} is entanglement breaking.

3.2 Positive Partial Transpose Channels and Complementary Channels

In this section, we discuss a related class of quantum channels [12].

Definition 3.2.1. *(Partial Transpose). The partial transpose of a density matrix ρ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}$ is the Hilbert space of system A and B respectively, is $\rho^{T_B} = (id \otimes T)(\rho)$ where* T *is the transpose map.*

Given a quantum state, a well known condition known as the Peres-Horodecki or Positive Partial Transpose (PPT) criterion can be used to detect the separability (or entanglement) of the given state by way of their density matrices. This separability condition introduced by Peres [24] is a necessary condition for a state to be separable. However, in Hilbert spaces with dimensions less or equal to 6, the criterion is also sufficient.

Theorem 3.2.2 (PPT Criterion)**.** *[12] If a state ρ is separable then, the partial transpose of ρ is positive. That is, $\rho^{T_B} \geq 0$.*

Proof. Let ρ be a separable state. By this,

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i.$$

Note that every density operator is a positive operator (that is, it has positive eigenvalues). Hence, both ρ_A^i and ρ_B^i have positive eigenvalues. Now, taking the partial transposition with respect to the second system, we have

$$\rho^{T_B} = \sum_i p_i \rho_A^i \otimes (\rho_B^i)^T.$$

The transpose map preserves the eigenvalues of any matrix, that is to say, the eigenvalues of ρ_B^i is equal to the eigenvalues of $(\rho_B^i)^T$ and a tensor product of positive operators is positive. By this, since the sum of positive operators is also a positive operator, $\rho^{T_B} \geq 0$. □

Definition 3.2.3 (PPT channel)**.** *[12] A channel $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ is PPT if for an arbitrary state $\rho \in M_n(\mathbb{C})$, $(id \otimes \Phi)(\rho)$ has a positive partial transpose.*

It is easy to see that every entanglement breaking channel Φ is PPT since $(id \otimes \Phi)(\rho)$ is always separable. Thus, by the PPT criterion, this holds true. However, not every PPT channel is entanglement breaking. We refer to these channels as Entanglement Binding.

Define a linear operator $V : \mathcal{H}_1 \longrightarrow \mathcal{H}_2 \otimes \mathcal{H}_3$

$$V = \sum_{k=1}^{n} V_k \otimes |k\rangle .$$

Now suppose we have a map $\Phi : L(\mathcal{H}_1) \longrightarrow L(\mathcal{H}_2)$ for finite dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and another map $\Psi : L(\mathcal{H}_1) \longrightarrow L(\mathcal{H}_3)$ such that

$$\Phi(X) = \mathrm{Tr}_{\mathcal{H}_3}(VXV^*).$$

Then the channel

$$\Psi(X) = \mathrm{Tr}_{\mathcal{H}_2}(VXV^*)$$

is **complementary** to Φ. If the operator V is an isometry, that is $V^*V = I_V$, both maps are considered to be trace preserving. Notice that

$$VXV^* = \sum_{k,l=1}^{n} V_k X V_l^* \otimes |k\rangle\langle l| .$$

Therefore, for a map $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ with Kraus Representation

$$\Phi(X) = \sum_{k=1}^{n} V_k X V_k^*,$$

the complementary channel $\Psi(X)$ reduces to

$$\Psi(X) = \sum_{k,l=1}^{n} \mathrm{Tr}(V_k X V_l^*) \, |k\rangle\langle l| \, .$$

Definition 3.2.4 (Schur Product Channel)**.** *[12] A quantum channel $\Phi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ is said to be a Schur product channel if there exist a matrix C satisfying*

$$\Phi(\rho) = C \circ \rho,$$

where C and ρ have the same dimension and $\circ$ performs element-wise multiplication.

To further highlight how this element-wise multiplication is done, let us consider two matrices

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix} \qquad \text{and} \qquad \rho = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{pmatrix}.$$

The Schur product channel is computed as

$$\begin{aligned} \Phi(\rho) &= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix} \circ \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{pmatrix} \\ &= \begin{pmatrix} c_{11}\, p_{11} & c_{12}\, p_{12} & \dots & c_{1n}\, p_{1n} \\ c_{21}\, p_{21} & c_{22}\, p_{22} & \dots & c_{2n}\, p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}\, p_{n1} & c_{n2}\, p_{n2} & \dots & c_{nn}\, p_{nn} \end{pmatrix}. \end{aligned}$$

Example 3.2.5. *[12] Consider an entanglement breaking channel with Kraus representation given by*

$$\Phi(\rho) = \sum_{k=1}^{\tilde{d}} |\varphi_k\rangle\langle\psi_k| \, \rho \, |\psi_k\rangle\langle\varphi_k| \, . \tag{3.4}$$

We have the complementary channel Ψ of the channel Φ to be given as

$$\begin{aligned}\Psi(\rho) &= \sum_{k,l=1}^{\tilde{d}} \mathrm{Tr}(|\varphi_k\rangle\langle\psi_k| \, \rho \, |\psi_k\rangle\langle\varphi_k|) \, |k\rangle\langle l| \\ &= \sum_{k,l=1}^{\tilde{d}} \langle\varphi_k|\varphi_l\rangle \, \langle\psi_k| \, \rho \, |\psi_k\rangle \, |k\rangle\langle l| \\ &= \sum_{k,l=1}^{\tilde{d}} c_{kl} \, \langle\psi_k| \, \rho \, |\psi_l\rangle \, |k\rangle\langle l| \, ,\end{aligned}$$

where $c_{kl} = \langle\varphi_k|\varphi_l\rangle$. Notice that if $\rho = \sum_{k,l} p_{k,l} \, |\psi_k\rangle\langle\psi_l|$ is a density matrix in the orthonormal basis set $\{|\psi_k\rangle\}$, we have that

$$\begin{aligned}\Psi(\rho) &= \sum_{k,l=1}^{\tilde{d}} c_{kl} \, p_{k,l} \, |k\rangle\langle l| \\ &= ((c_{kl})(p_{kl}))_{1\leq k,l\leq \tilde{d}} \\ &= C \circ \rho,\end{aligned}$$

where $\Psi(\rho) = C \circ \rho$ is defined as the Schur product channel. Thus, we see that every entanglement breaking channel has a corresponding complementary channel which is a Schur product channel.

3.3 Extreme Points in the Convex Set of Entanglement Breaking Channels

In this section, we will discuss some results of extreme entanglement breaking channels. Some results from [4], as stated in Section 2.4 will prove useful here.

Definition 3.3.1 (Convex Set). *[3] A set C is said to be a convex set if for every $x, y \in C$*

and $0 \leq \theta \leq 1$, we have that $\theta x + (1-\theta)y \in C$.

Definition 3.3.2 (Extreme Points of a Convex Set)**.** *[3] A point z in a set C is extreme if there does not exist $x, y \in C$ and $0 < \theta < 1$, such that $x \neq y \neq z$ and $z = \theta x + (1-\theta)y$.*

Theorem 3.3.3. *[4] The set of entanglement breaking channels is convex.*

Proof. Let Φ_1 and Φ_2 be entanglement breaking channels with Holevo form

$$\Phi_1(\rho) = \sum_i R_i \operatorname{Tr}(F_i \rho) \quad \text{and} \quad \Phi_2(\rho) = \sum_j R_j \operatorname{Tr}(F_j \rho),$$

with $\{R_i\}_i$ and $\{R_j\}_j$, density operators and $\{F_i\}_i$ and $\{F_j\}_j$ form POVMs respectively. For some $0 \leq \lambda \leq 1$, we have

$$(\lambda\Phi_1 + (1-\lambda)\Phi_2)(\rho) = \lambda \sum_i R_i \operatorname{Tr}(F_i \rho) + (1-\lambda) \sum_j R_j \operatorname{Tr}(F_j \rho). \tag{3.5}$$

Thus the channel $(\lambda\Phi_1 + (1-\lambda)\Phi_2)(\rho)$ has the Holevo form as shown in Equation 3.5 where the set with elements λF_i and $(1-\lambda)F_j$ for all i, j also form a POVM. Therefore, it is also entanglement breaking, thus the set is convex. □

We will make use of the following Lemma.

Lemma 3.3.4. *[15] Let ρ be a density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$. If ρ is separable, ρ has rank d and $\rho_A = \operatorname{Tr}_B \rho$ has rank d, then ρ can be written as a convex combination of products of pure states using at most d products.*

Proof. Suppose ρ is separable, then,

$$\sum_{i=1}^{k} p_i \left|\phi_i\right\rangle\!\left\langle\phi_i\right| \otimes \left|\psi_i\right\rangle\!\left\langle\psi_i\right|. \tag{3.6}$$

Let us assume that $k > d$ and ρ cannot be written as the convex combination seen in Equation 3.6 using less than k product states. Assume that ρ_A has rank d. Then without loss of generality we can assume that the vectors $|\phi_1\rangle, |\phi_2\rangle, ..., |\phi_d\rangle$ are linearly independent. Suppose ρ also has rank d, then $d < k$ and the first $d+1$ vectors $|\phi_i\rangle \otimes |\psi_i\rangle$ must be linearly dependent so that we can find λ_j such that

$$\sum_{j=1}^{d+1} \lambda_j \left|\phi_i\right\rangle \otimes \left|\psi_i\right\rangle = 0.$$

By taking $\{|e_k\rangle\}$ to be the orthonormal basis for $\mathcal{H}_B$, we have that $\forall\ k$

$$\sum_{j=1}^{d+1} \lambda_j \langle e_k|\psi_j\rangle |\phi_j\rangle = 0. \tag{3.7}$$

We know that the first d vectors $|\phi_j\rangle$ are linearly independent, and as such there is a vector $|x\rangle$ in $\mathbb{C}^{d+1}$ such that $\sum_j v_j |\phi_j\rangle = 0$ if and only if $|v\rangle$ is a multiple of $|x\rangle$. Applying this to the coefficients in Equation 3.7, there are numbers ν_k such that $\lambda_j \langle e_k|\psi_k\rangle = \nu_k x_j$. Let's take $|\nu\rangle = \sum_k \nu_k |e_k\rangle$, then we find that

$$\lambda_j |\psi_j\rangle = x_j |\nu_k\rangle .$$

Since $|\psi_j\rangle$ was chosen to have norm 1, it follows that when $\lambda_j \neq 0$, $\left|\frac{x_j}{\lambda_j}\right| = 1$ and $|\psi_j\rangle = e^{i\theta_j}|\nu\rangle$. Thus, one can rewrite Equation 3.6 as

$$\rho = \sum_{j:\lambda_j\neq 0} p_j |\phi_j\rangle\langle\phi_j| \otimes |\psi_j\rangle\langle\psi_j| + \sum_{j:\lambda_j=0} p_j |\phi_j\rangle\langle\phi_j| \otimes |\nu\rangle\langle\nu| . \tag{3.8}$$

Suppose that t of the λ_j are nonzero. Since the vectors $\{|\phi\rangle : \lambda_j \neq 0\}$ are linearly dependent, the density matrix $\widetilde{\rho}_A = \sum_{j:\lambda_j=0} p_j |\phi_j\rangle\langle\phi_j|$ has rank strictly $< t$ and can be rewritten in the form $\widetilde{\rho}_A = \sum_{j=1}^{t'} p'_j \left|\phi'_j\right\rangle\left\langle\phi'_j\right|$ using only $t' < t$ vectors. Substituting this in Equation 3.8 gives ρ as linear combination of products using strictly less than k contradicting the assumption that Equation 3.6 used the minimum number. □

Note that if Φ, a completely positive map can be written with fewer than d Kraus operators, then it is not an entanglement breaking map and if it requires more than d Kraus operators, it is not extreme.

Theorem 3.3.5. *[15] Let $\Phi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ be a quantum channel.*

1. *If Φ is an extreme classical-quantum (CQ) channel, then Φ is an extreme point in the set of entanglement breaking channels.*
2. *If Φ is an extreme CQ channel, then Φ is an extreme point in the set of completely positive maps if and only if $\langle\varphi_j|\varphi_k\rangle \neq 0,\ \forall\ j,k$ in Equation 3.2.*
3. *If Φ is both in the set of entanglement breaking channels and an extreme point of the completely positive maps, then Φ is an extreme CQ channel.*

4. *When $d = 2$, the extreme points of the set of entanglement breaking maps are precisely the extreme CQ channel. When $d \geq 3$, there are extreme entanglement breaking maps which are not CQ.*

Proof. 1. Assume Φ is an extreme CQ channel and Φ is not an extreme entanglement breaking map. Then for some entanglement breaking channel $\Phi_1, \Phi_2 \neq \Phi$ with $0 < a < 1$, we have

$$\Phi = a\Phi_1 + (1-a)\Phi_2.$$

That is,

$$\Phi(\rho) = \sum_i t_i \left|\phi_i\right\rangle\!\left\langle\phi_i\right| \left\langle f_i|\rho|f_i\right\rangle,$$

such that $\Phi_1(\rho)$ and $\Phi_2(p)$ can be written in the same Holevo form 3.1 with different $t_i \geq 0$. By assumption, since Φ is an extreme CQ, it can be written as

$$\Phi(\rho) = \sum_k \left|\varphi_k\right\rangle\!\left\langle\varphi_k\right| \left\langle e_k|\rho|e_k\right\rangle, \tag{3.9}$$

with $\{e_k\}$ orthonormal. Thus, we have that

$$\Phi(|e_k\rangle\!\langle e_k|) = |\varphi_k\rangle\!\langle\varphi_k| = \sum_i t_i |\left\langle e_k|f_i\right\rangle|^2 \left|\phi_i\right\rangle\!\left\langle\phi_i\right|,$$

since $t_i \geq 0$ and $|\left\langle e_k|f_i\right\rangle|^2 \geq 0$, each rank one projection $|\varphi_k\rangle\!\langle\varphi_k|$ is a linear combination of positive coefficients of the projections $|\phi_i\rangle\!\langle\phi_i|$. However, this is only possible if every projection $|\phi_i\rangle\!\langle\phi_i|$ is equal to one of the projections $|\varphi_k\rangle\!\langle\varphi_k|$. We can relabel the projections $|\varphi_{k'}\rangle\!\langle\varphi_{k'}|$ so they are distinct and let $E_{k'} = \sum_{i \in k'} |e_i\rangle\!\langle e_i|$ where $\{e_i\}$ is the orthonormal set of vectors associated with $|\varphi_k\rangle\!\langle\varphi_k|$ in Equation 3.9 . Then $\{E_{k'}\}$ gives a partition of I into mutually orthogonal projections. Then, we can have

$$\Phi(\rho) = \sum_k |\varphi_k\rangle\!\langle\varphi_k| \operatorname{Tr} E_k \rho.$$

Also,

$$\Phi_1(\rho) = \sum_k |\varphi_k\rangle\!\langle\varphi_k| \operatorname{Tr} F_k \rho,$$
$$\Phi_2(\rho) = \sum_k |\varphi_k\rangle\!\langle\varphi_k| \operatorname{Tr} G_k \rho,$$

with $\{F_k\}$ and $\{G_k\}$ each a POVM. However, $|\varphi_k\rangle\langle\varphi_k|$ are chosen such that they are distinct and E_k orthonormal. Therefore,

$$\Phi = a\Phi_1 + (1-a)\Phi_2 \quad \iff \quad E_k = aF_k + (1-a)G_k.$$

Since $\theta \leqslant F_k, G_k \leqslant I$, this is only true if $F_k = G_k = E_k$. Therefore, $\Phi_1 = \Phi_2 = \Phi$. Thus, Φ is an extreme entanglement breaking channel.

2. Assume Φ is an extreme CQ channel, then

$$\Phi(p) = \sum_k |\varphi_k\rangle\langle\varphi_k| \langle e_k|\rho|e_k\rangle ,$$

with $\{e_k\}$ orthonormal. We can choose Kraus operators $V_k = |\varphi_k\rangle\langle e_k|$ such that

$$V_j^* V_k = |e_j\rangle\langle\varphi_j| \, |\varphi_k\rangle\langle e_k| = \langle\varphi_j|\varphi_k\rangle \, |e_j\rangle\langle e_k| \, .$$

The set $\left\{V_j^* V_k\right\}$ is linearly independent if and only if $\langle\varphi_j|\varphi_k\rangle \neq 0$ (i.e, none of the $|\psi_j\rangle$'s are mutually orthogonal). However, Φ is an extreme point in the set of completely positive maps if and only if $\left\{V_j^* V_k\right\}$ is linearly independent. Therefore, Φ is an extreme completely positive map if and only if $\langle\varphi_j|\varphi_k\rangle \neq 0 \quad \forall\, j, k$.

3. Assume that Φ is an entanglement breaking channel and an extreme completely positive map, then we can choose $\text{rank}(|\beta\rangle\langle\beta|) = d$ with $|\beta\rangle = d^{-1/2} \sum_j |j\rangle \otimes |j\rangle$. Then, by Lemma 3.3.4, $|\beta\rangle\langle\beta|$ can be written as a convex combination of pure states. Thus, Φ must be an extreme CQ channel.

4. According to Ruskai [26], every extreme point in the set of entanglement breaking qubit maps is a CQ map. Thus, the extreme qubit entanglement breaking is also an extreme point in the set of CQ maps. The extreme qubit entanglement breaking channels are exactly the extreme CQ maps.

□

3.4 Single Qubit Entanglement Breaking Maps as the Convex Hull of Classical-Quantum Channels

The identity and Pauli matrices form a basis $\{I, \sigma_X, \sigma_Y, \sigma_Z\}$ for $M_2(\mathbb{C})$ so that any 2×2 matrix A can be written as

$$A = w_0 I + \mathbf{w} \cdot \sigma,$$

where σ denotes the vector of Pauli matrices and $\mathbf{w} \in \mathbb{C}^3$. Every completely positive map $\Phi : M_2(\mathbb{C}) \longrightarrow M_2(\mathbb{C})$ can be represented by a matrix $\mathbb{T}$ in this basis such that

$$\Phi\left(w_0 I + \mathbf{w} \cdot \sigma\right) = w_0 I + (\mathbf{t} + \mathbf{T}\mathbf{w}) \cdot \sigma,$$

where $\mathbf{t}$ is the column vector with elements $t_k = t_{0k}, k = 1, 2, 3$ and $\mathbf{T}$ is a 3×3 matrix, i.e., $\mathbb{T} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & \mathbf{T} \end{pmatrix}$. As shown in [20], without loss of generality, that is, after an appropriate change in bases, we can assume that $\mathbf{T}$ is a diagonal matrix with eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ so that $\mathbb{T}$ can be written in the canonical form

$$\mathbb{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}. \tag{3.10}$$

From here on, we will denote by $\mathcal{S}$, the closure of the set of extreme points of completely positive maps.

Theorem 3.4.1. *[26] Let Φ be a trace preserving map induced by the matrix $\mathbb{T}$ of the form 1. Then Φ belongs to $\mathcal{S}$ if and only if the matrix R_Φ is unitary where*

$$R_\Phi = \begin{pmatrix} \frac{t_1 + it_2}{(1+t_3+\lambda_3)^{1/2}(1-t_3-\lambda_3)^{1/2}} & \frac{\lambda_1+\lambda_2}{(1+t_3+\lambda_3)^{1/2}(1-t_3+\lambda_3)^{1/2}} \\ \frac{\lambda_1-\lambda_2}{(1+t_3-\lambda_3)^{1/2}(1-t_3-\lambda_3)^{1/2}} & \frac{t_1+it_2}{(1+t_3-\lambda_3)^{1/2}(1-t_3+\lambda_3)^{1/2}} \end{pmatrix}.$$

Geometrically, the convex set of unital trace preserving maps with $\mathbf{t} = \mathbf{0}$ forms a tetrahedron with four extreme points for which $[\lambda_1, \lambda_2, \lambda_3]$ is a permutation of $[\pm 1, \pm 1, \pm 1]$. The generalized extreme points of the set of completely positive maps as discussed by [20] , includes "quasi-extreme" points which correspond to the edges of the tetrahedron with the form $[\pm 1, s, \pm s]$. Thus, any arbitrary unital trace preserving map can be written as a convex combination of four true extreme points, or non-uniquely as a convex combination of two quasi-extreme points.

The channels corresponding to a permutation of $[\pm 1, 0, 0]$ belong to the subclass known as CQ channels.

Lemma 3.4.2. *[26]Every map Φ in $\mathcal{S}$ lies in one of two disjoint sets which allows it to be characterized as follows. Either*

1. *Φ is a generalized extreme point of $\mathcal{S}$, or*

2. *Φ is in the interior of a segment of a plane in $\mathcal{S}$.*

Sketch of proof

Suppose after reduction to canonical form Equation 3.10, for any map which is a generalized extreme point, the parameters λ_k must satisfy (up to permutation) $\lambda_3 = \lambda_1\lambda_2$. This is compatible with the sign change condition if and only if at least two of the $\lambda_k = 0$, which implies that Φ must be a CQ *map.*

When we consider maps for which R_Φ is not unitary, then we can assume, without loss of generality, that the singular values of R_Φ can be written as $\cos\theta_1$ and $\cos\theta_2$, that $\cos\theta_1 \geq \cos\theta_2$, and that $0 \leq \cos\theta_2 < 1$. We can use the singular value decomposition of R_Φ to write

$$R_\Phi = V \begin{pmatrix} \cos\theta_1 & 0 \\ 0 & \cos\theta_2 \end{pmatrix} W^\dagger = \frac{1}{2}U_+ + \frac{1}{2}U_-,$$

where $U_\pm = V \begin{pmatrix} e^{i\pm\theta_1} & 0 \\ 0 & e^{i\pm\theta_2} \end{pmatrix} W^\dagger$ and V, W are unitary. By this, Φ can be written as a non-trivial convex combination of at least two distinct completely positive maps, two of which are generalized extreme points. Hence, Φ can be written as a point on two distinct line segments in $\mathcal{S}$. Therefore, there is a segment of a plane in $\mathcal{S}$ which contains Φ and for which Φ does not lie on the boundary of the plane.

Let $\mathcal{T}$ denote the set of maps for which $\Phi \circ T$ is in $\mathcal{S}$. Since $\mathcal{T}$ is a convex set isomorphic to $\mathcal{S}$, its elements can also be broken into the two classes as above. The set of entanglement breaking maps is precisely $\mathcal{S} \cap \mathcal{T}$.

Theorem 3.4.3. *[26]Every extreme point of the set of entanglement-breaking qubit channels* *is a CQ map. Hence, the set of entanglement-breaking qubit maps is the convex hull of qubit CQ channels.*

Proof. Let Φ be in $\mathcal{S} \cap \mathcal{T}$ which is also a convex set. If Φ is a generalized extreme point of either $\mathcal{S}$ or $\mathcal{T}$, then the only possibility consistent with Φ being entanglement-breaking is that it is CQ. Thus we suppose that Φ belongs to class II for both $\mathcal{S}$ and $\mathcal{T}$. Then Φ lies within a plane in $\mathcal{S}$ and within a plane in $\mathcal{T}$. The intersection of these two planes is non-empty (since it contains Φ) and its intersection must contain a line segment in $\mathcal{S} \cap \mathcal{T}$ which contains Φ and for which Φ is not an endpoint. Therefore, Φ is not an extreme point of $\mathcal{S} \cap \mathcal{T}$. Thus all possible extreme points of $\mathcal{S} \cap \mathcal{T}$ must be generalized extreme points of $\mathcal{S}$ or $\mathcal{T}$, in which case they are CQ. Thus, the convex hull of CQ channels is $\mathcal{S} \cap \mathcal{T}$. □

Chapter 4

New Results on Entanglement Breaking Channels

In this chapter, we provide insight into how stochastic matrix representations are generated from the Holevo form of entanglement breaking channels. We will provide characterizations of these stochastic matrix representations and how this corresponds to the behaviour of their associated entanglement breaking channels. We will also show how our result of primitivity indices of an entanglement breaking channel and its stochastic matrix representations is optimal.

4.1 Stochastic Matrix Representation of Entanglement Breaking Channels

Let us consider the action of a map in the Holevo form as shown in Equation (3.1). Assume that ρ_0 is a density matrix such that the entanglement breaking map Φ acts on, that is $\rho = \Phi(\rho_0)$. Note that as ρ is a density matrix in the range of Φ, it belongs to the convex hull of the R_k's and can therefore be written as $\rho = \sum_m c_m R_m$ where the $c_m = \mathrm{Tr}(F_m \rho_0)$

are positive real numbers summing to one. As such, observe that

$$\begin{aligned}\Phi(\rho) &= \sum_k \operatorname{Tr}(F_k\rho)R_k \\ &= \sum_k \operatorname{Tr}\left(F_k\left(\sum_m c_m R_m\right)\right)R_k \\ &= \sum_k \Big(\sum_m c_m \operatorname{Tr}(F_k R_m)\Big)R_k,\end{aligned}$$

and hence $\Phi(\rho) = \sum_k a_k R_k$ where $a_k = \sum_m c_m \operatorname{Tr}(F_k R_m) \in \mathbb{R}$.

By a matrix multiplication $S\vec{c} = \vec{a}$ in which the (i,j) entry of the matrix S is $\operatorname{Tr}(F_i R_j)$, the vector $\vec{c} = (c_1\, c_2\, \ldots)$ is transformed to the vector $\vec{a} = (a_k)$, that is,

$$S = \big(\operatorname{Tr}(F_i R_j)\big)_{i,j}.$$

Notice that as all F_i and R_j are nonnegative, we have

$$\operatorname{Tr}(F_i R_j) = \operatorname{Tr}\Big(\sqrt{F_i}\sqrt{R_j}\Big)^*\Big(\sqrt{F_i}\sqrt{R_j}\Big) = \operatorname{Tr}\Big(\sqrt{F_i}R_j\sqrt{F_i}\Big) \geq 0,$$

and therefore, all entries of S are nonnegative. Also, notice that for each column j,

$$\sum_i \operatorname{Tr}(F_i R_j) = \operatorname{Tr}\left(\sum_i F_i R_j\right) = \operatorname{Tr}(R_j) = 1,$$

that is, the sum of all the entries in any column of this matrix equals one since $\{F_i\}$ forms a POVM and R_j is a density matrix. Therefore, we have observed that any representation of an entanglement breaking channel in the Holevo form is a (column) stochastic matrix $S = (s_{ij})_{i,j}$, that is,

$$S = \begin{pmatrix} s_{11} & s_{12} & \ldots & s_{1n} \\ s_{21} & s_{22} & \ddots & s_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ s_{n1} & s_{n2} & \ldots & s_{nn} \end{pmatrix},$$

where $s_{ij} = \operatorname{Tr}(F_i R_j)$ for all i,j. Therefore, as noted above, for ρ in the range of Φ we have $\Phi(\rho) = \sum_k \vec{a}_k R_k = \sum_k (\vec{s}_k \cdot \vec{c})R_k$, where $\vec{s}_k = (s_{kj})$ is the kth row vector of S.

A Holevo form of an entanglement breaking channel is determined exclusively by the choice of the set of $\{F_i\}$ and $\{R_i\}$ and as such, is not unique. Thus, there can be different Holevo forms of such a channel. The stochastic matrix depends on our choice of the Holevo form and therefore is not uniquely determined by the channel. This leads to a natural question: what properties of the stochastic matrix are invariant under the choice of Holevo form? The following result shows how the structure of any such stochastic matrix representation is closely related to the structure of the channel itself.

Theorem 4.1.1. *Let Φ be an entanglement breaking channel on $M_n(\mathbb{C})$. Suppose S is a stochastic matrix defined by operators F_k, R_k that define a Holevo form for Φ. Then the Jordan canonical forms of Φ and S are the same, except possibly on blocks that correspond to zero eigenvalues.*

Proof. Suppose we have the Holevo representation $\Phi(\rho) = \sum_{k=1}^{r} \mathrm{Tr}(F_k \rho) R_k$. We begin by looking at Φ as a linear operator on $M_n(\mathbb{C})$. Consider a matrix representation $[\Phi]$ of Φ in a fixed basis for $M_n(\mathbb{C})$ of the matrix units $E_{ij} = |i\rangle\langle j|$ defined by a fixed orthonormal basis for $\mathbb{C}^n$. For $X \in M_n(\mathbb{C})$, let $\mathrm{vec}(X)$ be the n^2-tuple of coordinates for X in this basis viewed as a column matrix. Put

$$R_k = \sum_{i,j} r_{ij}^{(k)} E_{ij} \quad \text{and} \quad F_k = \sum_{i,j} f_{ij}^{(k)} E_{ij},$$

for some scalars $r_{ij}^{(k)}, f_{ij}^{(k)}$, so that $\langle i| F_k |j\rangle = f_{ij}^{(k)}$, $\langle i| R_k |j\rangle = r_{ij}^{(k)}$, and $\mathrm{vec}(R_k) = (r_{ij}^{(k)})_{i,j}$, $\mathrm{vec}(F_k) = (f_{ij}^{(k)})_{i,j}$.

Now let A be the $n^2 \times r$ matrix whose kth column is $\mathrm{vec}(R_k)$ for all $1 \leq k \leq r$, and let B be the $r \times n^2$ matrix whose kth row is the transposed column matrix $\mathrm{vec}(F_k^T)^T$.

We first claim that $[\Phi] = AB$. Indeed, note that the $s_1 = (i_1, j_1)$, $s_2 = (i_2, j_2)$ entry of the $n^2 \times n^2$ matrix AB is, $(AB)_{s_1 s_2} = \sum_{k=1}^{r} r_{i_1 j_1}^{(k)} f_{j_2 i_2}^{(k)}$, whereas the s_1, s_2 entry of $[\Phi]$ is given by,

$$\begin{aligned} \langle \Phi(E_{i_2 j_2}) | E_{i_1 j_1} \rangle &= \sum_{k=1}^{r} \mathrm{Tr}(F_k E_{i_2 j_2}) \, \mathrm{Tr}(E_{j_1 i_1} R_k) \\ &= \sum_{k=1}^{r} \langle j_2| F_k |i_2\rangle \langle i_1| R_k |j_1\rangle \\ &= (AB)_{s_1 s_2}. \end{aligned}$$

On the other hand, we claim that the stochastic matrix S defined by the operators F_k, R_k

satisfies $S = BA$. To see this, fix a pair $1 \leq k_1, k_2 \leq r$ and observe that

$$(BA)_{k_1k_2} = \text{vec}(F_{k_1}^T)^T \text{vec}(R_{k_2}) = \sum_{i,j} f_{ij}^{(k_1)} r_{ji}^{(k_2)} = \text{Tr}(F_{k_1} R_{k_2}) = (S)_{k_1k_2}.$$

Thus we have $[\Phi] = AB$ and $S = BA$, and we can now apply the classical Flanders Theorem [8] that relates the Jordan forms of matrix products AB and BA as claimed in the theorem statement. □

We obtain a consequential result of Theorem 4.1.1 as follows;

Corollary 4.1.2. *Let Φ be an entanglement breaking channel on $M_n(\mathbb{C})$. Then, the non-zero spectrum (that is, the set of eigenvalues) of Φ and that of any of its stochastic matrix representations S are the same, including multiplicities.*

Example 4.1.3. *Given the completely depolarizing channel on $M_n(\mathbb{C})$ which satisfies*

$$\Phi_{\text{CD}}(\rho) = \frac{\text{Tr}(\rho)}{n} I_n$$

for all ρ, we have Holevo operators $F_1 = I_n$ and $R_1 = \frac{1}{n} I_n$. In this trivial case, with the notation of the theorem, A is the $n^2 \times 1$ matrix with n entries of $\frac{1}{n}$ corresponding to the diagonal matrix units and 0's elsewhere, and B is the $1 \times n^2$ matrix with n entries of 1 in the same coordinate positions and 0's elsewhere. Here $S = BA = (1)$ and $[\Phi] = AB$ is the corresponding rank-1 matrix representation of the channel.

Example 4.1.4. *Another illustrative example is given by the map-to-diagonal channel Λ on $M_n(\mathbb{C})$, written as $\Lambda(\rho) = \text{diag}(\rho)$ such that*

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \dots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \dots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \dots & \rho_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} \rho_{11} & 0 & \dots & 0 \\ 0 & \rho_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_{nn} \end{pmatrix}.$$

Viewed as a map on operators represented as matrices in a fixed orthonormal basis $\{|k\rangle\}_{k=1}^n$, this is a special type of quantum classical channel with Holevo form given by $F_k = |k\rangle\langle k| = R_k$. In this case the factored matrices are related as $B = A^*$, with A as the $n^2 \times n$ matrix whose n columns are the vec *representations of the projections $|k\rangle\langle k|$, and the stochastic matrix construction yields the $n \times n$ identity matrix $S = BA = I$.*

In a similar fashion, we seek to generate entanglement breaking channels from stochastic matrices, specifically, quantum-classical(QC) channels. Recall that a channel is QC if it satisfies the Holevo form in Equation (3.1) and each density operator $R_k = |k\rangle\langle k|$ is a rank one projection with the set $\{|k\rangle\}_{k=1}^n$ forming an orthonormal basis for $\mathbb{C}^n$.

Let $S = (s_{ij})$ be an $n \times n$ (column) stochastic matrix, with each $s_{ij} \geq 0$ and $\sum_{i=1}^n s_{ij} = 1$ for all $1 \leq j \leq n$. Let $\{|k\rangle\}_{k=1}^n$ be a fixed orthonormal basis for $\mathbb{C}^n$. For each k, let F_k be the operator on $\mathbb{C}^n$ with $n \times n$ diagonal matrix representation in the fixed basis and whose diagonal entries form the kth row of S. That is,

$$S = \begin{pmatrix} \text{—} & \operatorname{diag}(F_1) & \text{—} \\ \text{—} & \operatorname{diag}(F_2) & \text{—} \\ & \vdots & \\ \text{—} & \operatorname{diag}(F_k) & \text{—} \end{pmatrix},$$

and explicitly, $F_k = \sum_{j=1}^n s_{kj} |j\rangle\langle j|$ for each k. Observe that every F_k is a positive operator and $\sum_{k=1}^n F_k = I$ by construction.

Now let $R_k = |k\rangle\langle k|$ for $1 \leq k \leq n$ and define a QC channel on $M_n(\mathbb{C})$ by $\Phi_S(\rho) = \sum_{k=1}^n \operatorname{Tr}(F_k \rho) R_k$. Finally, observe that if we apply our stochastic matrix construction above to the entanglement breaking channel Φ_S, we get the matrix S back again as follows: for $1 \leq i, j \leq n$, we have

$$\operatorname{Tr}(F_i R_j) = \sum_{k=1}^n s_{ik} \operatorname{Tr}(|k\rangle\langle k|\, |j\rangle\langle j|) = s_{ij}.$$

4.2 Primitivity of Quantum Channels

In this section, we make use of the stochastic matrix representations above to derive a matrix theoretic characterization of when entanglement breaking channels are primitive. We begin by reviewing some background.

We note that a quantum channel from $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ maps the compact convex set of density matrices to itself. Therefore Brouwer's theorem guarantees that every such quantum channel must have a density operator fixed point. This fixed point may or may not be globally attractive; for entanglement breaking channels, we can use the theory of stochastic matrices and the connections developed above to study this question.

There is a well known sufficient condition for a stochastic matrix to have a unique globally attractive fixed point among the probability vectors.

Definition 4.2.1. *A nonnegative matrix F is said to be primitive if there exists a $m \in \mathbb{N}$ such that F^m has all entries positive. The smallest such $m \in \mathbb{N}$ that accomplishes this is called the index of primitivity of F and is denoted as $p(F)$.*

The following is one of the most important classical results on primitive matrices [13, 21].

Theorem 4.2.2 (Perron-Frobenius Theorem)**.** *Let F be an $n \times n$ primitive matrix. Then there is a positive number λ_{max} such that all other eigenvalues of F satisfy $|\lambda| < \lambda_{max}$ and the eigenspace associated with λ_{max} is one-dimensional. Moreover, there is an eigenvector v of F with eigenvalue λ_{max} such that all coordinates of v are positive, and any nonnegative eigenvector of F is a multiple of v.*

The corollary of this result is that if the associated stochastic matrix (which will satisfy $\lambda_{max} = 1$) of an entanglement breaking channel is primitive then its fixed point is globally attractive. We note the following equivalent condition for a nonnegative matrix to be primitive.

Proposition 4.2.1. *Let A be a nonnegative matrix inside $M_n(\mathbb{R})$. Then A is primitive if and only if for every nonnegative nonzero $x \in \mathbb{R}^n$, there exists $m > 0$ such that $A^m x$ has all of its entries strictly positive.*

Proof. Suppose A is primitive. Then by definition 4.2.1, there exist an $m \in \mathbb{N}$ such that A^m has all entries positive. By that, A^m is a strictly positive matrix $A^m > 0$. Thus, suppose that $x \in \mathbb{R}^n$ is nonnegative and nonzero. Then, the product $A^m x$ must be strictly positive. Therefore, it must have all its entries strictly positive.

Conversely, let $x \in \mathbb{R}^n$ be nonnegative and nonzero and take some $m > 0$ such that $A^m x$ has all its entries strictly positive. Choose $x = e_k$, where e_k is the vector with 1 in the kth entry and 0's elsewhere. Then $A^m e_k$ is just the kth column of A^m with all entries strictly positive. Thus, A^m becomes

$$A^m = \begin{pmatrix} | & | & & | \\ A^m e_1 & A^m e_2 & \cdots & A^m e_n \\ | & | & & | \end{pmatrix}.$$

As such, A^m has strictly positive entries and therefore implying that A is primitive. □

Motivated by this, the concept of a primitive quantum channel was introduced in [28] as follows (see also [25] for more recent work).

Definition 4.2.3. *Let Φ be a quantum channel, then Φ is said to be primitive if there exists $m > 0$ such that $\Phi^m(\rho)$ is positive definite for all density matrices ρ. The smallest $m \in \mathbb{N}$ which accomplishes this is called the index of primitivity of Φ and is denoted as $q(\Phi)$.*

We present the following result connecting these two notions of primitivity for entanglement breaking channels. The proof relies on the stochastic matrix representations discussed in the previous section. Recall that we have made the standard assumption that none of the F_k's in the POVM in the Holevo form are zero.

Theorem 4.2.4. *Let Φ be an entanglement breaking channel and S be the stochastic matrix representation associated to the Holevo form $\Phi(X) = \sum_k \mathrm{Tr}(F_k X) R_k$. Then Φ is a primitive channel if and only if S is a primitive stochastic matrix and $\sum_k R_k$ is positive definite.*

Proof. Suppose S is primitive with index of primitivity m and $\sum_k R_k$ is positive definite. Let ρ be an arbitrary density matrix and let w be the vector with entries $\mathrm{Tr}(F_k\rho)$ so $\Phi(\rho) = \sum_k w_k R_k$. Now let $v = S^m w$ and note that all entries of v must be strictly positive. Since we have the inequalities $\Phi^{m+1}(\rho) = \sum_k v_k R_k \geq (\min_k v_k) \sum_k R_k > 0$, it follows that Φ is a primitive quantum channel with index of primitivity at most $m + 1$.

For the converse suppose Φ is primitive and let $m = q(\Phi)$. Then for any j, $\Phi^m(R_j)$ is positive definite. Let $w = S^m e_j$, then $\Phi^m(R_j) = \sum_k w_k R_k$. Since

$$(\max_k w_k) \sum_k R_k \geq \sum_k w_k R_k > 0,$$

it follows that $\sum_k R_k$ is positive definite. Now let $x = Sw = S^{m+1}e_j$. Then

$$x_i = \sum_k \mathrm{Tr}(F_i R_k) w_k = \mathrm{Tr}\left(F_i (\sum_k w_k R_k) \right).$$

As $\sum_k w_k R_k = \Phi^m(R_j)$ is positive definite and F_i is (nonzero) positive semidefinite, we have $x_i > 0$ and $S^{m+1}e_j$ has all positive entries. Since j was arbitrary, it follows that S is primitive with index of primitivity less than or equal to $m + 1$. □

Remark 4.2.5. *Observe that both conditions in the hypotheses of the theorem are indeed required to describe primitivity of an entanglement breaking channel. If $R = \sum_k R_k$ is not positive definite, then its nullspace is non-zero, and hence the intersection of the nullspaces of the R_k is non-zero (as each $R_k \leq R$), which implies that for any ρ, the density operator $\Phi(\rho)$ is not invertible and Φ cannot be primitive. Moreover, the quantum-classical channels* *discussed in the previous section, those implemented by a stochastic matrix (which will also be the matrix obtained through the Holevo representation), will not be primitive if the matrix is not primitive, given how the channel's iterative behaviour is so closely tied to that of the matrix for that subclass of channels.*

Let us point out a large class of entanglement breaking channels that satisfy the conditions of the theorem.

Corollary 4.2.6. *Let $\Phi(\rho) = \sum_k \mathrm{tr}(F_k \rho) R_k$ be an entanglement breaking channel such that $\sum_k R_k$ and F_k, for all k, are positive definite operators. Then Φ is a primitive channel.*

Proof. Each F_k being positive definite (and hence also invertible) together with each R_k being a density matrix, implies $\mathrm{Tr}(F_j R_k) > 0$ for all j, k. Thus, the associated stochastic matrix S is primitive, and so by the theorem Φ is a primitive channel.

Note in this case the primitivity indices are both equal to one; the stochastic matrix S satisfies $p(S) = 1$ essentially by definition, and the channel satisfies $q(\Phi) = 1$ as $\Phi(\rho)$ will be positive definite for every density matrix ρ from the given properties of the F_k and $\sum_k R_k$. □

Based on the proof of the theorem above, we can also give the following statement on the relationship between the primitivity indices of a channel and its stochastic matrices.

Corollary 4.2.7. *Let Φ be a primitive entanglement breaking channel and let S be one of its (primitive) stochastic matrix representations. Then,*

$$|q(\Phi) - p(S)| \leq 1.$$

Proof. If S is primitive, and $\sum_k R_k$ is positive definite, then the proof of Theorem 4.2.4 shows that $q(\Phi) \leq p(S) + 1$. On the other hand, if Φ is primitive, then that proof also shows that $p(S) \leq q(\Phi) + 1$. □

Remark 4.2.8. *We note that equality of these two primitivity indices is satisfied for the channels of Corollary 4.2.6, where $q(\Phi) = p(S) = 1$. The same is true for the primitive channels from the special class of quantum-classical channels considered in the previous section, where primitivity of the channel is easily seen to be equivalent to primitivity of the defining stochastic matrix via Theorem 4.2.4. This was first observed in [28], where the two notions were also connected with the size of operator spaces spanned by products of the channel's Kraus operators.*

The two primitivity indices can be different, however, as the following pair of examples show.

Example 4.2.9. *Consider the 'single-qubit' entanglement breaking channel $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ with Holevo form defined by $R_1 = |0\rangle\langle 0|$, $R_2 = |1\rangle\langle 1|$, and $F_1 = |+\rangle\langle +|$, $F_2 = |-\rangle\langle -|$ where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Then observe that*

$$S = \left(\operatorname{Tr}(F_i R_j)\right)_{i,j} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and so S is primitive with $p(S) = 1$. We also have $R_1 + R_2 = I$ is positive definite, and hence by the theorem above we know Φ is primitive. However, note that $\Phi(|-\rangle\langle -|) = |1\rangle\langle 1|$ is not positive definite, and so $q(\Phi) \gneq 1$. In fact, we know $q(\Phi) = 2$ from the above corollary as $q(\Phi) \leq p(S) + 1 = 2$. We can observe this directly by computing that for all single-qubit density matrices ρ, we have $\Phi^2(\rho) = \frac{1}{2}I$, and hence Φ^2 is the completely depolarizing channel which is of course primitive.

On the other hand, we can show the other primitivity index inequality is sharp by using the following Holevo form for the single-qubit completely depolarizing channel $\Phi = \Phi_{\mathrm{CD}}$. Let $F_1 = \frac{1}{2}|0\rangle\langle 0|$, $F_2 = \frac{1}{2}|1\rangle\langle 1|$, $F_3 = \frac{1}{2}I$, and $R_1 = R_2 = |0\rangle\langle 0|$, $R_3 = |1\rangle\langle 1|$. Thus, $q(\Phi) = 1$. However, the stochastic matrix associated with this decomposition is

$$S = \left(\operatorname{Tr}(F_i R_j)\right)_{i,j} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and so $p(S) \gneq 1$. The inequality of the corollary tells us $p(S) = 2$ as $p(S) \leq q(\Phi) + 1 = 2$, and this can be observed directly by noting that $S^2 > 0$.

4.3 Wielandt's Inequality

For every $n \times n$ primitive matrix $A \in M_n(\mathbb{C})$ with index of primitivity $p(A)$. Wielandt's inequality [32] states that for every primitive matrix

$$p(A) \leq n^2 - 2n + 2.$$

This bound is optimal and solely depends on the dimension of A. Thus, it takes account of the matrix elements of A. Wielandt's inequality has been applied widely in different fields including number theory[2] and numerical analysis [29]. This idea was relayed to quantum channels to determine an upper bound to the number of times a quantum channel is applied to a density matrix in order to obtain another that is full rank. Here, we briefly discuss some of the results from [28].

Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a quantum channel with d independent Kraus operators $\{V_i \in M_n(\mathbb{C})\}$ such

$$\Phi(X) = \sum_{i=1}^{d} V_i X V_i^*.$$

We denote by $S_k(A) \subseteq M_n(\mathbb{C})$ the linear space spanned by all possible products of exactly k Kraus operators, $V_{i_1} V_{i_2} \cdots V_{i_n}$. By the Choi-Jamiołkowski isomorphism between the quantum channel Φ and its Choi matrix C_Φ, we have $\operatorname{rank}(C_{\Phi^k}) = \dim(S_k(A))$. We can also define $S_k(A, \phi) := S_k(A) |\phi\rangle \subseteq \mathbb{C}^n$ as the space spanned by the vectors $V_{i_1} V_{i_2} \cdots V_{i_n} |\phi\rangle$. Thus, it follows that $\operatorname{rank}(\Phi^k(|\phi\rangle\langle\phi|)) = \dim(S_k(A, \phi))$.

Proposition 4.3.1. *A quantum channel Φ is primitive if there exists some $k \in \mathbb{N}$ such that for all $|\phi\rangle \in \mathbb{C}^n$, $S_k(A, \phi) = \mathbb{C}^n$.*

This is an equivalent definition for that in Proposition 4.2.1. As such, if Φ is primitive, then for every $m \in \mathbb{N}$, the map Φ^m is also primitive for all $k \geq q(\Phi)$. We can denote by $i(\Phi)$, the minimum natural number k such that the linear space spanned by all possible products of exactly k Kraus operators is equal to $M_n(\mathbb{C})$. That is,

$$i(\Phi) = \min\{k \in \mathbb{N} : S_k(A) = M_n(\mathbb{C})\}.$$

This equally translates to mean that $i(\Phi)$ is the minimum number for which the Choi matrix of Φ^k has full rank for all $k \geq i(\Phi)$.

Through further analysis of the sets $S_k(A)$, one can also prove the following.

Proposition 4.3.2. *For every quantum channel Φ we have that $q(\Phi) \leq i(\Phi)$.*

It is shown in [32] that $i(\Phi) \leq (n^2 - d + 1)n^2$ and as such, we have the quantum bound

$$q(\Phi) \leq (n^2 - d + 1)n^2.$$

4.4 Holevo Rank of Entanglement Breaking Channels

It follows that our result relating the primitivity indices of an entanglement breaking channel and its stochastic matrix representations is optimal for the full class of such channels, in the sense that it identifies bounds between the indices which can be saturated. These indices are related to the following notion for entanglement breaking channels, which is natural to consider, but does not appear to have been formally investigated previously.

Definition 4.4.1. *Given an entanglement breaking channel Φ, we define the Holevo rank of the channel to be the minimal number r of pairs $\{F_k, R_k\}_{k=1}^r$ that make up a Holevo form for the channel, $\Phi(\rho) = \sum_{k=1}^r \mathrm{Tr}(F_k \rho) R_k$.*

As a class of channels, entanglement breaking channels can be characterized by properties of their Choi matrix; namely, a channel is entanglement breaking if and only if its Choi matrix is not entangled, or separable. Analogous to this property, we can also frame the Holevo rank in terms of properties of the Choi matrix. Indeed, observe that if Φ has a Holevo form determined by $\{F_k, R_k\}_{k=1}^r$, then the Choi matrix satisfies:

$$J(\Phi) = \sum_{k=1}^{r} \tau(F_k) \otimes R_k,$$

where τ is the transpose map. Thus, the Holevo rank is also the minimal size of such a sum decomposition of the Choi matrix, with the F_k forming a POVM and each R_k a density matrix.

We can use the stochastic matrix representations and our results above to bound the primitivity index of an entanglement breaking channel in terms of the Holevo rank as follows.

Corollary 4.4.2. *Let Φ be a primitive entanglement breaking channel on $M_n(\mathbb{C})$ with Holevo rank r. Then we have,*

$$q(\Phi) \leq r^2 - 2r + 3.$$

Proof. First recall the classical Wielandt inequality [32, 21, 13], which for a $r \times r$ primitive matrix S gives $p(S) \leq r^2 - 2r + 2$. The result thus immediately follows from Corollary 4.2.7. □

Remark 4.4.3. *One could compare this upper bound to the quantum version of Wielandt's inequality from [28], which was established for general (not necessarily entanglement breaking) channels as: $q(\Phi) \leq (n^2 - d + 1)n^2$, where d is the number of Kraus operators required to* *implement the channel. Additionally, further investigation is warranted on the relationship between these quantities and the recently studied notion of entanglement breaking rank [23].*

Chapter 5

Conclusion

We discussed the fundamental concepts of quantum information and saw the description of entanglement breaking channels and their Holevo form. We also considered some early significant results of these channels which motivated our results.

We demonstrated how the Holevo form induces stochastic matrices of such channels. We also discovered that structures such as the nonzero spectrum of the channels and their matrices are the same, while the Jordan canonical forms of the stochastic matrices and the channels are very much alike. By considering the iterative behaviour of these matrices, we described when these channels are primitive and found that the primitivity of the channels are dependent on the primitivity of the matrices. In addition to this, we realized that the primitivity index of entanglement breaking channels and their associated matrices differ by a unit. Previous studies showed that an upper bound could be found for the primitivity index of quantum channels. Motivated by that, we introduced the Holevo rank of entanglement breaking channels and obtained a new and improved bound for their primitivity indices. Ideas and concepts were illustrated with examples. The new results contained in this thesis were taken from the recently published paper [1].

It would be interesting to further investigate what the maximal Holevo rank of an entanglement breaking channel from $M_m(\mathbb{C})$ to $M_n(\mathbb{C})$ could be for fixed m and n. We can also extend our work by examining how the Holevo rank relates to the entanglement breaking rank of such quantum channels. This entanglement breaking rank has been defined to be the minimum number of rank one Kraus operators required to express an entanglement breaking channel in all possible Kraus representations. Equivalently, due to the Choi-Jamiołkowski isomorphism between quantum channels and states, the entanglement breaking rank is exactly equal to what is referred to as the separability rank of the Choi matrix, which is defined as the minimal number of product states in a sum decomposition of the Choi matrix. As we have already established that the Holevo rank can be expressed in terms of properties of the Choi matrix, it would be interesting to explore the properties of the Choi matrix that relates the entanglement breaking rank and the Holevo rank. From there, we can examine how this relationship affects the quantum bound for the primitivity of the quantum channels.

Finally, we know that the Holevo form of an entanglement breaking channel and the associated stochastic matrix are not unique. Nevertheless, this raises the question: Is there a choice of these stochastic matrices that makes the correspondence between the Holevo form and the matrices locally continuous? In other words, are two entanglement breaking channels close if and only if they have stochastic matrices that are close?

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