University of Guelph Numeracy Project

About Derivatives

UNIVERSITY of GUELPH

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About Derivatives

What is a DERIVATIVE?

- Using limits, we're going to find the rates of change and slopes of tangent curves which will be the building blocks to investigate the derivative.
- The derivative is the mathematical process of evaluating limits to find an instantaneous rate of change or the slope of a curve at any point.

Rates of Change

Average Rate of Change

- Calculus is the study of rates of change for variable quantities. An important tool in that is the derivative, but first we must discuss rates of change.
- The average rate of change of a quantity over an interval is given by the following ratio:

$$\text{Average rate of change} = \frac{\text{amount of change in the quantity over the interval}}{\text{amount of change in the interval}}$$

- We'll present the average rate of change now as a function:
- The average rate of change of a function f(x) between points (a, f(a)) and (b, f(b)) is given by:

$$\frac{f(b) - f(a)}{b-a}$$

Instantaneous Rate of Change

- We want to know the rate of change at one specific point, but now we don’t have two points to compare.
- The instantaneous rate of change of f(x) when x = a is given by:

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Slopes of Curves

- Unlike lines however, the slope of curves depends where on the curve you are. The slope at $x = 1$ can be drastically different than $x = -1$. For example, the graph $y = x2$, the slope at $x = -1$ will be the negative slope at $x = 1$.

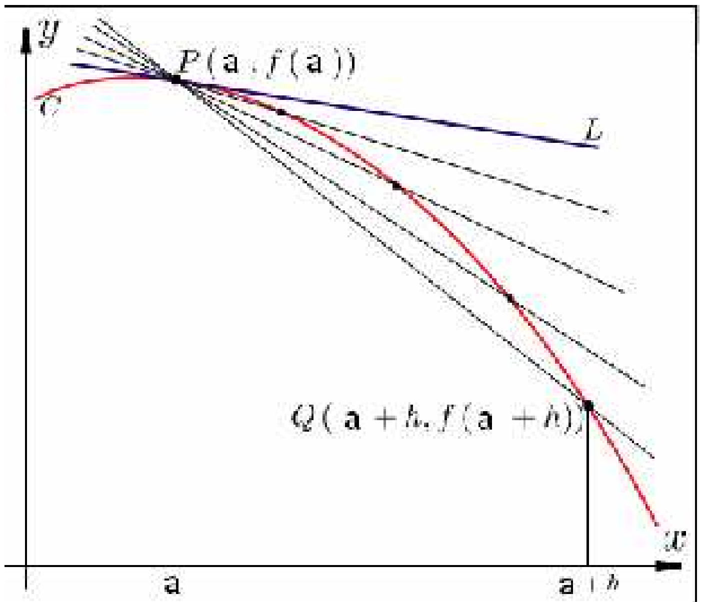
- Definition: The tangent line to a function at a point can touch the curve only once in an interval around that point.

- Definition: The slope of a curve at a point on the curve is the slope of the tangent line to the curve at that point

- We need two points to find the slope of the line, but with a tangent line we'll only know one set $(a, f(a))$.

- We set another point on the curve $(a + h, f(a + h))$, and we find the slope of the secant line.

- And just like last time, we calculate the slope as normal, and take the limit as h approaches infinity. Giving us the slope of the tangent.



- The slope of the curve f(x) at the point (a,f(a)) is given by:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Which is the exact same as the definition of instantaneous rate of change.

The Derivative

- Now that we've looked at instantaneous rate of change and the slope of the tangent, we're ready to introduce the Derivative.

- Definition: The derivative of f(x) is found by evaluating (if the limit exists)

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- If the derivative exists, then f(x) is said to be differentiable. The process of finding a derivative is called differentiation.

- The derivative of y = f(x) is represented by any one of the following notations:
  · f '(x)
  · y'
  · dy/dx
  · Dxy
  · d/dx f(x)

  In all these cases we're taking the derivative of the function y (or f(x)), with respect to the variable x.

Nonexistence of Derivatives

- There are times when the derivative doesn't exist, and it's similar to before with limits. If the limit doesn't exist then the derivative won't.
- Case1: If f(x) is discontinuous at a point then the derivative does not exist at that point.

- Case 2: If the graph of f(x) has a sharp corner at some point, the derivative does not exist at that point

- Case 3: If the graph of f(x) has a vertical tangent line at some point, the derivative does not exist at that point.

Properties of Derivatives

- We know how to calculate the derivative using limits, however this can be tiresome. We wouldn't want to find the derivative of f(x) = x6 + 5x3 - 3x + 2, things just get to messy and it takes too long, there must be a faster way!

- · The derivative of a constant function is zero because the graph of a constant function is a horizontal line, which has slope of zero.
  · The derivative of a linear function is its slope
  · For simplification, we've been finding the limits whose highest power was 2, and the derivative was always linear.

Power Rule

- Power Rule for Derivatives:
  If $f(x) = x^n$, then $f'(x) = D_x(x^n) = nx^{n-1}$, where n is any real number.

Properties

- We've seen the power rule, here are two important results from it:
  1) If f(x) = k, then f '(x) = 0 where k is a constant
  2) If f(x) = mx + b then f '(x) = m (the slope of the line)

- And here are three more important properties

  Sum Rule: $D_x(f(x) + g(x)) = f'(x) + g'(x)$
  Difference Rule: $D_x(f(x) - g(x)) = f'(x) - g'(x)$
  Constant Multiple Rule: $D_x(c * f(x)) = c * f'(x)$ (c is a constant)

Glossary

Average Rate of Change: Found by calculating the amount of change in the quantity over the interval over the amount of change in the interval

Instantaneous Rate of Change: The rate of change at a particular moment.

Secant Line: Line connecting two points of a function.

Slope of a Curve: The slope of a curve at a point on the curve is the slope of the tangent line to the curve at that point.

Tangent Line: The tangent line to a function at a point can touch the curve only once in an interval around that point

References

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