University of Guelph Numeracy Project

About Permutations and Combinations: Examples

UNIVERSITY of GUELPH

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About Permutations and Combinations: Examples

Basics

- Example: A standard Canadian postal code contains 3 letters and 3 numbers arranged as follows.

  Letter-Number-Letter Number-Letter-Number

  How many different postal codes are there?

- Answer: First we note that there are 26 letters of the alphabet to choose from, and 10 numbers to choose from. We can list some of the early ones.

  A0A 0A0
  A0A 0A1
  A0A 0A2
  .... etc

- This will take lots of time, luckily there is a quicker way.

  Since there are 26 possible letters, we have 26 outcomes for the first letter, and each letter after. Similarly there are 10 numbers to choose from. So then the number of possible postal codes becomes.

  26*10*26 * 10*26*10 = 17,576,000

  So there are 17,576,000 different possible postal codes.

Product Rule

- A student has a choice of 4 classes in the morning and 5 classes to choose from in the afternoon, how many different combinations of classes are there for the student?

  - Answer: There are 4 possibilities for the morning classes, and 3 for afternoon classes, thus $n_1 = 4$, and $n_2 = 5$ and by the product rule.

Total Combinations = $n_1 * n_2$ = 4*5 = 20

There are 20 possible combinations of classes.

Sum Rule

- A student has a choice of 4 classes in the morning and 2 classes to choose from in the afternoon, how many different classes can the student sign up for?

- Answer: There are 4 classes for morning classes, and 2 classes for afternoon classes. Letting e_1 being morning classes and e_2 being afternoon classes. Thus $n_1 = 4$ and $n_2 = 2$, and by the Sum Rule Theorem

  Total classes = $n_1 + n_2$ = 5 + 3 = 8

  There are 8 total classes for the student to take.

Permutations

- Example: How many different ways can we arrange the numbers 1,2,3?

- Answer: There aren't many, we can list them here: 123,132,213,231,312,321.

  There are 6 possibilities, note we had three numbers to arrange. For the first spot we had 3 possibilities, then 2 possibilities, then just 1 number left.

- 4! = 4*3*2*1 = 24
  6! = 6*5*4*3*2*1 = 720
  10! = 10*9*8*7*6*5*4*3*2*1 = 3628800

- How many permutations of ABCDEFG contain the block ABC?

- Answer: We already know there are 7! ways to arrange the 7 objects (! = 5040). However, how many of those have ABC together? ABDCEFG is a valid permutation, but it doesn't have ABC together.

- We consider ABC as one element, therefore the number of elements we have to order is 5, they are ABC, D, E, F, G. Since there are 5 elements, then there are 5! Ways to arrange them, 5! = 120.

  There are 120 ways to arrange those letters with ABC together.

- In a horse race with 6 horses, how many different ways are there to select the top 3 horses?

- Answer: Again we use the product rule, we have 6 ways to pick the first place horse, 5 ways to pick the second, and 4 ways to pick the 3^{rd} place horse.

  Total Combinations = 6*5*4 = 210

- There are 210 ways to arrange 3 objects when you have a total of 7 objects.

Combinations

- How many different committees of three students can be formed from a group of four students?

- Answer: For simplification, let’s label the students “ABCD”, A for student one, B for student two, etc. In this case picking student A then students B (AB) is the same as picking student B then student A (BA). In other words, order doesn’t matter.

  Thus, the total number of committees one can create is 4.

- The worst hockey team is allowed to draft any three players from the top 3 teams. The top 3 teams have 20, 23, 21 players respectively. How many different ways can the worst team choose 3 players?

- Solution: We have to first recognize that order doesn't matter. There is nothing in the question saying first we need a left winger. We are also free to choose any players we want, we don't have to pick one player from each team or anything, so the total number of players we have it 64, and we are choose 3, so:

  C(64,3) = 64! / (64-3)! (3!) = 64! / (61!)(3!) = 83328

  There are 83328 ways to select the 3 players.

Pascal's Triangle

Binomial Theorem

- Consider the following expansion

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)(x + y)\\ &= (xx + xy + yx + yy)(x + y)\\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

- With the Binomial Theorem it becomes:

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

- Which can of course be simplified by calculating C(3,0), C(3,1), etc.

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Pascal's Identity/Triangle

- $\binom{7}{2} + \binom{7}{3} = \binom{8}{3}$
- C(7,2) = 21 , C(7,3) = 35, C(8,3) = 56 (21+35 = 56)

Examples

Example 1

- In how many ways can we select two students from a group of seven students to stand in a line for a picture? In how many ways can we arrange all seven of these students in a line for a picture?

- Solution: We first note that order matters in this problem. For the first spot in the photo we have seven choices, once that student is chosen we have six, or simply:

  P(7,2) = 42

- For all seven students we keep going after the second spot, which gets us:

  7! = 5040.

Example 2

- How many ways are there to select 4 players from a 8-member tennis team to compete in an upcoming tournament?

- Solution: Order doesn't matter here, so n = 8 and r = 4, thus plugging into out equation.

  C(8,4) = 8!/(4!)(8-4)! = 8!/(4!)(4!) = 70

Example 3

- What's the coefficient of $x^{12}y^{13}$ in $(x + y)^{25}$

- Solution: Using the Binomial Theorem, n = 25, and j =13.

  And thus the coefficient is C(25,13) = 25!/(13!)(12!)
  = 5200300

Example 4

- What is the expansion of (x - y)4

- Solution: Note that the equation can be written as $(x + (-y))^4$
  And thus, by the Binomial Theorem
  $(x - y)^4 = C(4,0)(-1)^0x^4 + C(4,1)(-1)x^3y + C(4,2)(-1)^2x^2y^2 + C(4,3)(-1)^3xy^3 + C(4,4)(-1)^4y^4$

  $(x - y)^4 = x^4 - x^3y + x^2y^2 - xy^3 + y^4$

Example 5

- The English alphabet has 21 consonants and 5 vowels. How many strings of 6 letters contain:

- a) exactly one vowel
  b) exactly two vowels

- Solution a) We know there are 266 strings with length 6 and 216 with only consonants.

  There are 6 positions to place 1 vowel, and 5 vowels to pick. This will leave us with 5 free positions for consonants: 6*5*215

- Solution b) There are C(6,2) ways to select the 2 positions for the vowels, and 5 ways to pick each vowel. This leaves us 4 positions free for the consonants: 15*52214